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BOUNDARY BEHAVIORS OF THE POINCARÉ DENSITY AND ITS DERIVATIVES NEAR A NONISOLATED BOUNDARY POINT

To Masatsugu Tsuji (1894-1960)

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Abstract

Let $P_\Omega(z)|dz|$ be the Poincaré metric element with the constant Gauss curvature -4 of a hyperbolic domain Ω in the complex plane $\boldsymbol{C}$. We find some boundary properties of the Poincaré density P_Ω and its complex partial derivatives $(P_\Omega)_z$, $(P_\Omega)_{zz}$ and $(P_\Omega)_{z\bar{z}}$, in terms of the distance $\delta_\Omega(z)$ of $z\in\Omega$ and the boundary of Ω in $\boldsymbol{C}$. For the proof we make use of the sharp, lower estimates of $P_{\Omega(K)}$ of a domain $\Omega(K)\subset\boldsymbol{C}$ such that $K=\boldsymbol{C}\setminus\Omega(K)$ is a non-degenerate continuum. Several properties of the function $p(z, K)$, $z\in\Omega(K)$, are proposed.

1. Introduction

A domain Ω in the complex plane $\boldsymbol{C}=\{|z|<+\infty\}$ is called hyperbolic if its boundary $\partial\Omega$ in $\boldsymbol{C}$ contains at least two points. Each hyperbolic domain Ω has the Poincaré metric element $P_\Omega(z)|dz|$, $z\in\Omega$, that is, if f is an analytic, universal-covering projection from the disk $D=\{|z|<1\}$ onto Ω, $f\in\mathrm{Proj}(\Omega)$ in notation, then

$$1/P_\Omega(z)=(1-|w|^2)|f'(w)|$$

for the Poincaré density $P_\Omega>0$ at $z=f(w)$, $w\in D$. The choice of f and w is immaterial as far as $z=f(w)$ is satisfied.

It is familiar that $P_\Omega(z)$ tends to $+\infty$ as z tends to each point ζ of $\partial\Omega$ [J, p. 116]. This also follows from a more precise property:

$$\liminf_{z\to\zeta}[\delta_\Omega(z)\log(1/\delta_\Omega(z))]P_\Omega(z)\geqq 1/2, \tag{1.1}$$

where $\delta_\Omega(z)$ is the distance of $z\in\Omega$ and $\partial\Omega$; a proof is contained in Section 8 for completeness. In general, $\delta_\Omega(z)P_\Omega(z)\leqq 1$ at each point $z\in\Omega$; see [Kr, p. 45] and [Y2, p. 104, (IP)] for example. In the forthcoming paper [Y4] we shall

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mainly study behaviors of P_Ω, $(P_\Omega)_z=\overline{(P_\Omega)_{\bar z}}$, $(P_\Omega)_{zz}=\overline{(P_\Omega)_{\bar z\bar z}}$ and $(P_\Omega)_{z\bar z}=4^{-1}\Delta P_\Omega$ in the vicinity of an isolated boundary point b in terms of $\delta_\Omega(z)$ which is $|z-b|$ near b. Here, $\phi_z=\partial\phi/\partial z=(1/2)\{(\partial\phi/\partial x)-i(\partial\phi/\partial y)\}$, $\phi_{\bar z}=\overline{(\bar\phi)_z}$, $\phi_{zz}=\partial^2\phi/\partial z^2$, $\phi_{z\bar z}=\partial^2\phi/\partial z\partial\bar z$, *etc.*, are complex partial derivatives with respect to $z=x+iy$ and $\bar z=x-iy$. In the present paper we investigate behaviors of them near a general or a nonisolated boundary point. We begin with

THEOREM 1. *Let $\Omega\subset \boldsymbol{C}$ be a hyperbolic domain and let $\zeta\in\partial\Omega$. Suppose that there exists a connected component K of $\boldsymbol{C}\backslash\Omega$ which contains ζ and another point. Suppose further that there exists an open disk U of center ζ such that $U\cap(\boldsymbol{C}\backslash\Omega)=U\cap K$. Then,*

$$\liminf_{z\to\zeta}\delta_\Omega(z)P_\Omega(z)\geqq 1/4. \tag{1}$$

Note that the following fact

$$(P_\Omega)_{z\bar z}=P_\Omega^{-1}|(P_\Omega)_z|^2+P_\Omega^3>0 \tag{1.2}$$

in Ω is derived from the Gauss curvature identity

$$\Delta\log P_\Omega=4P_\Omega^2 \quad \text{in} \quad \Omega.$$

THEOREM 2. *Let $\Omega\subset\boldsymbol{C}$ be a hyperbolic domain and set*

$$\alpha_\Omega(\zeta)=\liminf_{z\to\zeta}\delta_\Omega(z)P_\Omega(z)$$

for each $\zeta\in\partial\Omega$. Then,

$$\liminf_{z\to\zeta}\delta_\Omega(z)^2|(P_\Omega)_z(z)|\leqq A(\alpha_\Omega(\zeta))\leqq 1/2, \tag{2}$$

where $A(x)=2(x-x^2)$, $0\leqq x\leqq 1$;

$$\liminf_{z\to\zeta}\delta_\Omega(z)^3|(P_\Omega)_{zz}(z)|\leqq B(\alpha_\Omega(\zeta))\leqq\beta, \tag{3}$$

where $B(x)=5x^3-16x^2+11x$, $0\leqq x\leqq 1$ and

$$\beta=(182\sqrt{91}-272)/675=2.169\cdots;$$

$$\liminf_{z\to\zeta}\delta_\Omega(z)^3(P_\Omega)_{z\bar z}(z)\leqq C(\alpha_\Omega(\zeta))\leqq 1, \tag{4}$$

where $C(x)=5x^3-8x^2+4x$, $0\leqq x\leqq 1$.

For our proof of Theorem 1 we need a detailed study of P_Ω in case $\boldsymbol{C}\backslash\Omega$ is a nondegenerate continuum, that is, a closed and connected set in $\boldsymbol{C}$ containing at least two points. For this purpose we introduce a function $p(z, K)$ of $z\in\boldsymbol{C}\backslash K$, where K is a bounded, nondegenerate continuum and $\boldsymbol{C}\backslash K$ is connected. The function p itself has some properties which would be worth proposing, and will be given mainly in the long Section 6 and in Section 7. Our investigation of p culminates in Theorem 4.

The existence condition of a disk U described in Theorem 1 cannot be dropped to obtain (1); see Remark 2 in Section 5. The absolute constant 1/4 in (1) is best possible. Actually let Ω_0 be the complement of the nonpositive real axis $(-\infty, 0]$ with respect to $\boldsymbol{C}$. Then for $r>0$ and for real α, $|\alpha|<\pi$, we have the expression:

$$\text{(1.3)}\qquad \delta_{\Omega_0}(re^{i\alpha})P_{\Omega_0}(re^{i\alpha})=1/\{4\cos(\alpha/2)\}, \quad \text{if } |\alpha|\leqq\pi/2;$$
$$=(1/2)\sin(|\alpha|/2), \quad \text{if } \pi/2<|\alpha|<\pi.$$

In particular, $\delta_{\Omega_0}(x)P_{\Omega_0}(x)=1/4$ for $x>0$, so that the lower limit in (1) at $\zeta=0$ for the present Ω_0 is just 1/4.

