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KODAI MATH. J.
17 (1994), 329—340

SOME ESTIMATES FOR MEROMORPHIC FUNCTIONS SHARING FOUR VALUES

Dedicated to Professor Nobuyuki Suita on the occasion of his 60th birthday

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1. Introduction

In this paper the term "meromorphic function" will mean a meromorphic function in $\boldsymbol{C}$. We will use the standard notations of Nevanlinna theory: $T(r, f)$, $m(r, c, f)$, $N(r, c, f)$, $\bar{N}(r, c, f)$, $N_k(r, c, f)$, $\bar{N}_k(r, c, f)$ $(c\in\boldsymbol{C}\cup\{\infty\},\ k=1, 2, \cdots)$, and we assume that the reader is familiar with the basic results in Nevanlinna theory as found in [3]. Further, we will use the notations and terminology defined in the following (i)-(vi):

(i) Let f and g be distinct nonconstant meromorphic functions. For $r>0$, put $T(r)=\max\{T(r, f), T(r, g)\}$. We write $\sigma(r)=S(r)$ for every function $\sigma:(0, \infty)\to(-\infty, \infty)$ satisfying $\sigma(r)/T(r)\to 0$ for $r\to\infty$ possibly outside a set of finite Lebesgue measure.

(ii) For two nonconstant meromorphic functions f, g and $c\in\boldsymbol{C}\cup\{\infty\}$ we denote by $\bar{n}(r, c)=\bar{n}(r, c; f, g)$ (resp. $\bar{n}_1(r, c)=\bar{n}_1(r, c; f, g)$, $\bar{n}_3(r, c)=\bar{n}_3(r, c; f, g)$) the number of distinct roots of at least one of the equations $f=c$ and $g=c$ in $|z|\leqq r$ (resp. the number of distinct common roots of $f=c$ and $g=c$ with the same multiplicities in $|z|\leqq r$, the number of distinct c-points of f or g which are not common to f and g in $|z|\leqq r$). We write

$$\bar{N}(r, c)=\bar{N}(r, c; f, g)=\int_0^r \{\bar{n}(t, c)-\bar{n}(0, c)\}/t\, dt+\bar{n}(0, c)\log r,$$

$$\bar{N}_j(r, c)=\bar{N}_j(r, c; f, g)=\int_0^r \{\bar{n}_j(t, c)-\bar{n}_j(0, c)\}/t\, dt+\bar{n}_j(0, c)\log r \quad (j=1, 3)$$

and

$$\bar{N}_2(r, c)=\bar{N}(r, c)-\bar{N}_1(r, c).$$

Further, for a complex number $a(\neq 0, 1)$ we write

$$\bar{N}(r)=\bar{N}(r, 0)+\bar{N}(r, 1)+\bar{N}(r, \infty)+\bar{N}(r, a),$$

$$\bar{N}_j(r)=\bar{N}_j(r, 0)+\bar{N}_j(r, 1)+\bar{N}_j(r, \infty)+\bar{N}_j(r, a) \qquad (j=1, 2).$$

AMS classification 30D35
Received May 12, 1993.

(iii) We say that *f and g share the value c* IM'' (resp. CM'') if $\bar{N}_3(r, c)=S(r)$ (resp. $\bar{N}_2(r, c)=S(r)$).

These notions IM'' and CM'' are slight generalizations of IM (See [2, p 545].) and CM (See [2, p 545].) (or "CM" (See [5, p 172].), respectively. It is easily seen that Theorem F, Corollary 1, Lemma 2 in [2] and Theorem B′ in [5] remain valid if IM and "CM" are replaced by IM'' and CM'', respectively. And so, using the argument of the proof of Theorem 1 in [5], we see that Theorems E and 2 in [2] are still true if IM and CM are replaced by IM'' and CM'', respectively. The function ϕ defined in [4, 5, 7] satisfies $N(r, \infty, \phi)=S(r)$ and $m(r, \infty, \phi)=S(r)$ if IM is replaced by IM'', and hence Lemma 2 in [7] remains true if IM is replaced by IM''. (For convenience sake we state in § 3 only a part of these facts without proof- which will be used to prove our results.)

(iv) For a given complex number $a(\neq 0, 1)$ and two nonconstant meromorphic functions f and g we write

$n^{(k)}(r, c)=\#\{z_c\in \boldsymbol{C};\ |z_c|\leqq r,\ z_c$ is a c-point with multiplicity p for f and with multiplicity q for g, where p and q satisfy $\max(p, q)\geqq k+1$. z_c is counted $\max(p, q)-k$ times.$\}$
$(c=0, 1, \infty, a;\ k=0, 1, 2)$, and

$$N^{(k)}(r, c)=\int_0^r \{n^{(k)}(t, c)-n^{(k)}(0, c)\}/t\, dt+n^{(k)}(0, c)\log r$$

$(c=0, 1, \infty, a;\ k=0, 1, 2)$.

(v) Let $\bar{N}(r, f=g=c$ with $(p, q))$ denote the counting function of the c-points with multiplicity p for f and with multiplicity q for g, each point counted once. Further, we denote such a c-point by $z_c(p, q)$.

