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ON RIEMANNIAN MANIFOLDS ADMITTING A FUNCTION WHOSE GRADIENT IS OF CONSTANT NORM

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Abstract

Let M be a complete Riemannian manifold which admits a smooth funotion f such that $\|\nabla f\| \equiv$const. holds for the gradient vector field of f. We show that the Ricci curvature of M controlls such f considerably.

1. Introduction

Let $f(x^1, \cdots, x^m)$ be a smooth function of real m variables which satisfies

$$(*) \qquad \left(\frac{\partial f}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x^m}\right)^2 \equiv \text{const}.$$

Then it is known that locally there is a variety of such functions. However, globally a smooth function $f: \boldsymbol{R}^m \to \boldsymbol{R}$ satisfies $(*)$ if and only if f is an affine function (see e. g. [Co] for these facts from a viewpoint of symplectic geometry).

Now in the present note we are concerned with a similar problem from a viewpoint of Riemannian geometry. Let M be a complete connected smooth Riemannian manifold. We ask when M admits a smooth function $f: M \to \boldsymbol{R}$ satisfying

$$(**) \qquad \|\nabla f\| \equiv \text{const.},$$

where ∇f denotes the gradient vector field of f. Locally again we have a variety of such f's (see Remark 2.2). Now recall that a function $f: M \to \boldsymbol{R}$ is said to be an affine function if $f \circ \gamma : \boldsymbol{R} \to \boldsymbol{R}$ is an affine function for any geodesic γ in M (see [In-1]). Then affine functions f are turned out to be of class C^∞ and satisfies $(**)$. Innami proved that M admits a nonconstant affine function if and only if M splits as a Riemannian product $M = N \times \boldsymbol{R}$. Now are there many smooth functions which satisfy $\|\nabla f\| \equiv$const. other than affine functions? In fact, Busemann functions on the hyperbolic spaces $\boldsymbol{H}^m$ (or on Hadamard manifolds, see [B-G-S]), and the signed distance functions to N for warped

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product manifolds $M=N\times_{\phi}\boldsymbol{R}$ give such examples (see Example 2.5).

The purpose of the present note is to show that the Ricci curvature controls the behavior of such f with $\|\nabla f\|\equiv$const. $(\neq 0)$ considerably. Namely, we get

THEOREM A. *Let M be a complete connected Riemannian manifold of nonnegative Ricci curvature. Then any smooth function which satisfies $\|\nabla f\|\equiv$const. is an affine function.*

THEOREM B. *Let M be an m-dimensional complete and connected Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ricci}_M\geqq-(m-1)$. Let f be a smooth function on M with $\|\nabla f\|\equiv 1$. Then we get $|\Delta f|\leqq m-1$, where $\Delta f(=-\mathrm{trace}\, D^2 f)$ denotes the Laplacian of f. Moreover, we have $|\Delta f|\equiv m-1$ everywhere if and only if M is isometric to the warped product $M=N\times_{\varphi}\boldsymbol{R}$, where $N:=f^{-1}(0)$ is a complete $(m-1)$-dimensional Riemannian manifold of nonnegative Ricci curvature and $\phi(t)=e^t$ (resp. e^{-t}). Furthermore, in this case f, which is the signed distance function to $N(=N\times\{0\}\subset M)$, is the Busemann function defined by asymptotic rays $t\mapsto(p,-t)$ (resp. $t\mapsto(p,t))\in N\times_{\phi}\boldsymbol{R}$, $p\in M$ up to the sign.*

In § 1 we give an elementary characterization of a smooth function f with $\|\nabla f\|\equiv$const. (>0) on a Riemannian manifold M as a signed distance function to a complete hypersurface of M. We also give some characterizations of affine functions for completeness. In § 2 we give proofs of the above theorems (see Theorem 3.1 and Theorem 3.5). In the first version of the present note the author gave a proof of Theorem A using the Cheeger-Gromoll splitting theorem ([C-G], [E-H]), and Theorem B was stated under the assumption on the sectional curvature. Then the referee pointed out that it is possible to give more direct proofs of Theorem A. Inspired by the referee's suggestion, Theorem B is also improved in the above form. I would like to express my hearty thanks to the referee for kind suggestions and giving alternate proofs of Theorem A, which considerably improved the first version. I also thank N. Innami for useful comments and pointing out Kanai's work ([K]), A. Katsuda and H. Takeuchi for discussions. We treat smooth functions in the following for the sake of simplicity, although the following argument requires only the differentiability of class C^2.

2. Preliminaries

Let (M, g) be a connected complete Riemannian manifold. Suppose we have a smooth function $f: M\to\boldsymbol{R}$ with $\|\nabla f\|\equiv a(>0)$, where ∇f denotes the gradient vector field of f, namely the vector field characterized by $g(\nabla f, X)=Xf$ for any vector field X on M. Now let φ_s be the flow generated by ∇f. Then $\varphi_s(p)$, $p\in M$ is defined for all $s\in\boldsymbol{R}$. Also note that $f(\varphi_s(p))=f(p)+a^2 s$ and $f(M)=\boldsymbol{R}$. Now since f admits no critical points, $f^{-1}(t)$, $t\in\boldsymbol{R}$ are complete hypersurfaces and M is diffeomorphic to $f^{-1}(0)\times\boldsymbol{R}$. Furthermore, for $p\in f^{-1}(t_0)$, a trajectory $\gamma_p: s\mapsto\varphi_s(p)$, $0\leqq s\leqq s_0$ is in fact a minimal geodesic between hypersurfaces $f^{-1}(t_0)$ and $f^{-1}(t_1)$, $t_1=t_0+a^2 s_0$, namely $d(f^{-1}(t_0), f^{-1}(t_1))=L(c|[0, s_0])=as_0$. In fact, $f:(M, g)\rightarrow(\boldsymbol{R}, a^2 g_0)$ is a Riemannian submersion, where g_0 denotes the canonical Riemannian metric on $\boldsymbol{R}$ with $g_0(\partial/\partial t, \partial/\partial t)=1$, and we have for any curve $c:[0, l]\rightarrow M$ parametrized by arc-length joining a point of $f^{-1}(t_0)$ to a point of $f^{-1}(t_1)$

