T. WATANABE
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SOME EXAMPLES ON UNIMODALITY OF LÉVY PROCESSES

By Toshiro Watanabe

§ 1. Introduction and results

Many works have been done on unimodality of Lévy processes on R^1. Sato [4] surveys the results of these works and indicates open problems. In this paper we answer some questions raised by Sato.

A measure μ on R^1 is said to be unimodal with mode a if $\mu(dx)=f(x)dx+c\delta_a(dx)$, where $c\geqq 0$, δ_a is the delta measure at a, and $f(x)$ is increasing on $(-\infty, a)$ and decreasing on (a, ∞). In this paper we use the words "increase" and "decrease" in the wide sense. A probability measure μ on R^1 is said to be strongly unimodal if, for every unimodal probability measure η, the convolution $\mu*\eta$ is unimodal. We say that a random variable is unimodal (resp. strongly unimodal) if its distribution is unimodal (resp. strongly unimodal). Let $\{X_t(\omega): t\geqq 0\}$ be a Lévy process on R^1 defined on a probability space $(\Omega, \mathcal{F}, P)$ (that is, a stochastically continuous process with stationary independent increments starting at the origin). Then the characteristic function of X_t is represented as

$$E\exp(izX_t)=\exp(t\psi(z)), \tag{1.1}$$

$$\psi(z)=i\gamma z-2^{-1}\sigma^2z^2+\int_{-\infty}^{\infty}(e^{izx}-1-1_{(-1,1)}(x)izx)\nu(dx),$$

where $\gamma\in R^1$, $\sigma^2\geqq 0$, $1_{(-1,1)}(x)$ is the indicator function of the interval $(-1, 1)$, and ν is a measure on R^1 satisfying $\nu(\{0\})=0$ and $\int(1\wedge x^2)\nu(dx)<\infty$, called Lévy measure of $\{X_t\}$. We say that a Lévy process $\{X_t\}$ is unimodal if X_t is unimodal for each t. A Lévy process $\{X_t\}$ is called self-decomposable if, for each $c\in(0, 1)$, there are a probability space $(\Omega', \mathcal{F}', P')$ and two Lévy processes $\{Y_t\}$ and $\{Z_t\}$ defined on it such that (i) $\{Y_t\}$ and $\{Z_t\}$ are independent, (ii) $\{Y_t\}$ and $\{cX_t\}$ are equivalent in law, and (iii) $\{Y_t+Z_t\}$ and $\{X_t\}$ are equivalent in law. A self-decomposable Lévy process is simply called a self-decomposable process. A Lévy process $\{X_t\}$ with Lévy measure ν is self-decomposable if and only if $\nu(dx)=|x|^{-1}k(x)dx$ with $k(x)$ increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Yamazato proves in the celebrated paper [11] that every self-decomposable process on R^1 is unimodal. The author proves in [9] the following

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theorem.

THEOREM 1.1. *If $\{X_t\}$ and $\{Y_t\}$ are independent unimodal increasing Lévy processes, then $\{X_t - Y_t\}$ is a unimodal Lévy process.*

Sato raises in [4] three related questions:

Question (i). Does Theorem 1.1 exhaust unimodal Lévy processes with Lévy measure satisfying $\int(1\wedge|x|)\nu(dx)<\infty$ without Gaussian part?

Question (ii). Can all unimodal Lévy processes be approximated by unimodal Lévy processes with Lévy measure satisfying $\int(1\wedge|x|)\nu(dx)<\infty$?

Question (iii). If $\{X_t\}$ with $\sigma^2>0$ in (1.1) is a unimodal Lévy process, is the Lévy process having the identical Lévy measure without Gaussian part a unimodal process?

Question (i) has answer 'no' in symmetric case. In symmetric case Questions (ii) and (iii) have answer 'yes' by Proposition 3.2 of Sato [4]. The purpose of this paper is to show that, in non-symmetric case, Question (i) has answer 'no' and Question (iii) has answer 'no' in general by giving concrete examples. However Qestion (ii) still remains unanswered in non-symmetric case. Further we give an interesting example on unimodality of Lévy processes. Namely our results are as follows.