If $\boldsymbol{C}\backslash\Omega$ is unbounded and it consists of a finite number of nondegenerate continua, then (1) is valid at each $\zeta\in\partial\Omega$. In particular we know further that $\inf_{z\in\Omega}\delta_\Omega(z)P_\Omega(z)>0$ [M, Lemma 2], or Ω is of finite type [Y1, Y2].

Let Ω be a hyperbolic domain in $\boldsymbol{C}$. In the course of the proof of Theorem 2 we actually have the following at each $z\in\Omega$:

$$\delta_\Omega(z)^2|(P_\Omega)_z(z)|\leqq 1/2;$$

$$\delta_\Omega(z)^3|(P_\Omega)_{zz}(z)|\leqq\beta;$$

$$\delta_\Omega(z)^3(P_\Omega)_{z\bar{z}}(z)\leqq 1.$$

The constants in the right-hand sides may not be sharp, yet the powers $k=2, 3$ of $\delta_\Omega(z)^k$ are sharp. Actually,

$$\lim_{z\to\zeta}\delta_D(z)^2|(P_D)_z(z)|=1/4;$$

$$\lim_{z\to\zeta}\delta_D(z)^3|(P_D)_{zz}(z)|=1/4;$$

$$\lim_{z\to\zeta}\delta_D(z)^3(P_D)_{z\bar{z}}(z)=1/4.$$

2. Simply or doubly connected domains

Given a nondegenerate continuum K in $\boldsymbol{C}$ we set

$$\delta_K(z)=\inf_{w\in K}|z-w| \quad\text{and}\quad \Delta_K(z)=\sup_{w\in K}|z-w|$$

for $z\in\boldsymbol{C}$. Then $0\leqq\delta_K\leqq\Delta_K\leqq+\infty$. If $\delta_K(z)=\Delta_K(z)$ at $z\in\boldsymbol{C}\backslash K$, then K lies on $\{w;\ |w-z|=\delta_K(z)\}$ and has the length $\delta_K(z)\Theta_K(z)$ with $0<\Theta_K(z)\leqq 2\pi$.

By $\Omega(K)$ we always mean a domain in $\boldsymbol{C}$ such that $\boldsymbol{C}\backslash\Omega(K)=K$ is a nondegenerate continuum. Thus, $\Omega(K)$ is hyperbolic, and further, $\Omega(K)$ is simply connected (doubly connected, respectively) if and only if K is unbounded (bounded, respectively). We have $\delta_{\Omega(K)}(z)=\delta_K(z)>0$ at each $z\in\Omega(K)$.

THEOREM 3. *For $\Omega(K)$ defined in the preceding paragraph we have the following propositions* (I) *and* (II):
(I) *If K of $\Omega(K)$ is unbounded, then*

$$\delta_K(z)P_{\Omega(K)}(z)\geqq 1/4 \quad \text{for all} \quad z\in\Omega(K). \tag{2.1}$$

(II) *Suppose that K of $\Omega(K)$ is bounded and let $z\in\Omega(K)$.*
(II.1) *If $\delta_K(z)<\Delta_K(z)$, then*

$$\delta_K(z)P_{\Omega(K)}(z)\geqq\frac{\delta_K(z)^{1/2}/\Delta_K(z)^{1/2}}{4\operatorname{arctanh}(\delta_K(z)^{1/2}/\Delta_K(z)^{1/2})}, \tag{2.2}$$

where $\operatorname{arctanh} x=(1/2)\log[(1+x)/(1-x)]$, $0\leqq x<1$.
(II.2) *If $\delta_K(z)=\Delta_K(z)$, then*

$$\delta_K(z)P_{\Omega(K)}(z)=[\cos^2(\Theta_K(z)/4)]/[2\log\{\operatorname{cosec}(\Theta_K(z)/4)\}]. \tag{2.3}$$

Actually Proposition (I) is well known because $\Omega(K)$ is simply connected; see [Kr, p. 45] and [Y2, p. 104, (IIP)] for example. As we have seen in Section 1, the equality in (2.1) holds at all points $x>0$ in $\Omega_0=\Omega((-\infty, 0])$. The "limiting" case where $\Delta_K(z)=+\infty$, that is, K is unbounded, "in" (2.2) is (2.1). An example of a pair z, K for which the equality in (2.2) holds will be proposed.

Fix $\delta>0$ and $z\in\boldsymbol{C}$. Let $V(z, \delta)=\{w;\ |w-z|<\delta\}$ and let K be a closed arc on the circle $\partial V(z, \delta)$ with $\Theta_K(z)<2\pi$. Then, it follows from (2.3) that

$$P_{\Omega(K)}(z)\longrightarrow 1/\delta=P_{V(z,\delta)}(z) \tag{2.4}$$

as $\Theta_K(z)\to 2\pi$. Namely, at the very moment when K separates z from ∞, we have a continuous "change" (2.4). On the other hand, $P_{\Omega(K)}(z)\to 0$ as $\Theta_K(z)\to 0$. Namely, at the very moment when $\Omega(K)$ becomes nonhyperbolic, we "lose" $P_{\Omega(K)}$.

3. Lemmata

Let $\mathscr{S}(p)$ for $0<p<1$ be the family of meromorphic and univalent functions f in D with their common pole at p and $f(0)=f'(0)-1=0$. A typical member of $\mathscr{S}(p)$ is

$$k_p(z)=pz/((p-z)(1-pz))$$

which maps, in particular, the punctured disk

$$D(p)=D\setminus\{p\}$$

onto the domain $\Omega(K(p))$, where

$$K(p)=[-p/(1-p)^2,\ -p/(1+p)^2]$$

is the closed real interval. Another typical one is k_p^* explained later in the proofs of Lemma 2 and Theorem 4. We begin with

LEMMA 1. *For* $f\in\mathscr{S}(p)$, $0<p<1$,

$$(C\cup\{\infty\})\backslash f(D)\subset\{z\,;\ p/(1+p)^2\leqq|z|\leqq p/(1-p)^2\}. \tag{3.1}$$

Both bounds $p/(1\pm p)^2$ in (3.1) are attained by k_p. Lemma 1 is due to W. Fenchel, W. E. Kirwan and G. Schober; see [F; KS] and [Go, p. 249, Theorem 41].

Suppose that K of $\Omega(K)$ is bounded. Then, for each $z\in\Omega(K)$ there exists a meromorphic function g in D which maps D univalently onto $\Omega(K)\cup\{\infty\}$ with $g(0)=z$, $g(p)=\infty$, $0<p<1$. Suppose that g_1 is another with $g_1(0)=z$, $g_1(q)=\infty$, $0<q<1$. Applying the Schwarz lemma to $g^{-1}\circ g_1$ and $g_1^{-1}\circ g$ one can easily observe that $p=q$ and hence $g=g_1$. Thus, g and p both are unique. We shall call g canonical for z and K and write $p=p(z, K)$. We may regard $(g-z)/g'(0)\in\mathscr{S}(p)$ for the canonical g for z and K. Here we consider a geometrical bound for $p(z, K)$ in (3.2) below.