$$\begin{aligned} L_g(c)&=\int_0^l\|\dot{c}(t)\|dt\geqq\left|\int_0^l<\dot{c}(t), \frac{1}{a}\nabla f(c(t))>dt\right| \\ &=\frac{1}{a}\left|\int_0^l\frac{d}{dt}f(c(t))dt\right|=\frac{1}{a}|f(t_1)-f(t_0)| \\ &=L_g(\gamma_p|[0, s_0]) \end{aligned}$$

(see also [E-H]).

Now we give a simple characterization of such a function f

PROPOSITION 2.1. *Let M be a connected complete Riemannian manifold. There exists a smooth function $f: M\rightarrow\boldsymbol{R}$ with $\|\nabla f\|\equiv a$ (>0) if and only if there exists a connected complete hypersurface N of M with a smooth unit normal vector field X such that the normal exponential map $\exp^{\perp}:\nu(N)\rightarrow M$ is a diffeomorphism, where $\nu(N)$ denotes the normal bundle of N. Further, in this case $(1/a)f$ coincides with the signed distance function to N, namely we have $f(\exp^{\perp}tX_p)=at$, $p\in N$.*

Proof. Let $f: M\rightarrow\boldsymbol{R}$ be a smooth function with $\|\nabla f\|\equiv a$ $(a>0)$ and set $N:=f^{-1}(0)$. $X:=\nabla f/a$ gives a smooth unit normal vector field to N. Then our first assertion is clear from the description preceding the proposition. In particular, we get

$$f(\exp^{\perp} sX_p)=f(\gamma_p(s/a))=f(\varphi_{s/a}(p))=as\,,$$

namely, f/a is the signed distance function to N.

Conversely, suppose we have a complete connected hypersurface N with a smooth unit normal vector field X to N such that $\exp^{\perp}:\nu(N)\rightarrow M$ is a diffeomorphism. Then define a map $\Phi: N\times\boldsymbol{R}\rightarrow M$ by $\Phi(p, t):=\exp^{\perp} tX_p$, which is a diffeomorphism. Then any unit speed geodesic $\gamma_p: s\mapsto\exp^{\perp} sX_p$ emanating from $p\in N$ perpendicularly to N is a minimal geodesic realizing the distance to N, and the signed distance function $\tilde{f}$ to N is given by $\tilde{f}(q)=pr_2\circ\Phi^{-1}(q)$, where pr_2 denotes the canonical projection $N\times\boldsymbol{R}\rightarrow\boldsymbol{R}$. It follows that $\tilde{f}$ is a smooth function. Next we show that $\|\nabla\tilde{f}\|\equiv 1$. In fact, for $q\in M$ take $(p, s)\in N\times\boldsymbol{R}$ such that $q=\Phi(p, s)$. Then for $s\geqq 0$ (resp. $s<0$) $\gamma_p|[0, s]$ (resp. $\gamma_p|[s, 0]$) is a minimal geodesic from N (resp. q) to q (resp. N), which emanates from (resp. ends at) $p\in N$ and perpendicular to N. Now let $u\in T_qM$ be given. Then there exists a unique N-Jacobi field Y along γ_p such that $Y(s)=u$ (see e.g. [Bi-Cr], [H-K], [S]), since $\exp^{\perp}$ is a diffeomorphism. From the first variation formula we get for any $u \in T_q M$

$$u\tilde{f}=\frac{d}{dt}|_{t=0}\,\tilde{f}(\exp_q tu)=\mathrm{sign}(s)\frac{d}{dt}|_{t=0}\,d(\exp_q tu,\,N)$$

$$=\mathrm{sign}(s)g(Y(s),\,\dot{\gamma}_p(s))=\mathrm{sign}(s)g(u,\,\dot{\gamma}_p(s))\,.$$

Namely, we get $\nabla\tilde{f}=\mathrm{sign}(s)\dot{\gamma}_p(s)$ and therefore $\|\nabla\tilde{f}\|\equiv 1$. $\square$

Remark 2.2. Let M be a not necessarily complete Riemannian manifold. If we take a (local) hypersurface N with a smooth unit normal vector field to N and consider a domain U of M which is a diffeomorphic image under $\exp^{\perp}$ of a disk bundle of the normal bundle $\nu(N)$ of N. Then the signed distance function f to N restricted to U gives a local smooth function with $\|\nabla f\|\equiv 1$. Therefore, we have a variety of local smooth functions f with $\|\nabla f\|\equiv$const. for any Riemannian manifold.

Now the simplest examples of f with $\|\nabla f\|\equiv$const. are affine functions. Geometrically it is natural to define that $f: M\to \boldsymbol{R}$ is an affine function if $f\circ\gamma: \boldsymbol{R}\to\boldsymbol{R}$ is an affine function for any geodesic γ in M. Then f is smooth due to the Innami splitting theorem ([In-1], see also [Shi]). We give a direct proof of smoothness of f, although it is a simplified version of Innami's argument, and give some elementary characterizations of affine functions, which seem to be folklore, for completeness.