PROPOSITION 1.2. *There are independent increasing Lévy processes $\{X_t\}$ and $\{Y_t\}$ such that $\{X_t\}$ is not a unimodal Lévy process, $\{Y_t\}$ is a unimodal Lévy process, and $\{X_t - Y_t\}$ is a non-symmetric unimodal Lévy process.*

PROPOSITION 1.3. *There are an increasing Lévy process $\{X_t\}$ and a Brownian motion $\{B_t\}$ such that $\{X_t\}$ and $\{B_t\}$ are independent and $\{X_t\}$ is not a unimodal Lévy process, but $\{X_t+\sigma B_t\}$ is a unimodal Lévy process for sufficiently large $\sigma>0$.*

PROPOSITION 1.4. *There are $t_1>0$, $t_2>0$, $c>1$, and an increasing Lévy process $\{X_t\}$ such that, for each integer n, X_t is unimodal at $t=c^n t_1$ and not unimodal at $t=c^n t_2$.*

We prove the propositions above in the following sections. Results of Sato [4], Sato-Yamazato [5], and Medgyessy [3] are employed for the proof of Proposition 1.2. Proofs of Propositions 1.3 and 1.4 are based on the integro-differential equations that the densities satisfy. Such equations are extensively used by Sato-Yamazato [5], Watanabe [8, 10], and Yamazato [11, 12]. Semi-stability of the process in the sense of Lévy [2] plays an essential role in the proof of Proposition 1.4.

§2. Proof of Proposition 1.2

Let $\{X_t\}$ be an increasing Lévy process such that in (1.1)

$$E\exp(izX_t)=\exp\left(\varepsilon t\int_0^\infty (e^{izx}-1)e^{-x}dx\right), \tag{2.1}$$

where ε is a constant satisfying $0<\varepsilon\leqq 1$. Then, as in the proof of Proposition 3.3 of Sato [4], $\{X_t\}$ is not a unimodal Lévy process and, if $0\leqq t\leqq 2\varepsilon^{-1}$, then X_t is unimodal with mode 0. Let $\{Y_t\}$ be an increasing Lévy process independent of $\{X_t\}$ such that in (1.1)

$$E\exp(izY_t)=\exp\left(t\int_0^\infty (e^{izx}-1)(\varepsilon+x^{-1})e^{-x}dx\right). \tag{2.2}$$

Let $k(x)=(\varepsilon x+1)e^{-x}1_{(0,\infty)}(x)$. The process $\{Y_t\}$ is self-decomposable and hence a unimodal Lévy process, since $k'(x)=(-\varepsilon x-1+\varepsilon)e^{-x}1_{(0,\infty)}(x)<0$ on $(0,\infty)$. Hence, if $0\leqq t\leqq 1$, then Y_t is unimodal with mode 0 by Theorem 1.3 of Sato-Yamazato [5], since $k(0+)=1$. Therefore, if $0\leqq t\leqq 1$, then X_t-Y_t is unimodal with mode 0 by Corollary 2.7 of Sato [4]. Let $\{W_t\}$ be a symmetric compound Poisson process such that in (1.1)

$$E\exp(izW_t)=\exp\left(\varepsilon t\int_{-\infty}^\infty (e^{izx}-1)e^{-|x|}dx\right). \tag{2.3}$$

Since Lévy measure of $\{W_t\}$ is symmetric unimodal, $\{W_t\}$ is a symmetric unimodal Lévy process by Medgyessy [3]. Let $\{Z_t\}$ be a gamma process independent of $\{W_t\}$ such that in (1.1)

$$E\exp(izZ_t)=\exp\left(t\int_0^\infty (e^{izx}-1)x^{-1}e^{-x}dx\right). \tag{2.4}$$

Then we can express the distribution η_t of Z_t as

$$\eta_t(dx)=1_{(0,\infty)}(x)\{\Gamma(t)\}^{-1}e^{-x}x^{t-1}dx, \tag{2.5}$$

where $\Gamma(t)$ is the gamma function. Hence, if $t\geqq 1$, then Z_t is strongly unimodal by Ibragimov [1] and hence W_t-Z_t is unimodal. Since $\{X_t-Y_t\}$ and $\{W_t-Z_t\}$ are equivalent in law, X_t-Y_t is unimodal for $t\geqq 1$. Thus $\{X_t\}$ is not a unimodal Lévy process, $\{Y_t\}$ is a unimodal Lévy process, and $\{X_t-Y_t\}$ is a non-symmetric unimodal Lévy process. The proof of Proposition 1.2 is complete.

