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KODAI MATH. J.
20 (1997), 1–7

ON SPACE FORMS OF REAL GRASSMANN MANIFOLDS WHICH ARE ISOSPECTRAL BUT NOT ISOMETRIC

Dedicated to Professor Kiyosato Okamoto on the Occasion of his sixtieth birthday

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1. Introduction

Let M and N be compact connected Riemannian manifolds. We say M is isospectral to N if the associated Laplace Beltrami operator have the same eigenvalue spectrum. Does the spectrum of M determine the Riemannian structure of the manifold? Milnor [5] gave first counter example for the problem, 16-dimensional tori which are isospectral but not isometric. Many examples have appeared in the past decade. For recent references, refer to [2].

In [3] and [4], we gave many examples of isospectral non-isometric spherical space forms, lens spaces in [3] and spherical space forms with non-cyclic fundamental groups of type I in [4]. There are another space forms of Riemannian symmetric spaces. Let $G_{q,n}(\boldsymbol{R})$ be the real Grassmann manifold consisting of all q-dimensional linear subspaces of $\boldsymbol{R}^n$. Note that the dimension of $G_{q,n}(\boldsymbol{R})$ is $q(n-q)$. In this paper we consider space forms $\Gamma\backslash G_{q,n}(\boldsymbol{R})$ for constructing isospectral non-isometric examples. The space forms of real Grassmann manifolds are classified by Wolf in [7]. There are only a few even dimensional space forms $\Gamma\backslash G_{q,n}(\boldsymbol{R})$, so in this paper we consider *odd* dimensional space forms $\Gamma\backslash G_{q,2d}(\boldsymbol{R})$ where q is odd. The classification states that there is a one to one correspondence of odd dimensional spherical space forms $\Gamma\backslash S^{2d-1}$ and odd dimensional space forms of real Grassmann manifolds $\Gamma\backslash G_{q,2d}(\boldsymbol{R})$, $\Gamma\subset O(2d)$ (there are more space forms of real Grassmann manifolds when $q=d$). The correspondence is given by $\Gamma\backslash S^{2d-1}\rightarrow\Gamma\backslash G_{q,2d}(\boldsymbol{R})$.

We raised the following question;

For isospectral spherical space forms $\Gamma_1\backslash S^{2d-1}$, $\Gamma_2\backslash S^{2d-1}$ given in [3] *or* [4], *corresponding space forms $\Gamma_1\backslash G_{q,2d}(\boldsymbol{R})$, $\Gamma_2\backslash G_{q,2d}(\boldsymbol{R})$ are isospectral?*

The main result in this paper is that the above question is *yes* in cases Γ_1, Γ_2 are non-cyclic fundamental groups of type I given in [4].

THEOREM 5. *Let $\Gamma_1\backslash G_{q,2d}(\boldsymbol{R})$ and $\Gamma_2\backslash G_{q,2d}(\boldsymbol{R})$ be odd dimensional space*

Received August 23, 1995; revised November 14, 1996.

forms of real Grassmann manifolds with non-cyclic fundamental groups of type I and Γ_1, $\Gamma_2 \subset O(2d)$. Suppose Γ_1 and Γ_2 are irreducible and Γ_1 is isomorphic to Γ_2. Then $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$ is isospectral to $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$.

2. Isospectral manifolds

In [6] Sunada gave a method of a general construction for isospectral manifolds. In this section we give a variation of Sunada's Theorem for constructing isospectral space forms of real Grassmann manifolds.

Let M be a compact Riemannian manifold and Δ the Laplacian acting on the space of smooth functions on M. We denote $E_\lambda(M)$ the eigenspace with eigenvalue λ of Δ. Let $I(M)$ be the isometry group of M. Let G be a finite subgroup of $I(M)$. We say G is fixed point free if for each $g(\neq identity) \in G$, g acts fixed point freely on M. For a fixed point free finite group G, we have a smooth compact Riemannian manifold $G \backslash M$ with induced metric from M. Let G_1 and G_2 be finite subgroups in $I(M)$. We say G_1 is *almost conjugate* to G_2 if there is a bijection ϕ of G_1 onto G_2 satisfying that $\phi(g)$ is conjugate to g in $I(M)$ for each $g \in G_1$.