LEMMA 2.3. *Let (M, g) be a complete connected smooth Riemannian manifold. Then the following* (1)-(4) *are equivalent to each other.*

(1) *f is an affine function.*
(2) *f is smooth and its gradient vector field ∇f is parallel.*
(3) *f is smooth and its Hessian $D^2 f$ vanishes everywhere.*
(4) *f is smooth and ∇f is a Killing vector field with* $\|\nabla f\|\equiv$const.

Proof. Recall that the Hessian $D^2 f$ of f is defined as

$$D^2 f(X,\,Y)=g(\nabla_X Y,\,\nabla f)=Xg(Y,\,\nabla f)-g(Y,\,\nabla_X\nabla f) \tag{2.1}$$

and is a symmetric tensor field on M. In particular, we get for any vector field X on M

$$D^2 f(\nabla f,\,X)=g(\nabla_X\nabla f,\,\nabla f)=\frac{1}{2}Xg(\nabla f,\,\nabla f), \tag{2.2}$$

$$D^2 f(X,\,Y)=-g(Y,\,\nabla_X\nabla f) \qquad \text{if } Y\perp\nabla f\,. \tag{2.3}$$

From these formulas we easily see the equivalence of (2)-(4). Note that ∇f belongs to the null space of $D^2 f$ if $\|\nabla f\|\equiv$const. (see (2.2)).

Finally, we show the equivalence of (1) and (2)-(4). In fact, it suffices to see that any affine function f is smooth. Since f is convex it is (locally) Lipschitz (see e.g. [Ba]), and therefore continuous and differentiable almost everywhere. Now to any $X \in TM$ we assign $a(X) := (d/dt)|_{t=0} f(\exp tX)$ and $b(X) := f(\pi(X))$, where $\pi : TM \to M$ denotes the canonical projection. Then the functions $a, b : TM \to \boldsymbol{R}$ are continuous, since f is continuous and $f(\exp tX) = a(X)t + b(X)$ holds. To see that f is differentiable it suffices to show that $a|T_pM$ is linear for any $p \in M$, which is clear because this is true for almost all p and a is continuous. Further note that we have $g(\nabla f, X) = a(X)$ and f is of class C^1.

Now if $\nabla f(p) = 0$ holds at some point $p \in M$, then f is constant along any geodesic emanating from p and therefore f is a constant function. Thus we may assume that ∇f vanishes nowhere. Then $f^{-1}(t)$ is a complete totally geodesic hypersurface of M for any $t \in \boldsymbol{R}$, since for any two points of $f^{-1}(t)$ any geodesic line through these points is contained in $f^{-1}(t)$. In particular, $f^{-1}(t)$ is a smooth hypersurface of M and $X_p := \nabla f(p)/\|\nabla f(p)\|$, $p \in f^{-1}(t)$ defines a smooth unit normal vector field to $f^{-1}(t)$. Next we set $N := f^{-1}(0)$ and let $\gamma(t) := \exp^{\perp} tX_p$, $p \in N$ be a unit speed geodesic perpendicular to N. We show that γ is a minimal geodesic from p to the level $f^{-1}(f(\gamma(t)))$ for any $t > 0$. In fact, otherwise we have a unit speed minimal geodesic $\gamma_1 : [0, l] \to M$ from p to $f^{-1}(f(\gamma(t)))$ with $l < t$. Note that $a(\dot\gamma_1(0)) \leqq a(X_p)$ holds, since $\|\nabla f\| X_p$ is the gradient vector of f. Therefore, we have $f(\gamma_1(l)) = a(\dot\gamma_1(0))l < a(X_p)t = f(\gamma(t))$, a contradiction. It follows that $a(\dot\gamma(t)) = a(X_p)$, $\dot\gamma(t) = a(X_p)^{-1}\nabla f(\gamma(t))$ hold for any $t \geqq 0$ and γ is perpendicular to all levels $f^{-1}(f(\gamma(t)))$. By the same argument we see that $\exp^{\perp} : \nu(N) \to M$ is a diffeomorphism. Next let $x(s)$ be any curve in $f^{-1}(0)$ with $x(0) = p$. Then $s \mapsto \exp^{\perp} tX_{x(s)}$ is orthogonal to ∇f everywhere. In fact, if we set $\alpha(t, s) := \exp^{\perp} tX_{x(s)}$ we have $d/dt\langle \partial\alpha/\partial t, \partial\alpha/\partial s\rangle = 1/2\, d/ds\langle \partial\alpha/\partial t, \partial\alpha/\partial t\rangle \equiv 0$ and $\langle \partial\alpha/\partial t, \partial\alpha/\partial s\rangle_{t=0} = 0$. Therefore $\langle \partial\alpha/\partial t, \partial\alpha/\partial s\rangle \equiv 0$. It follows that $s \mapsto \exp^{\perp} X_{x(s)}$ is contained in $f(\gamma(t))$. Namely, we have $a(X_{x(s)}) \equiv a(X_p)$ and $\|\nabla f\| \equiv a(=\text{const.})$. Then f/a is the signed distance function to a smooth hypersurface N, and therefore is a smooth function. □

Remark 2.4. (i) Let f be a smooth function on M and t a regular value of f. Then the second fundamental form S of $N := f^{-1}(t)$ at $p \in N$ with respect to the normal vector $\nabla f(p)$ is given by

$$S(X, Y) = D^2 f(X, Y) \qquad X, Y \in T_p f^{-1}(t), \tag{2.4}$$

because of $S(X, Y) = g(\nabla_X Y, \nabla f)$. Therefore, if f is a nontrivial affine function, namely if it satisfies one of (i)—(iv) and $\|\nabla f\| \equiv a > 0$, then $f^{-1}(t)$, $t \in \boldsymbol{R}$ are totally geodesic hypersurfaces. Conversely, if $\|\nabla f\| \equiv a > 0$ and $f^{-1}(t)$, $t \in \boldsymbol{R}$ are totally geodesic hypersurfaces, then $D^2 f$ vanishes everywhere and f is a nontrivial affine function.