§3. Proof of Proposition 1.3

Let $\{X_t\}$ be an increasing Lévy process such that in (1.1)

(3.1) $$E\exp(izX_t)=\exp\left(t\int_0^1(e^{izx}-1)x^{-1}(1+mx)dx\right),$$

where m is a constant satisfying $0<m<1$. Let $k(x)=(1+mx)1_{(0,1)}(x)$. Then X_t is strongly unimodal for $t\geqq 1$ by Theorem 1 of Yamazato [12], since $\log k(x)$ is concave on $(0, 1)$ and $k(0+)=1$. On the other hand, X_t is unimodal with mode 0 for $0\leqq t\leqq 1-m$ and $\{X_t\}$ is not a unimodal Lévy process by Example 4.2 of Watanabe [9]. Let $\{B_t\}$ be a Brownian motion independent of $\{X_t\}$ and let $\{Y_t\}=\{X_t+\sigma B_t\}$ for $\sigma>0$. Note that σB_t is strongly unimodal for every $\sigma>0$ and for every t by Ibragimov [1]. We shall show that $\{Y_t\}$ is a unimodal Lévy process for sufficiently large $\sigma>0$. If $t\geqq 1$, then Y_t is strongly unimodal, since X_t and σB_t are strongly unimodal. If $0\leqq t\leqq 1-m$, then Y_t is unimodal, since X_t is unimodal and σB_t is strongly unimodal. Hence, from now on, we assume $1-m<t<1$. Let μ_t be the distribution of Y_t. Then μ_t is absolutely continuous with a density function $f_t(x)$ of class $C^\infty(R^1)$ by positivity of σ. We obtain the following equation as in Theorem 2.1 of Sato-Yamazato [5]:

(3.2) $$\sigma^2 t f_t'(x)=t\int_0^1 f_t(x-u)(1+mu)du-xf_t(x).$$

By using integration by parts, we get that

(3.3) $$\sigma^2 t f_t'(x)=tF_t(x)-tF_t(x-1)(1+m)+mt\int_0^1 F_t(x-u)du-xf_t(x),$$

where $F_t(x)=\int_{-\infty}^x f_t(u)du$. Differentiating (3.3), we find that

(3.4) $$\sigma^2 t f_t''(x)=(t-1)f_t(x)-tf_t(x-1)(1+m)+mt\int_0^1 f_t(x-u)du-xf_t'(x).$$

Put $c=1-m$. Define the set $A=\{(x, y, t)\colon -1\leqq x,\ y\leqq 4,\ c<t<1\}$. We shall show that there is $\sigma_1>0$ such that, for every $\sigma\geqq\sigma_1$,

(3.5) $$\inf_{(x,y,t)\in A} f_t(y)/f_t(x)\geqq 1/2.$$

Let η_t and ρ_t be the distributions of X_t and σB_t, respectively. Then η_t is absolutely continuous by Tucker [7], since $\int_0^1 x^{-1}(1+mx)dx=\infty$. The distribution ρ_t is Gaussian with mean 0 and variance $\sigma^2 t$. Let $\eta_t(dx)=g_t(x)dx$ and $\rho_t(dx)=h_t(x)dx$. Then we see that

(3.6) $$f_t(x)=\int_0^\infty h_t(x-u)g_t(u)du.$$

Since $\{X_t\}$ is increasing,

(3.7) $$\int_0^N g_t(x)dx=P(X_t\leqq N)\geqq P(X_1\leqq N)$$

for $N>0$ and for $c<t<1$. Hence there is $N_1>0$ such that

$$\inf_{c<t<1}\int_0^{N_1} g_t(x)dx \geqq 3/4\,. \tag{3.8}$$

Define the set $B=\{(x, u, t): -1\leqq x\leqq 4,\ 0\leqq u\leqq N_1,\ c<t<1\}$. Then there is sufficiently large $\sigma_1>0$ such that, for every $\sigma\geqq\sigma_1$ and for every $(x, u, t)\in B$,

$$(2/3)h_t(0)\leqq h_t(x-u)\leqq h_t(0)\,. \tag{3.9}$$

Hence we obtain (3.5) from (3.6) and (3.8). We divide the proof of unimodality of Y_t for $c<t<1$ and for $\sigma\geqq\sigma_1$ into four cases.

(i) We find from (3.2) that

$$f_t'(x)>0 \tag{3.10}$$

for $x\leqq 0$.

