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THREE-SHEETED ALGEBROID SURFACES WHOSE PICARD CONSTANTS ARE FIVE

Dedicated to Professor Nobuyuki Suita on his 60th birthday

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§1. Introduction

The notion of Picard constant of a Riemann surface R was introduced in [3]. Let $\mathcal{M}(R)$ be the family of non-constant meromorphic functions on R. Let $P(f)$ be the number of values which are not taken by f in $\mathcal{M}(R)$. Now put

$$P(R)=\sup_{f\in\mathcal{M}(R)} P(f).$$

This $P(R)$ is called the Picard constant of R. If R is open, then $P(R)\geqq 2$. Further if R is an n-sheeted algebroid surface, which is the proper existence domain of an n-valued algebroid function, then $P(R)\leqq 2n$ by Selberg's theory of algebroid functions [7].

We now list up two results for the case of three-sheeted algebroid surfaces. The first one is the following: Let R be a regularly branched three-sheeted algebroid surface, that is, R is defined by $y^3=g(x)$, where $g(x)$ is an entire function with infinitely many simple or double zeros. Then $P(R)=6$, if and only if $g(x)=(e^H-\alpha)(e^H-\beta)^2$, $\alpha\beta(\alpha-\beta)\neq 0$, where H is a non-constant entire function with $H(0)=0$ and α, β are constants. Further there is no regularly branched three-sheeted surface R with $P(R)=5$ [1].

The second one is the following: Let R be a general three-sheeted algebroid surface. Then there are two kinds of surfaces R with $P(R)=6$. One is defined by

$$(1)\qquad y^3-(x_0e^H+x_1)y^2+(a_1x_0e^H+x_2)y-x_3=0,$$

where x_0 is a non-zero constant, $x_1=a_2+a_3+a_4$, $x_2=a_2a_3+a_3a_4+a_2a_4$ and $x_3=a_2a_3a_4$ with non-zero different complex numbers a_1, a_2, a_3, a_4 and H is a non-constant entire function with $H(0)=0$. The other is defined by

$$(2)\qquad y^3-(x_0e^H+x_1)y^2+\{(a_1+a_2)x_0e^H+x_2\}y-a_1a_2x_0e^H=0,$$

where x_0 is a non-zero constant, $x_1=a_3+a_4$, $x_2=a_3a_4$ with non-zero different complex numbers a_1, a_2, a_3, a_4 and H is a non-constant entire function with $H(0)=0$ [5].

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In general it is very difficult to decide the exact value of $P(R)$ of any given surface R. Our problem is the following one: Is there any method to prove $P(R)=5$ for three-sheeted algebroid surfaces? In the first place we shall determine several three-sheeted algebroid surfaces R:

$$y^3-S_1y^2+S_2y-S_3=0$$

with $P(y)=5$. Then we shall give a method to prove really $P(R)=5$.

§ 2. Surfaces with $P(y)=5$

Let us put

$$F(z, y)\equiv y^3-S_1y^2+S_2y-S_3.$$

By Rémoundos' theorem [6] we may consider firstly

$$\begin{pmatrix} F(z,0) \\ F(z,a_2) \\ F(z,a_3) \\ F(z,a_4) \end{pmatrix} = \begin{pmatrix} c_1 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix},$$

where c_1, β_1, β_2, β_3 are non-zero constants and H_1, H_2, H_3 are non-constant entire functions satisfying $H_1(0)=H_2(0)=H_3(0)=0$. The first one is the same as the following simultaneous equation:

$$\begin{cases} -S_3=c_1, \\ a_2^3-S_1a_2^2+S_2a_2-S_3=\beta_1e^{H_1}, \\ a_3^3-S_1a_3^2+S_2a_3-S_3=\beta_2e^{H_2}, \\ a_4^3-S_1a_4^2+S_2a_4-S_3=\beta_3e^{H_3}. \end{cases}$$

Then by Borel's unicity theorem [2]

$$c_1=-a_2a_3a_4, \qquad H_1=H_2=H_3\equiv H$$

and

$$a_2a_4(a_4-a_2)\beta_2-a_2a_3(a_3-a_2)\beta_3+a_3a_4(a_3-a_4)\beta_1=0.$$

Further

$$\begin{cases} S_1=\dfrac{a_4\beta_2-a_3\beta_3}{a_3a_4(a_4-a_3)}e^H+a_2+a_3+a_4, \\ S_2=\dfrac{a_4^2\beta_2-a_3^2\beta_3}{a_3a_4(a_4-a_3)}e^H+a_2a_3+a_3a_4+a_2a_4, \\ S_3=-c_1=a_2a_3a_4. \end{cases}$$

Let us compute $F(z, A)$. Then

$$F(z, A)=A^3-A^2S_1+AS_2-S_3=\frac{A\{(-Aa_4+a_4^2)\beta_2-(a_3^2-Aa_3)\beta_3\}}{a_3a_4(a_4-a_3)}e^H$$
$$+(A-a_2)(A-a_3)(A-a_4).$$

Suppose that $F(z, A)$ does not reduce to a non-zero constant for any non-zero constant A. Then there is no non-zero constant A for which $A(a_3\beta_3-a_4\beta_2)=a_3^2\beta_3-a_4^2\beta_2$. Hence we have either $a_3\beta_3=a_4\beta_2$ or $a_3^2\beta_3=a_4^2\beta_2$. In the former case

$$\text{(A)}\quad \frac{\beta_3}{a_4}=\frac{\beta_2}{a_3}=\frac{\beta_1}{a_2}$$

and in the latter case

$$\text{(B)}\quad \frac{\beta_3}{a_4^2}=\frac{\beta_2}{a_3^2}=\frac{\beta_1}{a_2^2}.$$

Case (A). Then

$$\begin{cases} S_1=a_2+a_3+a_4\equiv y_1, \\ S_2=y_0e^H+a_2a_3+a_3a_4+a_2a_4\equiv y_0e^H+y_2, \\ S_3=a_2a_3a_4\equiv y_3 \end{cases}$$

with $y_0=\beta_2/a_3$. Let us consider the discriminant of R: $y^3-S_1y^2+S_2y-S_3=0$. Let us denote it by Δ, then

$$\Delta=4S_1^3S_3-S_1^2S_2^2-18S_1S_2S_3+4S_2^3+27S_3^2$$
$$=4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0,$$

where $\zeta_2=12y_2-y_1^2$, $\zeta_1=12y_2^2-18y_1y_3-2y_1^2y_2$ and $\zeta_0=4y_1^3y_3-y_1^2y_2^2-18y_1y_2y_3+4y_2^3+27y_3^2$, which is equal to $-(a_2-a_3)^2(a_3-a_4)^2(a_2-a_4)^2\neq0$. We denote this surface by R_A.

Case (B). Then with the same notations y_1, y_2, y_3 as in (A)

$$\begin{cases} S_1=y_0e^H+y_1, \qquad y_0=-\beta_2/a_3^2, \\ S_2=y_2, \\ S_3=y_3. \end{cases}$$

In this case the discriminant Δ of R is

$$\Delta=4y_0^3e^{3H}y_3+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0,$$

where $\zeta_2=12y_1y_3-y_2^2$,, $\zeta_1=12y_1^2y_3-2y_1y_2^2-18y_2y_3$ and $\zeta_0=4y_1^3y_3-y_1^2y_2^2-18y_1y_2y_3+4y_2^3+27y_3^2$, which is equal to $-(a_2-a_3)^2(a_3-a_4)^2(a_2-a_4)^2\neq0$. We denote this surface by R_B.