(vi) Let $a(\neq 0, 1)$ be a complex number, and let f and g be two nonconstant meromorphic functions. For $c=0, 1, a, \infty$ we use the following notation:

$\bar{n}_{11}(r, c)=\#\{z_c\in \boldsymbol{C};\ |z_c|\leqq r,\ f(z_c)=g(z_c)=c,\ f'(z_c)=-g'(z_c)\neq 0,\ f''(z_c)=g''(z_c),\ f'''(z_c)=-g'''(z_c),\ 3f''(z_c)+2\kappa(c)(f'(z_c))^2=0\}$
$(c=0, 1, a)$, where

$$\kappa(c)=(a+1)(c-1)\{(a-2)c-a(a-1)\}/a^2(a-1)+(a-2)c\{(a+1)c-(a^2+1)\}/a(a-1)^2$$
$$=\begin{cases}(a+1)/a & (c=0)\\ (2-a)/(a-1) & (c=1),\\ (1-2a)/a(a-1) & (c=a)\end{cases}$$

$\bar{n}_{11}(r, \infty)=\#\{z_\infty\in \boldsymbol{C};\ |z_\infty|\leqq r,\ f(z)=\alpha/(z-z_\infty)+(a+1)/3+\beta(z-z_\infty)+\cdots,\ g(z)=-\alpha/(z-z_\infty)+(a+1)/3-\beta(z-z_\infty)+\cdots,\ \alpha\neq 0\}$
(α and β may depend on z_∞),

$$\bar{N}_{11}(r, c)=\int_0^r \{\bar{n}_{11}(t, c)-\bar{n}_{11}(0, c)\}/t\,dt+\bar{n}_{11}(0, c)\log r.$$

Further, we write

$$\bar{N}_{11}(r)=\bar{N}_{11}(r, 0)+\bar{N}_{11}(r, 1)+\bar{N}_{11}(r, \infty)+\bar{N}_{11}(r, a).$$

2. Results

In this section we give some estimates for meromorphic functions sharing four values IM''. Without loss of generality we may assume that these four shared values are 0, 1, ∞ and a.

In Theorems 1-4 we assume that f and g are distinct nonconstant meromorphic functions sharing four values 0, 1, ∞ and a IM''. Further, we assume in Theorems 1-3 that g is not any Möbius transformation of f.

The following Theorem 1 is a refinement of a well known theorem of R. Nevanlinna [6, p 122].

THEOREM 1. $2T(r)+3\bar{N}_{11}(r)\leqq 3\bar{N}_2(r)+S(r)$.

Our Theorem 2 contains a corresponding result to an author's uniqueness theorem for meromorphic functions sharing three values CM [9, Theorem 2].

THEOREM 2. $$2T(r)+3\{\bar{N}_{11}(r, 1)+\bar{N}_{11}(r, a)\}+\{N^{(2)}(r, 1)+N^{(2)}(r, a)\}$$
$$\leqq 3\{\bar{N}(r, 0)+\bar{N}(r, \infty)\}+2\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r).$$

Our Theorem 3 is a refinement of so called 2-2-Theorem of G. Gundersen [2].

THEOREM 3. (i) *There exists a positive constant* K_1 *satisfying*

$$2T(r)+3\bar{N}_{11}(r)+\{N^{(2)}(r, 1)+N^{(2)}(r, a)\}$$
$$\leqq K_1\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r).$$

We may take $K_1=11$ *if* $a=-1$, *and* $K_1=17$ *otherwise.*

(ii) *Particularly if* $\bar{N}_1(r, 0)=\bar{N}_{11}(r, 0)+S(r)$ *and* $\bar{N}_1(r, \infty)=\bar{N}_{11}(r, \infty)+S(r)$ *hold, then we have*

$$2T(r)+3\{\bar{N}_{11}(r, 1)+\bar{N}_{11}(r, a)\}+\{N^{(2)}(r, 1)+N^{(2)}(r, a)\}$$
$$\leqq(13/2)\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r).$$

Finally, using a method of E. Mues [4], we prove

THEOREM 4. (i) *There exist positive constants* K_2 *and* K_3 *satisfying*

$$T(r)\leqq K_2\{\bar{N}_1(r, 0)+\bar{N}_1(r, 1)+\bar{N}_1(r, a)\}+K_3\bar{N}(r, \infty)+S(r).$$

We may take $(K_2, K_3)=(11/2, 19/2)$ *if* $a=-1, 1/2$ *or* 2, *and* $(K_2, K_3)=(10, 17)$ *otherwise.*

(ii) *Particularly if* $\bar{N}(r, f=g=c$ *with* $(2, 1))+\bar{N}(r, f=g=c$ *with* $(1, 2))=S(r)$ *for* $c=0, 1$ *and* a, *then*

$$T(r)\leqq \bar{N}_1(r, 0)+\bar{N}_1(r, 1)+\bar{N}_1(r, a)+2\bar{N}(r, \infty)+S(r).$$

