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STEENROD OPERATIONS ON THE MODULAR INVARIANTS

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Introduction

Fix an odd prime p. Let A_{p^n} be the alternating group on p^n letters. Denote by $\Sigma_{p^n,p}$ a Sylow p-subgroup of A_{p^n} and E^n an elementary abelian p-group of rank n. Then we have the restriction homomorphisms

$$\begin{aligned}
&\operatorname{Res}(E^n, \Sigma_{p^n,p}) : H^*(B\Sigma_{p^n,p}) \longrightarrow H^*(BE^n),\\
&\operatorname{Res}(E^n, A_{p^n}) : H^*(BA_{p^n}) \longrightarrow H^*(BE^n),
\end{aligned}$$

induced by the regular permutation representation $E^n \subset \Sigma_{p^n,p} \subset A_{p^n}$ of E^n (see Mùi [4]). Here and throughout the paper, we assume that the coefficients are taken in the prime field $\mathbf{Z}/p$. Using modular invariant theory of linear groups, Mùi proved in [3], [4] that

$$\begin{aligned}
&\operatorname{ImRes}(E^n, \Sigma_{p^n,p}) = E(U_1, \ldots, U_n) \otimes P(V_1, \ldots, V_n),\\
&\operatorname{ImRes}(E^n, A_{p^n}) = E(\tilde{M}_{n,0}, \ldots, \tilde{M}_{n,n-1}) \otimes P(\tilde{L}_n, Q_{n,1}, \ldots, Q_{n,n-1}).
\end{aligned}$$

Here and in what follows, $E(., \ldots, .)$ and $P(., \ldots, .)$ are the exterior and polynomial algebras over $\mathbf{Z}/p$ generated by the variables indicated. $\tilde{L}_n, Q_{n,s}$ are the Dickson invariants of dimensions p^n, $2(p^n - p^s)$, and $\tilde{M}_{n,s}$, U_k, V_k are the Mùi invariants of dimensions $p^n - 2p^s$, p^{k-1}, $2p^{k-1}$ respectively (see §1).

Let A be the mod p Steenrod algebra and let τ_s, ξ_i be the Milnor elements of dimensions $2p^s - 1$, $2p^i - 2$ respectively in the dual algebra A_* of A. In [7], Milnor showed that, as an algebra

$$A_* = E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots).$$

Then A_* has a basis consisting of all monomials $\tau_S \xi^R = \tau_{s_1} \ldots \tau_{s_k} \xi_1^{r_1} \ldots \xi_m^{r_m}$, with $S = (s_1, \ldots, s_k)$, $0 \le s_1 < \ldots < s_k$, $R = (r_1, \ldots, r_m)$, $r_i \ge 0$. Let $St^{S,R} \in A$ denote the dual of $\tau_S \xi^R$ with respect to that basis. Then A has a basis consisting all operations $St^{S,R}$. For $S = \emptyset$, $R = (r)$, $St^{\emptyset,(r)}$ is nothing but the Steenrod operation P^r.

Since $H^*(BG)$, $G = E^n$, $\Sigma_{p^n,p}$ or A_{p^n}, is an A-module (see [13 ; Chap. VI]) and the restriction homomorphisms are A-linear, their images are A-submodules of $H^*(BE^n)$.

The purpose of the paper is to study the module structures of $\operatorname{ImRes}(E^n, \Sigma_{p^n,p})$ and $\operatorname{ImRes}(E^n, A_{p^n})$ over the Steenrod algebra A. More precisely, we prove a duality relation between $St^{S,R}(\tilde{M}_{n,s}^{\delta} Q_{n,s}^{1-\delta})$ and $St^{S',R'}(U_{k+1}^{\delta} V_{k+1}^{1-\delta})$ for $\delta = 0, 1$, $\ell(R) = k$ and $\ell(R') = n$.

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Here by the length of a sequence $T=(t_1,\dots,t_q)$ we mean the number $\ell(T)=q$. Using this relation we explicitly compute the action of the Steenrod operations P^r on U_{k+1}, V_{k+1}, $\tilde{M}_{n,s}$ and $Q_{n,s}$.

The analogous results for $p=2$ have been announced in [11].

The action of P^r on V_{k+1} and $Q_{n,s}$ has partially studied by Campbell [1], Madsen [5], Madsen-Milgram [6], Smith-Switzer [12], Wilkerson [14]. Eventually, this action was completly determined by Hung-Minh [10] and by Hai-Hung [8] , Hung [9] for the case of the coefficient ring $\mathbf{Z}/2$.

The paper contains 3 sections. After recalling some needed information on the invariant theory, the Steenrod homomorphism $d_n^*P_n$ and the operations $St^{S,R}$ in Section 1, we prove the duality theorem and its corollaries in Section 2. Finally Section 3 is an application of the duality theorem to determine the action of the Steenrod operations on the Dickson and Mùi invariants.

