D.J. HALLENBECK AND K. SAMOTIJ
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ON THE SHARP GROWTH OF ANALYTIC CAUCHY-STIELTJES TRANSFORMS

BY D. J. HALLENBECK AND K. SAMOTIJ*

Introduction

Let $\Delta=\{z: |z|<1\}$ and $\Gamma=\{z: |z|=1\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ. For each $\alpha\geqq 0$ the family $\mathcal{F}_\alpha$ of functions analytic in Δ is defined as follows. If $\alpha>0$ then $f\in\mathcal{F}_\alpha$ provided that there exists $\mu\in\mathcal{M}$ such that

$$(1)\qquad f(z)=f_\mu(z)=\int_\Gamma \frac{1}{(1-\bar{\zeta}z)^\alpha}\,d\mu(\zeta)$$

for $|z|<1$. Also, $f\in\mathcal{F}_0$ provided that there exists $\mu\in\mathcal{M}$ such that

$$(2)\qquad f(z)=f_\mu(z)=\int_\Gamma \log\frac{1}{(1-\bar{\zeta}z)}\,d\mu(\zeta)+f(0)$$

for $|z|<1$ (Here and throughout this paper every logarithm means the principal branch.). The classes $\mathcal{F}_\alpha$ for $\alpha\geqq 0$ were first studied in [3] and [4]. Of course, the case $\alpha=1$ is classical and well studied in the literature. The mapping from $\mathcal{M}$ to $\mathcal{F}_\alpha$ given by $\mu\to f_\mu$ is not one-to-one, i.e., the correspondence between measures and functions in $\mathcal{F}_\alpha$ is not unique. Suppose that $\mu\in\mathcal{M}$. Let $|\mu|$ denote the total variation norm of μ and let $\|\mu\|=|\mu|(\Gamma)$. For $|\zeta|=1$ and $0<x\leqq\pi$ let $I(\zeta, x)$ denote the closed arc on Γ centered at ζ and having length $2x$. A function w is defined on $[0, \pi]$ by

$$(3)\qquad w(x)=|\mu|(I(\zeta, x)) \quad \text{for } 0<x\leqq\pi \text{ and } w(0)=0.$$

To indicate the dependence of w on ζ and x we sometimes write $w(x)=w(x, \zeta, \mu)$ or $w(x)=w(x, \mu)$. As explained in [1] formula (1) is equivalent to

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^\alpha}\,dg(t)$$

where g is a complex-valued function of bounded variation on $[-\pi, \pi]$. Similar remarks apply to (2). We point out, that in the standard way, our measures may be regarded as being defined on $[-\pi, \pi]$ rather than on Γ. This is noth-

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ing more than a technical convenience which we will use in Theorems 1 and 2.

In [1] and [2] the authors examined the interplay between local and global aspects of radial and nontangential limits for functions in $\mathcal{F}_\alpha(\alpha\geqq 0)$ and how this depends on an analysis of properties of the representing measures.

In this paper we intend to describe precisely in what sense the following two theorems from [1] and [2] respectively can be said to be sharp.

THEOREM A. *Let* $\alpha\geqq 0$. *Suppose that* $f\in\mathcal{F}_\alpha$ *and* (1) *or* (2) *holds where* $\mu\in M$. *Let* w *be defined by* (3) *where* $\zeta=e^{i\theta}$ *and* $-\pi\leqq\theta\leqq\pi$. *Then there are positive constants* A *and* B *depending only on* α *such that*

$$|f(re^{i\theta})|\leqq A\|\mu\|+B\int_{1-r}^{\pi}\frac{w(x)}{x^{\alpha+1}}dx \qquad for \quad 0\leqq r<1.$$

THEOREM B. *Suppose that* $\alpha\geqq 1$, g *is a complex-valued function of bounded variation on* $[-\pi, \pi]$ *and let*

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^\alpha}dg(t)$$

for $|z|<1$. *Assume that* g *is differentiable at some* θ *in* $[-\pi, \pi]$. *If* $\alpha>1$ *then* $(1-e^{-i\theta}z)^{\alpha-1}f(z)$ *has the nontangential limit zero at* $e^{i\theta}$. *If* $\alpha=1$ *then* $f(z)/\log(1/(1-e^{-i\theta}z))$ *has the nontangential limit zero at* $e^{i\theta}$.

In Theorems 1 and 2 to follow we show that Theorem A [1] is sharp and in Theorem 3 we show that Theorem B [2] is sharp. We note that when $1\leqq\alpha\leqq 2$, Theorem B was shown to be "sharp" in [Theorem 5, 1].

In our Theorem 3 we strengthen this result by replacing "lim sup" by "lim" and extend the range of α to all $\alpha\geqq 1$. This result for $\mathcal{F}_\alpha$ classes is analogous to the result of G. D. Taylor [5] for H^p spaces.

PROPOSITION 1. *Let* $\alpha>0$ *and* μ *be a non-negative measure on* $[-\pi, \pi]$. *Then*

$$\frac{\min(\alpha, \pi^{-\alpha})}{2^{\alpha/2}}\left[\|\mu\|+\int_{1-r}^{\pi}\frac{w(x)}{x^{\alpha+1}}dx\right]\leqq\int_{-\pi}^{\pi}\frac{d\mu(t)}{|1-re^{-it}|^\alpha} \tag{4}$$

and

$$\int_{-\pi}^{\pi}\frac{d\mu(t)}{|1-re^{-it}|^\alpha}\leqq A\|\mu\|+B\int_{1-r}^{\pi}\frac{w(x)}{x^{\alpha+1}}dx \tag{5}$$

where $w(x)=w(x, 1, \mu)$ *and* $r\in[0, 1)$. *Also the constants* A *and* B *depend only on* α.

Proof. The proof of (5) can be found in [1]. To prove (4) we first remark that it is easy to use the identity $|1-re^{ix}|^2=(1-r)^2+4r\sin^2(x/2)$ and the inequality $\sin|x|\leqq|x|$ to prove that

$$\frac{1}{|1-re^{ix}|}>\frac{1}{\sqrt{2}}\frac{1}{1-r} \tag{6}$$

when $x\in I_1=[-(1-r), 1-r]$, and that

$$\frac{1}{|1-re^{ix}|}>\frac{1}{\sqrt{2}}\frac{1}{|x|} \tag{7}$$

when $x\in I_2=[-\pi, \pi]\setminus I_1$.