The second one is the same as the following simultaneous equation:

$$\begin{cases} -S_3=\beta_1 e^{H_1}, \\ a_2^3-S_1 a_2^2+S_2 a_2-S_3=c_1, \\ a_3^3-S_1 a_3^2+S_2 a_3-S_3=\beta_2 e^{H_2}, \\ a_4^3-S_1 a_4^2+S_2 a_4-S_3=\beta_3 e^{H_3}. \end{cases}$$

By Borel's unicity theorem

$$c_1=a_2(a_2-a_3)(a_2-a_4),$$

$$H_1=H_2=H_3\equiv H$$

and

$$a_2 a_4(a_4-a_2)\beta_2+(a_2-a_3)(a_4-a_2)(a_3-a_4)\beta_1+a_2 a_3(a_2-a_3)\beta_3=0.$$

Then we have

$$\begin{cases} S_1=\dfrac{e^H}{a_2 a_3(a_2-a_3)}(a_2\beta_2-(a_2-a_3)\beta_1)+a_3+a_4, \\ S_2=\dfrac{e^H}{a_2 a_3(a_2-a_3)}(a_2^2\beta_2-(a_2^2-a_3^2)\beta_1)+a_3 a_4, \\ S_3=-\beta_1 e^H. \end{cases}$$

Now we pose the following condition: There is no non-zero constant A, being different from a_2, such that $F(z, A)$ reduces to a non-zero constant. In this case

$$F(z, A)=\frac{e^H}{a_2 a_3(a_2-a_3)}\{-A^2(a_2\beta_2-(a_2-a_3)\beta_1)+A(a_2^2\beta_2-(a_2^2-a_3^2)\beta_1)$$
$$+a_2 a_3(a_2-a_3)\beta_1\}+A(A-a_3)(A-a_4)$$

dose not reduce to a non-zero constant excepting $A=a_2$.

Case (C). $-A^2(a_2\beta_2-(a_2-a_3)\beta_1)+A(a_2^2\beta_2-(a_2^2-a_3^2)\beta_1)+a_2 a_3(a_2-a_3)\beta_1=\alpha(A-a_2)^2$ with some constant $\alpha\neq 0$. Then

$$\{a_2^2\beta_2-(a_2^2-a_3^2)\beta_1\}^2=-4\{a_2\beta_2-(a_2-a_3)\beta_1\}a_2 a_3(a_2-a_3)\beta_1,$$

which implies

$$a_2^2\beta_2=(a_2-a_3)^2\beta_1.$$

Then

$$F(z, A)=\frac{\beta_1 e^H}{a_2^2}(A-a_2)^2+A(A-a_3)(A-a_4).$$

In this case we have

$$\begin{cases} S_1=y_0e^H+y_1, & y_0=-\beta_1/a_2^2, \\ S_2=2a_2y_0e^H+y_2, & y_1=a_3+a_4, \qquad y_2=a_3a_4, \\ S_3=a_2^2y_0e^H. \end{cases}$$

Then the discriminant Δ of $y^3-S_1y^2+S_2y-S_3=0$ is

$$\Delta=\xi_3y_0^3e^{3H}+\xi_2y_0^2e^{2H}+\xi_1y_0e^H+\xi_0,$$

where

$$\xi_3=4(a_2^2y_1-a_2^3-a_2y_2)=-4a_2(a_2-a_3)(a_2-a_4)\neq 0,$$

$$\xi_2=8a_2^2y_1^2-36a_2^3y_1+27a_2^4-8a_2y_1y_2+30a_2^2y_2-y_2^2,$$

$$\xi_1=4a_2^2y_1^3-4a_2y_1^2y_2-18a_2^2y_1y_2-2y_1y_2^2+24a_2y_2^2,$$

$$\xi_0=-y_2^2(y_1^2-2y_2)=-a_3^2a_4^2(a_3-a_4)^2\neq 0.$$

We denote this surface by R_C.

Case (D). $-A^2(a_2\beta_2-(a_2-a_3)\beta_1)+A(a_2^2\beta_2-(a_2^2-a_3^2)\beta_1)+a_2a_3(a_2-a_3)\beta_1=\alpha(A-a_2)$ with some non-zero constant α being independent of A. Then $a_2\beta_2=(a_2-a_3)\beta_1$ firstly and hence the above expression is equal to $-a_3(a_2-a_3)\beta_1(A-a_2)$. Then we have

$$F(z, A)=\frac{a_2-A}{a_2}\beta_1e^H+A(A-a_3)(A-a_4).$$

In this case we have

$$\begin{cases} S_1=y_1, & y_0=-\beta_1/a_2, \\ S_2=y_0e^H+y_2, & y_1=a_3+a_4, \qquad y_2=a_3a_4, \\ S_3=a_2y_0e^H. \end{cases}$$

Then the discriminant Δ of $y^3-S_1y^2+S_2y-S_3=0$ is

$$\Delta=4y_0^3e^{3H}+\xi_2y_0^2e^{2H}+\xi_1y_0e^H+\xi_0,$$

where

$$\xi_2=12y_2+27a_2^2-18a_2y_1-y_1^2,$$

$$\xi_1=12y_2^2-18a_2y_1y_2-6y_1^2y_2+4a_2y_1^3,$$

$$\xi_0=y_2^2(4y_2-y_1^2)=-a_3^2a_4^2(a_3-a_4)^2\neq 0.$$

We denote this surface by R_D.

We now consider

$$\begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \\ F(z,a_3) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ c_2 \\ \beta_2 e^{H_2} \end{pmatrix}$$

The first one is the following simultaneous equation:

$$\begin{cases} -S_3 = c_1, \\ a_1^3 - S_1 a_1^2 + S_2 a_1 - S_3 = c_2, \\ a_2^3 - S_1 a_2^2 + S_2 a_2 - S_3 = \beta_1 e^{H_1}, \\ a_3^3 - S_1 a_3^2 + S_2 a_3 - S_3 = \beta_2 e^{H_2}. \end{cases}$$

By Borel's unicity theorem $H_1 = H_2 \equiv H$, $a_3(a_3 - a_1)\beta_1 = a_2(a_2 - a_1)\beta_2$ and

$$c_1(a_3 - a_1)(a_2 - a_1) - c_2 a_2 a_3 + a_1 a_2 a_3(a_3 - a_1)(a_2 - a_1) = 0.$$

Then

$$\begin{cases} S_1 = -\dfrac{c_1}{a_2 a_3} + a_2 + a_3 - \dfrac{\beta_1 e^H}{a_2(a_2 - a_1)}, \\ S_1 = -\dfrac{(a_2 + a_3)c_1}{a_2 a_3} + a_2 a_3 - \dfrac{a_1 \beta_1 e^H}{a_2(a_2 - a_1)}, \\ S_3 = -c_1. \end{cases}$$

Now we pose the following condition: There is no non-zero constant B, being different from a_2 and a_3, such that $F(z, B)$ reduces to the form αe^X, where $\alpha \neq 0$ and X: non-constant entire function.

$$F(z, B) = \frac{B(B - a_1)}{a_2(a_2 - a_1)} \beta_1 e^H + \frac{(B - a_2)(B - a_3)}{a_2 a_3}(c_1 + B a_2 a_3).$$

Case (E). $c_1 + B a_2 a_3 = a_2 a_3(B - a_2)$. Then $c_1 = -a_2^2 a_3$ and $c_2 = -(a_3 - a_1)(a_2 - a_1)^2$. Further

$$\begin{cases} S_1 = 2a_2 + a_3 + y_0 e^H, \qquad y_0 = -\dfrac{\beta_1}{a_2(a_2 - a_1)}, \\ S_2 = a_2^2 + 2a_2 a_3 + a_1 y_0 e^H, \\ S_3 = -c_1 = a_2^2 a_3. \end{cases}$$

In this case the discriminant Δ of $y^3 - S_1 y^2 + S_2 y - S_3 = 0$ is

$$\Delta = y_0 e^H(-a_1^2 y_0^3 e^{3H} + A_2 y_0^2 e^{2H} + A_1 y_0 e^H + A_0),$$

where

$$A_2 = 4a_1^3 - 2(2a_2 + a_3)a_1^2 - 2(a_2 + 2a_3)a_2 a_1 + 4a_2^2 a_3,$$

$$A_1=(8a_2^2+20a_2a_3-a_3^2)a_1^2-(8a_2^3+38a_2^2a_3+8a_2a_3^2)a_1$$
$$-a_2^4+20a_2^3a_3+8a_2^2a_3^2,$$
$$A_0=4a_2(a_1-a_2)(a_2-a_3)^3\neq 0.$$

We denote this surface by R_E.