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§1. Preliminaries

As is well-known $H^*(BE^n)=E(x_1,\dots,x_n)\otimes P(y_1,\dots,y_n)$ where $\dim x_i=1$, $y_i=\beta x_i$ with β the Bockstein homomorphism. Following Dickson [2] and Mùi [3], we define

$$[e_1,\dots,e_k]=\det(y_i^{p^{e_j}}),$$

$$[1;e_2,\dots,e_k]=\begin{vmatrix} x_1 & \cdots & x_k \\ y_1^{p^{e_2}} & \cdots & y_k^{p^{e_2}} \\ \vdots & \cdots & \vdots \\ y_1^{p^{e_k}} & \cdots & y_k^{p^{e_k}} \end{vmatrix},$$

for every sequence of nonnegative integers $(e_1,\dots,e_k)$, $1\le k\le n$. We set

$$L_{k,s}=[0,\dots,\hat{s},\dots,k],\ L_k=L_{k,k}=[0,\dots,k-1],\ L_0=1,$$
$$M_{k,s}=[1;0,\dots,\hat{s},\dots,k-1],\ 0\le s<k\le n.$$

Then $\tilde{L}_n,Q_{n,s},\tilde{M}_{n,s},U_k,V_k$ are defined by

$$\tilde{L}_n=L_n^h,\ h=(p-1)/2,\ Q_{n,s}=L_{n,s}/L_n,\quad 0\le s\le n,$$
$$\tilde{M}_{n,s}=M_{n,s}L_n^{h-1},U_k=M_{k,k-1}L_{k-1}^{h-1},\ V_k=L_k/L_{k-1},\quad 1\le k\le n.$$

Note that $Q_{n,0}=\tilde{L}_n^2$, $Q_{n,n}=1$ for any $n\ge 0$.

Let X be a topological space. Then we have the Steenrod power map

$$P_n: H^q(X)\longrightarrow H^{p^nq}(EA_{p^n}\underset{A_{p^n}}{\times}X^{p^n}),$$

which sends u to $1\otimes u^{p^n}$ at the cochain level (see [13; Chap. VII]). We also have the diagonal homomorphism

$$d_n^*: H^*(EA_{p^n}\underset{A_{p^n}}{\times}X^{p^n})\longrightarrow H^*(BE^n)\otimes H^*(X)$$

induced by the diagonal map of X, the inclusion $E^n \subset A_{p^n}$ and the Künneth formula.

$d_n^* P_n$ has the following fundamental properties.

PROPOSITION 1.1 (Mùi [3], [4]). (i) *$d_n^* P_n$ is natural monomorphism preserving cup product up to a sign, more precisely*

$$d_n^* P_n(uv) = (-1)^{nhqr} d_n^* P_n u d_n^* P_n v,$$

where $q = \dim u, r = \dim v, h = (p-1)/2$.

(ii) $d_n^* P_n = d_{n-s}^* . P_{n-s} d_s^* P_s \quad , 0 \le s \le n.$

(iii) *For $H^*(E^1) = E(x) \otimes P(y)$, we have*

$$d_n^* P_n x = (-h!)^n U_{n+1} = (h!)^n (\tilde{L}_n x + \sum_{s=0}^{n-1} (-1)^{s+1} \tilde{M}_{n,s} y^{p^s}),$$

$$d_n^* P_n y = V_{n+1} = (-1)^n \sum_{s=0}^{n} (-1)^s Q_{n,s} y^{p^s},$$

where $U_{n+1} = U_{n+1}(x_1, \dots, x_n, x, y_1, \dots, y_n, y)$, $V_{n+1} = V_{n+1}(y_1, \dots, y_n, y)$.

The following is a description of $d_n^* P_n$ in terms of modular invariants and cohomology operations.

THEOREM 1.2 (Mùi [4; 1.3]). *Let $z \in H^q(X), \mu(q) = (h!)^q (-1)^{hq(q-1)/2}$. We then have*

$$d_n^* P_n z = \mu(q)^n \sum_{S,R} (-1)^{r(S,R)} \tilde{M}_{n,s_1} \dots \tilde{M}_{n,s_k} \tilde{L}_n^{r_0} Q_{n,1}^{r_1} \cdot \ Q_{n,n-1}^{r_{n-1}} \otimes St^{S,R} z.$$

Here the sum runs over all (S,R) with $S = (s_1, \dots, s_k)$, $0 \le s_1 < \dots < s_k$, $R = (r_1, \dots, r_n)$, $r_i \ge 0$, $r_0 = q - k - 2\,(r_1 + \dots + r_n) \ge 0$, $r(S,R) = k + s_1 + \dots + s_k + r_1 + 2r_2 + \dots + nr_n$.