It follows from (6) and (7) that

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{d\mu(t)}{|1-re^{-it}|^{\alpha}}&>\frac{1}{2^{\alpha/2}}\frac{1}{(1-r)^{\alpha}}\int_{I_1}d\mu(t)+\frac{1}{2^{\alpha/2}}\int_{I_2}\frac{1}{|t|^{\alpha}}d\mu(t)\\&=\frac{1}{2^{\alpha/2}}\frac{w(1-r)}{(1-r)^{\alpha}}+\frac{1}{2^{\alpha/2}}\int_{1-r}^{\pi}\frac{dw(x)}{x^{\alpha}}.\end{aligned} \tag{8}$$

An integration by parts gives

$$\int_{1-r}^{\pi}\frac{dw(x)}{x^{\alpha}}+\frac{w(1-r)}{(1-r)^{\alpha}}=\frac{w(\pi)}{\pi^{\alpha}}+\alpha\int_{1-r}^{\pi}\frac{w(x)}{x^{\alpha+1}}dx. \tag{9}$$

From (8), (9) and the fact that $w(\pi)=\|\mu\|$ we infer that (4) holds.

LEMMA 1. *Suppose* $0<\alpha<2$. *Then we have*

$$\mathrm{Re}\frac{e^{i\,\mathrm{sgn}(t)(\pi\alpha/4)}}{(1-re^{-it})}\geqq\cos\left(\frac{\pi\alpha}{4}\right)\frac{1}{|1-re^{-it}|^{\alpha}} \tag{10}$$

whenever $t\in[-\pi, \pi]$ *and* $r\in[0, 1)$.

Proof. Note that since both sides of (10) are even functions of t, it is enough to prove the lemma for $0<t\leqq\pi$. To this end note that for $r\in[0, 1)$ and $0<t\leqq\pi$ we have $-\pi/2\leqq\mathrm{Arg}(1/(1-re^{-it}))\leqq 0$, where Arg denotes here (and elsewhere in this paper) the principal argument. Hence we have $-(\pi/4)\alpha\leqq(\pi/4)\alpha+\alpha\,\mathrm{Arg}(1/(1-re^{-it}))\leqq(\pi/4)\alpha$. Using this fact, the fact that $0<\alpha<2$ and the identity

$$\mathrm{Re}\frac{e^{i\,\mathrm{sgn}(t)(\pi\alpha/4)}}{(1-re^{-it})^{\alpha}}=\frac{\cos(\alpha(\pi/4)+\alpha\,\mathrm{Arg}(1/(1-re^{-it})))}{|1-re^{-it}|^{\alpha}}$$

we obtain (10).

In Theorem 1 to follow we prove that Theorem A is sharp when $0\leqq\alpha<2$.

THEOREM 1. *Suppose* $0\leqq\alpha<2$ *and* μ *is a complex measure on* $[-\pi, \pi]$. *Then there exists a function* $f_\nu\in\mathcal{F}_\alpha$ *such that* $|\nu|=|\mu|$ *and a constant* A_α *depending only on* α *such that*

$$|f_\nu(r)|\geqq A_\alpha\left[\|\mu\|+\int_{1-r}^{\pi}\frac{w(x)}{x^{\alpha+1}}dx\right] \tag{11}$$

where $w(x)=w(x, 1, \mu)$ *and* $r\in[0, 1)$.

Proof. We suppose $\alpha>0$. The proof for $\alpha=0$ is similar and we do not give it here. For a complex measure μ on $[-\pi, \pi]$ define a measure ν by $d\nu(t)=e^{i\operatorname{sgn}(t)(\pi\alpha/4)}d|\mu|(t)$ and, consequently,

$$f_\nu(z)=\int_{-\pi}^{\pi}\frac{d\nu(t)}{(1-e^{-it}z)^\alpha} \tag{12}$$

for $z\in\Delta$. Note that $\|\nu\|=\|\mu\|$. It follows from (12) that

$$|f_\nu(r)|\geqq \operatorname{Re} f_\nu(r)=\int_{-\pi}^{\pi}\operatorname{Re}\left[\frac{e^{i\operatorname{sgn}(t)(\pi\alpha/4)}}{(1-re^{-it})^\alpha}\right]d|\mu|(t). \tag{13}$$

We infer from (4), (10) and (13) that (11) holds for $A_\alpha=(\min(\alpha, \pi^{-\alpha})/2^{\alpha/2})\cos(\pi\alpha/4)$.

Remark. Theorem 1 shows that for $0\leqq\alpha<2$, Theorem A in [1] is sharp in the most strict possible manner. For $\alpha>2$ we are not able to do quite as well. The case $\alpha=2$ (and also $\alpha=6, 10, 14, \cdots$) remains open.

LEMMA 2. *For any $\varepsilon>0$ there are positive constants a, b, and T such that $T\leqq\pi$ and for all $r\in[0, 1)$ we have*

$$\left|\operatorname{Arg}\frac{1}{(1-re^{-it})}\right|<\varepsilon \tag{14}$$

whenever $0\leqq t\leqq a(1-r)$ and

$$\left|\operatorname{Arg}\frac{1}{(1-re^{-it})}+\frac{\pi}{2}\right|<\varepsilon \tag{15}$$

whenever $b(1-r)\leqq t\leqq T$.

Proof. To verify the first inequality we note that without loss of generality we may assume that $\varepsilon<\tan^{-1}(\pi/2)$. Let $a=\tan\varepsilon$. Then for $0\leqq t\leqq a(1-r)$ we have

$$0\leqq\tan\left(-\operatorname{Arg}\frac{1}{(1-re^{-it})}\right)=\frac{r\sin t}{1-r\cos t}\leqq\frac{rt}{1-r}$$
$$\leqq ra<a=\tan\varepsilon\,,$$

which gives (14).

We may assume without loss of generality that $\varepsilon<(\pi/2)$ and is such that $A=\tan(\pi/2-\varepsilon)>1$. Let $b=2\pi A$ and $T=(2/b)$. Then we have $T=1/\pi A$. Noting that $b\geqq 2\pi$ and $T<\pi$ we see that the set of t's such that $b(1-r)\leqq t\leqq T$ when $0\leqq r<1/2$ is vacuous. So to prove (15) we may assume $r\in[1/2, 1)$. The inequalities $b(1-r)\leqq t\leqq T$ are equivalent to $2\pi A(1-r)\leqq t\leqq(1/\pi A)$, which implies $(1-r)/t<1/2\pi A$ and $rt/2<1/2\pi A$. Adding the last two inequalities we obtain $(1-r)/t+rt/2<1/\pi A$. This can be rewritten as $A<1/\pi(t/1-r+(rt^2/2))$. Since $1/2\leqq r<1$ we deduce that

$$\tan\left(\frac{\pi}{2}-\varepsilon\right)=A<\frac{2t}{\pi}\frac{r}{1-r(1-(t^2/2))}\leqq\frac{2t}{\pi}\frac{r}{1-r\cos t}<\frac{r\sin t}{1-r\cos t}. \tag{16}$$

It is clear that (15) follows from (16).