Case (F). $c_1+Ba_2a_3=a_2a_3(B-a_3)$. Then $c_1=-a_2a_3^2$ and $c_2=-(a_3-a_1)^2(a_2-a_1)$. Further

$$\begin{cases} S_1=a_2+2a_3+y_0e^H, & y_0=-\dfrac{\beta_1}{a_2(a_2-a_1)}, \\ S_2=2a_2a_3+a_3^2+a_1y_0e^H, & \\ S_3=-c_1=a_2a_3^2. & \end{cases}$$

In this case the discriminant Δ of $y^3-S_1y^2+S_2y-S_3=0$ is

$$\Delta=y_0e^H(-a_1^2y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0),$$

where

$$A_2=4a_1^3-2(a_2+2a_3)a_1^2-2(2a_2+a_3)a_3a_1+4a_2a_3^2,$$
$$A_1=(8a_3^2+20a_2a_3-a_2^2)a_1^2-(8a_3^3+38a_3^2a_2+8a_3a_2^2)a_1$$
$$-a_3^4+20a_3^3a_2+8a_2^2a_3^2,$$
$$A_0=-4a_3(a_1-a_3)(a_2-a_3)^3\neq 0.$$

We denote this surface by R_F.

The second one is the following simultaneous equation:

$$\begin{cases} -S_3=\beta_1e^{H_1}, \\ a_1^3-S_1a_1^2+S_2a_1-S_3=c_1, \\ a_2^3-S_1a_2^2+S_2a_2-S_3=c_2, \\ a_3^3-S_1a_3^2+S_2a_3-S_3=\beta_2e^{H_2}. \end{cases}$$

By Borel's unicity theorem we have $H_1=H_2\equiv H$, $\beta_2a_1a_2=\beta_1(a_3-a_1)(a_3-a_2)$ and

$$c_1a_2(a_3-a_2)-c_2a_1(a_3-a_1)=a_1a_2(a_2-a_1)(a_3-a_1)(a_3-a_2).$$

Then

$$\begin{cases} S_1=\dfrac{c_1}{a_1(a_3-a_1)}+a_1+a_3-\dfrac{\beta_1e^H}{a_1a_2}, \\ S_2=\dfrac{c_1a_3}{a_1(a_3-a_1)}+a_1a_3-\dfrac{(a_1+a_2)\beta_1e^H}{a_1a_2}, \\ S_3=-\beta_1e^H. \end{cases}$$

Now we pose the following condition: There is no non-zero constant B, being different from a_3, such that $F(z, B)$ reduces to the form αe^X, where $\alpha\neq 0$ and X: non-constant entire function. We have

$$F(z, B)=\frac{\beta_1 e^H}{a_1 a_2}(B-a_1)(B-a_2)+\frac{B(B-a_3)}{a_1(a_3-a_1)}((B-a_1)a_1(a_3-a_1)-c_1).$$

Case (G). $(B-a_1)a_1(a_3-a_1)-c_1=\gamma B$ with a non-zero constant γ, which is independent of B. Then $c_1=-a_1^2(a_3-a_1)$ and $c_2=-a_2^2(a_3-a_2)$. Further

$$\begin{cases} S_1=a_3-\dfrac{\beta_1}{a_1 a_2}e^H\equiv a_3+y_0 e^H, \\ S_2=-\dfrac{a_1+a_2}{a_1 a_2}\beta_1 e^H\equiv(a_1+a_2)y_0 e^H, \\ S_3=-\beta_1 e^H\equiv a_1 a_2 y_0 e^H. \end{cases}$$

Then the discriminant Δ of R is

$$\Delta=y_0 e^H(A_3 y_0^3 e^{3H}+A_2 y_0^2 e^{2H}+A_1 y_0 e^H+A_0),$$

where

$$A_3=4a_1 a_2-(a_1+a_2)^2=-(a_1-a_2)^2\neq 0,$$

$$A_2=-2(a_1^2-4a_1 a_2+a_2^2)a_3-2(a_1+a_2)(2a_1-a_2)(a_1-2a_2)$$

$$A_1=-(a_1^2-10a_1 a_2+a_2^2)a_3^2-18a_1 a_2(a_1+a_2)a_3+27a_1^2 a_2^2,$$

$$A_0=4a_1 a_2 a_3^3\neq 0.$$

We denote this surface by R_G.

Case (H). $(B-a_1)a_1(a_3-a_1)-c_1=\gamma(B-a_3)$. Then $c_1=a_1(a_3-a_1)^2$ and $c_2=a_2(a_3-a_2)^2$. Further

$$\begin{cases} S_1=2a_3+y_0 e^H, \quad y_0=\dfrac{-\beta_1}{a_1 a_2}, \\ S_2=a_3^2+(a_1+a_2)y_0 e^H, \\ S_3=a_1 a_2 y_0 e^H. \end{cases}$$

The discriminant Δ of R is

$$\Delta=y_0 e^H(A_3 y_0^3 e^{3H}+A_2 y_0^2 e^{2H}+A_1 y_0 e^H+A_0),$$

where

$$A_3=-(a_1-a_2)^2\neq 0,$$

$$A_2=-2a_3^2(a_1+a_2)-4a_3(a_1^2+4a_1 a_2+a_2^2)$$
$$+2(a_1+a_2)(2a_1-a_2)(a_1-2a_2),$$

$$A_1=-a_3^4-8a_3^3(a_1+a_2)+a_3^2(8a_1^2+46a_1a_2+8a_2^2)$$
$$-36a_1a_2(a_1+a_2)a_3+27a_1^2a_2^2,$$
$$A_0=-4a_3^3(a_3-a_1)(a_3-a_2)\neq 0.$$

We denote this surface by R_H.

§ 3. Riemann surfaces of $P(R)=6$

In introduction we have listed up two kinds of Riemann surfaces of six Picard constant. We briefly introduce how to construct them. Let R be the Riemann surface defined by

$$F(z,\ y)\equiv y^3-S_1y^2+S_2y-S_3=0,$$

where S_1, S_2, S_3 are entire functions. Suppose that $P(R)=6$. By Rémoundos' theorem [6] we may consider the following two cases:

$$\text{(i)}\quad \begin{pmatrix} F(z,0)\\ F(z,b_1)\\ F(z,b_2)\\ F(z,b_3)\\ F(z,b_4)\end{pmatrix}=\begin{pmatrix} c_1\\ c_2\\ \beta_1e^{L_1}\\ \beta_2e^{L_2}\\ \beta_3e^{L_3}\end{pmatrix},\qquad \text{(ii)}\quad =\begin{pmatrix} \beta_1e^{L_1}\\ c_1\\ c_2\\ \beta_2e^{L_2}\\ \beta_3e^{L_3}\end{pmatrix}.$$

Here c_1, c_2, β_1, β_2, β_3 are non-zero constants. L_j are non-constant entire functions with $L_j(0)=0$ for $j=1, 2, 3$. Further b_1, b_2, b_3, b_4 are different non-zero complex numbers.

Case (i). $L_1=L_2=L_3=L$ follows easily. Then

$$\begin{cases} S_1=x_0e^L+x_1, \\ S_2=b_1x_0e^L+x_2, \\ S_3=x_3 \end{cases}$$

with $x_0=\beta_1/b_2(b_1-b_2)$, $x_1=b_2+b_3+b_4$, $x_2=b_2b_3+b_3b_4+b_2b_4$ and $x_3=b_2b_3b_4$. Hence the surface is defined by

$$y^3-(x_0e^L+x_1)y^2+(b_1x_0e^L+x_2)y-x_3=0.$$

Its discriminant D is

$$D=-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0,$$

where

$$\eta_3=4b_1^3-2b_1^2x_1-2b_1x_2+4x_3,$$

$$\eta_2=12x_1x_3-18b_1x_3-x_2^2-4b_1x_1x_2+12b_1^2x_2-b_1^2x_1^2,$$

$$\eta_1=12x_1^2x_3-18b_1x_1x_3-18x_2x_3-2x_1x_2^2+12b_1x_2^2-2b_1x_1^2x_2,$$

$$\eta_0=4x_1^3x_3-x_1^2x_2^2+27x_3^2-18x_1x_2x_3+4x_2^3$$

$$=-(b_2-b_3)^2(b_2-b_4)^2(b_3-b_4)^2\neq 0.$$

This surface is denoted by X_1.