LEMMA 2. *At each point z of $\Omega(K)$ with bounded K we have*

$$p(z, K)\geqq\frac{1-(\delta_K(z)/\Delta_K(z))^{1/2}}{1+(\delta_K(z)/\Delta_K(z))^{1/2}}. \tag{3.2}$$

In case $\delta_K(z)=\Delta_K(z)$ we have

$$p(z, K)=\sin(\Theta_K(z)/4). \tag{3.3}$$

Proof. Apply Lemma 1 to $(g-z)/g'(0)\in\mathscr{S}(p)$ for the canonical g for z and K with $p=p(z, K)$. Then,

$$\delta_K(z)/|g'(0)|\geqq p/(1+p)^2\,; \tag{3.4}$$

$$\Delta_K(z)/|g'(0)|\leqq p/(1-p)^2, \tag{3.5}$$

so that

$$\delta_K(z)/\Delta_K(z)\geqq(1-p)^2/(1+p)^2$$

shows (3.2). To see the sharpness let a constant p with $0<p<1$ be given. Then, k_p is canonical for 0 and $K(p)$, and $\delta_K(0)=p/(1+p)^2$ and $\Delta_K(0)=p/(1-p)^2$. Now the equality in (3.2) holds for the pair $z=0$ and $K=K(p)$.

In case $\delta=\delta_K(z)=\Delta_K(z)$ we let $z+\delta e^{i\alpha}$ and $z+\delta e^{i\beta}$ be the initial and terminal points of the arc K, so that $\beta-\alpha=\Theta_K(z)<2\pi$. Set

$$c=\tan(\Theta_K(z)/4) \quad\text{and}\quad b=((c^2+1)^{1/2}-1)/c.$$

Let $\zeta=g(w)$ be the composed function of the following four:

$$w_1=(w-b)/(1-bw),\qquad w\in D\,;$$

$$w_2=(c/2)(w_1-w_1^{-1})\,;$$

$$w_3=(1-w_2)/(1+w_2)\,;$$

$$\zeta=z+\delta w_3 e^{i(\alpha+\beta)/2}.$$

Then g is canonical for z and K with $g(b)=z-\delta e^{i(\alpha+\beta)/2}$ and $g(p)=\infty$, where

$$p=p(z, K)=2b/(b^2+1)=\sin(\Theta_K(z)/4).$$

To be more explicit, we define for general p, $0<p<1$, the function

$$k_p^*(z)=pz(1-pz)/(p-z), \qquad z\in D. \tag{3.6}$$

Then $k_p^*\in \mathscr{S}(p)$, and, for the specified $p=\sin((\beta-\alpha)/4)$, we have the exact form of g:

$$g=z-(p^{-1}\delta e^{i(\alpha+\beta)/2})k_p^* \quad \text{in } D. \tag{3.7}$$

4. Proof of Theorem 3

If K of $\Omega(K)$ is bounded, then

$$\delta_K(z)P_{\Omega(K)}(z)\geqq(p-1)/(2(1+p)\log p) \tag{4.1}$$

at each $z\in\Omega(K)$ with $p=p(z, K)$. For the proof we let g be canonical for z and K, and further, $f\in\mathrm{Proj}(D(p))$ with $f(0)=0$. Then $g\circ f\in\mathrm{Proj}(\Omega(K))$ with $z=g\circ f(0)$. Hence

$$1/P_{\Omega(K)}(z)=|g'(0)f'(0)|=|g'(0)|/P_{D(p)}(0), \tag{4.2}$$

so that

$$\delta_K(z)P_{\Omega(K)}(z)=P_{D(p)}(0)\delta_K(z)/|g'(0)|.$$

Since

$$P_{D(p)}(0)=(p^2-1)/(2p\log p),$$

one obtains (4.1) with the aid of (3.4). By the way, (3.5) yields for $p=p(z, K)$ that

$$\Delta_K(z)P_{\Omega(K)}(z)\leqq(1+p)/(2(p-1)\log p).$$

An exact form of $f\in\mathrm{Proj}(D(p))$ with $f(0)=0$ is, for example, $f(w)=\phi_p(w+w_p)/(1+\overline{w_p}w))$, where

$$\phi_p(w)=\left[p+\exp\left(\frac{w+1}{w-1}\right)\right]\Big/\left[1+p\exp\left(\frac{w+1}{w-1}\right)\right], \qquad w\in D \tag{4.3}$$

with

$$w_p=(\log p+\pi i+1)/(\log p+\pi i-1).$$

The function ϕ_p will be considered again.

Proof of (II.1). We now have (2.2) by (4.1) and (3.2) because the right-hand side of (4.1) is an increasing function of p, $0<p<1$.

The function k_p is canonical for 0 and $K(p)$ with $k_p'(0)=1$, so that (4.2) yields:

$$1/P_{\Omega(K(p))}(0)=1/P_{D(p)}(0)=(2p\log p)/(p^2-1).$$

It is not difficult to prove that the equality in (2.2) holds for $K=K(p)$ and $z=0$.

Proof of (II.2). Let g be the function of (3.7) considered in the proof of (3.3). Again, $g\circ\psi_p\in\mathrm{Proj}(\Omega(K))$, where $p=\sin(\Theta_K(z)/4)$ by (3.3) and

$$|g'(0)|=\delta_K(z)\operatorname{cosec}(\Theta_K(z)/4).$$

Hence

$$1/P_{\Omega(K)}(z)=(1/P_{D(p)}(0))|g'(0)|$$

yields (2.3).

Remark. Suppose that Ω is hyperbolic, unbounded, and $\partial\Omega$ is bounded. A typical example of Ω is $\Omega(K)$ with bounded K. Fix $a\in\partial\Omega$. Then, 0 is an isolated boundary point of

$$\Omega^*=\{1/(z-a);\ z\in\Omega\}$$

and for $z\in\Omega$,

$$(|z-a|^{-1}\log|z-a|)P_{\Omega^*}(1/(z-a))=(|z-a|\log|z-a|)P_\Omega(z).$$

Since the left-hand side of the above equality tends to 1/2 as $|z-a|\to+\infty$ (see the end of Section 8), the right-hand side has the limit 1/2 as $|z|\to+\infty$. Since $\delta_\Omega(z)/|z-a|\to 1$ as $|z|\to+\infty$ it follows that

$$\lim_{|z|\to+\infty}(\delta_\Omega(z)\log\delta_\Omega(z))P_\Omega(z)=1/2. \tag{4.4}$$

In particular,

$$\lim_{|z|\to+\infty}\delta_\Omega(z)P_\Omega(z)=0. \tag{4.5}$$

We cannot drop the boundedness of $\partial\Omega$ to have (4.4). Actually, with the aid of (1.3) one observes that

$$(\delta_{\Omega_0}(z)\log\delta_{\Omega_0}(z))P_{\Omega_0}(z)\longrightarrow+\infty$$

as $|z|\to+\infty$ along each half line in Ω_0 emanating from the origin. Furthermore, (4.5) is false for Ω_0.

5. Proofs of Theorems 1 and 2

Proof of (1). For U we may further assume that

$$U=\{z;\ |z-\zeta|<3\varepsilon\}\qquad(\varepsilon>0)$$

satisfies $(\boldsymbol{C}\backslash U)\cap K\neq\emptyset$. Then, for each z of

$$U(\zeta, \varepsilon)=\{z\in\Omega;\ |z-\zeta|<\varepsilon\}$$

we have

$$\delta_\Omega(z)=\delta_K(z)\leqq\varepsilon<2\varepsilon\leqq\Delta_K(z)\leqq+\infty.$$

Here we remember that if $\Omega_1\subset\Omega_2$ then $P_{\Omega_1}\geqq P_{\Omega_2}$ in Ω_1; see [Gl, p. 337]. Thus, $P_\Omega(z)\geqq P_{\Omega(K)}(z)$ at each $z\in\Omega$.

If K is unbounded, then it immediately follows from (2.1) that

$$\delta_\Omega(z)P_\Omega(z)\geqq\delta_K(z)P_{\Omega(K)}(z)\geqq 1/4$$

at all $z\in U(\zeta, \varepsilon)$. Hence (1).