LEMMA 3. *Let $\alpha>0$ with $(\alpha+2)/4\notin \boldsymbol{Z}$. Let $\beta=((\pi/2)\alpha-2k\pi)/2$, where k is the greatest integer less than or equal to $(\alpha+2)/4$. Then there exist positive constants a, b, T, $0<T<\pi$, and c_1 such that*

$$\operatorname{Re}\frac{e^{-i\beta}}{(1-re^{-it})^{\alpha}}\geqq\frac{c_1}{|1-re^{-it}|^{\alpha}} \tag{17}$$

for every $r\in[0, 1)$ and every t such that $0\leqq t\leqq a(1-r)$ or $b(1-r)\leqq t\leqq T$ or $t=\pi$.

Proof. By the definition of β it follows that $\beta\in(-\pi/2, \pi/2)$. Choose $\varepsilon>0$ such that $[\beta-\alpha\varepsilon, \beta+\alpha\varepsilon]\subset(-\pi/2, \pi/2)$. Let a, b and T be such that Lemma 2 holds. We have the equality

$$\operatorname{Re}\frac{e^{-i\beta}}{(1-re^{-it})^{\alpha}}=\frac{\cos(\beta-\alpha\operatorname{Arg}(1/(1-re^{-it})))}{|1-re^{-it}|^{\alpha}}. \tag{18}$$

Suppose $0\leqq t\leqq a(1-r)$. Then (14) gives

$$-\frac{\pi}{2}<\beta-\alpha\varepsilon<\beta-\alpha\operatorname{Arg}\frac{1}{(1-re^{-it})}<\beta+\alpha\varepsilon<\frac{\pi}{2}. \tag{19}$$

Let $d=\max\{|\beta-\alpha\varepsilon|, |\beta+\alpha\varepsilon|\}$ and note that $-\pi/2<d<\pi/2$. Therefore (18) and (19) imply (17) with $c_1=\cos d$.

Now suppose $b(1-r)\leqq t\leqq T$. Then (15) implies

$$-\frac{\pi}{2}\alpha-\alpha\varepsilon<\alpha\operatorname{Arg}\frac{1}{(1-re^{-it})}<-\frac{\pi}{2}\alpha+\alpha\varepsilon. \tag{20}$$

Using the definition of β and a short computation, (20) gives

$$\beta-\alpha\varepsilon<\alpha\operatorname{Arg}\frac{1}{(1-re^{-it})}-\beta-2k\pi<\beta+\alpha\varepsilon. \tag{21}$$

Again (18) and (21) give (17) with $c_1=\cos d$.

Finally, if $t=\pi$ then, since $\cos\beta>\cos d$, (17) holds also in this case.

LEMMA 4. *Suppose $0\leqq r<1$, $0\leqq r\leqq\pi$, $0<a<1<b$ and $a(1-r)<\tau<b(1-r)$. Then*

$$\frac{\tau}{b}\leqq|1-re^{-it}|\leqq\frac{2}{a}\tau \tag{22}$$

whenever $0\leqq t\leqq\tau$ and

$$\frac{1}{2}|1-e^{it}|\leqq|1-re^{-it}|\leqq 2|1-e^{it}| \tag{23}$$

whenever $t\geqq(\pi/a)\tau$.

Proof. We note that if $0 \leqq t \leqq \tau$ then

$$1-r \leqq |1-re^{-it}| \leqq (1-r)+t \leqq \frac{\tau}{a}+\tau \leqq \frac{2}{a}\tau \tag{24}$$

and (24) clearly implies (22) since $\tau/b<1-r$. By the inequality $\sin x \geqq (2/\pi)x$, $0 \leqq x<\pi/2$, we have

$$\frac{1}{2}|1-e^{it}|=\sin\frac{t}{2} \geqq \frac{t}{\pi}. \tag{25}$$

When $t \geqq (\pi/a)\tau$, (25) implies

$$\frac{1}{2}|1-e^{it}| \geqq \frac{\tau}{a}. \tag{26}$$

Hence

$$\begin{aligned}|1-re^{-it}|&=|e^{it}-r| \geqq |e^{it}-1|-(1-r)\\ &\geqq |e^{it}-1|-\frac{\tau}{a} \geqq \frac{1}{2}|e^{it}-1|.\end{aligned}$$

It also follows from (26) that

$$\begin{aligned}|1-re^{-it}|&=|e^{it}-r| \leqq |e^{it}-1|+(1-r)\\ &\leqq |e^{it}-1|+\frac{\tau}{a} \leqq |e^{it}-1|+\frac{1}{2}|e^{it}-1|\\ &<2|e^{it}-1|.\end{aligned}$$

So (23) holds.

We remark that in this paper we assume the notation $\sum_{l=k+1}^{l=\infty} a_l=\sum_{l=k+1}^{\infty} a_l+a_\infty$.

LEMMA 5. *Let $\alpha>0$. Then for each positive sufficiently small number q there is an $\eta>0$ so that for each sequence $\mu_0, \cdots, \mu_k, \cdots, \mu_\infty$ of non-negative numbers with $\sum_{l=0}^{l=\infty} \mu_l<+\infty$ there are sequences $\nu_0, \nu_1, \cdots, \nu_k, \cdots, \nu_\infty$ of non-negative numbers, and $\tau_0, \tau_1, \cdots, \tau_k, \cdots, \tau_\infty$ satisfying*

$$\nu_0=\mu_0, \quad \textit{and} \quad \eta^2\sum_{l=k}^{l=\infty}\mu_l \leqq \sum_{l=k}^{l=\infty}\nu_l \leqq \sum_{l=k}^{l=\infty}\mu_l, \qquad k=0, 1, \cdots \tag{27}$$

and

$$\tau_0=\pi,\ \tau_\infty=0,\ \tau_k=q^{2k}\pi \quad \textit{or} \quad q^{2k-1}\pi, \qquad k=1, 2, \cdots, \tag{28}$$

such that for each $k \geqq 1$ we have either

$$\frac{a^\alpha\nu_k}{2^\alpha\tau_k^\alpha} \geqq 2\left[2^\alpha\sum_{l=0}^{k-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{b^\alpha}{\tau_k^\alpha}\sum_{l=k+1}^{l=\infty}\nu_l\right] \tag{29}$$

or

$$\frac{b^\alpha\nu_k}{\tau_k^\alpha} \leqq \frac{c_1}{2}2^{-\alpha}\left[\sum_{l=0}^{k-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{a^\alpha}{\tau_k^\alpha}\sum_{l=k+1}^{l=\infty}\nu_l\right], \tag{30}$$

where a, b *and* c_1 *are numbers from Lemma* 3.