Case (ii). $L_1=L_2=L_3=L$ follows easily. Then

$$\begin{cases} S_1=x_0e^L+x_1, \\ S_2=(b_1+b_2)x_0e^L+x_2, \\ S_3=b_1b_2x_0e^L \end{cases}$$

with $x_0=-\beta_1/b_1b_2$, $x_1=b_3+b_4$, $x_2=b_3b_4$. Hence the surface is defined by

$$y^3-(x_0e^L+x_1)y^2+\{(b_1+b_2)x_0e^L+x_2\}y-b_1b_2x_0e^L=0.$$

Its discriminant D is

$$D=(b_1-b_2)^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0,$$

where

$$\eta_3=(2b_1^2-8b_1b_2+2b_2^2)x_1+2(b_1+b_2)x_2$$

$$-2(b_1+b_2)(2b_1^2-5b_1b_2+2b_2^2),$$

$$\eta_2=(b_1^2-10b_1b_2+b_2^2)x_1^2+4(b_1+b_2)x_1x_2+x_2^2$$

$$+18(b_1+b_2)b_1b_2x_1-(12b_1^2+6b_1b_2+12b_2^2)x_2-27b_1^2b_2^2,$$

$$\eta_1=-4b_1b_2x_1^3+2(b_1+b_2)x_1^2x_2+2x_1x_2^2+18b_1b_2x_1x_2$$

$$-12(b_1+b_2)x_2^2,$$

$$\eta_0=x_1^2x_2^2-4x_2^3=b_3^2b_4^2(b_3-b_4)^2\neq 0.$$

This surface is denoted by X_2.

§ 4. A lemma

It is necessary to give an explicit proof of the following.

LEMMA. *Let R be the Riemann surface R_A defined by*

$$y^3-S_1y^2+S_2y-S_3=0$$

with $S_1=x_1$, $S_2=x_0e^H+x_2$, $S_3=x_3$, where x_0, x_1, x_2 and x_3 are constants, $x_0\neq 0$,

$x_1=a_2+a_3+a_4$, $x_2=a_2a_3+a_3a_4+a_2a_4$, $x_3=a_2a_3a_4$. *Let F be a regular function on R_A. Then F is representable as*

$$F=f_1+f_2y+f_3y^2,$$

where f_1, f_2 and f_3 are meromorphic functions in $|z|<\infty$, all of which are regular at any points z satisfying $H'(z)\neq 0$.

Proof. Let z_0 be a point satisfying $H'(z_0)\neq 0$.

Case 1). There are two different points of R_A over z_0. Of course one is a branch point and the other is an ordinary point. Then y has two branches y_1 and y_2 for which

$$y_1=A_0+A_1(z-z_0)^{p/2}+A_2(z-z_0)^{(p+1/2)}+\cdots$$

with $A_0A_1\neq 0$ and

$$y_2=B_0+B_1(z-z_0)^q+B_2(z-z_0)^{q+1}+\cdots$$

with $B_0B_1\neq 0$. $A_0B_0\neq 0$, since y does not vanish. If $p\geqq 3$, then $y_1^3-x_1y_1^2+(x_0e^{H(z)}+x_2)y_1-x_3=0$ gives

$$A_0^3+3A_0^2A_1(z-z_0)^{p/2}+\cdots-x_1(A_0^2+2A_0A_1(z-z_0)^{p/2}+\cdots)$$
$$+[x_0e^{H(z_0)}\{1+\varepsilon_1(z-z_0)+\cdots\}+x_2](A_0+A_1(z-z_0)^{p/2}+\cdots)-x_3=0$$

with $\varepsilon_1\neq 0$. This gives $\varepsilon_1x_0e^{H(z_0)}A_0=0$, which is absurd. If $p=2$, then there is the smallest index s for which

$$y_1=A_0+A_1(z-z_0)+\cdots+A_s^*(z-z_0)^{s/2}+\cdots$$

with an odd s and a non-zero constant A_s^*. Then we have

$$A_0^3+3A_0^2A_1(z-z_0)+\cdots+3A_0^2A_s^*(z-z_0)^{s/2}+\cdots$$
$$-x_1(A_0^2+2A_0A_1(z-z_0)+\cdots+2A_0A_s^*(z-z_0)^{s/2}+\cdots)$$
$$+[x_0e^{H(z_0)}\{1+\varepsilon_1(z-z_0)+\cdots\}+x_2](A_0+A_1(z-z_0)+\cdots+A_s^*(z-z_0)^{s/2}$$
$$+\cdots-x_3=0.$$

Hence from the coefficient of $(z-z_0)^{s/2}$,

$$\{3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\}A_s^*=0,$$

which gives

$$3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2=0.$$

The coefficient of $z-z_0$ is

$$\{3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\}A_1+x_0e^{H(z_0)}\varepsilon_1A_0=0.$$

Hence $x_0 e^{H(z_0)} \varepsilon_1 A_0 = 0$, which is absurd. Hence

$$y_1 = A_0 + A_1(z-z_0)^{1/2} + A_2(z-z_0) + \cdots .$$

Then from the coefficient of $(z-z_0)^{1/2}$ of $y_1^3 - x_1 y_1^2 + (x_0 e^{H(z)} + x_2) y_1 - x_3 = 0$.

$$\{3A_0^2 - 2x_1 A_0 + x_0 e^{H(z_0)} + x_2\} A_1 = 0 .$$

Hence

$$3A_0^2 - 2x_1 A_0 + x_0 e^{H(z_0)} + x_2 = 0 .$$

We shall make use of this relation later.

Similarly for the one-valued branch y_2 we have

$$y_2 = B_0 + B_1(z-z_0) + \cdots .$$

Assume that $F_1 = f_1 + f_2 y_1 + f_3 y_1^2$ and $F_2 = f_1 + f_2 y_2 + f_3 y_2^2$ are pole-free at z_0. Then put

$$f_1 = \frac{\alpha_n}{(z-z_0)^n} + \frac{\alpha_{n-1}}{(z-z_0)^{n-1}} + \cdots ,$$

$$f_2 = \frac{\beta_n}{(z-z_0)^n} + \frac{\beta_{n-1}}{(z-z_0)^{n-1}} + \cdots ,$$

$$f_3 = \frac{\gamma_n}{(z-z_0)^n} + \frac{\gamma_{n-1}}{(z-z_0)^{n-1}} + \cdots .$$

with $(\alpha_n, \beta_n, \gamma_n) \neq (0, 0, 0)$.