(ii) Let us prove that if $f_t'(x_1)=0$ for some x_1 with $0<x_1\leqq 3$, then $f_t''(x_1)<0$. Indeed, we see from (3.4) and (3.5) that

$$\begin{aligned}\sigma^2 t f_t''(x_1)&=(t-1)f_t(x_1)-tf_t(x_1-1)(1+m)+mt\int_0^1 f_t(x_1-u)du\\ &<-tf_t(x_1-1)(1-m)<0\,,\end{aligned} \tag{3.11}$$

noting from (3.5) that $\int_0^1 f_t(x_1-u)du\leqq 2f_t(x_1-1)$.

(iii) We obtain from (3.2) and (3.5) that

$$\begin{aligned}\sigma^2 t f_t'(x)&=t\int_0^1 f_t(x-u)(1+mu)du-xf_t(x)\\ &\leqq f_t(x)\{(2+m)t-x\}<0\end{aligned} \tag{3.12}$$

for $3\leqq x\leqq 4$.

(iv) Let us prove that $f_t'(x)<0$ on $(4, \infty)$. Suppose. on the contrary, that $f_t'(x_2)=0$ for some $x_2>4$. Define $x_3=\inf\{x: f_t'(x)=0,\ x>4\}$. Since $f_t'(u)<0$ for $3\leqq u<x_3$ by (iii), and $x_3\geqq 4$,

$$f_t(x_3-u)<f_t(x_3-1) \tag{3.13}$$

for $0<u<1$ and

$$f_t''(x_3)\geqq 0\,. \tag{3.14}$$

Hence we get by (3.4) that

$$\begin{aligned}\sigma^2 t f_t''(x_3)&=(t-1)f_t(x_3)-tf_t(x_3-1)(1+m)+mt\int_0^1 f_t(x_3-u)du\\ &<-tf_t(x_3-1)<0\,,\end{aligned} \tag{3.15}$$

which contradicts (3.14). Consequently $f_t'(x)<0$ on $(4, \infty)$. It follows from (i), (ii), (iii), and (iv) that Y_t is unimodal for every $\sigma \geqq \sigma_1$ and for every t satisfying $c<t<1$. Thus we have proved that $\{Y_t\}$ is a unimodal Lévy process for every $\sigma \geqq \sigma_1$, but $\{X_t\}$ is not a unimodal Lévy process.

§ 4. Proof of Proposition 1.4

Let $0<m<2^{-1}$, $0<\lambda<1$, $b>1$ and $c=b^{\lambda}$. Let $\zeta(x)$ be a positive right continuous periodic function on R^1 with period $\log b$ defined by

$$\zeta(x)=e^{\lambda x}(1+me^{x}) \tag{4.1}$$

on $[-\log b, 0)$. Let $\{X_b(t)\}$ be an increasing Lévy process such that in (1.1)

$$\begin{aligned} &E\exp(izX_b(t))=\exp(t\psi(z)), \\ &\psi(z)=\int_0^{\infty}(e^{izx}-1)\zeta(\log x)x^{-\lambda-1}dx\,. \end{aligned} \tag{4.2}$$

Then the distribution $\mu_t^{(b)}$ of $X_b(t)$ is semi-stable in the sense of Lévy [2] and satisfies the following equation:

$$\mu_{ct}^{(b)}(dx)=\mu_t^{(b)}(b^{-1}dx)\,. \tag{4.3}$$

By (4.1) $\psi(z)$ in (4.2) is represented as

$$\psi(z)=\sum_{n=-\infty}^{\infty}\int_{b^n}^{b^{n+1}}(e^{izx}-1)b^{-\lambda(n+1)}(1+mb^{-n-1}x)x^{-1}dx\,. \tag{4.4}$$

We note that

$$\begin{aligned} &\left|\sum_{n=0}^{\infty}\int_{b^n}^{b^{n+1}}(e^{izx}-1)b^{-\lambda(n+1)}(1+mb^{-n-1}x)x^{-1}dx\right| \\ &\leqq 2\sum_{n=0}^{\infty}\int_{b^n}^{b^{n+1}}b^{-\lambda(n+1)}(1+mb^{-n-1}x)x^{-1}dx \\ &=2(b^{\lambda}-1)^{-1}\{\log b+m(1-b^{-1})\} \longrightarrow 0 \end{aligned} \tag{4.5}$$

as $b\to\infty$ and similarly

$$\begin{aligned} &\left|\sum_{n=-\infty}^{-2}\int_{b^n}^{b^{n+1}}(e^{izx}-1)b^{-\lambda(n+1)}(1+mb^{-n-1}x)x^{-1}dx\right| \\ &\leqq |z|(b^{1-\lambda}-1)^{-1}\{1-b^{-1}+2^{-1}m(1-b^{-2})\} \longrightarrow 0 \end{aligned} \tag{4.6}$$

as $b\to\infty$, where we use $|e^{izx}-1|\leqq|zx|$. On the other hand, we see that

$$\int_{b^{-1}}^{1}(e^{izx}-1)(1+mx)x^{-1}dx \longrightarrow \int_0^1(e^{izx}-1)(1+mx)x^{-1}dx \tag{4.7}$$

as $b\to\infty$. Hence we find from (4.5), (4.6) and (4.7) that $\mu_t^{(b)}$ converges weakly to the distribution η_t of X_t defined in the proof of Proposition 1.3, as $b\to\infty$.