THEOREM 1. *Let G_1 and G_2 be finite fixed point free subgroups of $I(M)$. Suppose G_1 is almost conjugate to G_2. Then $G_1 \backslash M$ is isospectral to $G_2 \backslash M$.*

Proof. We consider $E_\lambda(M)$ as a representation space of $I(M)$.

$$\pi : I(G) \to \mathrm{Aut}(E_\lambda(M))$$

We denote $E_\lambda^i(M)(i=1,2)$ the subspace of $E_\lambda(M)$ consisting of functions fixed by the G_i. Then the eigenspace $E_\lambda(G_1 \backslash M)$ (resp. $E_\lambda(G_2 \backslash M)$) can be naturally identified with $E_\lambda^1(M)$ (resp. $E_\lambda^2(M)$). Then

$$\dim E_\lambda^i(M) = \frac{1}{|G|} \sum_{g \in G_i} \mathrm{Trace}(\pi(g)) \qquad (i=1,2).$$

Let ϕ be an almost conjugate map of G_1 onto G_2. Since g is conjugate to $\phi(g)$, we have

$$\mathrm{Trace}(\pi(g)) = \mathrm{Trace}(\pi(\phi(g))).$$

Thus

$$\dim E_\lambda^1(M) = \dim E_\lambda^2(M),$$

which means

$$\dim E_\lambda(G_1 \backslash M) = \dim E_\lambda(G_2 \backslash M). \qquad \square$$

3. Spherical space form with non-cyclic fundamental groups of type I

In this section, we review spherical space forms with non-cyclic fundamental groups of type I according to [7] and describe the pairs of almost conjugate non-cyclic groups of type I obtained in [4].

DEFINITION 1. A finite subgroup G of the orthogonal group $O(n)$ is said to be fixed point free if for any $g \in G(g \neq 1_n)g$ has not 1 for eigenvalue. A finite fixed point orthogonal representation of a finite group is fixed point free if it is faithful and its image is a fixed point subgroup of the orthogonal group. A finite group K is said to be fixed point free if K has a finite fixed point free orthogonal representation.

The following proposition is a fundamental property for the classification program of spherical space forms.

PROPOSITION 1 (See [7]). *Let K be a finite fixed point free group. Let π_1 and π_2 be fixed point free representations of degree $2d$. Then the spherical space forms $\pi_1(K)\backslash S^{2d-1}$ is isometric to $\pi_2(K)\backslash S^{2d-1}$ if and only if π_1 is equivalent to π_2 modulo automorphisms of K.*

A finite fixed point free group G is said to be of type I if all the Sylow subgroups of G are cyclic. A finite fixed point free group of type I is not so special becourse of the following.

PROPOSITION 2 (See [7]). *The fundamental group of every $(4k+1)$-dimensional spherical space form is of type I.*

For any integer m, K_m denotes the multiplicative group of residue classes modulo m of integers prime to m. The order of K_m is denoted by $\phi(m)$, so called Euler function. For two integers a and b, we denote by (a, b) the greatest common divisor of a and b.

We describe finite fixed point free groups of type I. Let m, n, d, n' and r be positive integers satisfying

$$(1) \qquad \begin{cases} ((r-1)n, m) = 1, \\ r^n \equiv 1 \pmod{m}, \\ d \text{ is the order of the residue class of } r \text{ in } K_m, \\ n = n'd, \\ n' \text{ is divisible by any prime divisor of } d. \end{cases}$$

For such integers m, n, d, n' and r, we have the finite group $\Gamma_d(m, n, r)$ of order $N = mn$ generated by two elements A and B with defining relations

$$(2) \qquad A^m = B^n = 1 \text{ and } BAB^{-1} = A^r.$$

Remark. The following four conditions are equivalent for the $\Gamma_d(m, n, r)$ (i) $\Gamma_d(m, n, r)$ is cyclic, (ii) $A = 1$, (iii) $r \equiv 1 \pmod{m}$, and (iv) $d = 1$. □

We define automorphisms of $\Gamma_d(m, n, r)$. Whenever s, t and u are integers with $(s, m) = 1 = (t, n)$ and $t \equiv 1 \pmod{d}$, we put

$$\psi_{s,t,u}(A) = A^s \text{ and } \psi_{s,t,u}(B) = B^t A^u. \tag{3}$$

Then we can see easily $\psi_{s,t,u}$ defines an automorphism of $\Gamma_d(m, n, r)$.

Proposition 3 (see [7]). *Let $K = \Gamma_d(m, n, r)$, and let $R(\theta)$ denote the rotation matrix on the plane;*

$$R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{4}$$

Given integers k, l with $(k, m) = 1 = (l, n)$, let $\pi_{k,l}$ be the representation of degree $2d$ of K defined by

$$\pi_{k,l}(A) = \begin{bmatrix} R(k/m) & 0 & \cdots & 0 \\ 0 & R(kr/m) & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & R(kr^{d-1}/m) \end{bmatrix} \tag{5}$$

and

$$\pi_{k,l}(B) = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & \ddots & I & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & & I \\ R(l/n') & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6}$$

where each matrix is a block matrix consisting of 2×2-matrices, I is the unit 2×2-matrices and all other components are zero. Then the $\pi_{k,l}$ is irreducible and a real representation of K is fixed point free if and only if it is equivalent to a sum of these representations $\pi_{k,l}$. $\pi_{k,l}$ is equivalent to $\pi_{k',l'}$ if and only if there exist numbers $e = \pm 1$ and $c = 0, 1, \ldots, d-1$ such that $k' \equiv kr^c$ (mod m) and $l' \equiv el \pmod{n'}$. $\pi_{k,l} \circ \psi_{s,t,u}$ is equivalent to $\pi_{sk,tl}$ where $\psi_{s,t,u}$ is the automorphism of K.