Proof. Let q be any positive number such that

$$\frac{2^{\alpha+1}(2^\alpha+b^\alpha)q^\alpha b^\alpha}{a^\alpha}<c_1\frac{2^{-\alpha}}{2}. \tag{31}$$

We introduce the auxiliary sequence $\{\theta_k\}$ defined by

$$\sum_{l=k}^{l=\infty}\nu_l=\theta_k\sum_{l=k}^{l=\infty}\mu_l, \qquad k=0,1,\cdots.$$

In the case when $\sum_{l=k}^{l=\infty}\mu_l=0$ we put $\theta_k=1$. Conditions (27), (29) and (30) may by rewritten as the following conditions on (θ_k)

$$\eta^2\leqq\theta_k\leqq 1, \qquad k=0,1,\cdots, \qquad \theta_0=\theta_1=1, \tag{32}$$

$$\frac{a^\alpha\nu_k}{2^\alpha\tau_k^\alpha}\geqq 2\left[2^\alpha\sum_{l=0}^{k-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{b^\alpha}{\tau_k^\alpha}\theta_{k+1}\sum_{l=k+1}^{l=\infty}\mu_l\right] \tag{33}$$

and

$$\frac{b^\alpha\nu_k}{\tau_k^\alpha}\leqq\frac{c_1}{2}2^{-\alpha}\left[\sum_{l=0}^{k-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{a^\alpha}{\tau_k^\alpha}\theta_{k+1}\sum_{l=k+1}^{l=\infty}\mu_l\right] \tag{34}$$

with

$$\nu_k=\theta_k\sum_{l=k}^{l=\infty}\mu_l-\theta_{k+1}\sum_{l=k+1}^{l=\infty}\mu_l\geqq 0,\ k=0,1,\cdots \text{ and with } \nu_\infty=0 \text{ if } \mu_\infty=0. \tag{35}$$

If $\mu_\infty>0$ then (35) implies $\sum_{k=0}^\infty(\theta_k-\theta_{k+1})^-\leqq(1/\mu_\infty)\sum_{k=0}^\infty\mu_k$. This fact together with the inequality $\sum_{k=0}^\infty(\theta_k-\theta_{k+1})^+\leqq\sum_{k=0}^\infty(\theta_k-\theta_{k+1})^-+\sup_k\theta_k-\inf_k\theta_k$ implies $\sum_{k=0}^\infty|\theta_k-\theta_{k+1}|<+\infty$. Therefore $\lim_{k\to\infty}\theta_k$ exists and we define $\nu_\infty=(\lim_{k\to\infty}\theta_k)\mu_\infty$ in this case.

We will construct sequences (θ_k) and (τ_k) inductively. To start this induction set $\theta_0=\theta_1=1$, $\tau_0=\pi$. Next suppose that $n\geqq 1$ and that $\theta_0,\cdots,\theta_n$ and $\tau_0,\cdots,\tau_{n-1}$ are already selected so that (28), (32), (33) or (34), and (35) hold for $k=1,\cdots,n-1$. If (33) or (34) is satisfied with $k=n$, $\tau_n=q^{2n}\pi$, and $\theta_{n+1}=\theta_n$; then, naturally, set $\tau_n=q^{2n}\pi$ and $\theta_{n+1}=\theta_n$. Then $\nu_n=\theta_n\mu_n\geqq 0$. Now assume that this is not the case, i.e., we have

$$\frac{a^\alpha\theta_n\mu_n}{2^\alpha(q^{2n}\pi)^\alpha}<2\left[2^\alpha\sum_{l=0}^{n-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{b^\alpha\theta_n}{(q^{2n}\pi)^\alpha}\sum_{l=n+1}^{l=\infty}\mu_l\right] \tag{36}$$

and

$$\frac{b^\alpha\theta_n\mu_n}{(q^{2n}\pi)^\alpha}>\frac{c_1}{2}2^{-\alpha}\left[\sum_{l=0}^{n-1}\frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}+\frac{a^\alpha\theta_n}{(q^{2n}\pi)^\alpha}\sum_{l=n+1}^{l=\infty}\mu_l\right]. \tag{37}$$

We now consider 4 cases.

Case 1. If

$$\sum_{l=0}^{n-1} \frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha} \geqq \frac{\theta_n}{(q^{2n}\pi)^\alpha} \sum_{l=n+1}^{l=\infty} \mu_l, \tag{38}$$

then set $\tau_n = q^{2n-1}\pi$, $\theta_{n+1}=\theta_n$, and, consequently, $\nu_n=\theta_n\mu_n$. Then (38) combined with (36) gives

$$\frac{a^\alpha \nu_n}{2^\alpha \tau_n^\alpha q^\alpha} < 2\left[(2^\alpha+b^\alpha)\sum_{l=0}^{n-1} \frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}\right],$$

which can be rewritten as

$$\frac{b^\alpha \nu_n}{\tau_n^\alpha} < \frac{2^{\alpha+1}(2^\alpha+b^\alpha)q^\alpha b^\alpha}{a^\alpha} \sum_{l=0}^{n-1} \frac{\nu_l}{|1-e^{-i\tau_l}|^\alpha}. \tag{39}$$

Clearly, by our choice of q, (31) and (39) imply (34) with $k=n$.

Case 2. If (38) does not hold and $1 \geqq \theta_n > \eta$, then we set $\tau_n = q^{2n}\pi$ and

$$\theta_{n+1} = \eta\theta_n. \tag{40}$$

Note that (40) gives $\eta^2 < \eta\theta_n = \theta_{n+1} \leqq \eta < 1$ and that

$$\nu_n = \theta_n \sum_{l=n}^{l=\infty} \mu_l - \eta\theta_n \sum_{l=n+1}^{l=\infty} \mu_l \geqq \theta_n\mu_n \geqq 0. \tag{41}$$

Combining the negation of (38) with (37) we obtain

$$\frac{b^\alpha \theta_n \mu_n}{(q^{2n}\pi)^\alpha} > \frac{c_1}{2} 2^{-\alpha}(1+a^\alpha) \frac{\theta_n}{(q^{2n}\pi)^\alpha} \sum_{l=n+1}^{l=\infty} \mu_l. \tag{42}$$

Inequality (42), together with (40) and (41) implies

$$\frac{a^\alpha \nu_n}{\tau_n^\alpha} > \frac{1}{\eta} \frac{a^\alpha(1+a^\alpha)c_1}{2 \cdot 2^\alpha b^{2\alpha}} \frac{b^\alpha \theta_{n+1}}{\tau_n^\alpha} \sum_{l=n+1}^{l=\infty} \mu_l. \tag{43}$$

Clearly (43) implies (33) (with $k=n$) if only η is sufficiently small.