§2. The duality theorem

Let $\tilde{m}_{m,s}$, $\tilde{\ell}_m$, $q_{m,s}$, $m = n$ or k, (resp. u_{k+1}, v_{k+1}) be the dual of $\tilde{M}_{m,s}$, $\tilde{L}_m$, $Q_{m,s}$ (resp. U_{k+1}, V_{k+1}) in

$$E(\tilde{M}_{m,0}, \dots, \tilde{M}_{m,m-1}) \otimes P(\tilde{L}_m, Q_{m,1}, \dots, Q_{m,m-1})$$

(resp. $E(U_{k+1}) \otimes P(V_{k+1})$) with respect to the basis consisting of all monomials

$$\tilde{M}_S \tilde{Q}^H = \tilde{M}_{m,s_1} \dots \tilde{M}_{m,s_k} \tilde{L}_m^{h_0} Q_{m,1}^{h_1} \dots Q_{m,m-1}^{h_{m-1}},$$

with $S = (s_1, \dots, s_k), 0 \le s_1 < \dots < s_k$, $H = (h_0, \dots, h_{m-1}), h_i \ge 0$, (resp. $U_{k+1}^e V_{k+1}^j$; $e = 0, 1$, $j \ge 0$). Let $\Gamma(\tilde{\ell}_m, q_{m,1}, \dots, q_{m,m-1})$ (resp. $\Gamma(v_{k+1})$) be the divided polynomial algebra with divided power $\gamma_i, i \ge 0$ generated by $\tilde{\ell}_m, q_{m,1}, \dots, q_{m,m-1}$ (resp. v_{k+1}). We set

$$\tilde{m}_S \tilde{q}_H = \tilde{m}_{m,s_1} \dots \tilde{m}_{m,s_k} \gamma_{h_0}(\tilde{\ell}_m) \gamma_{h_1}(q_{m,1}) \dots \gamma_{h_{m-1}}(q_{m,m-1}).$$

For $q \ge 0$ and $R = (r_1, \dots, r_m)$, set

$$R_q^* = (q - 2(r_1 + \dots + r_m), r_1, \dots, r_{m-1}).$$

Let V be a vector space over $\mathbf{Z}/p$ and V^* its dual. Denote by

$$\langle .,. \rangle : V \otimes V^* \longrightarrow \mathbf{Z}/p$$

the dual pairing.

The main result of the section is

THEOREM 2.1. *Suppose given* e, $\delta = 0, 1$, $j \geq 0$, (S, R), *and* (S', R') *with* $\ell(R) = k$, $\ell(R') = n$, $\ell(S) = t < k$, $\ell(S') = t' < n$. *Set* $\sigma = r(S,R)+r(S',R')+s+\delta+(t+[-2p^s])t'+nhk\delta$, *with* $-\delta \leq s \leq n-\delta$. *Then we have*

$$\begin{aligned}&\langle \tilde{m}_S \tilde{q}_{R^*_{(2-\delta)p^n-e-2j-t}} \otimes u^e_{k+1}\gamma_j(v_{k+1}), St^{S',R'}(U^\delta_{k+1}V^{1-\delta}_{k+1})\rangle \\ &= \begin{cases} (-1)^\sigma \langle \tilde{m}_{S'}\tilde{q}_{R'^*_{(2-\delta)p^k-t'}}, St^{S,R}(\tilde{M}^\delta_{n,s}Q^{1-\delta}_{n,s})\rangle, & e+2j = -[-2p^s], \\ 0 \qquad , & otherwise. \end{cases}\end{aligned}$$

Here, by convention, $\tilde{M}_{n,-1} = \tilde{L}_n$.

Proof. We prove the theorem for $\delta = 1$. For $\delta = 0$, it is similarly proved. We set

$$\begin{aligned}U &= U_{n+k+1}(x_1, \dots, x_k, x'_1, \dots, x'_n, x, y_1, \dots, y_k, y'_1, \dots, y'_n, y)\\ U' &= U_{n+k+1}(x'_1, \dots, x'_n, x_1, \dots, x_k, x, y'_1, \dots, y'_n, y_1, \dots, y_k, y).\end{aligned}$$

It is easy to verify that

(a) $$U = (-1)^{nkh} U' .$$

Computing directly from Proposition 1.1 gives

(b) $$\begin{aligned}U &= (-h!)^{-k} d^*_k P_k U_{n+1}(x'_1, \dots, x'_n, x, y'_1, \dots, y'_n, y)\\ &= (-h!)^{-k}(-1)^n d^*_k P_k \Big(\sum_{s=-1}^{n-1} (-1)^{s+1}\tilde{M}_{n,s} y^{p^s}\Big)\\ &= (-1)^n \sum_{s=-1}^{n-1} (-h!)^{-(s+1)k/(|s|+1)}(-1)^{s+1}(d^*_k P_k \tilde{M}_{n,s})V^{p^s}_{k+1}.\end{aligned}$$

Here by convention, $y^{1/p} = x$, and $V^{1/p}_{k+1} = U_{k+1}$.