(iii) *Assume that* $\bar{N}_2(r, c, f)+\bar{N}_2(r, c, g)=S(r)$ *for* $c=0, 1$ *and* a, *and that for each* $c=0, 1$ *and* a $\bar{N}(r, f=g=c$ *with* $(2, 1))=S(r)$ *or* $\bar{N}(r, f=g=c$ *with* $(1, 2))=S(r)$. *Suppose further that if* $\#\{z_c(2, 1)\}\geqq 2$ (*resp.* $\#\{z_c(1, 2)\geqq 2$), *then* $g'(z_c(2, 1))=$const. $C_{c(2,1)}$ *for all* $z_c(2, 1)$*'s* (*resp.* $f'(z_c(1, 2))(=$const. $C_{c(1,2)}$ *for all* $z_c(1, 2)$*'s*). *Then we have*

$$T(r)\leqq 2\{\bar{N}_1(r, 0)+\bar{N}_1(r, 1)+\bar{N}_1(r, a)+\bar{N}(r, \infty)\}+S(r).$$

Remark 1. The estimates of Theorems 1, 2 and 3 (ii) are sharp in the case of $a=-\omega$, where $\omega(\neq 1)$ is a third root of 1. Consider the two functions F, G and two values A, B in [8, p 94]. If we put $f=F/A$, $g=G/A$, and $a=B/A(=-\omega)$, then f and g are distinct meromorphic functions sharing four values 0, 1, a, ∞ *IM*, and g is not any Möbius transformation of f. Further, f and g satisfy the estimates of Theorems 1, 2 and 3 (ii) with equality.

Remark 2. The estimate of Theorem 4 (ii) is sharp in the case of $a=-1$, 1/2 or 2. For $a=-1$ this is illustrated by the pair of F and G in [7, Theorem 1]. For $a=1/2$ or 2 we can obtain an example from these F and G with the aid of a Möbius transformation.

Remark 3. The estimate of Theorem 4 (iii) is sharp in the case of $a=-1/8$, -8, 1/9, 9, 8/9 or 9/8. For $a=-1/8$ this is illustrated by the pair of $f=(e^z+1)/(e^z-1)^2$ and $g=(e^z+1)^2/8(e^z-1)$. This example is due to G. Gundersen [1]. For $a=-8$, 1/9, 9, 8/9 or 9/8 we can obtain an example from these f and g with the aid of a Möbius transformation.

3. Lemmas

In this section, we assume that f and g are distinct nonconstant meromorphic functions sharing four values 0, 1, ∞, a *IM″*. Then the following (3.1) (3.8) hold:

(3.1) $T(r, f)=T(r)+S(r)$, $\quad T(r, g)=T(r)+S(r)$;

(3.2) $\bar{N}(r)=2T(r)+S(r)$;

(3.3) $N(r, 0, f-g)=\bar{N}(r, 0, f-g)+S(r)=\bar{N}(r, 0)+\bar{N}(r, 1)+\bar{N}(r, a)+S(r)$;

(3.4) $N^{(0)}(r, \infty)=N(r, \infty, f-g)+S(r)$,

$N(r, \infty, f)+N(r, \infty, g)=N^{(0)}(r, \infty)+\bar{N}(r, \infty)+S(r)$;

(3.5) *If* $N_0(r, 0, f')$ *refers only to those roots of* $f'=0$ *such that* $f\neq 0$, 1 *and* a, *and if* $N_0(r, 0, g')$ *is similarly defined, then* $N_0(r, 0, f')=S(r)$ *and* $N_0(r, 0, g')=S(r)$ (cf. [2, Lemma 2]);

(3.6) *The function*

$$\phi_5=\frac{f'g'(f-g)^2}{fg(f-1)(g-1)(f-a)(g-a)}$$

satisfies $T(r, \phi_5)=S(r)$ (cf. [4, p 113]);

(3.7) $m(r, \infty, f'/(f-g))+N_1(r, \infty, f'/(f-g))=S(r)$,

$m(r, \infty, g'/(g-f))+N_1(r, \infty, g'/(g-f))=S(r)$ (cf. [7, Lemma 2]);

(3.8) *Suppose further that* f *and* g *share two values of* 0, 1, ∞, a *CM"*. *Then* $a=-1$, 1/2 *or* 2. *In this case* f *and* g *is connected with exactly one of the following relations*: $f+g\equiv 0\,(a=-1)$, $fg\equiv 1\,(a=-1)$, $f+g\equiv 1\,(a=1/2)$, $(f-1/2)(g-1/2)\equiv 1/4\,(a=1/2)$, $f+g\equiv 2\,(a=2)$, $(f-1)(g-1)\equiv 1\,(a=2)$ (cf. [5, Theorem 1]).

For (3.1), (3.2) and the second equation of (3.4), see [4, Hilfssatz 1]. (3.3) and the first equation of (3.4) are easily verified by using the function ϕ-which we denote by ϕ_5 in this paper-in the proof of [4, Hilfssatz 1].