Suppose next that K is bounded. Since

$$\delta_K(z)/\Delta_K(z)\leqq\delta_K(z)/(2\varepsilon)\leqq|z-\zeta|/(2\varepsilon)\longrightarrow 0$$

as $z\to\zeta$ in $U(\zeta, \varepsilon)$, it follows from (2.2) that

$$\liminf_{z\to\zeta}\delta_\Omega(z)P_\Omega(z)\geqq\liminf_{z\to\zeta}\delta_\Omega(z)P_{\Omega(K)}(z)\geqq 1/4.$$

This is (1).

Proof of (2). We remember that for generel Ω,

$$2+|(P_\Omega^{-1})_z|\leqq 2\delta_\Omega^{-1}P_\Omega^{-1};$$

see [Y2, p. 116, (7.3)]. It then follows that

$$\delta_\Omega^2|(P_\Omega)_z|\leqq 2(\delta_\Omega P_\Omega-\delta_\Omega^2P_\Omega^2)=A(\delta_\Omega P_\Omega) \tag{5.1}$$

in Ω. We now have (2) by $A(x)\leqq 1/2$.

Proof of (3). For $f\in\mathrm{Proj}(\Omega)$ with $z=f(w)$ we have

$$\begin{aligned}&P_\Omega(z)^{-1}|(P_\Omega^{-1})_{zz}(z)|\\&=(1/2)(1-|w|^2)^2|f'''(w)/f'(w)-(3/2)(f''(w)/f'(w))^2|\\&\leqq 3(\delta_\Omega(z)^{-2}P_\Omega(z)^{-2}-1);\end{aligned}$$

see [Y1, p. 168, (3.3); Y2, p. 113, (6.2)]. Hence in Ω,

$$\delta_\Omega^3|(P_\Omega)_{zz}|\leqq 2|\delta_\Omega^2(P_\Omega)_z|^2\delta_\Omega^{-1}P_\Omega^{-1}+3\delta_\Omega P_\Omega-3\delta_\Omega^3P_\Omega^3.$$

Combining this with (5.1) one observes that the right-hand side is not greater than $B(\delta_\Omega P_\Omega)\leqq\beta$. Hence (3).

Proof of (4). It follows from (1.2) that

$$\delta_\Omega^3(P_\Omega)_{z\bar z}=|\delta_\Omega^2(P_\Omega)_z|^2\delta_\Omega^{-1}P_\Omega^{-1}+(\delta_\Omega P_\Omega)^3.$$

The right-hand side is not greater than $C(\delta_\Omega P_\Omega)\leqq 1$. Hence (4).

Remark 1. The forthcoming property (6.12), together with (4.1), also proves (1) in case K is bounded.

Remark 2. For $n\geqq 9$ we set $a_n=1-n^{-1}$, $r_n=2^{-n}$, and we denote the closed real intervals by $I_n=[a_n-r_n,\ a_n+r_n]$. Then

$$\Omega=D\setminus\left(\bigcup_{n=9}^{\infty} I_n\right)$$

is a hyperbolic domain and for $n\geqq 9$,

$$A_n\equiv\{z\,;\ r_n<|z-a_n|<a_{n+1}-a_n-r_{n+1}\}\subset\Omega$$

with

$$\operatorname{mod} A_n\equiv(2\pi)^{-1}\log((a_{n+1}-a_n-r_{n+1})/r_n)\longrightarrow+\infty$$

as $n\to+\infty$. It then follows from [BP, p. 478, Corollary 1] that

$$\inf_{z\in\Omega}\delta_\Omega(z)P_\Omega(z)=0\,.$$

At each $\zeta\in\partial\Omega\setminus\{1\}$ we may apply Theorem 1 to have (1). Hence

$$\liminf_{z\to 1}\delta_\Omega(z)P_\Omega(z)=0\,. \tag{5.2}$$

There exists no U described in Theorem 1 for $1\in\partial\Omega$, where K is the circle ∂D. Here we further note that

$$\liminf_{z\to 1}|z-1|\,P_\Omega(z)\geqq 1/(2c_H)\,; \tag{5.3}$$

$$c_H=\Gamma(1/4)^4/(4\pi^2)=4.376\cdots\,;$$

see [Y4, Example 1 in Section 3]. The inequality $\delta_\Omega(z)<|z-1|$ for each $z\in\Omega$ near 1 with some unknown factors might yield this delicate difference between (5.2) and (5.3).

6. Further about $p(z, K)$

The disk D is a metric space with the distance (a bad terminology is the pseudodistance):

$$d(\zeta,\ \eta)=|\zeta-\eta|/|1-\bar{\zeta}\eta|,\ \zeta,\ \eta\in D\,;$$

see [T, p. 511] for the proof of the triangle inequality. Each conformal mapping from D onto D preserves the distance d. Throughout in the present section we assume that K of $\Omega(K)$ is bounded. We set

$$d_*(z,\ w)=d(f(z),\ f(w)) \tag{6.1}$$

for $z,\ w\in\Omega^*(K)\equiv\Omega(K)\cup\{\infty\}$, where f is a conformal mapping from $\Omega^*(K)$

onto D. The right-hand side of (6.1) is independent of the specified choice of f. In particular, for g canonical for z and K, we have

$$p(z) \equiv p(z, K) = d(0, p(z)) = d(g^{-1}(z), g^{-1}(\infty)) = d_*(z, \infty).$$

Since $\lim_{z\to\infty} p(z)=0$, we set $p(\infty)=0$. Hence, p is a C^∞ function in $\Omega(K)$; actually as will be observed, p is real-analytic. Furthermore,

$$|p(z)-p(w)| \leqq d_*(z, w) \quad \text{for} \quad z, w \in \Omega^*(K). \tag{6.2}$$

In other words, p is a contraction from the metric space $(\Omega^*(K), d_*)$ into the real interval $[0, 1)$. The Poincaré distance of z and w in $\Omega^*(K)$ is

$$d_{\Omega^*(K)}(z, w) = \operatorname{arctanh} d_*(z, w).$$

Again, $(\Omega^*(K), d_{\Omega^*(K)})$ is a metric space. Hence, for $z, w \in \Omega^*(K)$,

$$|\operatorname{arctanh} p(z) - \operatorname{arctanh} p(w)| \leqq d_{\Omega^*(K)}(z, w). \tag{6.3}$$

Fix $z_0 \in \Omega(K)$ and let g_0 be canonical for z_0 and K. Then,

$$p(z) = d_*(z, \infty) = d(g_0^{-1}(z), p(z_0)), \quad z \in \Omega^*(K). \tag{6.4}$$

Setting $h_0 = g_0^{-1}$ in $\Omega^*(K)$ and then partially differentiating (6.4) with respect to z in $\Omega(K)$, together with some calculations, we have

$$2|p_z(z)|/(1-p(z)^2) = |h_0'(z)|/(1-|h_0(z)|^2), \quad z \in \Omega(K). \tag{6.5}$$

Note that $|\operatorname{grad} \phi| = 2|\phi_z|$ for a real function ϕ. Letting $z \to z_0$ in (6.5) we then have

$$|\operatorname{grad} p(z_0)|/(1-p(z_0)^2) = 1/|g_0'(0)|, \quad \text{or}$$

$$|\operatorname{grad} p(z)|/(1-p(z)^2) = 1/|g'(0)|, \tag{6.6}$$

where g in (6.6), this time, is canonical for z and K.