We observe that $\dim \tilde{M}_{n,s} = p^n + [-2p^s]$. According to Theorem 1.2 we have

(c) $$d^*_k P_k \tilde{M}_{n,s} = \mu(p^n + [-2p^s])^k \sum_{S,R} (-1)^{r(S,R)} \tilde{M}_S \tilde{Q}^{R^*_{p^n+[-2p^s]-t}} St^{S,R} \tilde{M}_{n,s}.$$

A simple computation shows that

(d) $$(-h!)^{-(s+1)/(|s|+1)}\mu(p^n + [-2p^s]) = (-1)^{nh} .$$

Combining (b), (c) and (d) we get

$$U = \sum_{s=-1}^{n-1} \Big(\sum_{S,R} (-1)^{n(kh+1)+r(S,R)+s+1} \tilde{M}_S \tilde{Q}^{R^*_{p^n+[-2p^s]-t}} St^{S,R}\tilde{M}_{n,s}\Big) V^{p^s}_{k+1}.$$

From this, we see that it implies

(e) $$(-1)^{r(S,R)+n(hk+1)+s+1}\langle \tilde{m}_S\tilde{q}_{R^*{}_{p^n-e-2j-t}} \otimes \tilde{m}_{S'}\tilde{q}_{R'^*_{p^k-t'}} \otimes u^e_{k+1}\gamma_j(v_{k+1}), U\rangle$$
$$= \begin{cases} (-1)^{tt'}\langle \tilde{m}_{S'}\tilde{q}_{R'^*_{p^k-t'}}, St^{S,R}(\tilde{M}_{n,s}\rangle, & e+2j=-[-2p^s], \\ 0 \quad , & \text{otherwise.} \end{cases}$$

On the other hand, from Proposition 1.1 and Theorem 1.2 we have
$$\begin{aligned} U' &= (-h!)^{-n}d^*_n P_n U_{k+1}(x_1,\dots,x_k,x,y_1,\dots,y_k,y) \\ &= (-h!)^{-n}\mu(p^k)^n \sum_{S',R'} (-1)^{r(S',R')}\tilde{M}_{S'}\tilde{Q}^{R'^*_{p^k-t'}} St^{S',R'}U_{k+1}. \end{aligned}$$

From this and the fact that $(-h!)^{-1}\mu(p^k) = (-1)^{hk}$, we get

(f) $$(-1)^{r(S',R')+n(hk+1)}\langle \tilde{m}_S\tilde{q}_{R^*_{p^n-e-2j-t}} \otimes \tilde{m}_{S'}\tilde{q}_{R'^*_{p^k-t'}} \otimes u^e_{k+1}\gamma_j(v_{k+1}), U'\rangle$$
$$= (-1)^{t'e}\langle \tilde{m}_S\tilde{q}_{R^*_{p^n-e-2j-t}} \otimes u^e_{k+1}\gamma_j(v_{k+1}), St^{S',R'}U_{k+1}\rangle.$$

Comparing (e) with (f) and using (a), we obtain the theorem for $\delta = 1$.

Since the basis $\{\tilde{M}_{S'}\tilde{Q}^{H'}\}$ of $E(\tilde{M}_{n,0},\dots,\tilde{M}_{n,n-1})\otimes P(\tilde{L}_n,Q_{n,1},\dots,Q_{n,n-1})$ is dual to the basis $\{\tilde{m}_{S'}\tilde{q}_{H'}\}$ of $E(\tilde{m}_{n,0},\dots,\tilde{m}_{n,n-1})\otimes\Gamma(\tilde{\ell}_n,q_{n,1},\dots,q_{n,n-1})$. Hence, we easily obtain from Theorem 2.1

COROLLARY 2.2. *Set*
$$C_{S',R'} = \langle \tilde{m}_S\tilde{q}_{R^*_{(2-\delta)p^n+[-2p^s]-t}} \otimes \gamma_{p^s}(v_{k+1}), St^{S',R'}(U^\delta_{k+1}V^{1-\delta}_{k+1})\rangle.$$
We have
$$St^{S,R}(\tilde{M}^\delta_{n,s}Q^{1-\delta}_{n,s}) = \sum_{S',R'}(-1)^\sigma C_{S',R'}\tilde{M}_{S'}Q^{R'^*_{(2-\delta)p^k-t'}},$$
Here, by convention, $\gamma_{1/p}(v_{k+1}) = u_{k+1}$.

By an analogous argument we obtain

COROLLARY 2.3. *Set* $C_{s,S,R} = \langle \tilde{m}_{S'}\tilde{q}_{R'^*_{(2-\delta)p^k-t'}}, St^{S,R}(\tilde{M}^\delta_{n,s}Q^{1-\delta}_{n,s})\rangle$. *We have*
$$St^{S',R'}(\tilde{U}^\delta_{k+1}V^{1-\delta}_{k+1}) = \sum_{s=-\delta}^{n-\delta}\Big(\sum_{S,R}(-1)^\sigma C_{s,S,R}\tilde{M}_S\tilde{Q}^{R^*-(2-\delta)p^n+[-2p^s]-t}\Big)V^{p^s}_{k+1},$$
Here, by convention, $V^{1/p}_{k+1} = U_{k+1}$.