Case 3. If (38) does not hold, $\theta_n \leqq \eta$ and

$$\frac{1}{\eta} \sum_{l=n+1}^{l=\infty} \mu_l < \sum_{l=n}^{l=\infty} \mu_l, \tag{44}$$

then we set $\tau_n = q^{2n}\pi$, and $\theta_{n+1} = (1/\eta)\theta_n$. Note that $\eta^2 \leqq (\theta_n/\eta) = \theta_{n+1} \leqq 1$. Moreover, by (44), we have $\nu_n = \theta_n \sum_{l=n}^{l=\infty} \mu_l - (\theta_n/\eta)\sum_{l=n+1}^{l=\infty} \mu_l \geqq 0$. Also since we may require that $\eta<1$, we have $\nu_n = \theta_n\mu_n + \theta_n(1-(1/\eta))\sum_{l=n+1}^{l=\infty}\mu_l \leqq \theta_n\mu_n$. This last inequality together with the negation of (38) applied to (36) yields

$$\frac{a^\alpha \nu_n}{2^\alpha \tau_n^\alpha} \leqq 2\eta(2^\alpha+b^\alpha) \frac{\theta_{n+1}}{\tau_n^\alpha} \sum_{l=n+1}^{l+\infty} \mu. \tag{45}$$

Inequality (45) may be rewritten as

$$\frac{b^{\alpha}\nu_n}{\tau_n^{\alpha}} \leqq 2^2\eta \frac{2^{\alpha}b^{\alpha}(2^{\alpha}+b^{\alpha})}{a^{2\alpha}c_1} \frac{a^{\alpha}c_1}{2\cdot 2^{\alpha}} \frac{\theta_{n+1}}{\tau_n^{\alpha}} \sum_{l=n+1}^{l=\infty} \mu_l$$

which clearly implies (34) with $k=n$ if η is sufficiently small.

Case 4. It only remains to deal with the case when $\theta_n \leqq \eta$ and neither (38) nor (44) hold. In this case put

$$\tau_n = q^{2n}\pi \quad \text{and} \quad \theta_{n+1} = \frac{\sum_{l=n}^{l=\infty} \mu_l}{\sum_{l=n+1}^{l=\infty} \mu_1} \theta_n .$$

Since (44) is not true we have $\eta^2 \leqq \theta_n \leqq \theta_{n+1} \leqq (1/\eta)\theta_n \leqq (\eta/\eta) = 1$. A simple calculation shows that $\nu_n = 0$ and so (34) is trivially satisfied.

LEMMA 6. *Suppose that $\alpha > 0$ is such that $(\alpha+2)/4$ is not an integer. Let c_1 be the constant from Lemma* 3, *and let a, b, and T be the constants from Lemma* 3. *Let $0 < q < \min(a/b, T/\pi, a/\pi)$. Then there is a constant $c_2 > 0$ such that for any sequences*: $(\tau_k$ *satisfying* (28) *of Lemma* 5, *and non-negative $\nu_0, \cdots, \nu_k, \cdots, \nu_\infty$ which for each k, $k=1, 2, \cdots$, satisfies either* (29) *or* (30) *of Lemma* 5 *we have*

$$\left| \sum_{l=0}^{l=\infty} \frac{\nu_l}{(1-re^{-i\tau_l})^{\alpha}} \right| \geqq c_2 \sum_{l=0}^{l=\infty} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}, \tag{46}$$

for $0 \leqq r \leqq 1$.

Proof. Note that for any positive integer k the inequality (29) of Lemma 5 implies that

$$\frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}} \geqq 2 \sum_{l \neq k} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}, \tag{47}$$

when $a(1-r) < \tau_k < b(1-r)$, while the inequality (30) of Lemma 5 implies that

$$\frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}} \leqq \frac{c_1}{2} \sum_{l \neq k} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}} \tag{48}$$

where $a(1-r) < \tau_k < b(1-r)$. Indeed, since $\tau_k \leqq q\tau_l \leqq (a/\pi)\tau_l$, the first inequality of (23) in Lemma 4 gives

$$2^{\alpha} \sum_{l=0}^{k-1} \frac{\nu_l}{|1-e^{-i\tau_l}|^{\alpha}} \geqq \sum_{l=0}^{k-1} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}} \geqq 2^{-\alpha} \sum_{l=0}^{k-1} \frac{\nu_l}{|1-e^{-i\tau_l}|^{\alpha}} .$$

Then, since $\tau_k < b(1-r)$ we have

$$\frac{b^{\alpha}}{\tau_k^{\alpha}} \sum_{l=k+1}^{l=\infty} \nu_l \geqq \sum_{l=k+1}^{l=\infty} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}} \geqq \frac{2^{-\alpha}a^{\alpha}}{\tau_k^{\alpha}} \sum_{l=k+1}^{l=\infty} \nu_l . \tag{50}$$

Finally, applying the second inequality of (22) Lemma 4 we get

$$\frac{a^{\alpha}\nu_k}{2^{\alpha}\tau_k^{\alpha}} \leqq \frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}} \leqq \frac{b^{\alpha}\nu_k}{\tau_k^{\alpha}}. \tag{51}$$

Combining (49), (50) and (51) with (29) we obtain (47). The proof of (48) which we do not give here can be given in a similar fashion. If $\{\tau_k\}_{k=1}^{k=\infty} \cap (a(1-r), b(1-r)) = \emptyset$, then since $\tau_1 \leqq q\pi \leqq T$ we have $\{\tau_k\}_{k=0}^{k=\infty} \subset [0, a(1-r)] \cup [b(1-r), T] \cup \{\pi\}$. Then, by Lemma 3 we have

$$\left|\sum_{l=0}^{l=\infty} \frac{\nu_l}{(1-re^{-i\tau_l})^{\alpha}}\right| \geqq \sum_{l=0}^{l=\infty} \nu_l \operatorname{Re} \frac{e^{-i\beta}}{(1-re^{-i\tau_l})^{\alpha}} \geqq c_1 \sum_{l=0}^{l=\infty} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}.$$

with β from Lemma 3. If this is not the case, then since $(\tau_{l+1}/\tau_l) \leqq q < (a/b)$, $l=0, 1, \cdots$, we have $a(1-r) < \tau_k < b(1-r)$ for exactly one k, $k \geqq 1$. By what was proved earlier, either (47) or (48) holds for this case. If (47) holds then we have

$$\left|\sum_{l=0}^{l=\infty} \frac{\nu_l}{(1-re^{-i\tau_l})^{\alpha}}\right| \geqq \frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}} - \sum_{l\neq k} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}$$
$$\geqq \frac{1}{3} \sum_{l=0}^{l=\infty} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}.$$

If (48) holds, then, since all terms of (τ_l) except for τ_k are in $[0, a(1-r)] \cup [b(1-r), T] \cup \{\pi\}$, applying Lemma 3 and the fact that $0 < c_1 < 1$ we get

$$\left|\sum_{l=0}^{l=\infty} \frac{\nu_l}{(1-re^{-i\tau_l})^{\alpha}}\right| \geqq \left|\sum_{l\neq k} \frac{\nu_l}{(1-re^{-i\tau_l})^{\alpha}}\right| - \frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}}$$
$$\geqq \sum_{l\neq k} \nu_l \operatorname{Re} \frac{e^{-i\beta}}{(1-re^{-i\tau_l})^{\alpha}} - \frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}}$$
$$\geqq c_1 \sum_{l\neq k} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}} - \frac{\nu_k}{|1-re^{-i\tau_k}|^{\alpha}}$$
$$\geqq \frac{c_1}{3} \sum_{l=0}^{l=\infty} \frac{\nu_l}{|1-re^{-i\tau_l}|^{\alpha}}$$

which completes the proof.