4. Proof of Theorems

4.1. *Proof of Theorem* 1. Consider the functions

$$\phi_1=\frac{f'(f-1)}{f(f-a)}-\frac{g'(g-1)}{g(g-a)}, \tag{4.1}$$

$$\phi_2=\frac{f'(f-a)}{f(f-1)}-\frac{g'(g-a)}{g(g-1)}, \tag{4.2}$$

$$\phi_3=\frac{f'f}{(f-1)(f-a)}-\frac{g'g}{(g-1)(g-a)}, \tag{4.3}$$

$$\phi_4=\frac{f'}{f(f-1)(f-a)}-\frac{g'}{g(g-1)(g-a)}. \tag{4.4}$$

If $\phi_k\equiv 0$ for $k=1, 2, 3$ or 4, then it is easily seen that f and g share four values 0, 1, a, ∞ *CM"*. Hence by (3.8) g is a Möbius transformation of f, which contradicts our assumption.

Now, consider the case $\phi_k\not\equiv 0$ for all $k=1, 2, 3, 4$. We first note from the fundamental estimate of the logarithmic derivative and (3.1) that

$$m(r, \phi_k)=S(r) \quad \text{for} \quad k=1, 2, 3, 4. \tag{4.5}$$

By substituting into (4.1) the Taylor expansions of f and g at a $z_1\,(1, 1)$ which is counted into $\bar{n}_{11}(r, 1)$, we see that ϕ_1 has a zero whose multiplicity is at least four at this point. Hence from (4.1), (3.1), the first fundamental theorem and (4.5) it follows that

$$\text{(4.6)} \qquad \bar{N}(r, 1)+3\bar{N}_{11}(r, 1)\leqq N(r, 0, \phi_1)+S(r)\leqq T(r, \phi_1)+S(r)$$
$$=N(r, \infty, \phi_1)+S(r)\leqq \bar{N}_2(r, 0)+\bar{N}_2(r, a)+\bar{N}_2(r, \infty)+S(r).$$

In the same way, we deduce from (4.2), (4.3) and (4.4) that

$$\text{(4.7)} \qquad \bar{N}(r, a)+3\bar{N}_{11}(r, a)\leqq \bar{N}_2(r, 0)+N_2(r, 1)+\bar{N}_2(r, \infty)+S(r),$$

$$\text{(4.8)} \qquad \bar{N}(r, 0)+3\bar{N}_{11}(r, 0)\leqq \bar{N}_2(r, 1)+\bar{N}_2(r, a)+\bar{N}_2(r, \infty)+S(r)$$

and

$$\text{(4.9)} \qquad \bar{N}(r, \infty)+3\bar{N}_{11}(r, \infty)\leqq \bar{N}_2(r, 0)+\bar{N}_2(r, 1)+\bar{N}_2(r, a)+S(r),$$

respectively. Taking (3.2) into consideration, the combination of (4.6)-(4.9) yields the estimate of Theorem 1. ■

4.2. *Proof of Theorem* 2. Define ϕ_j $(j=1, 2)$ by (4.j). From the argument of the first part of the proof of Theorem 1 we deduce that $\phi_j\not\equiv 0$ $(j=1, 2)$. In this case, (4.6) and (4.7) hold, and so,

$$\text{(4.10)} \qquad \bar{N}_1(r, 1)+\bar{N}_1(r, a)+3\{\bar{N}_{11}(r, 1)+\bar{N}_{11}(r, a)\}$$
$$\leqq 2\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r).$$

Next, by (3.1), (3.2) and the first fundamental theorem

$$\text{(4.11)} \qquad N(r, 1, f)+N(r, a, f)+N(r, 1, g)+N(r, a, g)\leqq 4T(r)+S(r)$$
$$=2\{\bar{N}(r, 0)+\bar{N}(r, 1)+\bar{N}(r, a)+\bar{N}(r, \infty)\}+S(r).$$

Since $2\{\bar{N}(r, 1)+\bar{N}(r, a)\}+N^{(1)}(r, 1)+N^{(1)}(r, a)\leqq N(r, 1, f)+N(r, a, f)+N(r, 1, g)+N(r, a, g)+S(r)$ by the definition of IM'', it follows from (4.11) that

$$\text{(4.12)} \qquad N^{(1)}(r, 1)+N^{(1)}(r, a)\leqq 2\{\bar{N}(r, 0)+\bar{N}(r, \infty)\}+S(r).$$

Further, from (3.3) and the definition of $N^{(k)}(r, c)$ we deduce that

$$\text{(4.13)} \qquad \bar{N}_2(r, 1)+\bar{N}_2(r, a)+N^{(2)}(r, 1)+N^{(2)}(r, a)$$
$$=N^{(1)}(r, 1)+N^{(1)}(r, a)+S(r).$$

Thus the combination of (3.2), (4.10), (4.13) and (4.12) yields the estimate of Theorem 2. ■

4.3.1. *Proof of Theorem* 3. (i) Define ϕ_j $(j=1, 2)$ by (4.j), and further define ϕ_j $(j=5, 6, 7, 8, 9)$ as follows:

$$\phi_5=\frac{f'g'(f-g)^2}{fg(f-1)(g-1)(f-a)(g-a)},$$

(4.14) $$\phi_6=\left\{\frac{f''}{f'}-2\frac{f'}{f}-\frac{f'}{f-1}-\frac{f'}{f-a}\right\}-\left\{\frac{g''}{g'}-2\frac{g'}{g}-\frac{g'}{g-1}-\frac{g'}{g-a}\right\},$$