§3. Applications

Fix a nonnegative integer r. Let $\alpha_i = \alpha_i(r)$ denote the i-th coefficient in p-adic expansion of r. That means
$$r = \alpha_0 p^0 + \alpha_1 p^1 + \dots$$
with $0 \le i < p$, $i \ge 0$. Set $\alpha_i = 0$ for $i < 0$.

The aim of the section is to prove the following four theorems:

THEOREM 3.1. *Set* $c = \frac{(h-1)!}{(h-\alpha_{k-1})!\prod_{0\le i<k}(\alpha_i-\alpha_{i-1})!}$, $t_i = \alpha_i - \alpha_{i-1}$, $0 \le i < k$. *We have*

$$P^r U_{k+1} = \begin{cases} c\Big(hU_{k+1} + \sum_{u=0}^{k-1} t_u V_{k+1}\tilde{M}_{k,u}Q_{k,u}^{-1}\Big)\prod_{i=0}^{k-1} Q_{k,i}^{t_i}, & 2r < p^k,\ t_i \ge 0,\ i < k, \\ 0, & \textit{otherwise}. \end{cases}$$

THEOREM 3.2. *Set* $c = \frac{(h-1)!}{(h-\alpha_{n-1})!(\alpha_s+1-\alpha_{s-1})!\prod_{s\ne i<n}(\alpha_i-\alpha_{i-1})!}$, $t_i = (h-\alpha_s)(\alpha_i - \alpha_{i-1})$, $-1 \le i < s$, $t_s = (h-\alpha_s)(\alpha_s + 1 - \alpha_{s-1})$, $t_i = \frac{(s+1)(\alpha_s+1)}{|s|+1}(\alpha_i - \alpha_{i-1})$, $i > s$, *with* $-1 \le s \le n-1$. *We have*

$$P^r\tilde{M}_{n,s} = \begin{cases} c\sum_{u=0}^{n-1} t_u\tilde{M}_{n,u}Q_{n,u}^{t_u-1}\prod_{u\ne i<n} Q_{n,i}^{t_i}, & 2r \le p^n + [-2p^s], \alpha_i \ge \alpha_{i-1}, \\ & s \ne i < n,\ \alpha_s + 1 \ge \alpha_{s-1}, \\ 0, & \textit{otherwise}. \end{cases}$$

The following two theorems were first proved in [10] by another method.

THEOREM 3.3 (Hung-Minh [10]).

$$P^r V_{k+1} = \begin{cases} V_{k+1}^p, & r = p^k, \\ \dfrac{(-1)^{\alpha_{k-1}}\alpha_{k-1}!}{\prod_{0\le i<k}(\alpha_i - \alpha_{i-1})!}V_{k+1}\prod_{i=0}^{k-1} Q_{k,i}^{\alpha_i-\alpha_{i-1}}, & r < p^k,\ \alpha_i \ge \alpha_{i-1},\ i < k, \\ 0, & \textit{otherwise}. \end{cases}$$

THEOREM 3.4 (Hung-Minh [10]). *Set* $c = \frac{(-1)^{\alpha_{n-1}}\alpha_{n-1}!(\alpha_s+1)}{(\alpha_s+1-\alpha_{s-1})!\prod_{s\ne i<n}(\alpha_i-\alpha_{i-1})!}$. *Then*

$$P^r Q_{n,s} = \begin{cases} Q_{n,s}^p, & r = p^n - p^s, \\ cQ_{n,s}\prod_{0\le i<n} Q_{n,i}^{\alpha_i-\alpha_{i-1}}, & r < p^n - p^s,\ \alpha_i \ge \alpha_{i-1}, \\ & s \ne i < n,\ \alpha_s + 1 \ge \alpha_{s-1}, \\ 0, & \textit{otherwise}. \end{cases}$$

To prove these theorems we need

NOTATION 3.5. Let $R = (r_1, \dots, r_n)$ be a sequence of arbitrary integers and $b \ge 0$. Denote by $|R| = \sum_{i=1}^n (p^i - 1)r_i$, and $\binom{b}{R}$ the coefficient of $y_1^{r_1}\dots y_n^{r_n}$ in $(1+y_1+\dots+y_n)^b$. That means,

$$\binom{b}{R} = \begin{cases} \dfrac{b!}{(b-r_1-\dots-r_n)!r_1!\dots r_n!}, & r_i \ge 0,\ r_1 + \dots + r_n \le b, \\ 0, & \text{otherwise}. \end{cases}$$

The proofs of Theorems 3.1 and 3.3 are based on the duality theorem and the following

LEMMA 3.6. *Let b be a nonnegative integer and $\varepsilon = 0, 1$. We then have*

$$St^{S,R}(x^\varepsilon y^b) = \begin{cases} \binom{b}{R} x^\varepsilon y^{b+|R|}, & S = \emptyset, \\ \varepsilon \binom{b}{R} y^{b+|R|+p^u}, & S = (u),\ u \geq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Here x and y are the generators of $H^*(BE^1) = E(x) \otimes P(y)$,