Then we have

$$\begin{aligned} F_1 &= f_1 + f_2 y_1 + f_3 y_1^2 \\ &= \frac{\alpha_n}{(z-z_0)^n} + \frac{\alpha_{n-1}}{(z-z_0)^{n-1}} + \cdots \\ &\quad + \left\{ \frac{\beta_n}{(z-z_0)^n} + \frac{\beta_{n-1}}{(z-z_0)^{n-1}} + \cdots \right\} \{A_0 + A_1(z-z_0)^{1/2} + A_2(z-z_0) + \cdots\} \\ &\quad + \left\{ \frac{\gamma_n}{(z-z_0)^n} + \frac{\gamma_{n-1}}{(z-z_0)^{n-1}} + \cdots \right\} \{A_0^2 + 2A_0A_1(z-z_0)^{1/2} \\ &\quad + (A_1^2 + 2A_0A_2)(z-z_0) + \cdots\} . \end{aligned}$$

Then

$$\alpha_n + \beta_n A_0 + \gamma_n A_0^2 = 0 ,$$

$$\beta_n A_1 + \gamma_n 2A_0 A_1 = 0 .$$

Similarly for $F_2 = f_1 + f_2 y_2 + f_3 y_2^2$ we have

$$\alpha_n + \beta_n B_0 + \gamma_n B_0^2 = 0 .$$

Hence $\{\beta_n+\gamma_n(A_0+B_0)\}(A_0-B_0)=0$. If $A_0\neq B_0$, then $\beta_n+\gamma_n(A_0+B_0)=0$. On the other hand we have $\beta_n+2\gamma_nA_0=0$. And if $\gamma_n\neq 0$, we have $A_0=B_0$, which is absurd. If $\gamma_n=0$ then we have $\beta_n=\alpha_n=0$, which is absurd. Therefore $A_0=B_0$. By $y_2^3-x_1y_2^2+(x_0e^{H(z)}+x_2)y_2-x_3=0$ we have

$$\{3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\}B_1+x_0e^{H(z_0)}\varepsilon_1A_0=0.$$

Hence we have an absurdity relation $x_0e^{H(z_0)}\varepsilon_1A_0=0$.

Case 2). There is only one point of R_A over z_0. Then

$$y(z)=A_0+A_p(z-z_0)^{p/3}+\cdots.$$

If $p\geqq 4$, then the coefficient of $z-z_0$ of $y^3-x_1y^2+(x_0e^{H(z)}+x_2)y-x_3=0$ is equal to $x_0e^{H(z_0)}\varepsilon_1A_0$. Hence this vanishes, which is impossible. If $p=3$, then there is the smallest index s for which

$$y=A_0+A_3(z-z_0)+\cdots+A_s^*(z-z_0)^{s/3}+\cdots$$

with $s\not\equiv 0 \bmod 3$ and non-zero A_s^*. Then the coefficient of $(z-z_0)^{s/3}$ in the Puisseux expansion of $y^3-x_1y^2+(x_0e^{H(z)}+x_2)y-x_3=0$ is equal to

$$3A_0^2A_s^*-2x_1A_0A_s^*+(x_0e^{H(z_0)}+x_2)A_s^*=0.$$

Hence

$$3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2=0.$$

On the other hand the coefficient of $z-z_0$ is equal to

$$(3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2)A_1+x_0e^{H(z_0)}\varepsilon_1A_0=0.$$

This is evidently impossible. Therefore $p=2$ or $p=1$.

Suppose that $p=1$ and further that $y=A_0+A_1(z-z_0)^{1/3}+A_3(z-z_0)+A_4(z-z_0)^{4/3}+\cdots$ with $A_1\neq 0$. Then

$$\begin{aligned}F&=f_1+f_2y+f_3y^2\\&=\frac{\alpha_n}{(z-z_0)^n}+\frac{\alpha_{n-1}}{(z-z_0)^{n-1}}+\cdots\\&\quad+\left\{\frac{\beta_n}{(z-z_0)^n}+\frac{\beta_{n-1}}{(z-z_0)^{n-1}}+\cdots\right\}\{A_0+A_1(z-z_0)^{1/3}+A_3(z-z_0)+\cdots\}\\&\quad+\left\{\frac{\gamma_n}{(z-z_0)^n}+\frac{\gamma_{n-1}}{(z-z_0)^{n-1}}+\cdots\right\}\{A_0^2+2A_0A_1(z-z_0)^{1/3}+A_1^2(z-z_0)^{2/3}+\cdots\}.\end{aligned}$$

Since F is pole-free at z_0,

$$\alpha_n+\beta_nA_0+\gamma_nA_0^2=0,$$

$$\beta_nA_1+2\gamma_nA_0A_1=0$$

and

$$\gamma_nA_1^2=0.$$

Then $\gamma_n=0$ implies $\beta_n=0$ and $\alpha_n=0$. This holds for all $n\geqq 1$. Hence we arrive at a contradiction.

Suppose that $p=1$ and further that $y=A_0+A_1(z-z_0)^{1/3}+A_2(z-z_0)^{2/3}+A_3(z-z_0)+\cdots$ with $A_1A_2\neq 0$. Similarly we have

$$\alpha_n+\beta_n A_0+\gamma_n A_0^2=0,$$

$$(\beta_n+2\gamma_n A_0)A_1=0$$

and

$$\beta_n A_2+\gamma_n(2A_0A_2+A_1^2)=0.$$

These relations contain a contradiction similarly.

Suppose that $p=2$. Then $y=A_0+A_2(z-z_0)^{2/3}+A_3(z-z_0)+\cdots$, $A_0A_2\neq 0$. In this case

$$\begin{aligned}0&=y^3-x_1y^2+(x_0e^{H(z)}+x_2)y-x_3\\&=A_0^3+3A_0^2A_2(z-z_0)^{2/3}+3A_0^2A_3(z-z_0)+\cdots\\&\quad-x_1(A_0^2+2A_0A_2(z-z_0)^{2/3}+2A_0A_3(z-z_0)+\cdots)\\&\quad+\{x_0e^{H(z_0)}(1+\varepsilon_1(z-z_0)+\cdots)+x_2\}(A_0+A_2(z-z_0)^{2/3}+A_3(z-z_0)+\cdots)-x_3.\end{aligned}$$

Hence we have

$$(3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2)A_2=0$$

and

$$(3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2)A_3+x_0e^{H(z_0)}\varepsilon_1A_0=0.$$

Therefore we have a contradiction.

Case 3). There are three ordinary points of R_A over z_0. Then there are three different branches of y around these points. Suppose that

$$y_1=A_0+A_1(z-z_0)^p+\cdots$$

with $p\geqq 2$, $A_0A_1\neq 0$. Then by $y_1^3-x_1y_1^2+(x_0e^{H(z)}+x_2)y_1-x_3=0$ we have $x_0e^{H(z_0)}\varepsilon_1A_0=0$, which is absurd. Hence $y_1=A_0+A_1(z-z_0)+A_2(z-z_0)^2+\cdots$. Similarly

$$y_2=B_0+B_2(z-z_0)+B_2(z-z_0)^2+\cdots,\qquad B_0B_1\neq 0$$

and

$$y_3=C_0+C_1(z-z_0)+C_2(z-z_0)^2+\cdots,\qquad C_0C_1\neq 0.$$

Let us put

$$f_1=\frac{\alpha_n}{(z-z_0)^n}+\frac{\alpha_{n-1}}{(z-z_0)^{n-1}}+\cdots,$$

$$f_2=\frac{\beta_n}{(z-z_0)^n}+\frac{\beta_{n-1}}{(z-z_0)^{n-1}}+\cdots,$$

$$f_3=\frac{\gamma_n}{(z-z_0)^n}+\frac{\gamma_{n-1}}{(z-z_0)^{n-1}}+\cdots.$$

Then $F=f_1+f_2y+f_3y^2$ should be pole-free at z_0 for any branch of y. Hence

$$\alpha_n+\beta_nA_0+\gamma_nA_0^2=0,$$
$$\alpha_n+\beta_nB_0+\gamma_nB_0^2=0,$$
$$\alpha_n+\beta_nC_0+\gamma_nC_0^2=0.$$

Then

$$(\beta_n+\gamma_n(A_0+B_0))(A_0-B_0)=0$$

and

$$(\beta_n+\gamma_n(A_0+C_0))(A_0-C_0)=0.$$

If $A_0\neq B_0$ and $A_0\neq C_0$, then $\beta_n+\gamma_n(A_0+B_0)=\beta_n+\gamma_n(A_0+C_0)=0$. Hence $\gamma_n(B_0-C_0)=0$. If $B_0\neq C_0$, then $\gamma_n=0$ and $\beta_n=0$, $\alpha_n=0$. This gives a contradiction. Hence $B_0=C_0$. Therefore we have either $A_0=B_0$ or $A_0=C_0$ or $B_0=C_0$. Suppose now $A_0=B_0$.