(ii) Let f be a nontrivial affine function. Then M splits as the Riemannian direct product $M = f^{-1}(0) \times \boldsymbol{R}$, where $f^{-1}(0)$ carries the induced metric and $\boldsymbol{R}$ carries the Riemannian metric with $\langle \partial/\partial t, \partial/\partial t\rangle \equiv a^2 (=\|\nabla f\|^2)$. This fact is the Innami splitting theorem and also follows easily from Lemma 2.3, since ∇f is a Killing vector field (see e. g. [Y-N]).

Next we give examples of f with $\|\nabla f\| \equiv$const. other than affine functions.

Example 2.5 (warped product). Let $(N, h=ds_N^2)$ be a complete Riemannian manifold and define a Riemannian metric $g=ds_M^2$ on $M=N\times \boldsymbol{R}$ by

$$ds_M^2=dt^2+\psi^2(t)ds_N^2\,, \tag{2.5}$$

where $\psi(t)$ is a positive smooth function defined on $\boldsymbol{R}$ with $\psi(0)=1$. Then (M, g) is called a warped product and denoted by $M=N\times_{\psi}\boldsymbol{R}$. Note that g is complete (see e. g. [Bi-O]). Now since the canonical projection $f:=\pi_2: M\times \boldsymbol{R}\to \boldsymbol{R}$ is a Riemannian submersion, $t\mapsto (p, t)$, $p\in N$ is a minimal geodesic between $f^{-1}(0)$ and $f^{-1}(t)$ for any $t\in \boldsymbol{R}$. Then f is the signed distance function to $N=N\times\{0\}$ and therefore a smooth function with $\|\nabla f\|\equiv 1$.

Finally, we briefly mention the Busemann functions. Let $\gamma:[0, +\infty)\to M$ be a geodesic ray, namely a unit speed geodesic with $d(\gamma(s), \gamma(t))=t-s$ for any $t>s>0$. Then the Busemann function b_γ corresponding to γ is defined as

$$b_\gamma(q):=\lim_{t\to+\infty}(d(q, \gamma(t))-t)\,.$$

If M is an Hadamard manifold, then it is known that b_γ is a convex function of class C^2 with $\|\nabla f\|\equiv 1$ and these conditions characterize Busemann functions (see [B-G-S], [Im-H]).

3. Proof of theorems

First we generalize the fact that any smooth function $f:\boldsymbol{R}^m\to \boldsymbol{R}$ with $(\partial f/\partial x^1)^2+\cdots+(\partial f/\partial x^m)^2\equiv$const. is an affine function, to a complete Riemannian manifold of nonnegative Ricci curvature. Namely, we have

THEOREM 3.1. *Let M be a complete connected Riemannian manifold of nonnegative Ricci curvaure. Then any smooth function $f: M\to \boldsymbol{R}$ with $\|\nabla f\|\equiv$const. is an affine function.*

Proof. If $\|\nabla f\|\equiv 0$, then f is a constant function and we may assume that $\|\nabla f\|\equiv 1$. The result easily follows from the Cheeger-Gromoll splitting theorem (see [C-G] and also [E-H] for a simple proof). Recall that any trajectory γ of ∇f is a geodesic line which is perpendicular to each level $f^{-1}(t)$. We fix such a γ. Now Cheeger-Gromoll showed that M splits as a Riemannian product $M=N\times \boldsymbol{R}$ and at each point of γ, the second factor $\boldsymbol{R}$ corresponds to γ. It follows that for the curvature tensor R of M we have

$$R(\dot\gamma(t), X)=0 \qquad \text{for } X\in\dot\gamma(t)^{\perp}\,. \tag{3.1}$$

We will show that for any $t_0 \in \boldsymbol{R}$, $N := f^{-1}(t_0)$ is a totally geodesic hypersurface. Let $p \in N$ and γ be the trajectory of ∇f emanating from p, which is a geodesic line with $\gamma(0)=0$ and $\dot{\gamma}(0)=\nabla f(p)$. Suppose we have a nonzero principal curvature λ of N at $p \in N$ with respect to the unit normal $\nabla f(p)$, namely λ is a nonzero eigenvalue of the shape operator $A_{\dot{\gamma}(0)}$ with a unit eigenvector $e \perp \dot{\gamma}(0)$. Here we define the shape operator as $g(A_{\dot{\gamma}(0)}X, Y)=-S(X, Y)$. Then we have an N-Jacobi field $Y(t)$ along γ which is determined by the initial conditions $Y(0)=e$, $\nabla Y(0)=\lambda e$ (see e. g. [Bi-Cr], [H-K], [S]). We defined the shape operator so that N-Jacobi fields satisfy the initial conditions $Y(0) \in T_pN$, $\nabla Y(0)=A_{\dot{\gamma}(0)}Y(0)$. Note that $Y(t)$ is perpendicular to γ and can not vanish because the normal exponetial map $\exp^{\perp}$ of N is a diffeomorphism (see Proposition 2.1). On the other hand, we get from (2.1) and the Jacobi equation

$$0=\nabla\nabla Y(t)+R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t)=\nabla\nabla Y(t). \tag{3.2}$$

Therefore $\nabla Y(t)$ is parallel along γ and we get $\nabla Y(t)=\lambda e(t)$, where $e(t)$ denotes the parallel translation of e along γ. Solving this equation we get $Y(t)=(\lambda t+1)e(t)$ and therefore we have $Y(-1/\lambda)=0$, which is a contradiction. Namely, all the principal curvatures of $N=f^{-1}(t_0)$ vanish and N is a totally geodesic hypersurfaces for any $t_0 \in \boldsymbol{R}$. Then f is an affine function by Remark 1.4. $\square$

In the following we give alternate proofs of the above theorem pointed out by the referee.