PROPOSITION 1. *For each $z \in \Omega(K)$ we have*

$$p(1-p)/((1+p)\delta_K) \leqq |\operatorname{grad} p| \leqq p(1+p)/((1-p)\Delta_K), \tag{6.7}$$

where $p = p(z)$, $\delta_K = \delta_K(z)$, $\Delta_K = \Delta_K(z)$.

Proof. We have (6.7) from (6.6), (3.4) and (3.5). Consider

$$k_q(w) = qw/((q-w)(1-qw))$$

for $w \in D$ and $0<q<1$, which is canonical for 0 and the closed real interval $K(q)$. It then follows from (6.4) with $g_0 = k_q$ and $z_0 = 0$ that

$$p(z) = p(z, K(q)) = d(k_q^{-1}(z), q), \quad z \in \Omega(K(q)).$$

Hence,

$$p(0)=q \quad\text{and}\quad |\mathrm{grad}\ p(0)|=1-q^2;$$

the latter follows from (6.6). It is now easy to prove that the equalities in (6.7) hold at $z=0$ for $K=K(q)$. Q. E. D.

An application of (6.7) will be described in the remark after Corollary 1 to Theorem 4.

PROPOSITION 2. *For each* $z\in\Omega(K)$ *we have*

$$P_{\Omega(K)}(z)=(1/2)|\mathrm{grad}\{\log(-\log p)\}|=|\mathrm{grad}\ p|/(-2p\log p), \tag{6.8}$$

where $p=p(z)$.

Proof. Remember (4.2): $P_{\Omega(K)}(z)=P_{D(p)}(0)/|g'(0)|$, where g is canonical for z and K. This, together with (6.6), yields (6.8). Q. E. D.

PROPOSITION 3. *Let* $f: D\to\Omega(K)$ *be analytic. Then*

$$(1-|z|^2)|(\partial/\partial z)p(f(z))|\leqq -p(f(z))\log p(f(z)) \tag{6.9}$$

at each $z\in D$.

Note that $(\partial/\partial z)p(f(z))=p_\zeta(\zeta)f'(z)$, $\zeta=f(z)$.

Proof. We choose $F\in\mathrm{Proj}(\Omega(K))$ with $F(0)=f(z)$. Apply the Schwarz lemma: $|h'(0)|\leqq 1$ to a branch $h(w)$, with $h(0)=0$, of the function $F^{-1}\circ f((w+z)/(1+\bar{z}w))$ of $w\in D$. Then, since $1/|F'(0)|=P_{\Omega(K)}(f(z))$, it follows that

$$(1-|z|^2)|f'(z)|P_{\Omega(K)}(f(z))\leqq 1, \tag{6.10}$$

which, combined with (6.8), shows (6.9). The equality in (6.9) at z (actually, then at all $z\in D$) holds if and only if $h(w)\equiv \varepsilon w$, $\varepsilon\in\partial D$, and hence, if and only if $f\in\mathrm{Proj}(\Omega(K))$. Q. E. D.

It immediately follows from (6.9) that,

$$2|(\partial/\partial z)\log(-\log p(f(z)))|\leqq 2/(1-|z|^2), \qquad z\in D,$$

for analytic $f: D\to\Omega(K)$. Hence for $z,\ w\in D$,

$$|\log(\{\log p(f(z))\}/\{\log p(f(w))\})|\leqq 2\ \mathrm{arctanh}\ d(z,\ w).$$

Another consequence of (6.8) is that

$$|\log(\{\log p(z)\}/\{\log p(w)\})|\leqq 2d_{\Omega(K)}(z,\ w)$$

for $z,\ w\in\Omega(K)$, where

$$d_{\Omega(K)}(z, w)=\inf_{\gamma}\int_{\gamma} P_{\Omega(K)}(\zeta)|d\zeta|,$$

γ ranging over all recctifiable curves connecting z and w in $\Omega(K)$. We remember that $d_{\Omega(K)}$ is the Poincaré distance in $\Omega(K)$. There exists a not necessarily unique curve $\gamma_0=\gamma_0(z, w)\subset\Omega(K)$ connecting z and w with $z\neq w$ in $\Omega(K)$ such that

$$d_{\Omega(K)}(z, w)=\int_{\gamma_0} P_{\Omega(K)}(\zeta)|d\zeta|.$$

Note that $\log(-\log p)$ is superharmonic in $\Omega(K)$. In fact, for $p=p(z)$, $z\in\Omega(K)$,

$$\Delta \log(-\log p)=-|\text{grad}\, p|^2/(p \log p)^2=-4P_{\Omega(K)}(z)^2.$$

Since the derivative of g_0^{-1} in (6.4) never vanishes in $\Omega(K)$, the function $\log|\text{grad}\, p|$ is harmonic in $\Omega(K)$. Consider the metric $\lambda_{\Omega(K)}(z)|dz|$ with the density

$$\lambda_{\Omega(K)}=|\text{grad}\ p|/(1-p^2)=|\text{grad arctanh}\ p|,\quad p=p(z),$$

in $\Omega(K)$. In view of (6.5), $\lambda_{\Omega(K)}(z)|dz|$ is actually the restriction to $\Omega(K)$ of the Poincaré metric element of $\Omega^*(K)$. In particular, with the aid of (6.5) one observes that

$$-\lambda_{\Omega(K)}(z)^{-2}\Delta \log \lambda_{\Omega(K)}(z)\equiv-4,$$

or the Gauss curvature of $\lambda_{\Omega(K)}(z)|dz|$ at each $z\in\Omega(K)$ is constantly -4. Remember that the Gauss curvature of $P_{\Omega(K)}(z)|dz|$ is also constantly -4.

PROPOSITION 4. *For $z\neq w$ in $\Omega(K)$,*

$$|\text{arctanh}\ p(z)-\text{arctanh}\ p(w)|<d_{\Omega(K)}(z, w). \tag{6.11}$$

Proof. Since $\Omega(K)\subset\Omega^*(K)$ we have $\lambda_{\Omega(K)}<P_{\Omega(K)}$ in $\Omega(K)$, so that (6.11) is immediate. However, we shall give a self-contained proof. First,

$$P_{\Omega(K)}(z)/\lambda_{\Omega(K)}(z)=P_{D(p(z))}(0)>1 \qquad \text{in} \quad \Omega(K).$$

Hence,

$$|\text{arctanh}\ p(z)-\text{arctanh}\ p(w)|$$

$$\leqq\int_{\gamma_0(z,w)}\lambda_{\Omega(K)}(\zeta)|d\zeta|<\int_{\gamma_0(z,w)}P_{\Omega(K)}(\zeta)|d\zeta|=d_{\Omega(K)}(z, w).$$

Q. E. D.

Now, we have $\delta_K(z)/\Delta_K(z)\to 0$ as $z\to b\in\partial\Omega(K)$ in $\Omega(K)$. For, $\Delta_K(z)$ is bounded away from zero as $z\to b$. Hence (3.2) with $p<1$ yields that

$$\lim_{z\to b} p(z)=1, \qquad b\in\partial\Omega(K). \tag{6.12}$$

Suppose that there exists a rectifiable curve $\gamma\subset\Omega(K)$ with a starting point

$w \in \Omega(K)$ and an ending point $b \in \partial\Omega(K)$. Then,

$$\int_{\gamma} P_{\Omega(K)}(\zeta)|d\zeta| = \int_{\gamma} \lambda_{\Omega(K)}(\zeta)|d\zeta| = +\infty. \tag{6.13}$$