(4.15) $$\phi_7=\left\{\frac{f''}{f'}+2\frac{f'}{f}-\frac{f'}{f-1}-\frac{f'}{f-a}\right\}-\left\{\frac{g''}{g'}+2\frac{g'}{g}-\frac{g'}{g-1}-\frac{g'}{g-a}\right\},$$

(4.16) $$\phi_8=\phi_6^2-(1+a)^2\phi_5$$

and

(4.17) $$\phi_9=\phi_7^2-(1+a)^2\phi_5.$$

Consider the case that $\phi_j\equiv 0$ for $j=8$ or 9. Then from (4.16), (4.17) and (3.6) we have $T(r, \phi_j)=S(r)$ for $j=6$ or 7. Noting that $N(r, \infty, \phi_j)=\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)+S(r)$ ($j=6, 7$) from (4.14), (4.15) and (3.5), we obtain $\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)=S(r)$, which implies that g is a Möbius transformation of f in view of (3.8).

Now, assume that $\phi_j\not\equiv 0$ for $j=8$ and 9. The substitution into (4.16) (resp. (4.17)) of the Taylor (resp. Laurent) expansions of f and g at a $z_0(1, 1)$ (resp. $z_\infty(1, 1)$) gives $\phi_8(z_0(1, 1))=0$ (resp. $\phi_9(z_\infty(1, 1))=0$). (See [5, pp. 174-175].) Especially if $z_0(1, 1)$ (resp. $z_\infty(1, 1)$) is counted into $\bar{n}_{11}(r, 0)$ (resp. $\bar{n}_{11}(r, \infty)$), then ϕ_8 (resp. ϕ_9) has a zero whose multiplicity is at least two at this point. Hence, from (3.1), (3.3)-(3.6), the first fundamental theorem and the fundamental estimate of the logarithmic derivative it follows that

(4.18) $$\bar{N}_1(r, 0)+\bar{N}_{11}(r, 0)\leqq N(r, 0, \phi_8)+S(r)\leqq T(r, \phi_8)+S(r)$$
$$=2T(r, \phi_6)+S(r)=2\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r)$$

and

(4.19) $$\bar{N}_1(r, \infty)+\bar{N}_{11}(r, \infty)\leqq N(r, 0, \phi_9)+S(r)\leqq T(r, \phi_9)+S(r)$$
$$=2T(r, \phi_7)+S(r)=2\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r).$$

Combining (4.18) and (4.19) with the estimate of Theorem 2, we have

(4.20) $$2T(r)+3\bar{N}_{11}(r)+N^{(2)}(r, 1)+N^{(2)}(r, a)$$
$$\leqq K_1\{\bar{N}_2(r, 0)+\bar{N}_2(r, \infty)\}+S(r)$$

with $K_1=17$.

Finally, we consider the case $a=-1$. In this case we note that $\phi_6(z_0(1, 1))=0$ and $\phi_7(z_\infty(1, 1))=0$ (Especially if $z_0(1, 1)$ (resp. $z_\infty(1, 1)$) is counted into $\bar{n}_{11}(r, 0)$ (resp. $\bar{n}_{11}(r, \infty)$), then ϕ_6 (resp. ϕ_7) has a zero whose multiplicity is at least two at this point.), and so the above estimates (4.18) and (4.19) can be replaced by

(4.18)′ $$\overline{N}_1(r, 0)+\overline{N}_{11}(r, 0)\leqq N(r, 0, \phi_6)+S(r)\leqq T(r, \phi_6)+S(r)$$
$$=\overline{N}_2(r, 0)+\overline{N}_2(r, \infty)+S(r)$$

and

(4.19)′ $$\overline{N}_1(r, \infty)+\overline{N}_{11}(r, \infty)\leqq N(r, 0, \phi_7)+S(r)\leqq T(r, \phi_7)+S(r)$$
$$=\overline{N}_2(r, 0)+\overline{N}_2(r, \infty)+S(r),$$

respectively. Thus the combination of (4.18)′, (4.19)′ and the estimate of Theorem 2 yields (4.20) with $K_1=11$. ■

4.3.2. *Proof of Theorem* 3. (ii) Define ϕ_{10} and ϕ_{11} by

(4.21) $$\phi_{10}=\left\{\frac{f''}{f'}+\frac{f'}{f}-\frac{f'}{f-1}-\frac{f'}{f-a}\right\}-\left\{\frac{g''}{g'}+\frac{g'}{g}-\frac{g'}{g-1}-\frac{g'}{g-a}\right\},$$

(4.22) $$\phi_{11}=\left\{\frac{f''}{f'}-\frac{f'}{f}-\frac{f'}{f-1}-\frac{f'}{f-a}\right\}-\left\{\frac{g''}{g'}-\frac{g'}{g}-\frac{g'}{g-1}-\frac{g'}{g-a}\right\},$$

respectively. Assume first that $\phi_{10}\equiv 0$. In this case we have $f'f/(f-1)(f-a)\equiv Lg'g/(g-1)(g-a)$ with a nonzero constant L. If $L=1$, then f and g share four values 0, 1, ∞, a CM'', and so by (3.8) g is a Möbius transformation of f. Unless $L=1$, then $\overline{N}(r, 0)=\overline{N}_1(r, 0)+S(r)=\overline{N}_{11}(r, 0)+S(r)=S(r)$ and $\overline{N}_1(r, 1)+\overline{N}_1(r, a)+\overline{N}_1(r, \infty)=S(r)$. Applying Theorem 4 (i) -which will be proved later- to $1/f$ and $1/g$ with shared four values ∞, 1, 0, $1/a$ IM'', we conclude that there does not exist such a pair of f and g. The case of $\phi_{11}\equiv 0$ can be handled in the same way as the one of $\phi_{10}\equiv 0$.