Alternate proofs given by the referee. We may again assume that $\|\nabla f\| \equiv 1$ and show that f is harmonic. To see this note that $\lambda := \nabla f(p)/(m-1)$, $p \in M$ is equal to the mean curvature of the level $f^{-1}(f(p))$ at p, since $\nabla f(p)$ belongs to the null space of the Hessian $D^2 f(p)$. Now take a hypersurface $\tilde{N}$ of $\boldsymbol{R}^m$ such that $\tilde{N}$ is totally umbilic at a point $\tilde{p} \in \tilde{N}$ with λ as the mean curvature. Now if $\lambda \neq 0$ then $\tilde{N}$ posseses a focal point, as is well known and may be proved by an argument as in the above proof. It follows from the Heintze-Karcher comparison theorem ([H-K], § 3.2, [S]) that N also posseses a focal point. This contradicts Proposition 2.1 and f is harmonic. Now recall the following identity which holds for any smooth function u on M (see e. g. [S]):

$$\frac{1}{2}\Delta\|\nabla u\|^2=\langle \nabla u, \nabla \Delta u\rangle-\mathrm{Ric}\,(\nabla u, \nabla u)-\|D^2 u\|^2. \tag{3.3}$$

Applying it especially to f, for which we have $\|\nabla f\| \equiv 1$ and $\Delta f \equiv 0$, we immediately get $D^2 f \equiv 0$ because of the assumption on the Ricci curvature.

Further, there is another proof of $\Delta f \equiv 0$ in which the comparison theorem of Heintze-Karcher is not used either apparently. Suppose again $\|\nabla f\| \equiv 1$. Then the formula (3.3) tells that

$$\langle \nabla f, \nabla \Delta f\rangle \geqq \|D^2 f\|^2 .$$

Take an arbitrary integral curve $\gamma(t)$ ($t \in \boldsymbol{R}$) of the gradient vector field ∇f. It then follows that

$$\frac{d}{dt}(\Delta f)(\gamma(t)) \geqq \|D^2 f\|^2(\gamma(t)).$$

Meanwhile, we have

$$\|D^2 f\|^2 \geqq \frac{1}{m-1}(\Delta f)^2,$$

because of $\Delta f = -\mathrm{trace}\, D^2 f$ and $D^2 f(\nabla f, \nabla f) \equiv 0$. In consequence, we are led to the differential inequality

$$\frac{d}{dt}(\Delta f)(\gamma(t)) \geqq \frac{1}{m-1}(\Delta f(\gamma(t)))^2,$$

which implies that $\Delta f(\gamma(t)) \geqq -(m-1)/(t+a)$ unless $\Delta f(\gamma(t)) \equiv 0$. Since Δf is defined entirely on M, it trurns out that $\Delta f = 0$. $\square$

Remark 3.2. Innami pointed out that his Theorem D and the argument of §5 of his paper [In-2] also implies that the above f is an affine function.

Next we are concerned with the case where there are smooth functions f with $\|\nabla f\| \equiv 1$ other than affine functions. For instance, the warped products in Example 2.5 give such examples. Here we first consider the hyperbolic space $(\boldsymbol{H}^m, g_0)$ of constant curvature -1. We show that there are many smooth functions f with $\|\nabla f\| \equiv 1$ other than affine functions, and give a characterization of the Busemann function from this viewpoint. Then we generalize these facts to the variable curvature case.

PROPOSITION 3.3. (1) *Let $M = (\boldsymbol{H}^m, g_0)$ be the hyperbolic space of dimension m and let f be a smooth function f on M with $\|\nabla f\| \equiv 1$. Then any eigenvalue λ of the Hessian $D^2 f(p)$, $p \in M$ satisfies $|\lambda| \leqq 1$. In particular, we get $|\Delta f| \leqq m-1$ for the Laplasian $\Delta f = -\mathrm{trace}\, D^2 f$ of f. Further, $|\Delta f| \equiv m-1$ holds everywhere if and only if f is a Busemann function up to the sign.*

(2) *Conversely, let N be an oriented complete connected smooth hypersurface of M whose principal curvatures λ satisfy $|\lambda| \leqq 1$. Then the normal exponential map $\exp^\perp : \nu(N) \to M$ of N is a diffeomorphism and the signed distance function f to N satisfies $\|\nabla f\| \equiv 1$.*

Proof. (1) Set $N := f^{-1}(s_0)$, $s_0 \in \boldsymbol{R}$. Let γ be the trajectory of ∇f through $p \in N$. Now let λ be a principal curvature of N, namely an eigenvalue of the shape operator A_p of N with respect to the unit normal $\nabla f(p)$, and e a unit eigenvector of A_p with the eigenvalue λ. Let $Y(t)$ be the N-Jacobi field along γ with the initial conditions $Y(0) = e$, $\nabla Y(0) = Ae = \lambda e$. Solving the Jacobi equation we get $Y(t) = (\cosh t + \lambda \sinh t) e(t)$, where $e(t)$ denotes the parallel translation of e along γ. Note that $Y(t_0) = 0$ occurs for some t_0 if and only if $|\lambda| > 1$. Since the normal exponential map $\exp^\perp : \nu(N) \to M$ is a diffeomorphism (see Proposition 2.1), we see that any geodesic emanating perpendiculary from

N is free of focal points. Therefore we have $|\lambda|\leqq 1$ for all the principal curvatures of N. Recalling that $A_p=-D^2f|T_pN\times T_pN$ and $\nabla f(p)$ belongs to the null space of $D^2f(p)$, we see that all eigenvalues λ of the Hessian D^2f should satisfy $|\lambda|\leqq 1$. Then we have $|\Delta f|\leqq m-1$ because of $\Delta f=-\mathrm{trace}\, D^2f$.