Proof. A direct computation using Proposition 1.1 shows that

$$\begin{aligned} d_m^* P_m(x^\varepsilon y^b) &= (-1)^{mb}(h!)^{m\varepsilon}\big(\tilde{L}_m^\varepsilon x^\varepsilon + \varepsilon \sum_{u=0}^{m-1} (-1)^{u+1} \tilde{M}_{m,u} y^{p^u}\big) \\ &\quad \big(\sum_{R=(r_1,\dots,r_m)} (-1)^{r(\emptyset,R)} \binom{b}{R} \tilde{Q}^{R^*_{2b}} y^{b+|R|}\big) \\ &= \mu(2b+\varepsilon)^m \Big(\sum_{R=(r_1,\dots,r_m)} (-1)^{r(\emptyset,R)} \binom{b}{R} \tilde{Q}^{R^*_{2b+\varepsilon}} x^\varepsilon y^{b+|R|} \\ &\quad + \varepsilon \sum_{u=0}^{m-1} \sum_{R=(r_1,\dots,r_m)} (-1)^{r((u),R)} \binom{b}{R} \tilde{M}_{m,u} \tilde{Q}^{R^*_{2b}} y^{b+|R|+p^u} \Big). \end{aligned}$$

The lemma now follows from Theorem 1.2.

Proof of Theorem 3.1. Since $\dim U_{k+1} = p^k$, it is clear that $P^r U_{k+1} = 0$ for $2r > p^k$. Suppose $r \leq (p^k - 1)/2$. Applying Corollary 2.3 with $\delta = n = 1$ and using Lemma 3.6 we obtain

$$\begin{aligned} P^r U_{k+1} &= \sum_{R=(r_1,\dots,r_k)} (-1)^{r(\emptyset,R)+r+hk} \langle \tilde{q}_{(r)^*_{p^k}}, St^{\emptyset,R} \tilde{L}_1 \rangle U_{k+1} \tilde{Q}^{R^*_{p-1}} \\ &\quad + \sum_{u=0}^{k-1} \sum_R (-1)^{r((u),R)+r+kh+1} \langle \tilde{q}_{(r)^*_{p^k}}, St^{(u),R} \tilde{M}_{1,0} \rangle V_{k+1} \tilde{M}_{k,u} \tilde{Q}^{R^*_{p-3}}\ . \end{aligned}$$

Set $\bar{r}_i = \alpha_i - \alpha_{i-1}$, $i < k$, $\bar{r}_k = h - \alpha_k$, $\bar{R}_0 = (\bar{r}_1, \dots, \bar{r}_k)$, $\bar{R}_u = (\bar{r}_1, \dots, \bar{r}_u - 1, \dots, \bar{r}_k)$, $1 \leq u \leq k$. Computing directly from Lemma 3.6 with $\varepsilon = 0$, $b = h$ or $\varepsilon = 1$, $b = h - 1$ gives

$$\langle \tilde{q}_{(r)^*_{p^k}}, St^{\emptyset,R} \tilde{L}_1 \rangle = \begin{cases} \dfrac{h!}{\bar{r}_0 \dots \bar{r}_k}, & R = \bar{R}_0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\langle \tilde{q}_{(r)^*_{p^k}}, St^{(u),R} \tilde{M}_{1,0} \rangle = \begin{cases} \dfrac{(h-1)!\bar{r}_u}{\bar{r}_0 \dots \bar{r}_k}, & R = \bar{R}_u \\ 0, & \text{otherwise.} \end{cases}$$

A simple computation shows that

$$r(\emptyset, \bar{R}_0) + r = r((u), \bar{R}_u) + r + 1 = hk \pmod 2.$$

Hence, the theorem is proved.

Proof of Theorem 3.3. Since $\dim V_{k+1} = 2p^k$, we have only to prove the theorem for $r < p^k$. Note that $Q_{1,1} = 1$. Hence

$$St^{S,R}Q_{1,1} = \begin{cases} 1, & S = \emptyset,\ R = (0,\dots,0), \\ 0, & \text{otherwise}. \end{cases}$$

So, $\langle \tilde{q}_{(r)^*_{2p^k}}, St^{S,R}Q_{1,1}\rangle = 0$ for any S, R. Remember that $Q_{1,0} = y^{p-1}$. So, applying Corollary 2.3 with $\delta = 0$, $n = 1$ and using Lemma 3.6 with $\varepsilon = 0$, $b = p-1$, we get

(a) $$P^r V_{k+1} = \sum_R (-1)^{r(\emptyset,R)+r} \langle \tilde{q}_{(r)^*_{2p^k}}, St^{\emptyset,R}Q_{1,0}\rangle V_{k+1}\tilde{Q}^{R^*_{2p-2}}.$$

From Lemma 3.6, we see that it implies

(b)
$$\langle \tilde{q}_{(r)^*_{2p^k}}, St^{\emptyset,R}Q_{1,0}\rangle = \begin{cases} \begin{pmatrix} p-1 \\ R \end{pmatrix}, & R = (\alpha_1 - \alpha_0, \dots, \alpha_{k-1} - \alpha_{k-2}, p-1-\alpha_{k-1}), \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $R = (\alpha_1 - \alpha_0, \dots, \alpha_{k-1} - \alpha_{k-2}, p-1-\alpha_{k-1})$. Then we can easily observe that