Then by $y_1^3-x_1y_1^2+(x_0e^{H(z)}+x_2)y_1-x_3=0$ we have

$$A_0^3-x_1A_0^2+(x_0e^{H(z_0)}+x_2)A_0-x_3=0,$$
$$(3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2)A_1+x_0e^{H(z_0)}\varepsilon_1A_0=0.$$

Similarly for y_2 we have

$$(3B_0^2-2x_1B_0+x_0e^{H(z_0)}+x_2)B_1+x_0e^{H(z_0)}\varepsilon_1B_0=0.$$

By $A_0=B_0$ we have

$$(3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2)(A_1-B_1)=0.$$

Suppose that $A_1\neq B_1$. Then $3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2=0$, whence follows $x_0e^{H(z_0)}\varepsilon_1A_0=0$, which is impossible. Hence $A_1=B_1$. In general

$$\{3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\}A_m+P_m(A_0, A_1, \cdots, A_{m-1}, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)=0$$

and

$$\{3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\}B_m+P_m(A_0, A_1, \cdots, A_{m-1}, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)=0,$$

if $A_0=B_0$, $A_1=B_1$, $\cdots$, $A_{m-1}=B_{m-1}$, where ε_j, $j=1, \cdots, m$ are defined by

$$x_0e^{H(z_0)}+x_2=x_0e^{H(z_0)}+x_2+x_0e^{H(z_0)}\sum_{j=1}^{\infty}\varepsilon_j(z-z_0)^j.$$

Since $3A_0^2-2x_1A_0+x_0e^{H(z_0)}+x_2\neq0$, we have $A_m=B_m$. Thus we have $y_1\equiv y_2$, which is absurd.

Similar lemma hold for the surfaces X_1, R_B and R_E. Proofs are quite similar. Further it is sufficient to prove Lemma for the surfaces R_A, R_B and

R_E. In §7 we show that, when e^H is commonly appeared, $R_D \sim R_A$, $R_C \sim R_B$ and $R_F \sim R_G \sim R_H \sim R_E$, where $\sim$ means the conformal equivalence by a suitable linear transformation $Y = \alpha y + \beta$. Evidently $X_1 \sim X_2$ too, if e^L is common.

§5. Transformation formula of discriminants

Let R be the surface R_A: $y^3 - S_1 y^2 + S_2 y - S_3 = 0$ with $S_1 = y_1$, $S_2 = y_0 e^H + y_2$, $S_3 = y_3$, where y_0 is a non-zero constant and $y_1 = a_2 + a_3 + a_4$, $y_2 = a_2 a_3 + a_3 a_4 + a_2 a_4$, $y_3 = a_2 a_3 a_4$ and H is an entire function.

From now on we shall assume that the surface is of finite order, that is,

$$H \text{ is a polynomial.}$$

The same assumption holds in §6 too.

Now suppose that $P(R) = 6$. Then there exists an entire function f on R with $P(f) = 6$. We can make use of Lemma in §4. Then f is representable as

$$f = f_1 + f_2 y + f_3 y^2$$

as in Lemma.

For simplicity's sake we put $F = f_1 - f$. Then

$$F + f_2 y + f_3 y^2 = 0,$$

$$f_3 S_3 + (F - f_3 S_2) y + (f_2 + f_3 S_1) y^2 = 0,$$

$$(f_2 + f_3 S_1) S_3 + (f_3 S_3 - f_3 S_1 S_2 - f_2 S_2) y + (F + f_2 S_1 + f_3 (S_1^2 - S_2)) y^2 = 0.$$

By eliminating y and y^2 we have

$$F^3 + Y_0 F^2 + Y_1 F + Y_2 = 0,$$

where

$$\begin{cases} Y_0 = f_2 S_1 + f_3 (S_1^2 - 2S_2), \\ Y_1 = f_2^2 S_2 + f_2 f_3 (S_1 S_2 - 3S_3) + f_3^2 (S_2^2 - 2S_1 S_3), \\ Y_2 = f_2^3 S_3 + f_2^2 f_3 S_1 S_3 + f_2 f_3^2 S_2 S_3 + f_3^3 S_3^2. \end{cases}$$

This gives

$$f^3 - f^2 U_1 + f U_2 - U_3 = 0$$

with

$$\begin{cases} U_1 = 3f_1 + Y_0, \\ U_2 = 3f_1^2 + 2f_1 Y_0 + Y_1, \\ U_3 = f_1^3 + f_1^2 Y_0 + f_1 Y_1 + Y_2, \end{cases}$$

U_1, U_2 and U_2 are all entire, since f is a three-valued entire algebroid function. Let g be $f - U_1/3$. Then $g^3 + Ag + B = 0$ with

$$A=\frac{1}{3}(-U_1^2+3U_2),$$

$$B=\frac{1}{27}(-2U_1^3+9U_1U_2-27U_3).$$

Then the discriminant D is equal to $4A^3+27B^2$. Hence

$$D=4U_1^3U_3-U_1^2U_2^2-18U_1U_2U_3+4U_2^3+27U_3^2.$$

For simplicity's sake we put

$$A=\frac{1}{3}(\alpha_1 f_2^2+\alpha_2 f_2 f_3+\alpha_3 f_3^2),$$

$$\begin{cases} \alpha_1=3S_2-S_1^2, \\ \alpha_2=-2S_1^3+7S_1S_2-9S_3, \\ \alpha_3=-S_1^4+4S_1^2S_2-6S_1S_3-S_2^2 \end{cases}$$

and

$$B=\frac{1}{27}(\beta_1 f_2^3+\beta_2 f_2^2 f_3+\beta_3 f_2 f_3^2+\beta_4 f_3^3),$$

$$\begin{cases} \beta_1=-2S_1^3+9S_1S_2-27S_3, \\ \beta_2=-6S_1^4+30S_1^2S_2-54S_1S_3-18S_2^2, \\ \beta_3=-6S_1^5+33S_1^3S_2-45S_1^2S_3-33S_1S_2^2+27S_2S_3, \\ \beta_4=-2S_1^6+12S_1^4S_2-18S_1^3S_3-15S_1^2S_2^2+36S_1S_2S_3-2S_2^3-27S_3^2 \end{cases}$$

Then

$$\begin{aligned} D=4A^3+27B^2=\frac{1}{27}\{ & f_2^6(4\alpha_1^3+\beta_1^2)+f_2^5 f_3(12\alpha_1^2\alpha_2+2\beta_1\beta_2) \\ & +f_2^4 f_3^2(12\alpha_1^2\alpha_3+12\alpha_1\alpha_2^2+2\beta_1\beta_3+\beta_2^2) \\ & +f_2^3 f_3^3(24\alpha_1\alpha_2\alpha_3+4\alpha_2^3+2\beta_1\beta_4+2\beta_2\beta_3) \\ & +f_2^2 f_3^4(12\alpha_1\alpha_3^2+12\alpha_2^2\alpha_3+2\beta_2\beta_4+\beta_3^2) \\ & +f_2 f_3^5(12\alpha_2\alpha_3^2+2\beta_3\beta_4)+f_3^6(4\alpha_3^3+\beta_4^2)\} \end{aligned}$$

$$\begin{aligned} =\Delta\{ & f_2^6+4S_1 f_2^5 f_3+2(3S_1^2+S_2)f_2^4 f_3^2+(4S_1^3+6S_1S_2-2S_3)f_2^3 f_3^3 \\ & +(S_1^4+6S_1^2S_2-4S_1S_3+S_2^2)f_2^2 f_3^4+2(S_1^3S_2-S_1^2S_3+S_1S_2^2-S_2S_3)f_2 f_3^5 \\ & +(S_1^2S_2^2-2S_1S_2S_3+S_3^2)f_3^6\} \end{aligned}$$

$$=\Delta\{f_2^3+2S_1 f_2^2 f_3+(S_1^2+S_2)f_2 f_3^2+(S_1S_2-S_3)f_3^3\}^2,$$

where Δ is the discriminant of $y^3-S_1y^2+S_2y-S_3=0$, that is,

$$\Delta=\frac{4}{27}\alpha_1^3+\frac{1}{27}\beta_1^2$$
$$=4S_1^3S_3-S_1^2S_2^2-18S_1S_2S_3+4S_2^3+27S_3^2.$$

Let us put the above formula as

$$(3) \qquad D=\Delta\cdot G^2.$$

G may have poles at most at zeros of H'.