Next we show that $D^2f\geqq 0$ (resp. $D^2f\leqq 0$) holds if and only if all the principal curvatures of $f^{-1}(t)$, $t\in\boldsymbol{R}$ are equal to -1 (resp. 1). This clearly occurs if and only if $|\Delta f|\equiv m-1$. To see this, let λ_i $(i=1, \cdots, m-1)$ be the eigenvalues of A_p, $p\in N=f^{-1}(s_0)$ with eigenvectors e_i which form an orthonormal basis of T_pN. Let γ be the trajectory of ∇f through p. Then any N-Jacobi field $Y(t)$ along γ, which is perpendicular to γ, may be written as

$$Y(t)=\sum_{i=1}^{m-1} a_i(\cosh t+\lambda_i \sinh t)e_i(t),$$

where $e_i(t)$ denote the parallel translations of e_i along γ. Note that these $Y(t)$'s span $T_{\gamma(t)}f^{-1}(f(\gamma(t)))$ for any t, and if λ is a principal curvature of $f^{-1}(f(\gamma(t)))$ with respect to $\nabla f(\gamma(t))$, then there exists an N-Jacobi field $Y(t)$ such that $\nabla Y(t)=\lambda Y(t)$. It follows that for at least one of $1\leqq i\leqq m-1$ we have $\lambda=(\sinh t+\lambda_i \cosh t)/(\cosh t+\lambda_i \sinh t)$. Now note that $(\sinh t+\lambda_i \cosh t)/(\cosh t+\lambda_i \sinh t)\leqq 0$ (resp. $\geqq 0$) hold for $i=1, \cdots, m-1$ and all $t\in\boldsymbol{R}$ if and only if $\lambda_i=-1$ (resp. $=1$) hold for $i=1, \cdots, m-1$. Then recalling that ∇f belongs to the null space of D^2f and $A_p=-D^2f(p)|T_pN\times T_pN$, the last condition is equivalent to $D^2f\geqq 0$ (resp. $D^2f\leqq 0$) everywhere, namely f is convex (resp. concave). This occurs if and only if f (resp. $-f$) is a Busemann function (see [B-G-S]). Note also that this holds if and only if we have $K_N\equiv 0$ because of the Gauss formula.

(2) Conversely, let N be an oriented connected complete hypersurface whose principal curvatures λ satisfy $|\lambda|\leqq 1$. Then as we showed above any nontrivial N-Jacobi filed $Y(t)$ along a normal geodesic γ emanating perpendicularly from N can not vanish. Therefore, the normal exponential map $\exp^\perp: \nu(N)\to M$ is a regular smooth map, and surjective since N is closed. We show that $\exp^\perp$ is injective. In fact, suppose we have $r=\exp^\perp u=\exp^\perp v\in M$ for some $u, v\in \nu(N)\backslash o(N)$, $u\neq v$. Set $p=\nu(u)$, $q=\nu(v)\in N$, where ν denotes the projection of the normal bundle. Let γ_p, γ_q be the geodesics given by $t\mapsto \exp^\perp tu$, $\exp^\perp tv$, respectively, which emanate from p, q, respectively, and are perpendicular to N. Take a curve $c:[0, 1]\to N$ joining p to q and consider a homotopy $\alpha:[0,1]\times[0, 1]\to M$ from γ_p to γ_q by taking geodesics in $\boldsymbol{H}^m$ joining $c(s)$ to r for $s\in[0, 1]$. Then since $\exp^\perp$ is regular we may lift curves $\alpha_s:[0, 1]\to M$, which are defined as $\alpha_s(t):=\alpha(t, s)$ and join $c(s)$ to q, to a family of curves $\tilde{\alpha}_s$ in the normal bundle $\nu(N)$ so that $\tilde{\alpha}_s(0)=o_{c(s)}$ and $\tilde{\alpha}_s(1)$ is a fixed point. However, this is impossible because $\tilde{\alpha}_0$, $\tilde{\alpha}_1$ are given by $t\mapsto tu$ and $t\mapsto tv$, respectively and can not end at the same point since $u\neq v$. It follows that $\exp^\perp$ is a diffeomorphism. Then the remaining assertion follows from Proposition 2.1. □

Remark 3.4. Other typical examples of the above $N\subset\boldsymbol{H}$ which satisfy the condition of (2) of the above Proposition are complete simply connected totally geodesic hypersurfaces which are isometric to $(\boldsymbol{H}^{m-1}, g_0)$. Then slight C^∞-deformations of $\boldsymbol{H}^{m-1}$ give examples of N satisfying the condition of (2). Therefore, we have a variety of smooth functions f with $\|\nabla f\| \equiv$const. which are neither affine functions nor Busemann functions.

Now we turn to the variable curvature case.