(c)
$$\begin{aligned} &r(\emptyset, R) + r = 0 \pmod 2, \\ &\begin{pmatrix} p-1 \\ R \end{pmatrix} = \frac{(-1)^{\alpha_{k-1}}\alpha_{k-1}!}{\prod_{0\le i<k}(\alpha_i - \alpha_{i-1})}, \\ &R^*_{2p-2} = (2\alpha_0, \alpha_1 - \alpha_0, \dots, \alpha_{k-1} - \alpha_{k-2}). \end{aligned}$$

Theorem 3.3 now follows from (a), (b) and (c).

Following Corollary 2.2, to determine $P^r\tilde{M}_{n,s}$ and $P^rQ_{n,s}$ we need to compute the action of $St^{S,R}$ on U_2 and V_2.

PROPOSITION 3.7. *Suppose given $R = (r_1, \dots, r_n)$, and $0 \le u < n$. Set $r_{u,s} = r_{s+1} + \dots + r_n$, for $s \ge u$, and $r_{u,s} = r_{s+1} + \dots + r_n - h$, for $s < u$. Then we have*

$$St^{S,R}U_2 = \begin{cases} \left(\dfrac{h}{R}\right)\left(\tilde{L}_1^{|R|/h}U_2 + \displaystyle\sum_{s=0}^{n-1} h^{-1}r_{0,s}\tilde{M}_{1,0}\tilde{L}_1^{(|R|-p^{s+1}+1)/h}V_2^{p^s}\right), & S = \emptyset, \\ \left(\dfrac{h}{R}\right)\displaystyle\sum_{s=0}^{n-1} h^{-1}r_{u,s}\tilde{L}_1^{(|R|-p^{s+1}+p^u+h)/h}V_2^{p^s}, \quad S = (u),\ u < n, & \\ 0, & \textit{otherwise}. \end{cases}$$

Here, $|R|$ and $\left(\frac{h}{R}\right)$ are defined in Notation 3.5.

The proposition will be proved by using Theorem 1.2 and the following

LEMMA 3.8. *Let u, v be nonnegative integers with $u \le v$. We have*

(i) $[u, v] = \sum_{s=u}^{v-1} V_1^{p^v - p^{s+1} + p^u} V_2^{p^s}$.

(ii) $[1; v] = V_1^{p^v - h}U_2 + M_{1,0}\sum_{s=0}^{v-1} V_1^{p^v - p^{s+1}}V_2^{p^s}$.

Here $[u, v]$ and $[1; v]$ are defined in §1.

The proof is straightforward.

Proof of Proposition 3.7. Recall that $M_{2,1} = x_1y_2 - x_2y_1$. From Proposition 1.1 we directly obtain

$$d_n^* P_n M_{2,1} = (-h!)^n \sum_{v=0}^{n} (-1)^v \tilde{L}_n Q_{n,v}[1;v] + \sum_{\substack{0\le u<n \\ 0\le v\le n}} (-1)^{u+v+1} \tilde{M}_{n,u} Q_{n,v}[u,v].$$

Since $L_1 = y_1$ and $2(h-1) = p-3$, using Proposition 1.1(iii) with $y = y_1$ and Notation 3.5 we get

$$d_n^* P_n L_1^{h-1} = (-1)^{n(h-1)} \sum_{R'} (-1)^{r(\emptyset,R')} \binom{h-1}{R'} \tilde{Q}^{R'^*_{p-3}} y_1^{|R'|+h-1}.$$

We have $U_2 = M_{2,1}L_1^{h-1}$, $\dim U_2 = p$ and $\mu(p) = (-1)^h h!$. So, it implies from the above equalities and Proposition 1.1 that

$$\begin{aligned} d_n^* P_n U_2 = {} & \mu(p)^n \Big(\sum_R (-1)^{r(\emptyset,R)} \tilde{Q}^{R^*_p} \binom{h}{R} \sum_{v=0}^{n} h^{-1} r_v y_1^{|R|+h-p^v} [1;v] \\ & + \sum_{u=0}^{n-1} \sum_R (-1)^{r((u),R)} \tilde{M}_{n,u} \tilde{Q}^{R^*_{p-1}} \binom{h}{R} \sum_{v=u}^{n} h^{-1} r_v y_1^{|R|+h-p^v} [u;v] \Big). \end{aligned}$$

Then by Theorem 1.2 we have

$$St^{S,R} U_2 = \begin{cases} h^{-1} \binom{h}{R} \sum_{v=0}^{n} r_v y_1^{|R|+h-p^v} [1;v], & S = \emptyset, \\ h^{-1} \binom{h}{R} \sum_{v=u}^{n} r_v y_1^{|R|+h-p^v} [u;v], & S = (u),\ u < n, \\ 0, & \text{otherwise.} \end{cases}$$

Now the proposition follows from Lemma 3.8.