Remark. Any irreducible fixed point free representation of $\Gamma_d(m, n, r)$ has the same degree $2d$. □

The following two Lemmas are obtained in [4]. Their proofs are necessary for constructing explicit examples of isospectral non-isometric space forms of real Grassmann manifolds. So we give their proofs.

Lemma 1 (See [4]). *Let $K = \Gamma_d(m, n, r)$ be a finite fixed point free group of* *type I with $n' = d$. Then the number of isometry classes in $(2d-1)$-dimensional spherical space forms with the same fundamental groups K is at least 2 if and only if $d = 5$ or $d > 6$.*

Proof. Let $\pi_{k,l}$ and $\pi_{k',l'}$ be fixed point free representations of K. Then $\pi_{k,l}$ is equivalent to $\pi_{k',l'}$ modulo automorphisms if and only if there exists an integer t with $(t, n) = 1$, $t \equiv 1 \pmod d$ and $\ell \equiv \pm t\ell' \pmod{n'}$. Since $n' = d$, the number of isometry classes in $(2d-1)$-dimensional spherical space forms with the fundamental groups K is $\phi(d)/2$. Let $d = p^{e_1} p^{e_2} \ldots p^{e_k}$ be the prime decomposition of d. It is well known that

$$\phi(d) = (p^{e_1} - p^{e_1-1})(p^{e_2} - p^{e_2-1}) \ldots (p^{e_k} - p^{e_k-1}).$$

From this formula, it is easy to see that $\phi(d)/2 \geq 2$ if and only if $d = 5$ or $d > 6$. □

Remark. The proof of Lemma 2.5 in [4] is incorrect.

LEMMA 2 (See [4]). *For fixed $d \geq 2$, there are infinitely many finite fixed point free groups $\Gamma_d(m, n, r)$ of type I with $n' = d$.*

Proof. It is well known that there are infinitely many prime numbers of the form $kd+1$. Let $m = kd+1$ be a prime number. Then K_m is a cyclic group of order kd. So there exists an integer r whose order in K_m is d. Put $n = d^2$, then we have a finite fixed point free group of type I, $\Gamma_d(m, n, r) = \Gamma_d(m, d^2, r)$.□

THEOREM 2. *Let G, G' be finite fixed point free non-cyclic groups of type I in $O(2d)$. Suppose G, G' are irreducible and that G is isomorphic to G'. Then G is almost conjugate to G'.*

Proof. By Proposition 3, G, G' are isomorphic to a finite fixed point free group $\Gamma_d(m, n, r)$. We may assume $G = \pi_{1,\ell}(K)$ and $G' = \pi_{1,1}(K)$, where $\pi_{1,1}$ and $\pi_{1,\ell}$ are fixed point free representations of K as in Proposition 3. We define the map ϕ of G into G' by

$$\phi(\pi_{1,\ell}(A^s B^t)) = \phi(\pi_{1,1}(A^s B^{\ell t})).$$

Then ϕ is clearly one to one onto map. Then by the proof of Theorem 1 in [7], for each $g \in G$ the characteristic polynomial of g is identical to the characteristic polynomial of $\phi(g)$;

$$\det(z1_{2d} - g) = \det(z1_{2d} - \phi(g)). \tag{7}$$

This means that g is conjugate to $\phi(g)$ in $O(2d)$. □

Combining Lemma 1, Lemma 2 and Theorem 2, there are many pairs of almost conjugate but not conjugate non-cyclic groups of type I.

4. Odd dimensional space forms of real Grassmann manifolds which are isospectral but not isometric

The classification of space forms of real Grassmann manifolds is obtained in [7]. Let n and q be integers with $n \geq 4$, $0 < q < n$. Then $G_{q,n}(\boldsymbol{R})$ denotes the real Grassmann manifold over $\boldsymbol{R}$ of all q-dimentional linear subspaces of $\boldsymbol{R}^n$. The orthogonal group $O(n)$ acts naturally on $G_{q,n}(\boldsymbol{R})$. Furthermore we always have an isometry β:

$$(8) \qquad \beta : G_{q,n}(\boldsymbol{R}) \to G_{n-q,n}(\boldsymbol{R}) \quad \text{by} \quad \beta(P) = P^{\perp}.$$

In particular, $\beta \in I(G_{d,2d}(\boldsymbol{R}))$. There are only a few space forms of real Grassmann manifolds in even dimension, so we consider *odd* dimensional space forms $\Gamma \backslash G_{q,n}(\boldsymbol{R})$. Since the dimension of $\Gamma \backslash G_{q,n}(\boldsymbol{R})$ is $q(n-q)$, we denote an odd dimensional real Grassmann manifold by $G_{q,2d}(\boldsymbol{R})$ where q is odd.