For the proof we let $\gamma(z)$ be a subarc of γ with the starting point w and an ending point $z \in \gamma$. Then, the estimate:

$$|\operatorname{arctanh} p(z) - \operatorname{arctanh} p(w)| \leqq \int_{\gamma(z)} \lambda_{\Omega(K)}(\zeta)|d\zeta|,$$

together with (6.12), proves (6.13). The situation is different for $\lambda_{\Omega(K)}$ and for a curve in $\Omega(K)$ ending at ∞. For each $z \in \Omega(K)$ we have an analytic curve $\Lambda(z) \subset \Omega(K)$ starting at $z \in \Omega(K)$ and ending at ∞ such that

$$\int_{\Lambda(z)} \lambda_{\Omega(K)}(\zeta)|d\zeta| < +\infty. \tag{6.14}$$

For the proof we remember (6.5). Let $\Gamma(z)$ be the geodesic line segment between $h_0(z)$ and $p(z_0)$. Then for $\Lambda(z) = g(\Gamma(z))$ we have

$$\int_{\Lambda(z)} \lambda_{\Omega(K)}(\zeta)|d\zeta| = \int_{\Gamma(z)} (1-|\eta|^2)^{-1}|d\eta| = d_{\Omega^*(K)}(z, \infty) < +\infty.$$

This is (6.14).

Let $\Lambda_0(z) \subset \Omega(K)$ be a locally rectifiable curve starting at $z \in \Omega(K)$ and ending at ∞. Then,

$$\int_{\Lambda_0(z)} P_{\Omega(K)}(\zeta)|d\zeta| = +\infty. \tag{6.15}$$

For the proof we have only to let $w \to \infty$ along $\Lambda_0(z)$ in

$$|\log(\{\log p(w)\}/\{\log p(z)\})| \leqq 2\int_{\Lambda_0(z)} P_{\Omega(K)}(\zeta)|d\zeta|.$$

Since $p(w) \to 0$ we have (6.15).

An important consequence of (6.4) is that $\log p$ is harmonic in $\Omega(K)$ and $\Delta p = 4|p_z|^2/p$ in $\Omega(K)$; in fact, $-\log p (=+\infty$ at $\infty)$ in $\Omega^*(K)$ is the Green function [N, p. 28 *et seq.*, p. 123] of $\Omega^*(K)$ with its pole at ∞. Another consequence of (6.4) is that the level set $\mathcal{L}(p, c) \equiv \{z \in \Omega(K);\ p(z) = c\}$ $(0 < c < 1)$ is the analytic Jordan curve which is the image by g_0, canonical for z_0 and K, of the Apollonius circle:

$$\{w;\ d(w, p(z_0)) = c\}.$$

It follows from (6.12) that $\mathcal{L}(p, c)$ "separates" ∞ and K: For each c, $0 < c < 1$, $\{z \in \Omega(K);\ p(z) \leqq c\}$ is unbounded.

Set $\mathcal{D}(p, c) = \{z \in \Omega(K);\ p(z) < c\}$ for $0 < c < 1$, and let h_0 be the inverse of g_0 in (6.4). Then, $h_0(\mathcal{D}(p, c)) = \{w;\ 0 < d(w, p(z_0)) < c\}$ has the non-Euclidean area $\pi c^2/(1-c^2)$. It then follows from (6.5) that

$$\iint_{\mathcal{D}(p,c)} \lambda_{\Omega(K)}(z)^2 dx\,dy = \pi c^2/(1-c^2).$$

Since in $\mathcal{D}(p, c)$,

$$P_{\Omega(K)}(z) > ((c^2-1)/(2c\log c))\lambda_{\Omega(K)}(z),$$

it follows that

$$\iint_{\mathcal{D}(p,c)} P_{\Omega(K)}(z)^2 dx\,dy > \pi(1-c^2)/(2\log c)^2.$$

In particular,

$$\liminf_{c\to 1}(1-c)\iint_{\mathcal{D}(p,c)} P_{\Omega(K)}(z)^2\,dx\,dy \geqq \pi/2\,;$$

note that $\mathcal{D}(p, c)\uparrow\Omega(K)$ as $c\uparrow 1$.

Similarly,

$$\int_{\mathcal{L}(p,c)} \lambda_{\Omega(K)}(z)|dz| = 2\pi c/(1-c^2)$$

and

$$P_{\Omega(K)}(z) = ((c^2-1)/(2c\log c))\lambda_{\Omega(K)}(z)$$

on $\mathcal{L}(p, c)$, so that

$$\int_{\mathcal{L}(p,c)} P_{\Omega(K)}(z)|dz| = \pi/(-\log c),$$

whence

$$\lim_{c\to 1}(1-c)\int_{\mathcal{L}(p,c)} P_{\Omega(K)}(z)|dz| = \pi.$$

This section now ends with a theorem and its corollaries.

THEOREM 4. *Suppose that K of $\Omega(K)$ is bounded and let $C(K)$ be the capacity* [N, p. 123] *of K. Then,*

(6.16) $$C(K)^{-1}p^2(1-p^2) \leqq |\operatorname{grad} p| \leqq C(K)^{-1}p^2(1-p^2)^{-1},$$

where $p = p(z, K)$, $z\in\Omega(K)$.

Proof. Let g be canonical for z and K. Since $-\log p(\zeta) = -\log d(g^{-1}(\zeta), p(z))$ is the Green function of $\Omega^*(K)$ with its pole at ∞, it follows on setting $\operatorname{Res}(g, p) = \lim_{w\to p}(w-p)g(w)$, $p = p(z)$, that

$$\mathcal{R}(K) = -\lim_{\zeta\to\infty}(\log d(g^{-1}(\zeta), p) + \log|\zeta|) = \log((1-p^2)/|\operatorname{Res}(g, p)|)$$

is the Robin constant [N, p. 123] of $\Omega^*(K)$, so that $C(K) = e^{-\mathcal{R}(K)} = |\operatorname{Res}(g, p)|/(1-p^2)$ by definition; in particular, $|\operatorname{Res}(g, p(z))|/(1-p(z)^2)$ is independent of $z\in\Omega(K)$. Since $(g-z)/g'(0)\in\mathcal{S}(p)$, it follows from [Ko, p. 278, (4.4)] (see also [Go, p. 263]) that

$$|g'(0)|p^2(1-p^2)\leqq|\mathrm{Res}(g,\ p)|\leqq|g'(0)|p^2(1-p^2)^{-1}.$$

Hence

$$|g'(0)|p^2\leqq C(K)\leqq|g'(0)|p^2(1-p^2)^{-2},$$

which, combined with (6.6), yields (6.16). Consider k_q canonical for 0 and $K(q)$, $0<q<1$, for which we have $p(0)=q$ and $|\mathrm{grad}\ p(0)|=1-q^2$. Since $\mathrm{Res}(k_q,\ q)=q^2(q^2-1)^{-1}$, it follows that $C(K(q))=q^2(1-q^2)^{-2}$. This also follows from the fact that a rectilinear segment of length a has the capacity $a/4$; see [L, p. 172]. Hence the right equality in (6.16) holds at $z=0$ in $\Omega(K(q))$. We next consider the function k_q^* of $\mathcal{S}(q)$, $0<q<1$; see (3.6). The function k_q^* is then canonical for 0 and the circular arc

$$K^*(q)=\{-qe^{it};\ |t|\leqq 2\Theta_q\},$$

where $\Theta_q=\arcsin q\in(0,\ \pi/2)$; in fact, $\Theta_{K^*(q)}(0)=4\Theta_q$. Then, $p(0)=q$ and $|\mathrm{grad}\ p(0)|=1-q^2$ by (6.6). Since $\mathrm{Res}(k_q^*,\ q)=q^2(q^2-1)$, it follows that $C(K^*(q))=q^2$. Thus, the left-hand side equality in (6.16) holds at $0\in\Omega(K^*(q))$. Q. E. D.