Now, consider the case of $\phi_{10}\not\equiv 0$ and $\phi_{11}\not\equiv 0$. By substituting the Taylor (resp. Laurent) expansions of f and g at a z_0 (1, 1) (resp. z_∞ (1, 1)) which is counted into $\bar{n}_{11}(r, 0)$ (resp. $\bar{n}_{11}(r, \infty)$) into (4.21) (resp. (4.22)), we see that ϕ_{10} (resp. ϕ_{11}) has a zero whose multiplicity is at least two at this point. Hence from (4.21) and (4.22) it follows that

$$2\overline{N}_1(r, 0)=2\overline{N}_{11}(r, 0)+S(r)\leqq N(r, 0, \phi_{10})+S(r)$$
$$\leqq T(r, \phi_{10})+S(r)=\overline{N}_2(r, 0)+S(r)$$

and

$$2\overline{N}_1(r, \infty)=2\overline{N}_{11}(r, \infty)+S(r)\leqq N(r, 0, \phi_{11})+S(r)$$
$$\leqq T(r, \phi_{11})+S(r)=\overline{N}_2(r, \infty)+S(r),$$

respectively. Combining these with the estimate of Theorem 2, we obtain the estimate of Theorem 3 (ii). ■

4.4.1. *Proof of Theorem* 4. (i) By (3.1) we easily see that $T(r, f')+T(r, g')\leqq 2T(r)+2\overline{N}(r, \infty)+S(r)$, and from (3.3) and (3.5) it follows that $\overline{N}(r, 0, f')+\overline{N}(r, 0, g')=\overline{N}_2(r, 0)+\overline{N}_2(r, 1)+\overline{N}_2(r, a)+S(r)$. Hence using (3.2), we get

(4.23) $$m(r, 0, f')+m(r, 0, g')+N_1(r, 0, f')+N_1(r, 0, g')$$
$$\leqq \overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)+3\overline{N}(r, \infty)+S(r).$$

Now, we define two auxiliary functions ϕ_{12} and ϕ_{13}:

(4.24) $$\phi_{12}=2\frac{f''}{f'}-3\left\{\frac{f'}{f}+\frac{f'}{f-1}+\frac{f'}{f-a}\right\}-2\frac{g''}{g'}+2\left\{\frac{g'}{g}+\frac{g'}{g-1}+\frac{g'}{g-a}\right\}$$
$$+\frac{f'-2g'}{f-g}$$

and

(4.25) $$\phi_{13}=2\frac{g''}{g'}-3\left\{\frac{g'}{g}+\frac{g'}{g-1}+\frac{g'}{g-a}\right\}-2\frac{f''}{f'}+2\left\{\frac{f'}{f}+\frac{f'}{f-1}+\frac{f'}{f-a}\right\}$$
$$+\frac{g'-2f'}{g-f}.$$

Making use of (3.3), (3.4), (3.5) and (3.7), we obtain the following estimates for $N(r, \infty, \phi_{12})$ and $N(r, \infty, \phi_{13})$:

(4.26) $$\overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)+\overline{N}_2(r, \infty)+\sum_{p\geqq 3}\overline{N}(r, f=g=0, 1 \text{ or } a$$
$$\text{with } (p, 1))+S(r)\leqq N(r, \infty, \phi_{12})\leqq \overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)$$
$$+\overline{N}(r, \infty)+\sum_{p\geqq 3}\overline{N}(r, f=g=0, 1 \text{ or } a \text{ with } (p, 1))+S(r),$$

(4.27) $$\overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)+\overline{N}_2(r, \infty)+\sum_{q\geqq 3}\overline{N}(r, f=g=0, 1 \text{ or } a$$
$$\text{with } (1, q))+S(r)\leqq N(r, \infty, \phi_{13})\leqq \overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)$$
$$+\overline{N}(r, \infty)+\sum_{q\geqq 3}\overline{N}(r, f=g=0, 1 \text{ or } a \text{ with } (1, q))+S(r).$$

According to E. Mues' calculations (See [4, pp. 116-117].) we have

$$(\phi_{12}^2/2\phi_5)(z_0(2, 1))=(a+1)^2, \qquad (\phi_{13}^2/2\phi_5)(z_0(1, 2))=(a+1)^2,$$
$$(\phi_{12}^2/2\phi_5)(z_1(2, 1))=(2-a)^2, \qquad (\phi_{13}^2/2\phi_5)(z_1(1, 2))=(2-a)^2,$$
$$(\phi_{12}^2/2\phi_5)(z_a(2, 1))=(2a-1)^2 \text{ and } (\phi_{13}^2/2\phi_5)(z_a(1, 2))=(2a-1)^2.$$