Before we state Theorem 2 we recall the notation

$$w(t, \nu) = \begin{cases} 0 & t=0 \\ |\nu|([-t, t]) & 0 < t \leqq \pi \end{cases}$$

and likewise for μ. Note that this implies

$$\int_0^{\pi} f(t)\,dw(t, \nu) = \int_{-\pi}^{\pi} f(|t|)\,d|\nu|(t)$$

for each f continuous on $[0, \pi]$.

THEOREM 2. *For each $\alpha > 2$ with $(\alpha+2)/4$ not an integer there is a positive* *constant c_3 depending only on α such that for any measure μ on $[-\pi, \pi]$ there is a measure ν on $[0, \pi]$ such that*

$$w(t, \nu) \leqq w(t, \mu) \tag{52}$$

for $0 \leqq t \leqq \pi$, and

$$|f_\nu(r)| \geqq c_3 \int_{-\pi}^{\pi} \frac{d|\mu|(t)}{|1-re^{-it}|^\alpha} \tag{53}$$

for $r \in [0, 1)$ where f_ν is defined by (1).

Remark. Note that by Proposition 1, the right-hand side of (53) in the theorem may be replaced by

$$c_3\left[\|\mu\| + \int_{1-r}^{\pi} \frac{w(t, \mu)}{t^{\alpha+1}} dt\right]$$

and so Theorem 2 proves that Theorem A in [1] is sharp.

Proof. Without loss of generality we may and so assume that μ is a nonnegative measure with support contained in $[0, \pi]$. Let a sequence $\mu_0, \mu_1, \cdots, \mu_\infty$ be defined by the formula

$$\mu_k = \mu((q^{2k+2}\pi, q^{2k}\pi]), \qquad k \geqq 0$$

and $\mu_\infty = \mu(\{0\})$, where q is a positive number for which both Lemma 5 and Lemma 6 hold. Note that q depends only on α. Let (τ_k) and (ν_k) be sequences constructed in Lemma 5 for this q. Define measures $\tilde{\mu}$ and ν by the formulas

$$\tilde{\mu} = \sum_{l=0}^{l=\infty} \mu_l \delta_{\tau_l}$$

and

$$\nu = \sum_{l=0}^{l=\infty} \nu_l \delta_{\tau_l}.$$

Since the measure $\tilde{\mu}$ is obtained by "sweeping" the mass from the interval $(q^{2k+2}\pi, q^{2k}\pi]$ to the point $\tau_k (k=0, 1, \cdots)$, which is given by (28) of Lemma 5, and which is located to the right of the interval, without moving the mass concentrated at 0 and at π we have

$$w(t, \tilde{\mu}) \leqq w(t, \mu) \tag{54}$$

for $0 \leqq t \leqq \pi$. Moreover, since $|1-re^{-ist}| \leqq s|1-re^{-it}|$ for $0<t$, $s>1$, $st<\pi$ and $r \in [0, 1)$, we have

$$\begin{aligned} \int_{[0,\pi]} \frac{d\mu(t)}{|1-re^{-it}|^\alpha} &= \frac{\mu_\infty}{(1-r)^\alpha} + \sum_{k=0}^{\infty} \int_{(q^{2k+2}\pi, q^{2k}\pi]} \frac{d\mu(t)}{|1-re^{-it}|^\alpha} \\ &\leqq \frac{\mu_\infty}{(1-r)^\alpha} + \sum_{k=0}^{\infty} \int_{(q^{2k+2}\pi, q^{2k}\pi]} \frac{d\mu(t)}{|1-re^{-iq^{2k+2}\pi}|^\alpha} \end{aligned} \tag{55}$$

$$=\frac{\mu_\infty}{(1-r)^\alpha}+\sum_{k=0}^{\infty}\frac{\mu_k}{|1-re^{-iq^{2k+2}\pi}|^\alpha}$$

$$=\frac{\mu_\infty}{(1-r)^\alpha}+\sum_{k=0}^{\infty}\Big(\frac{\tau_k}{q^{2k+2}\pi}\Big)^\alpha\frac{\mu_k}{|1-re^{-i\tau_k}|^\alpha}$$

$$\leqq q^{-3\alpha}\frac{\mu_\infty}{(1-r)^\alpha}+q^{-3\alpha}\sum_{k=0}^{\infty}\frac{\mu_k}{|1-re^{-i\tau_k}|^\alpha}$$

$$=q^{-3\alpha}\int_{[0,\pi]}\frac{d\tilde{\mu}(t)}{|1-re^{-it}|^\alpha}.$$

Note that (54) and the second inequality in (27) of Lemma 5 give (52).

We next prove that

$$\int_{[0,\pi]}\frac{d\nu(t)}{|1-re^{-it}|^\alpha}\geqq\frac{\eta^2q^{3\alpha}}{2}\int_{[0,\pi]}\frac{d\tilde{\mu}(t)}{|1-re^{-it}|^\alpha} \tag{56}$$

for $0\leqq r<1$. First note, it follows from (27), the choice of τ_k and the fact that $w(t,\tilde{\mu})$ is nondecreasing that $w(t,\nu)\geqq\eta^2w(q^3t,\tilde{\mu})$ for $0\leqq t<\pi$ and $w(\pi,\nu)\geqq\eta^2w(\pi,\tilde{\mu})$. Using the foregoing facts, the inequality $(1/|1-re^{-iq^{-3}t}|^\alpha)\geqq q^{3\alpha}(1/|1-re^{-it}|^\alpha)$ whenever $0\leqq t\leqq q^3\pi$ and the fact that $1/|1-re^{-iq^3t}|^\alpha$ is nondecreasing, it is readily proved that

$$\int_0^\pi\frac{dw(t,\nu)}{|1-re^{-it}|^\alpha}\geqq\eta^2\int_0^\pi\frac{dw(q^3t,\tilde{\mu})}{|1-re^{-it}|^\alpha}=\eta^2\int_0^{q^3\pi}\frac{dw(t,\tilde{\mu})}{|1-re^{-iq^{-3}t}|^\alpha}\geqq\eta^2q^{3\alpha}\int_0^{q^3\pi}\frac{dw(t,\tilde{\mu})}{|1-re^{-it}|^\alpha}$$

and

$$\int_0^\pi\frac{dw(t,\nu)}{|1-re^{-it}|^\alpha}\geqq\eta^2q^{3\alpha}\int_{q^3\pi}^\pi\frac{dw(t,\tilde{\mu})}{|1-re^{-it}|^\alpha}.$$

These last two inequalities imply (56).