We need more precise result on $D=\Delta\cdot G^2$. Evidently the poles of G are finite in number. Let us put

$$D=-b_1^2(x_0e^L-\gamma_1)(x_0e^L-\gamma_2)(x_0e^L-\gamma_3)(x_0e^L-\gamma_4)$$

and

$$\Delta=4(y_0e^H-\delta_1)(y_0e^H-\delta_2)(y_0e^H-\delta_3).$$

Case 1). The counting function of simple zeros of Δ satisfies

$$N_2(r, 0, \Delta)\sim 3T(r, e^H),$$

that is, $\delta_i\neq\delta_l$ for $i\neq l$. Then

$$N_2(r, 0, \Delta)=N_2(r, 0, D)\sim mT(r, e^L)$$

with $m=1, 2, 4$. Then L should be a polynomial, whose degree coincides with the one of H. In this case we can return back y from f. Then we have

$$\Delta=D\cdot K^2.$$

The number of poles of K is finite again. This gives that the zeros of G is finite in number. Hence

$$(4) \qquad D=\Delta\cdot\beta^2\cdot e^{2M}$$

with a rational function β. In this case we have $\gamma_j\neq\gamma_k$ for $j\neq k$.

Case 2). $N_2(r, 0, \Delta)\sim T(r, e^H)$, that is, $\delta_1\neq\delta_2=\delta_3$. Then

$$N_2(r, 0, \Delta)=N_2(r, 0, D)\sim mT(r, e^L)$$

with $m=1, 2, 4$. Then L should be a polynomial. Again we can return back y from f. Then $\Delta=D\cdot K^2$. Similarly we have a finite number of zeros of G. Hence

$$D=\Delta\beta^2e^{2M}.$$

Then the counting function of double zeros of Δ satisfies $N_1(r, 0, \Delta)\sim 2T(r, e^H)$ and $N_1(r, 0, D)\sim 2T(r, e^L)$. Hence $T(r, e^H)\sim T(r, e^L)$. On the other hand $T(r, e^H)\sim 2T(r, e^L)$, because that $N_2(r, 0, \Delta)=N_2(r, 0, D)$ and $N_1(r, 0, D)\sim$

$2T(r, e^L)$. This is a contradiction.

Case 3). Δ has no simple zero. Then

$$-b_1^2(x_0e^L-\gamma_1)(x_0e^L-\gamma_2)(x_0e^L-\gamma_3)(x_0e^L-\gamma_4)$$
$$=4(y_0e^H-\gamma_1)^3\cdot G^2.$$

This is a contradiction.

§6. Theorems

We shall prove the following

THEOREM 1. *Let R_A be the Riemann surface defined in §2. Assume that its discriminant Δ_{R_A} satisfies*

$$\Delta_{R_A}=4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0$$

with either $\zeta_2\neq0$ or $\zeta_1\neq0$, where $\zeta_2=12y_2-y_1^2$, $\zeta_1=12y_2^2-18y_1y_3-2y_1^2y_2$. Then $P(R_A)=5$.

THEOREM 2. *Let R_B be the Riemann surface defined in §2. Assume that its discriminant Δ_{R_B} has the form*

$$\Delta_{R_B}=4y_3y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0$$

with either $\zeta_2=12y_1y_3-y_2^2\neq0$ or $\zeta_1=12y_1^2y_3-2y_1y_2^2-18y_2y_3\neq0$. Then $P(R_B)=5$.

THEOREM 3. *Let R_C be the Riemann surface defined in §2. Assume that its discriminant Δ_{R_C} has the form*

$$\Delta_{R_C}=\xi_3y_0^3e^{3H}+\xi_2y_0^2e^{2H}+\xi_1y_0e^H+\xi_0$$

with either $\xi_2=8a_2^2y_1^2-36a_2^3y_1+27a_2^4-8a_2y_1y_2+30a_2^2y_2-y_2^2\neq0$ or $\xi_1=4a_2^2y_1^3-4a_2y_1^2y_2-18a_2^2y_1y_2-2y_1y_2^2+24a_2y_2^2\neq0$. Then $P(R_C)=5$.

THEOREM 4. *Let R_D be the Riemann surface defined in §2. Assume that its discriminant Δ_{R_D} has the form*

$$\Delta_{R_D}=4y_0^3e^{3H}+\xi_2y_0^2e^{2H}+\xi_1y_0e^H+\xi_0$$

with either $\xi_2=12y_2+27a_2^2-18a_2y_1-y_1^2\neq0$ or $\xi_1=12y_2^2-6y_1^2y_2-18a_2y_1y_2+4a_2y_1^3\neq0$. Then $P(R_D)=5$.

Proof of Theorem 1. Suppose that $P(R_A)=6$. Then on R_A there is an entire algebroid function f for which $P(f)=6$. Suppose that f defines the surface X_1. Then by (4)

$$D=\Delta_{R_A}\cdot\beta^2e^{2M}$$

which is just the following identity:

$$-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$
$$=(4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0)\beta^2e^{2H}.$$

Now we shall make use of the unicity theorem of Borel, which plays the decisive role in our proof. Evidently we have

$$4T(r, e^L)\sim N_2(r, 0, D)=N_2(r, 0, \Delta_{R_A})\sim 3T(r, e^H).$$

We already proved that it is enough to consider this case. Hence

$$T(r, e^H)\sim\frac{4}{3}T(r, e^L).$$

This relation makes our discussion simpler. Firstly assume that $M\equiv 0$. Then

$$-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$
$$=4\beta^2y_0^3e^{3H}+\beta^2\zeta_2y_0^2e^{2H}+\beta^2\zeta_1y_0e^H+\beta^2\zeta_0.$$

There remains only one possibility: $\eta_0=\beta^2\zeta_0$, $-b_1^2x_0^4=4\beta^2y_0^3$, $4L\equiv 3H$ and $\eta_3=\eta_2=\eta_1=\zeta_2=\zeta_1=0$. However at least one of ζ_1, ζ_2 does not vanish by our assumption. Thus we arrive at a contradiction.

Next assume that $M\not\equiv 0$. Then

$$-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$
$$=4\beta^2y_0^3e^{3H+2M}+\beta^2\zeta_2y_0^2e^{2H+2M}+\beta^2\zeta_1y_0e^{H+2M}+\beta^2\zeta_0e^{2M}.$$

Now suppose that $3H+2M=0$. Then

$$-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$
$$=4\beta^2y_0^3+\beta^2\zeta_2y_0^2e^{-H}+\beta^2\zeta_1y_0e^{-2H}+\beta^2\zeta_0e^{-3H}.$$

There remains only one possible case:

$$\eta_0=4\beta^2y_0^3,\quad -b_1^2x_0^4=\beta^2\zeta_0,\quad 4L=-3H,\quad \eta_3=\eta_2=\eta_1=\zeta_2=\zeta_1=0.$$

This is again a contradiction. Still there are several subcases to be discussed. However all of them lead to contradictions easily.

Suppose that f defines the surface X_2. Then we have

$$D=\Delta_{R_A}\cdot\beta^2e^{2M}$$

by (4), which is just the following identity:

$$-(b_1-b_2)^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$
$$=(4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0)\beta^2e^{2M}.$$

There appear only two possible cases: Either $\eta_0=\beta^2\zeta_0$, $-(b_1-b_2)^2x_0^4=4\beta^2y_0^3$, $M\equiv0$, $4L\equiv3H$ and $\eta_3=\eta_2=\eta_1=\zeta_2=\zeta_1=0$ or $\eta_0=4\beta^2y_0^3-(b_1-b_2)^2x_0^4=\beta^2\zeta_0$, $2M\equiv-3H$, $4L\equiv-3H$ and $\eta_3=\eta_2=\eta_1=\zeta_2=\zeta_1=0$. These two cases give the same contradiction $\zeta_2=\zeta_1=0$. Therefore $P(R_A)=5$.