As will be shown later

(4.28) $$\phi_{12}^2/2\phi_5\not\equiv(a+1)^2, (2-a)^2, (2a-1)^2$$

and

(4.29) $$\phi_{13}^2/2\phi_5\not\equiv(a+1)^2, (2-a)^2, (2a-1)^2$$

hold. Hence from (3.6), (3.7), (4.26), (4.27) and (4.23) we deduce that

$$(4.30)\quad \bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (2,\ 1))+\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (1,\ 2))\leqq N(r,\ (a+1)^2,\ \phi_{12}{}^2/2\phi_5)+N(r,\ (2-a)^2,\ \phi_{12}{}^2/2\phi_5)$$
$$+N(r,\ (2a-1)^2,\ \phi_{12}{}^2/2\phi_5)+N(r,\ (a+1)^2,\ \phi_{13}{}^2/2\phi_5)$$
$$+N(r,\ (2-a)^2,\ \phi_{13}{}^2/2\phi_5)+N(r,\ (2a-1)^2,\ \phi_{13}{}^2/2\phi_5)$$
$$\leqq 6\{N(r,\ \infty,\ \phi_{12})+N(r,\ \infty,\ \phi_{13})\}+S(r)$$
$$\leqq 12\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)+\bar{N}(r,\ \infty)\}$$
$$+6\textstyle\sum_{p\geq 3}\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (p,\ 1))$$
$$+6\textstyle\sum_{q\geq 3}\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (1,\ q))+S(r)$$
$$\leqq 18\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)\}+30\bar{N}(r,\ \infty)+S(r).$$

Together with (3.2) and (4.23), this yields

$$2T(r)=\bar{N}(r)+S(r)=\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)\}$$
$$+\{\bar{N}_2(r,\ 0)+\bar{N}_2(r,\ 1)+\bar{N}_2(r,\ a)\}+\bar{N}(r,\ \infty)+S(r)$$
$$\leqq\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)\}+\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (2,\ 1))$$
$$+\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (1,\ 2))+N_1(r,\ 0,\ f')+N_1(r,\ 0,\ g')+$$
$$\bar{N}(r,\ \infty)+S(r)\leqq 20\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)\}+34\bar{N}(r,\ \infty)+S(r).$$

This gives the estimate of Theorem 4 (i) with $(K_2,\ K_3)=(10,\ 17)$.

Consider the case of $a=-1$. Then, $a+1=0$ and $(2-a)^2=(2a-1)^2$ hold, so that (4.30) is replaced by

$$\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (2,\ 1))+\bar{N}(r,\ f=g=0,\ 1 \text{ or } a \text{ with } (1,\ 2))$$
$$\leqq 3\{N(r,\ \infty,\ \phi_{12})+N(r,\ \infty,\ \phi_{13})\}+S(r)$$
$$\leqq 9\{\bar{N}_1(r,\ 0)+\bar{N}_1(r,\ 1)+\bar{N}_1(r,\ a)\}+15\bar{N}(r,\ \infty)+S(r).$$

Combining this with (3.2) and (4.23), we obtain the estimate of Theorem 4 (i) with $(K_2,\ K_3)=(11/2,\ 19/2)$. The cases of $a=1/2$ and 2 can be handled in the same way.

It remains to show (4.28) and (4.29). We prove only that $\phi_{12}{}^2/2\phi_5\not\equiv(a+1)^2$ since the other cases can be handled in the same way. We assume that $\phi_{12}{}^2\equiv 2(a+1)^2\phi_5$, and will seek a contradiction. Consider first the case of $a\neq -1$. By the symmetry of ϕ_5 on f and g we have

$$(4.31)\qquad \phi_{12}{}^2\equiv 2(a+1)^2\phi_5\equiv\phi_{13}{}^2.$$

If $\bar{N}_1(r,\ \infty)\neq S(r)$, then by (3.4) there exists a $z_\infty(1,\ 1)$ satisfying $\mathrm{Res}(z_\infty(1,\ 1),$

$f)\neq \mathrm{Res}(z_\infty(1, 1), g)$. Let $\alpha=\mathrm{Res}(z_\infty(1, 1), f)$ and $\alpha'=\mathrm{Res}(z_\infty(1, 1), g)$. Simple computations on (4.31) give $\{(2\alpha-\alpha')/(\alpha-\alpha')\}^2=0=\{(2\alpha'-\alpha)/(\alpha'-\alpha)\}^2$, which is impossible. Hence $\overline{N}_1(r, \infty)=S(r)$, so that equality (up to an $S(r)$ term) must hold everywhere in (4.26) and (4.27), and further by (3.6) and (4.31) all sides of (4.26) and (4.27) are equal to $S(r)$. Together with (4.23) this yields