THEOREM 3.5. *Let M be an m-dimensional complete connected Riemannian manifold whose Ricci curvature satisfies* $\mathrm{Ricci}_M \geqq -(m-1)$ *everywhere. Let $f: M \to \boldsymbol{R}$ be a smooth function with $\|\nabla f\| \equiv 1$. Then we get $|\Delta f| \leqq m-1$ for the Laplacian of f. Furthermore if $|\Delta f| \equiv m-1$ holds everywhere, then M is isometric to the warped product $f^{-1}(0) \times_\phi \boldsymbol{R}$, where $f^{-1}(0)$ carries the induced metric with nonnegative Ricci curvature and $\phi(t) = e^{\pm t}$. Further f is the Busemann function corresponding to a line $\boldsymbol{R}$ up to the sign.*

Conversely, let N be a complete connected Riemannian manifold of nonnegative Ricci curvature. Let $M := N \times_\phi \boldsymbol{R}$ be warped product manifold with $\phi(t) = e^{\pm t}$. Then the Ricci curvature Ricci_M *of M satisfies* $\mathrm{Ricci}_M \geqq -(m-1)$ *everywhere and the signed distance function f to N satisfies $\|\nabla f\| \equiv 1$ and $|\Delta f| \equiv m-1$.*

Proof. Suppose that at some point $p \in M$ the absolute value of $\Delta f(p)$ is greater than $m-1$. Recall that $\lambda := \Delta f(p)/(m-1)$ is the mean curvature of the hypersurface $N := f^{-1}(f(p))$ at p. Let γ be the trajectory of ∇f through p, which is a geodesic perpendicular to N. Firstly we consider the case $\lambda < -1$. Let $M := (\boldsymbol{H}^m, g_0)$ be the hyperbolic space and take a (local) hypersurface $\tilde{N}$ through a point $\tilde{p} \in \tilde{M}$ such that $\tilde{N}$ is totally umbilic at $\tilde{p}$ with λ as the mean curvature. Then the argument of the previous Proposition implies that there exists a focal point of $\tilde{N}$ along the normal geodesic $\tilde{\gamma}$ emanting from $\tilde{p}$ perpendicularly to $\tilde{N}$. It follows from a comparison theorem (see [H-K] § 3.2, [S]) that there appears also a focal point of N along γ, which is a contradiction because the normal exponential map is a diffeomorphism. Considering the geodesic reversing the orientation of γ we get a contradiction for the case $\lambda > 1$, namely $|\Delta f| \leqq m-1$.

Next suppose $|\Delta f| \equiv m-1$ holds. Then checking the equality case of the Heintze-Karcher theorem, it follows that there exists an orthonormal basis $\{e_i\}_{i=1}^{m-1}$ of the orthogonal complement of $\nabla f(p)$ in $T_pM (p \in f^{-1}(0))$ such that $Y_i(t) := (\cosh t + \lambda \sinh t) e_i(t)$ $(i = 1, \cdots, m-1)$ are N-Jacobi fields along γ, where $N := f^{-1}(0)$, $\lambda = \pm 1$ and $e_i(t)$ denotes the parallel translation of e_i along γ. Then we see that all the principal curvatures of the levels $f^{-1}(t)$, $t \in \boldsymbol{R}$ with respect to the unit normal vector ∇f are equal to 1 (resp. -1). Let φ_t be the flow generated by ∇f. Then since all trajectories $t \mapsto \varphi_t(p)$, $p \in N := f^{-1}(0)$ are normal geodesics perpendicular to the levels $f^{-1}(t)$, $Y(t) := D\varphi_t(u)$, $u \in T_pN$, $N := f^{-1}(0)$, $p \in N$ are N-Jacobi fields along $t \mapsto \varphi_t(p)$ with $\nabla Y(0) = Y(0) = u$ (resp. $\nabla Y(0) = -Y(0) = -u$) and we get further $\nabla Y(t) = A_{\nabla f(\varphi_t(p))} Y(t) = Y(t)$ (resp. $-Y(t)$). Then it follows that

$$Y(t)=\nabla\nabla Y(t)=-R(Y(t), \nabla f)\nabla f,$$

and we get $g(R(Y(t), \nabla f)\nabla f, Y(t))\equiv-\|Y(t)\|^2$, namely $K_M(Y(t), \nabla f)\equiv-1$ for the sectional curvature.

Now we define a smooth map $\Phi: f^{-1}(0)\times_{\psi} \boldsymbol{R}\to M$ by $\Phi(p, t):=\varphi_t(p)$, which may be easily seen to be a diffeomorphism. We show that Φ is an isometry. Note that $D\Phi((p, t))(\partial/\partial t)=\nabla f(\Phi(p, t))$ and $D\Phi((p, t))u=D\varphi_t(u)(:=Y(t))$ for $u\in T_pN$. Then $Y(t)$ is an N-Jacobi field along geodesic $\gamma: t\mapsto\varphi_t(p)$ with $Y(0)=\nabla Y(0)=u$ and perpendicular to γ. It follows that $\|D\Phi(p, t)(\partial/\partial t)\|\equiv 1$ and $g(D\Phi((p, t))(\partial/\partial t), D\Phi(p, t))(u))=0$ for $u\in T_pN$. On the other hand, we get for $u, v\in T_pN$

$$g(D\Phi((p, t))(u), D\Phi((p, t))(v))=g(Y(t), Z(t))$$

as above. Then we have

$$\frac{d}{dt}g(Y(t), Z(t))=g(\nabla Y(t), Z(t))+g(Y(t), \nabla Z(t)) \tag{3.4}$$
$$=2g(Y(t), Z(t)) \quad (\text{resp. } -2g(Y(t), Z(t)))$$

and it follows that $g(D\Phi((p, t))(u), D\Phi((p, t))(v))=g(Y(t), Z(t))=e^{\pm 2t}g(u, v)$, which shows that Φ is an isometry, if we define $\psi(t):=e^{\pm t}$. Finally as for the curvature we recall the following formula (see e.g. [Bi-O]) for a warped product metric