Proof of Theorem 3.2. For simplicity, we assume that $0 \le s < n$. Applying Corollary 2.2 with $\delta = k = 1$ and using Proposition 3.7 we get

$$\text{(a)} \qquad P^r \tilde{M}_{n,s} = \sum_{\substack{0\le u<n \\ R=(r_1,\dots,r_n)}} (-1)^{r((u),R)+r+s+1+nh} C_{(u),R} \tilde{M}_{n,u} \tilde{Q}^{R^*_{p-1}}.$$

Here $C_{(u),R} = \langle \tilde{q}_{(r)}{}^*_{p^n-2p^s} \otimes \gamma_{p^s}(v_2), St^{(u),R} U_2 \rangle$.

If $2r > p^n - 2p^s - 1$ then $P^r \tilde{M}_{n,s} = 0$ since $\dim \tilde{M}_{n,s} = p^n - 2p^s$. Suppose $2r \le p^n - 2p^s - 1$. Set $\bar{r}_i = \alpha_i - \alpha_{i-1}$, for $0 \le i \ne s,\ n$, $\bar{r}_s = \alpha_s + 1 - \alpha_{s-1}$, $r_n = h - \alpha_{n-1}$, $\bar{R}_0 = (\bar{r}_1, \dots, \bar{r}_n)$, $\bar{R}_u = (\bar{r}_1, \dots, \bar{r}_u - 1, \dots, \bar{r}_n)$, $1 \le u \le n$. From Propositton 3.7 we have

$$C_{(u),R} = \begin{cases} ct_u, & R = \bar{R}_u, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$r((u), \bar{R}_u) + r + s + 1 = nh \pmod 2.$$

Theorem 3.2 now is proved by combining the above equalities.

Now we return to the proof of Theorem 3.4. It is proved by the same argument as given in the proof of Theorem 3.2. We only compute $St^{S,R} V_2$.

PROPOSITION 3.9. *For $R=(r_1,\dots,r_n)$, $r_0=p-r_1-\dots-r_n$, we have*

$$St^{\emptyset,R}V_2=\begin{cases} V_2^{p^s}, & r_s=p,\ r_i=0,\ i\neq s,\\ \displaystyle\sum_{s=0}^{n-1}\frac{(p-1)!(r_{s+1}+\dots+r_n)}{r_0!\dots r_n!}V_1^{|R|+p-p^{s+1}}V_2^{p^s}, & 0\le r_i<p,\ 0\le i\le n,\\ 0, & \textit{otherwise}.\end{cases}$$

Proof. Recall that $V_2=y_2^p-y_2y_1^{p-1}$. Applying Proposition 1.1 and Lemma 3.6 with $y=y_1$ or $y=y_2$ we get

(a)
$$\begin{aligned} d_n^*P_nV_2 &= \sum_{s=0}^{n}(-1)^{n+s}Q_{n,s}^py_2^{p^{s+1}}\\ &\quad -(-1)^n\sum_{u=0}^{n}\sum_{R'}(-1)^{u+r(\emptyset,R')}\binom{p-1}{R'}Q_{n,u}\tilde{Q}^{R'^*_{2p-2}}y_1^{|R'|+p-1}y_2^{p^u}\\ &= (-1)^n\sum_{s=0}^{n}(-1)^sQ_{n,s}^p(y_2^{p^{s+1}}-y_2^{p^s}y_1^{(p-1)p^s})\\ &\quad -(-1)^n\sum_{u=0}^{n}\sum_{R}(-1)^{r(\emptyset,R)}\binom{p-1}{R_u}\tilde{Q}^{R^*_{2p}}y_1^{|R|+p-p^u}y_2^{p^u}\Big). \end{aligned}$$

Here the last sum runs over all $R=(r_1,\dots,r_n)$ with $0\le r_i<p$, $0\le i\le n$, $R_0=R$, $R_u=(r_1,\dots,r_u-1,\dots,r_n)$, $1\le u\le n$.

Let v be the greatest index such that $r_v>0$. A simple computation shows

(b)
$$y_1^{|R|+p-p^u}y_2^{p^u}=-y_1^{|R|+p-p^u-p^v}[u,v]+y_1^{|R|+p-p^v}y_2^{p^v}.$$

Combining (a), (b), Lemma 3.8 and the fact that $\sum_{u=0}^{n}\binom{p-1}{R_u}=0$ we obtain

$$\begin{aligned} d_n^*P_nV_2 &= \mu(2p)^n\Big(\sum_{s=0}^{n}(-1)^sQ_{n,s}V_2^{p^s}\\ &\quad +\sum_{R}(-1)^{r(\emptyset,R)}\tilde{Q}^{R^*_{2p}}\sum_{s=0}^{n}\sum_{u=s+1}^{n}\binom{p-1}{R_u}y_1^{|R|+p-p^{s+1}}V_2^{p^s}\Big). \end{aligned}$$

The proposition now follows from this equality and Theorem 1.2.

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