COROLLARY 1. *Let $C(K)$ be the capacity of K and set $p(\zeta)=p(\zeta,\ K)$, $\zeta\in\Omega(K)$. Then, $p+p^{-1}$ is Lipschitz continuous:*

(6.17) $$|(p(z)+p(z)^{-1})-(p(w)+p(w)^{-1})|\leqq C(K)^{-1}|z-w|,$$

$z,\ w\in\Omega(K)$.

Proof. For $\Phi=p+p^{-1}$ in $\Omega(K)$ the upper estimate of $|\mathrm{grad}\ p|$ in (6.16) yields that

(6.18) $$|\mathrm{grad}\ \Phi|\leqq C(K)^{-1}.$$

For $z,\ w\in\Omega(K)$, $z\neq w$, we consider the directed line from w to z:

$$l(w,\ z)=\{\varphi(t)\equiv w+t(z-w);\ 0\leqq t\leqq 1\}.$$

Suppose first that $l(w,\ z)\cap K\neq\emptyset$ and then let

$$l(1)=\{\varphi(t);\ 0\leqq t<t_1\}\quad\text{and}\quad l(2)=\{\varphi(t);\ t_2<t\leqq 1\},\quad(t_1\leqq t_2)$$

be the connected components of $l(w,\ z)\cap\Omega(K)$ containing w and z, respectively. Since $\Phi(\zeta)\to 2$ as $\zeta\in\Omega(K)$ tends to a point of $\partial\Omega(K)$ by (6.12), it follows that $\Phi(\varphi(t))\to 2$ as $t\to t_j$ along $l(j)$, $j=1,\ 2$. Hence,

$$2-\Phi(w)=\int_{l(1)}(\Phi_\xi(\zeta)d\xi+\Phi_\eta(\zeta)d\eta),$$

$$\Phi(z)-2=\int_{l(2)}(\Phi_\xi(\zeta)d\xi+\Phi_\eta(\zeta)d\eta),$$

where $\zeta=\xi+i\eta$. In view of (6.18) one now obtains

$$|\Phi(z)-\Phi(w)|=\left|\int_{l(1)\cup l(2)}(\Phi_{\xi}(\zeta)d\xi+\Phi_{\eta}(\zeta)d\eta)\right|$$

$$\leqq\int_{l(1)\cup l(2)}|\operatorname{grad}\Phi(\zeta)|\,|d\zeta|\leqq C(K)^{-1}\int_{l(w,z)}|d\zeta|,$$

whence (6.17). The case $l(w, z)\cap K=\emptyset$ is now obvious. Q. E. D.

Let $w\to b\in\partial\Omega(K)$ in (6.17). It then follows from the resulting estimate that

$$(1-p(z))^2/p(z)\leqq C(K)^{-1}\delta_K(z)\equiv R(z), \quad \text{or}$$

$$p(z)\geqq 2^{-1}R(z)+1-2^{-1}(R(z)^2+4R(z))^{1/2},$$

$z\in\Omega(K)$; the right-hand side is positive and tends to 1 as z tends to a point of $\partial(\Omega(K))$. The equality holds at $z=0$ for $K=K(q)$, $0<q<1$, because $\delta_{K(q)}(0)=q/(1+q)^2$ and $C(K(q))=q^2/(1-q^2)^2$.

Remark. It is not difficult to prove that

$$\Delta(K)\equiv\inf_{z\in\Omega(K)}\Delta_K(z)>0.$$

The upper estimate in (6.7) then yields that $|\operatorname{grad}\Psi|\leqq\Delta(K)^{-1}$ in $\Omega(K)$, where $\Psi=\log(p(1+p)^{-2})$. We now have

$$|\Psi(z)-\Psi(w)|\leqq\Delta(K)^{-1}|z-w|, \qquad z,\ w\in\Omega(K),$$

by the similar manner as in the paragraph just after the proof of Corollary 1. It is not difficult to have

$$p(z)^{-1}(1+p(z))^2\leqq 4\exp(\Delta(K)^{-1}\delta_K(z)),$$

whence

$$p(z)\geqq 2Q(z)-1-2(Q(z)^2-Q(z))^{1/2},$$

where $Q(z)=\exp(\Delta(K)^{-1}\delta_K(z))$, $z\in\Omega(K)$; the right-hand side is positive and tends to 1 as z tends to a point of $\partial\Omega(K)$.

COROLLARY 2. *At each $z\in\Omega(K)$ with $p=p(z, K)$, we have*

$$C(K)^{-1}p(p^2-1)(2\log p)^{-1}\leqq P_{\Omega(K)}(z)\leqq C(K)^{-1}p(2(p^2-1)\log p)^{-1}.$$

Proof. This follows from (6.8) and (6.16). The right-hand side equality holds at 0 for $K=K(q)$ and the equality in the left holds at 0 for $K^*(q)$.

Q. E. D.

Remark. If K of $\Omega(K)$ is further, convex, then the lower estimate in (6.16) can be replaced by

$$C(K)^{-1}p^2(1+p^2)^{-1}\leqq|\operatorname{grad} p|, \tag{6.19}$$

where $p=p(z, K)$, $z\in\Omega(K)$. It is open whether or not the equality holds in (6.19). For the proof of (6.19), as in the proof of Theorem 4, we have only to make use of the estimate

$$p^2(1+p^2)^{-1}\leqq|\mathrm{Res}(f, p)|, \tag{6.20}$$

where $(\boldsymbol{C}\cup\{\infty\})\setminus f(D)$ for $f\in\mathscr{S}(p)$, $0<p<1$, is supposed to be convex (in the usual sense in $\boldsymbol{R}^2\equiv\boldsymbol{C}$) and again $\mathrm{Res}(f, p)=\lim_{z\to p}(z-p)f(z)$. Set $M=\mathrm{Res}(f, p)$ and consider the function:

$$F(z)=M^{-1}(p^2-1)f((p-z)/(1-pz))=z^{-1}+a_0+a_1z+\cdots$$

in D. Then $(\boldsymbol{C}\cup\{\infty\})\setminus F(D)$ is again convex. It then follows from [PP, p. 128, Corollary 5.1] (or [Go, p. 235, (45)] for $F(1/z)$) that

$$|M|^{-1}(p^2-1)^2|1-pz|^{-2}|f'((p-z)/(1-pz))|=|F'(z)|\leqq 1+|z|^{-2},$$

$z\in D\setminus\{0\}$. Setting $z=p$ we now have $|M|^{-1}\leqq 1+p^{-2}$ or (6.20).

7. Once more on $p(z, K)$

Again in this section we suppose that K of $\Omega(K)$ is bounded. We prove the strict inequality

$$\delta_K(z)P_{\Omega(K)}(z)<4\sigma_K(z)/(\sigma_K(z)+1)^2(<1), \qquad z\in\Omega(K), \tag{7.1}$$

where

$$\sigma_K(z)=\pi^{-1}[\log p+(\pi^2+(\log p)^2)^{1/2}] \quad (<1) \tag{7.2}$$

with $p=p(z, K)$.