Proofs of Theorems 2, 3 and 4 are quite similar as in the one of Theorem 1. So we shall omit their proofs.

THEOREM 5. *Let R_E be the Riemann surface defined in* §2. *Assume that its discriminant Δ_{R_E} has the form*

$$\Delta_{R_E}=y_0e^H(-a_1^2y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0),$$

where either $A_2=4a_1^3-2(2a_2+a_3)a_1^2-2(a_2+2a_3)a_2a_1+4a_2^2a_3\neq0$ *or* $A_1=(8a_2^2+20a_2a_3-a_3^2)a_1^2-(8a_2^3+38a_2^2a_3+8a_2a_3^2)a_1-a_2^4+20a_2^3a_3+8a_2^2a_3\neq0$. *Then* $P(R_E)=5$.

THEOREM 6. *Let R_F be the Riemann surface defined in* §2. *Assume that its discriminant Δ_{R_F} has the form*

$$\Delta_{R_F}=y_0e^H(-a_1^2y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0)$$

with either $A_2=4a_1^3-2(a_2+2a_3)a_1^2-2(2a_2+a_3)a_3a_1+4a_2a_3^2\neq0$ *or* $A_1=(8a_3^2+20a_2a_3-a_2^2)a_1^2-(8a_3^3+38a_3^2a_2+8a_3a_2^2)a_1-a_3^4+20a_3^3a_2+8a_2^2a_3^2\neq0$. *Then* $P(R_F)=5$.

THEOREM 7. *Let R_G be the Riemann surface defined in* §2. *Assume that its discriminant Δ_{R_G} has the form*

$$\Delta_{R_G}=y_0e^H(-(a_1-a_2)^2y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0)$$

with either

$$A_2=-2(a_1^2-4a_1a_2+a_2^2)a_3-2(a_1+a_2)(2a_1^2-5a_1a_2+2a_2^2)\neq0$$

or

$$A_1=-(a_1^2-10a_1a_2+a_2^2)a_3^2-18a_1a_2(a_1+a_2)a_3+27a_1^2a_2^2\neq0.$$

Then $P(R_G)=5$.

THEOREM 8. *Let R_H be the Riemann surface defined in* §2. *Assume that its discriminant Δ_{R_H} has the form*

$$\Delta_{R_H}=y_0e^H(-(a_1-a_2)^2y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0)$$

with either

$$\begin{aligned}A_2=&-2a_3^2(a_1+a_2)-4a_3(a_1^2+4a_1a_2+a_2^2)\\&+2(a_1+a_2)(2a_1^2-5a_1a_2+2a_2^2)\\&\neq0\end{aligned}$$

or

$$A_1 = -a_3^4 - 8a_3^3(a_1+a_2) + a_3^2(8a_1^2 + 46a_1a_2 + 8a_2^2)$$
$$-36a_1a_2(a_1+a_2)a_3 + 27a_1^2a_2^2$$
$$\neq 0.$$

Then $P(R_H)=5$.

Proof of Theorem 5. Suppose that $P(R_E)=6$. Then on R_E there is an entire algebroid function f for which $P(f)=6$. Suppose that f defines the surface X_1. Then we have

$$D = \Delta_{R_E} \cdot \beta^2 e^{2M}$$

by (4). This is just the following identity:

$$-b_1^2 x_0^4 e^{4L} + \eta_3 x_0^3 e^{3L} + \eta_2 x_0^2 e^{2L} + \eta_1 x_0 e^L + \eta_0$$
$$= y_0 e^H(-a_1^2 y_0^3 e^{3H} + A_2 y_0^2 e^{2H} + A_1 y_0 e^H + A_0)\beta^2 e^{2M}.$$

There remain only two possible cases: Either $2M \equiv -H$, $3H \equiv 4L$, $\eta_3=\eta_2=\eta_1=A_2=A_1=0$ or $2M \equiv -4H$, $4L \equiv -3H$, $\eta_3=\eta_2=\eta_1=A_2=A_1=0$. These contradict our assuption: Either $A_2 \neq 0$ or $A_1 \neq 0$.

Similarly we have a contradiction, when f defines the surface X_2.

Proofs of Theorems 6, 7 and 8 are quite similar as in the one of Theorem 5.

§7. Unsolved problems and Remarks

i) Let R_A be the Riemann surface defined in §2. Assume that its discriminant Δ_{R_A} has the following form:

$$\Delta_{R_A} = 4y_0^3 e^{3H} + \zeta_0.$$

Is $P(R_A)$ still five?

Of course there are corresponding unsolved problems for R_X ($x=B, C, D, E, F, G, H$).

ii) Let R_X and R_Y be the surfaces $P(R_X)=5$ and $P(R_Y)=5$. Can we list up all the analytic mappings of R_X into R_Y?

iii) Let R and S be the surfaces of $P(R)=6$ and $P(S)=5$. Is there any analytic mapping of R into S?

We shall now give some remarks. Let

$$F(z, y) \equiv y^3 - S_1 y^2 + S_2 y - S_3 = 0$$

and

$$\alpha^3 G(z, Y) \equiv F(z, \alpha Y + \beta)$$

$$=\alpha^3[Y^3-T_1Y^2+T_2Y-T_3]=0$$

with $A_2\alpha=-a_4$, $A_3\alpha=a_2-a_4$, $A_4\alpha=a_3-a_4$ and $\beta=a_4$.

R_A is defined by $F(z, y)=0$ with

$$\begin{cases} S_1=a_2+a_3+a_4\,, \\ S_2=y_0e^H+a_2a_3+a_3a_4+a_2a_4\,, \\ S_3=a_2a_3a_4\,. \end{cases}$$

Then

$$\begin{cases} T_1=A_3+A_4\,, \\ T_2=Y_0e^H+A_3A_4\,, \\ T_3=A_2Y_0e^H \end{cases}$$

with $Y_0=y_0/\alpha^2$. Then $G(z, Y)=0$ defines the surface R_D. Evidently inverse process is possible. Hence R_A coincides with R_D.

Similarly we can show that R_B coincides with R_C.

Next we put

$$A_1\alpha=-a_3, \quad A_2\alpha=a_1-a_3, \quad A_3\alpha=a_2-a_3, \quad \beta=a_3$$

R_E is defined by $F(z, y)=0$ with

$$\begin{cases} S_1=2a_2+a_3+y_0e^H\,, \\ S_2=a_2^2+2a_2a_3+a_1y_0e^H\,, \\ S_3=a_2^2a_3\,. \end{cases}$$

Then

$$\begin{cases} T_1=Y_0e^H+2A_3\,, \\ T_2=(A_1+A_2)Y_0e^H+A_3^2\,, \\ T_3=A_1A_2Y_0e^H \end{cases}$$

with $Y_0=y_0/\alpha$. $G(z, Y)=0$ defines the surface R_H. Hence R_E and R_H are coincident with each other.

Similarly we can show that R_F and R_G are coincident with each other. Next we put $A_1\alpha=-a_1$, $A_2\alpha=a_3-a_1$, $A_3\alpha=a_2-\alpha_1$ and $\beta=a_1$. R_E is defined by $F(z, y)=0$ with

$$\begin{cases} S_1=2a_2+a_3+y_0e^H\,, \\ S_2=a_2^2+2a_2a_3+a_1y_0e^H\,, \\ S_3=a_2^2a_3\,. \end{cases}$$

Then

$$\begin{cases} T_1 = Y_0 e^H + A_2 + 2A_3, \\ T_2 = A_1 Y_0 e^H + 2A_2 A_3 + A_3^2, \\ T_3 = A_2 A_3^2 \end{cases}$$

with $Y_0 = y_0/\alpha$. Hence $G(z, Y)=0$ defines the surface R_F. This shows that R_E coincides with R_F.

Therefore there are three types of Riemann surfaces of five Picard constant.

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