Combining (55) and (56) with (46) of Lemma 6 we obtain (53) with $c_3=q^{6\alpha}\eta^2c_2/2$.

The following two lemmas are technical results needed for the proof of Theorem 3.

LEMMA 7. *Let $\alpha>1$. Then there is a $\delta>0$ such that for each positive nondecreasing C^1 function $\varepsilon(t)$, $0<t\leqq\pi$, satisfying*

$$\frac{d\log\varepsilon(t)}{d\log t}\leqq\delta,\qquad 0<t\leqq\pi, \tag{57}$$

we have

$$|h(s)|\geqq\frac{\varepsilon(s)}{4s^{\alpha-1}(\alpha-1)}$$

for all sufficiently small $s>0$, where

$$h(s)=\int_0^\pi(s+it)^{-\alpha}\varepsilon(t)dt,\qquad s>0.$$

Proof. Note that

(58) $$\int_0^\infty (1+iu)^{-\alpha} du = \frac{-i}{\alpha-1}.$$

Choose u_1 and u_2, $0<u_1<1<u_2<+\infty$, so that

(59) $$\int_0^{u_1} |1+iu|^{-\alpha} du \leqq \frac{1}{8(\alpha-1)}$$

and

(60) $$u_2^{1-\alpha} < \frac{1}{16}.$$

Note that (58), (59) and (60) imply that

(61) $$\left|\int_{u_1}^{u_2} (1+iu)^{-\alpha} du\right| \geqq \frac{3}{4}\frac{1}{\alpha-1}.$$

Choose δ so that $0<\delta<(\alpha-1)/2$ and that

(62) $$\left(1-\left(\frac{u_1}{u_2}\right)^\delta\right)\int_0^\infty |1+iu|^{-\alpha} du \leqq \frac{1}{8(\alpha-1)}.$$

Note that (57) implies that for any $t_0 \in (0, 1)$ we have

(63) $$\bar{\varepsilon}(t) \leqq \left(\frac{t}{t_0}\right)^\delta \bar{\varepsilon}(t_0), \qquad t_0 \leqq t \leqq \pi,$$

and

(64) $$\bar{\varepsilon}(t) \geqq \left(\frac{t}{t_0}\right)^\delta \bar{\varepsilon}(t_0), \qquad 0 \leqq t \leqq t_0.$$

If $s<(\pi/u_2)$, then we may write

(65) $$h(s) = s^{1-\alpha}[I_1+I_2+I_3]$$

where

$$I_1 = \int_0^{u_1} (1+iu)^{-\alpha} \bar{\varepsilon}(su) du, \qquad I_2 = \int_{u_1}^{u_2} (1+iu)^{-\alpha} \bar{\varepsilon}(su) du$$

and

$$I_3 = \int_{u_2}^{\pi/s} (1+iu)^{-\alpha} \bar{\varepsilon}(su) du.$$

Since $\bar{\varepsilon}(t)$ is non-decreasing we obtain, by (59),

(66) $$|I_1| \leqq \int_0^{u_1} |1+iu|^{-\alpha} \bar{\varepsilon}(su) du \leqq \frac{\bar{\varepsilon}(su_2)}{8(\alpha-1)}.$$

While (63), with $t_0 = su_2$, and (60) yield

(67)
$$\begin{aligned}|I_3| &\leqq \int_{u_2}^{\pi/s} |1+iu|^{-\alpha}\tilde{\varepsilon}(su)du \\ &\leqq \int_{u_2}^{\pi/s} |1+iu|^{-\alpha}\tilde{\varepsilon}(su_2)\left(\frac{u}{u_2}\right)^{\delta} du \\ &\leqq \int_{u_2}^{\infty} u^{-\alpha}\tilde{\varepsilon}(su_2)u^{\delta}u_2^{-\delta}du \\ &= \frac{1}{\alpha-1-\delta} u_2^{1-\alpha}\tilde{\varepsilon}(su_2) \leqq \frac{\tilde{\varepsilon}(su_2)}{8(\alpha-1)}.\end{aligned}$$

By (64), with $t_0 = su_2$, and by (62), we have

(68)
$$\begin{aligned}&\left|\tilde{\varepsilon}(su_2)\int_{u_1}^{u_2}(1+iu)^{-\alpha}du - I_2\right| \\ &= \left|\int_{u_1}^{u_2}(1+iu)^{-\alpha}[\tilde{\varepsilon}(su_2)-\tilde{\varepsilon}(su)]du\right| \\ &\leqq \int_{u_1}^{u_2}|1+iu|^{-\alpha}\tilde{\varepsilon}(su_2)\left(1-\left(\frac{u_1}{u_2}\right)^{\delta}\right)du \\ &\leqq \frac{\tilde{\varepsilon}(su_2)}{8(\alpha-1)}.\end{aligned}$$

Combining (65), (66), (67) and (68) we obtain

$$\left|h(s)-s^{1-\alpha}\tilde{\varepsilon}(su_2)\int_{u_1}^{u_2}(1+iu)^{-\alpha}du\right| \leqq \frac{3}{8}s^{1-\alpha}\frac{\tilde{\varepsilon}(su_2)}{\alpha-1}.$$

This, together with (61) and the fact that $\tilde{\varepsilon}$ is nondecreasing gives

$$|h(s)| \geqq \frac{3\tilde{\varepsilon}(su_2)}{8(\alpha-1)s^{\alpha-1}} \geqq \frac{\tilde{\varepsilon}(s)}{4s^{\alpha-1}(\alpha-1)}$$

when $0<s<(\pi/u_2)$.