For the proof we let g be canonical for z and K, and we remember ψ_p of (4.3) for $p=p(z, K)$. Then $f=g\circ\psi_p\in\mathrm{Proj}(\Omega(K))$ with $f(w_p)=z$. The supremum $\sigma_K(z)$ of r, $0<r<1$, for which f is univalent in

$$\{w;\ |w-w_p|/|1-\overline{w_p}w|<r\}$$

is just that of r, $0<r<1$, for which the function e^ζ is univalent in an Apollonius disk:

$$\{\zeta;\ |\zeta-\log p-\pi i|/|\zeta+\log p-\pi i|<r\}$$

whose Euclidean diameter is

$$(4r\log p)/(r^2-1).$$

Equating this with 2π one has (7.2). The estimate (7.1) is just [Y2, p. 116, (7.4) for $\rho_{\Omega(K)}(z)=\sigma_K(z)$] which is strict in the present case.

It follows from [Y3, Theorem] that, for each f analytic and univalent in $\Omega(K)$, the strict inequality holds:

$$\sigma_K(z)P_{\Omega(K)}(z)^{-1}|f''(z)/f'(z)|<8, \qquad z\in\Omega(K).$$

Combining this with (6.8) we have

$$\tau_K(z)|f''(z)/f'(z)|<8, \qquad z\in\Omega(K), \tag{7.3}$$

where

$$\tau_K(z)=(-2p\log p)[\log p+(\pi^2+(\log p)^2)^{1/2}]/|\pi \operatorname{grad} p|$$
$$=(-p\log p)[\log p+(\pi^2+(\log p)^2)^{1/2}]/|\pi p_z|,$$

with $p=p(z, K)$, $z\in\Omega(K)$.

For g_0 in (6.4) we set

$$f=(g_0{}^{-1}-p(z_0))/(1-p(z_0)g_0{}^{-1})$$

in $\Omega(K)$. Then f is univalent and nonvanishing in $\Omega(K)$ with $p=|f|$. Consequently,

$$f'/f=2p_z/p, \qquad f''/f'-f'/f=p_{zz}/p_z-p_z/p,$$

so that, we may consider (7.3) for

$$f''/f'=p_{zz}/p_z+p_z/p$$

to have

$$\tau_K(z)|p_{zz}(z, K)/p_z(z, K)+p_z(z, K)/p(z, K)|<8, \qquad z\in\Omega(K).$$

Furthermore, it follows from [B, Corollary 3] (*note*: The last inequality in [B, Corollary 3] & [Hj, Theorem 1] $\Rightarrow$ [BG, Theorem 1] $\Rightarrow$ The last inequality in [B, Corollary 3]) that

$$|f'''(z)/f'(z)-(3/2)(f''(z)/f'(z))^2|\leqq 12P_{\Omega(K)}(z)^2$$

for all $z\in\Omega(K)$. A simple calculation, together with (6.8), now yields that, at each $z\in\Omega(K)$,

$$|p_{zzz}/p_z-(3/2)\{(p_{zz}/p_z)^2+(p_z/p)^2\}|\leqq 12|p_z|^2/(p\log p)^2,$$

where $p=p(z, K)$.

We finish our study of $p(z)=p(z, K)$ with a proposition.

PROPOSITION 5. *Suppose that K of $\Omega(K)$ is bounded. Then,*

$$\limsup_{z\to b}(1-p(z))^2P_{\Omega(K)}(z)\leqq 4^{-1}C(K)^{-1} \tag{7.4}$$

et each $b\in\partial\Omega(K)$;

$$\lim_{z\to\infty} p(z)^{-1}(-\log p(z))P_{\Omega(K)}(z)=2^{-1}C(K)^{-1}; \tag{7.5}$$

$$\limsup_{z\to b}(1-p(z))^4|(P_{\Omega(K)})_z(z)|\leqq 8^{-1}C(K)^{-2} \tag{7.6}$$

at each $b \in \partial\Omega(K)$;

(7.7) $$\limsup_{z\to\infty} p(z)^{-2}(-\log p(z))|(P_{\Omega(K)})_z(z)| \leqq \pi^{-1}C(K)^{-2}.$$

Proof. Both (7.4) and (7.5) are consequences of Corollary 2 to Theorem 4. It follows from [Y2, p. 116, (7.1)] that

$$|(P_{\Omega(K)})_z(z)|/P_{\Omega(K)}(z)^2 < 2\sigma_K(z)^{-1},$$

which, combined with (6.8), yields that

$$|(P_{\Omega(K)})_z(z)| < 2^{-1}|\mathrm{grad}\, p|^2(p\log p)^{-2}\sigma_K(z)^{-1}, \qquad p=p(z).$$

Thus, from the upper estimate in (6.16), the strict inequality follows:

$$|(P_{\Omega(K)})_z(z)| < 2^{-1}\pi C(K)^{-2}p^2(p^2-1)^{-2}(\log p)^{-2}[\log p+(\pi^2+(\log p)^2)^{1/2}]^{-1}.$$

Both (7.6) and (7.7) now follow from this estimate. Q. E. D.

8. Behavior of P_Ω without any restriction on $\partial\Omega$

In this section we prove (1.1) for a hyperbolic domain Ω in $\boldsymbol{C}$. Let $a, b \in \partial\Omega$, $a \neq b$, and let $w=\Phi(z)=(b-a)/(z-a)$. Then $\Phi(\Omega)\subset R\equiv \boldsymbol{C}\setminus\{0, 1\}$. It then follows from J. A. Hempel's result [Hm1, p. 443, (4.1)]:

$$1/P_R(w) \leqq 2|w|(|\log|w||+c_H), \qquad w\in R,$$

where c_H is defined in (5.3), together with

$$1/P_\Omega(z)=|z-a|^2/(|b-a|P_{\Phi(\Omega)}(w)), \qquad P_{\Phi(\Omega)}(w)\geqq P_R(w),$$

that

(8.1) $$1/P_\Omega(z) \leqq 2|z-a|(|\log(|b-a|/|z-a|)|+c_H)$$

at each $z\in\Omega$.

For the proof of (1.1) we choose $b\in\partial\Omega\setminus\{\zeta\}$. Let

$$V_1=\{z\in\Omega;\ |z-\zeta|<2|b-\zeta|/3\};$$

$$V_2=\{z\in\Omega;\ |z-\zeta|<|b-\zeta|/3\}.$$

Then for each $z\in V_2$ there exists $a=a(z)\in V_1\cap\partial\Omega$ (possibly ζ itself) such that

$$\delta_\Omega(z)=|z-a|\leqq|z-\zeta|<|b-\zeta|/3.$$

Since

$$|b-\zeta|/3\leqq|b-a|\leqq 5|b-\zeta|/3,$$

it follows that

$$1 \leqq |b-a|/\delta_\Omega(z) \leqq 5|b-\zeta|/(3\delta_\Omega(z)),$$

which, combined with (8.1), shows that

$$1/P_\Omega(z) \leqq 2\delta_\Omega(z)(\log[5|b-\zeta|/(3\delta_\Omega(z))]+c_H)$$

for $z \in V_2$. Hence (1.1).

We have no reasonable estimate for the partial derivatives of P_Ω.

As we have seen in [Y4], if $\zeta \in \partial\Omega$ is isolated, then

$$\lim_{z \to \zeta}[\delta_\Omega(z) \log(1/\delta_\Omega(z))]P_\Omega(z)=1/2.$$

See [Ha, Section 9.4.3] and [Hm2, p. 104, Lemma 5.2]; the present author regrets overlooking the cited article [Hm2] in [Y4].

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