THEOREM 3 (See [7]). *The isometry group $I(G_{q,2d}(\boldsymbol{R}))$ of odd dimensional real Grassmann manifold $G_{q,2d}(\boldsymbol{R})$ is*

$$I(G_{q,2d}(\boldsymbol{R})) = \begin{cases} O(2d) & \text{if } q \neq d, \\ O(2d) \cup \beta \cdot O(2d) & \text{if } q = d. \end{cases}$$

THEOREM 4 (See [7]). *Let M be an odd dimensional real Grassmann manifold. Then the isometry classes of manifolds $\Gamma \backslash M$, $\Gamma \in O(2d)$, are in one to one correspondence with the isometry classes of $(2d-1)$-dimensional spherical space forms. The correspondence is given by $\Gamma \backslash M \to \Gamma \backslash S^{2d-1}$.*

By Theorem 3 and Theorem 4, if $q \neq d$ then the isometry classes of odd dimensional space forms $\Gamma \backslash G_{q,2d}(\boldsymbol{R})$ are in one to one correspondence with the isometry classes of spherical space forms $\Gamma \backslash S^{2d-1}$. If $q = d$, there are another space forms $\Gamma \backslash G_{q,2d}(\boldsymbol{R})$ with Γ not containing in $O(2d)$. For details, see [7]. In this paper, we consider only $\Gamma \backslash G_{q,2d}(\boldsymbol{R})$ with $\Gamma \subset O(2d)$.

Combining Theorem 1, Theorem 2 and Theorem 4, we have

THEOREM 5. *Let $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$ and $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$ be odd dimensional space forms of real Grassmann manifolds with non-cyclic fundamental groups of type I and Γ_1, $\Gamma_2 \subset O(2d)$. Suppose Γ_1 and Γ_2 are irreducible and Γ_1 is isomorphic to Γ_2. Then $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$ is isospectral to $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$.*

THEOREM 6. *Let $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$ and $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$ be odd dimensional space forms of real Grassmann manifolds with non-cyclic fundamental groups of type I and Γ_1, $\Gamma_2 \subset O(2d)$. Suppose Γ_1 is isomorphic to Γ_2 and d is odd prime. Then $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$ is isospectral to $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$.*

Proof. If d is odd, then $2d-1 \equiv 1 \pmod 4$. Combining Proposition 2 with Theorem 4, Γ_1, Γ_2 are of type I. Moreover if d is odd prime and Γ_1, Γ_2 are not cyclic, then Γ_1, Γ_2 are irreducible by Proposition 3. Hence by Theorem 5, we have $\Gamma_l \backslash G_{q,2d}(\boldsymbol{R})$ is isospectral to $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$. □

Combining Theorem 5 with Lemma 1 and Lemma 2, we have

THEOREM 7. *Let $d=5$ or $d>6$ and let q be an odd number with $1<q\leq d$. For fixed such integers d and q, there are infinitely many pairs of space forms $\Gamma_1 \backslash G_{q,2d}(\boldsymbol{R})$, $\Gamma_2 \backslash G_{q,2d}(\boldsymbol{R})$ which are isospectral but not isometric.*

5. Examples

Using proofs of Lemma 1 and Lemma 2, We can easily construct explicit examples of pairs of space forms of real Grassmann manifolds which are isospectral but not isometric. Here we give two examples.

(1) $d=5$ and $q=3$, 5.
Put $K=\Gamma_5(11, 25, 3)$, $\Gamma_1=\pi_{1,1}(K)$ and $\Gamma_2=\pi_{1,2}(K)$.
Then $\Gamma_1 \backslash G_{q,10}(\boldsymbol{R})$ is isospectral to $\Gamma_2 \backslash G_{q,10}(\boldsymbol{R})$ $(q=3, 5)$.

(2) $d=7$ and $q=3$, 5, 7.
Put $K=\Gamma_7(29, 49, 4)$, $\Gamma_1=\pi_{1,1}(K)$, $\Gamma_2=\pi_{1,2}(K)$ and $\Gamma_3=\pi_{1,3}(K)$.
Then the three space forms of real Grassmann manifolds $\Gamma_1 \backslash G_{q,14}(\boldsymbol{R})$, $\Gamma_2 \backslash G_{q,14}(\boldsymbol{R})$ and $\Gamma_3 \backslash G_{q,14}(\boldsymbol{R})$ $(q=3, 5, 7)$ are mutually isospectral but not isometric to each other.

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