$$m(r, 0, f)+m(r, 1, f)+m(r, a, f)+m(r, 0, g) +m(r, 1, g)+m(r, a, g)=S(r),$$

$$N_1(r, 0, f')+N_1(r, 0, g')=S(r)$$

and

$$\overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)=S(r),$$

so that for $c=0$, 1 and a we have

$$\begin{aligned}T(r)&=T(r, 1/(f-c))+S(r)=m(r, c, f)+N(r, c, f)+S(r)\\ &\leqq m(r, c, f)+\overline{N}_1(r, c)+2\overline{N}(r, f=g=c \text{ with } (2, 1))\\ &\quad +\overline{N}(r, f=g=c \text{ with } (1, 2))+3N_1(r, 0, f')+3N_1(r, 0, g')+S(r)\\ &=2\overline{N}(r, f=g=c \text{ with } (2, 1))+\overline{N}(r, f=g=c \text{ with } (1, 2))+S(r)\\ &\leqq 2\overline{N}(r, f=g=c \text{ with } (2, 1))+N(r, c, g)/2+S(r)\\ &\leqq 2\overline{N}(r, f=g=c \text{ with } (2, 1))+T(r)/2+S(r).\end{aligned}\tag{4.32}$$

(4.32) guarantees the existence of all of $z_0(2, 1)$, $z_1(2, 1)$ and $z_a(2, 1)$ (cf. [4, p. 116]), which implies that $(a+1)^2=(2-a)^2=(2a-1)^2$. This is impossible. Next, consider the case of $a=-1$. $\phi_{12}\equiv 0$ yields $f'/f+f'/(f-1)+f'/(f+1)-(f'-2g')/(f-g)+g'/g+g'/(g-1)+g'/(g+1)-(g'-2f')/(g-f)\equiv 0$, i.e.,

$$(f-g)^3/f(f-1)(f+1)g(g-1)(g+1)\equiv A \tag{4.33}$$

with a nonzero constant A. If $\overline{N}_1(r, \infty)\neq S(r)$, then by (3.4) there exists a $z_\infty(1, 1)$ satisfying $\mathrm{Res}(z_\infty(1, 1), f)\neq\mathrm{Res}(z_\infty(1, 1), g)$. By substituting the Laurent expansions of f and g at such a point $z_\infty(1, 1)$ into (4.33) we obtain $A=0$, which is a contradiction. Hence $\overline{N}_1(r, \infty)=S(r)$, and so using the same argument as in the case of $a\neq -1$, we arrive at a contradiction.

This completes the proof of Theorem 4 (i). ■

4.4.2. *Proof of Theorem* 4. (ii) From our assumption and (4.23) we have $\overline{N}_2(r, 0)+\overline{N}_2(r, 1)+\overline{N}_2(r, a)=\overline{N}(r, 0, f')+\overline{N}(r, 0, g')+S(r)=\overline{N}_1(r, 0, f')+\overline{N}_1(r, 0, g')+S(r)\leqq\overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)+3\overline{N}(r, \infty)+S(r)$. Hence by (3.2) $2T(r)\leqq 2\{\overline{N}_1(r, 0)+\overline{N}_1(r, 1)+\overline{N}_1(r, a)\}+4\overline{N}(r, \infty)+S(r)$. ■

4.4.3. *Proof of Theorem* 4. (iii) We make use of the proof of Theorem 4 (i). Simple calculations give $\phi_{12}(z_0(2, 1))=-2g'(z_0(2, 1))(1+1/a)$, $\phi_{12}(z_1(2, 1))=$

$2g'(z_1(2, 1))\{1+1/(1-a)\}$, $\phi_{12}(z_a(2, 1))=2g'(z_a(2, 1))\{1/a+1/(a-1)\}$, $\phi_{13}(z_0(1, 2))=-2f'(z_0(1, 2))(1+1/a)$, $\phi_{13}(z_1(1, 2))=2f'(z_1(1, 2))\{1+1/(1-a)\}$ and $\phi_{13}(z_a(1, 2))=2f'(z_a(1, 2))\{1/a+1/(a-1)\}$. Hence, if neither ϕ_{12} nor ϕ_{13} is constant, using our assumptions the above estimate (4.30) can be replaced by

$$\bar{N}(r, f=g=0, 1 \text{ or } a \text{ with } (2, 1))+\bar{N}(r, f=g=0, 1 \text{ or } a \text{ with } (1, 2))\leqq 3\{\bar{N}_1(r, 0)+\bar{N}_1(r, 0)+\bar{N}_1(r, 1)+\bar{N}_1(r, a)+\bar{N}(r, \infty)\}. \tag{4.30$'$}$$

Further, under our assumption, $\bar{N}_2(r, 0)+\bar{N}_2(r, 1)+\bar{N}_2(r, a)=\bar{N}(r, f=g=0, 1$ or a with $(2, 1))+\bar{N}(r, f=g=0, 1$ or a with $(1, 2))+S(r)$ holds. Thus the estimate of Theorem 4 (iii) follows from (4.30)′ and (3.2). It remains to consider the case that ϕ_{12} or ϕ_{13} is constant. In each case, we easily obtain $(f-g)^3/\{fg(f-1)(g-1)(f-a)(g-a)\}\equiv e^{Az+B}$ with two constants A and B. This implies that $\bar{N}_1(r, 0)+\bar{N}_1(r, 1)+\bar{N}_1(r, a)+\bar{N}(r, \infty)=S(r)$. But by Theorem 4 (i) there does not exist such a pair of f and g. ■

Acknowledgement. The author is very grateful to the referee for valuable comments.

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