LEMMA 8. *Let* $\alpha>1$. *Then*

$$\lim_{s\to 0^+} s^{\alpha-1}\int_0^{\pi} |(s+it)^{-\alpha}-(s+(1-e^{-it}))^{-\alpha}|dt = 0.$$

Proof. Denote the integrand in the preceding expression by $k(s, t)$. Note that

(69)
$$k(s,t) \leqq \frac{2}{|1-e^{-it}|^{\alpha}} \leqq \frac{c}{t^{\alpha}}, \qquad 0<t\leqq\pi$$

for some constant $c>0$. On the other hand since $|(s+it)-(s+(1-e^{-it}))| \leqq t^2$, $0\leqq t\leqq\pi$, we may write

(70)
$$k(s,t) \leqq t^2 \sup_z \alpha|z|^{-\alpha-1}, \qquad 0\leqq t\leqq\pi,$$

where the supremum is taken over the closed line segment joining $s+it$ and $s+(1-e^{-it})$. If $s\leqq(1/4)$ and $0\leqq t\leqq(\sqrt{s}/2)$ then for each z in this interval we have

$$|z-(s+it)|\leqq|(s+it)-(s+(1-e^{-it}))|\leqq t^2\leqq\frac{1}{4}\min(t,\ s).$$

Hence for those z's we have

$$|z|^{-\alpha-1}\leqq\begin{cases}\left(\dfrac{3}{4}t\right)^{-\alpha-1}, & s\leqq t\leqq\dfrac{\sqrt{s}}{2}\\ \left(\dfrac{3}{4}s\right)^{-\alpha-1}, & 0\leqq t\leqq s.\end{cases}\tag{71}$$

Combining (69), (70) and (71) we get for $0<s\leqq(1/4)$,

$$k(s,\ t)\leqq c\begin{cases}t^{-\alpha}, & \dfrac{\sqrt{s}}{4}\leqq t\leqq\pi\\ t^{-\alpha+1}, & s\leqq t\leqq\dfrac{\sqrt{s}}{2}\\ s^{-\alpha+1}, & 0\leqq t\leqq s.\end{cases}$$

Applying these estimates to $\int_0^\pi k(s,\ t)dt$ we easily obtain the Lemma.

Our next result shows that Theorem B is sharp. When $1\leqq\alpha\leqq2$ a weaker sharpness result containing a limit superior was obtained in [2].

THEOREM 3. *Let* $\alpha>1$ *and let* $\varepsilon(r)$ *be a positive function on* $0\leqq r<1$ *with* $\lim_{r\to1^-}\varepsilon(r)=0$. *Then there is a differentiable function* $g(t)$, $-\pi\leqq t\leqq\pi$, *so that*

$$\lim_{r\to1^-}\frac{\left|\int_{-\pi}^{\pi}(1-re^{-it})^{-\alpha}dg(t)\right|}{\varepsilon(r)(1-r)^{1-\alpha}}=+\infty.$$

Proof. Denote $\tilde{\varepsilon}(t)=\sqrt{\varepsilon(1-t)}$, $0<t\leqq1$, and $\tilde{\varepsilon}(t)=\sqrt{\varepsilon(0)}$, $1<t\leqq\pi$. Note that if the assertion of the theorem is true with $\varepsilon(r)=\varepsilon_1(r)$ and $\varepsilon_2(r)\leqq\varepsilon_1(r)$, then it is also true with $\varepsilon(r)=\varepsilon_2(r)$. Hence, by replacing $\varepsilon(r)$ with a larger function we may assume additionally that:

i) $\tilde{\varepsilon}$ is C^1 and nondecreasing on $(0,\ \pi]$,

ii) $\displaystyle\lim_{s\to0^+}\frac{s^{\alpha-1}}{\tilde{\varepsilon}(s)}\int_0^\pi|(s+it)^{-\alpha}-(s+(1-e^{-it}))^{-\alpha}|dt=0$,

(Lemma 8 is used to ensure this),

iii) $(d\log\tilde{\varepsilon}(t)/d\log t)\leqq\delta$, $0<t\leqq\pi$, with δ being the positive constant from Lemma 7.

To obtain iii) note that for each bounded above real function $a(s)$ defined on a semifinite right-bounded interval with $\lim_{s\to-\infty}a(s)=-\infty$ there is a C^1 function a_1 defined on the same interval, and such that $\lim_{s\to-\infty}a_1(s)=-\infty$ and

$(da_1(s)/ds) \leqq \delta$. Then take $a(s) = \log \tilde{\varepsilon}(e^s)$ with $\tilde{\varepsilon}$ satisfying ii) and replace $\tilde{\varepsilon}(t)$ with $\exp(a_1(\log t))$.

Let us define

$$g(t) = \begin{cases} 0, & t \leqq 0 \\ \int_0^t \tilde{\varepsilon}(u)du, & 0 \leqq t \leqq \pi. \end{cases}$$

Clearly $g \in C^1[-\pi, \pi]$.

Note that

$$f(r) = \int_{-\pi}^{\pi} (1-re^{-it})^{-\alpha} dg(t) = \int_0^{\pi} (1-re^{-it})^{-\alpha} \tilde{\varepsilon}(t)dt.$$

Let $h(s) = \int_0^{\pi} (s+it)^{-\alpha} \tilde{\varepsilon}(t)dt$. By Lemma 7 we have

$$s^{\alpha-1}|h(s)| \geqq \frac{\tilde{\varepsilon}(s)}{4(\alpha-1)} \tag{72}$$

for all sufficiently small positive s.

Observe now that

$$\left| f(r) - r^{-\alpha} h\left(\frac{1-r}{r}\right) \right| \leqq \int_0^{\pi} \left| (1-re^{-it})^{-\alpha} - r^{-\alpha}\left(\frac{1-r}{r} + it\right)^{-\alpha} \right| \tilde{\varepsilon}(t)dt$$
$$\leqq \tilde{\varepsilon}(\pi) r^{-\alpha} \int_0^{\pi} \left| \left(\frac{1-r}{r} + (1-e^{-it})\right)^{-\alpha} - \left(\frac{1-r}{r} + it\right)^{-\alpha} \right| dt.$$

By (ii) above, the last expression multiplied by $(1-r)^{\alpha-1}/\tilde{\varepsilon}((1-r))/r$ tends to 0 when r approaches 1. Hence, by (72) with $s = (1-r)/r$, for all r sufficiently close to 1 we have

$$(1-r)^{\alpha-1}|f(r)| \geqq \frac{\tilde{\varepsilon}((1-r)/r)}{8(\alpha-1)} \geqq \frac{\tilde{\varepsilon}(1-r)}{8(\alpha-1)} = \frac{\sqrt{\varepsilon(r)}}{8(\alpha-1)}$$

Therefore, $(1-r)^{\alpha-1}|f(r)|/\varepsilon(r) \geqq 1/(8(\alpha-1)\sqrt{\varepsilon(r)})$ for such r's. But since $\lim_{r \to 1^-} \varepsilon(r) = 0$, the proof is complete.

Remark. When $\alpha = 1$ it is possible to prove that for any $\varepsilon(r)$ as in Theorem 3, there is a differentiable function $g(t)$, $-\pi \leqq t \leqq \pi$, so that

$$\lim_{r \to 1^-} \frac{\left| \int_{-\pi}^{\pi} \log(1/(1-re^{-it}))dg(t) \right|}{\varepsilon(r) \log 1/(1-r)} = +\infty.$$

We do not give the details. Such a result with a limit superior replacing the limit was obtained in [2].

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Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716
USA

Instytut Matematyki, Politechnika Wroclawska
Wybrzeże St. Wyspiańskiego 27
50-370 Wroclaw
Poland