$$K_M\left(\left(y, a\frac{\partial}{\partial t}\right), \left(z, b\frac{\partial}{\partial t}\right)\right) \tag{3.5}$$
$$=\psi^2(t)\{K_N(y, z)-\psi'^2(t)\}h(y\wedge z, y\wedge z)$$
$$-\psi(t)\{a^2h(z, z)-2ab\cdot h(y, z)+b^2h(y, y)\}\psi''(t),$$

where $\{(y, a(\partial/\partial t)), (z, b(\partial/\partial t))\}$ are orthonormal vectors in $T_{(p,t)}M$, $p\in N$. Applying (3.5) to the case where $\psi(t)=e^t$, $a=b=0$ and $\|y\|=\|z\|=e^{-t}$ we get

$$K_M((y, 0), (z, 0))=e^{2t}\{K_N(y, z)-e^{2t}\}e^{-4t},$$

Now suppose that the Ricci curvature of M satisfies $\mathrm{Ricci}_M\geqq-(m-1)$. Then we get for $y\in U_pN$

$$e^{2t}\,\mathrm{Ricci}_N(y, y)=m-2+\mathrm{Ricci}_M((y, 0), (y, 0))-K_M\left(y, \frac{\partial}{\partial t}\right)\geqq 0.$$

The case where $\psi(t)=e^{-t}$ may be treated in the same manner. Furthermore, when $\psi(t)=e^t$ (resp. $\psi(t)=e^{-t}$), geodesic rays $\boldsymbol{R}^+\ni t\mapsto\Phi(p, -t)$ (resp. $\Phi(p, t)$) $\in M$, $p\in N$, are asymptotic to each other and the Busemann functions corresponding these rays are in fact the signed distance functions to N up to the sign.

Now we turn to the last assertion. Firstly, we check that $\mathrm{Ricci}_M\geqq-(m-1)$. In fact, for a plane section σ of $T_{(p,t)}M$ with an orthonormal basis $\{u:=(y, a(\partial/\partial t)), v:=(z, b(\partial/\partial t))\}$, we get for the case where $\psi(t)=e^t$

$$K_M(\sigma)=e^{2t}\{K_N(y, z)-e^{2t}\}\{h(y, y)h(z, z)-h^2(y, z)\}$$
$$-e^{2t}\{a^2h(z, z)-2ab\cdot h(y, z)+b^2h(y, y)\}$$
$$=K_N(y, z)e^{-2t}(1-a^2-b^2)-1\,.$$

Now for $u:=(y, a(\partial/\partial t))\in U_{(p,t)}M$, taking an orthonormal basis of $T_{(p,t)}M$ given by $\{e_1=u, e_i=(y_i, 0), e_m\}$, where $i=2, \cdots, m-1$ and $e_m=(-(ae^{-t}/\|y\|)y, \|y\|e^t(\partial/\partial t))$, we may easily check that $\mathrm{Ricci}_M\geqq-(m-1)$. The same computation also works for the case where $\phi(t)=e^{-t}$. Secondly, let f be the signed distance function to N. We only consider the case where $\phi(t)=e^t$. Then the trajectry γ of ∇f through $p\in N$ is given by $t\mapsto(p, t)\in M=N\times_\phi \boldsymbol{R}$, which is a geodesic perpendicular to the levels $f^{-1}(t)$ of f. Note that for $u\in T_pN$, $Y: t\mapsto(u, 0)\in T_{\gamma(t)}M$ is an $f^{-1}(s)$-Jacobi field along γ for any $s\in\boldsymbol{R}$. Now we show that $\nabla Y(t)=\nabla_{\partial/\partial t}Y(t)=Y(t)$. In fact, for any $v\in T_pN$ we set $Z(t):=(v, 0)\in T_{\gamma(t)}M$. Then from the definition of the covariant derivative we get

$$g(\nabla Y(t), Z(t))=\frac{1}{2}\frac{\partial}{\partial t}g(Y(t), Z(t))=\frac{1}{2}\frac{\partial}{\partial t}e^{2t}h(u, v)$$
$$=e^{2t}h(u, v)=g(Y(t), Z(t))\,.$$

Since $Y(t)$ is perpendicular to γ, we get $\nabla Y(t)=Y(t)$, which implies that all eigenvalues of D^2f restricted to the tangent spaces to $f^{-1}(t)$ are equal to -1 and therefore $\Delta f=m-1$. □

Remark 3.6. From Theorems 3.1 and 3.6 we have the following: Let M be an m-dimensional connected complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ricci}_M\geqq-(m-1)c^2$, $c\geqq0$. Suppose we have a smooth function f on M with $\|\nabla f\|\equiv1$. Then we get $|\Delta f|\leqq(m-1)c^2$.

Remark 3.7. For the warped product $M=N\times_\phi \boldsymbol{R}$, if the Ricci curvature of M is nonnegative then ϕ is a (positive) constant. This follows from the computation of Ricci curvatures of $(0, \partial/\partial t)$ by (3.4), or Theorem 3.1.

Remark 3.8. In [K] (see also [T]) it is shown that if a complete Riemannian manifold (M, g) admits a nontrivial solution $\tilde{f}$ without critical points for a differential equation $D^2f-fg=0$, then (M, g) is isometric to the warped product $N\times_\varphi \boldsymbol{R}$ with $\ddot{\varphi}=\varphi$ $(\varphi>0)$. The relation between their $\tilde{f}$ and our f in Theorem 3.5 seems to be $\tilde{f}=e^f$.

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