Since $\{X_t\}$ is not a unimodal Lévy process, there is $b_0>1$ such that, for every $b\geqq b_0$, $\{X_b(t)\}$ is not a unimodal Lévy process. For nonnegative integers n, let $k_n(x)$ be functions on R^1 defined by

$$k_0(x)=0 \quad \text{for } x\leqq 0, \qquad =1+mx \quad \text{for } 0<x<1, \qquad =b^{-\lambda(k+1)}(1+mb^{-k-1}x) \quad \text{for } b^k\leqq x<b^{k+1} \tag{4.8}$$

for every nonnegative integer k and, for $n\geqq 1$,

$$k_n(x)=1_{(0,b^{-n})}(x)b^{\lambda(n-1)}\{b^\lambda-1+mx(b^{n+\lambda}-b^{n-1})\}. \tag{4.9}$$

For nonnegative integers n, let $\{Z_n(t)\}$ be increasing Lévy processes without drift such that processes $\{Z_n(t)\}$ are independent and Lévy measures ν_n of $\{Z_n(t)\}$ are expressed as

$$\nu_n(dx)=x^{-1}k_n(x)dx. \tag{4.10}$$

Note that $\{X_b(t)\}$ and $\left\{\sum_{n=0}^{\infty} Z_n(t)\right\}$ are equivalent in law. Let $b_1=3^{1/\lambda}$. If $b\geqq b_1$, then $\log k_n(x)$ is concave on $(0, b^{-n})$ and $2^{-1}k_n(0+)=2^{-1}b^{\lambda(n-1)}(b^\lambda-1)\geqq 1$ for every $n\geqq 1$. Hence $Z_n(2^{-1})$ is strongly unimodal for $n\geqq 1$ and $b\geqq b_1$ by Theorem 1 of Yamazato [12]. We shall show that $Z_0(2^{-1})$ is unimodal with mode 0 for $b\geqq b_1$. If this is true, then $X_b(2^{-1})$ is unimodal for $b\geqq b_1$. The distribution μ of $Z_0(2^{-1})$ is absolutely continuous by Tucker [7], since $\int_0^1 \nu_0(dx)=\infty$. Let $\mu=f(x)dx$. Put $t=2^{-1}$ and $a=1+m-b^{-\lambda}(1+mb^{-1})$. Then we have a relation by Steutel [6]:

$$xf(x)=t\int_0^1 f(x-u)(1+mu)du + t\sum_{n=0}^{\infty}\int_{b^n}^{b^{n+1}} f(x-u)b^{-\lambda(n+1)}(1+mb^{-n-1}u)du \tag{4.11}$$

for $x>0$. By using integration by parts, we get that

$$xf(x)=tF(x)+mt\int_0^1 F(x-u)du-at\sum_{n=0}^{\infty}F(x-b^n)b^{-\lambda n} + mt\sum_{n=0}^{\infty}\int_{b^n}^{b^{n+1}} F(x-u)b^{-(\lambda+1)(n+1)}du \tag{4.12}$$

for $x>0$, where $F(x)=\int_{-\infty}^{x} f(u)du$. Differentiating (4.12), we find that

$$xf'(x)=(t-1)f(x)+mt\int_0^1 f(x-u)du-at\sum_{n=0}^{\infty}f(x-b^n)b^{-\lambda n} + mt\sum_{n=0}^{\infty}\int_{b^n}^{b^{n+1}} f(x-u)b^{-(\lambda+1)(n+1)}du \tag{4.13}$$

for $x>0$ except at $x=b^n$ with nonnegative integers n. Let $E_0=(0, 1)$ and $E_k=(b^{k-1}, b^k)$ for integers $k\geqq 1$. We shall prove by induction in k that

$$f'(x)<0 \qquad \text{on } \bigcup_{k=0}^{\infty} E_k \tag{4.14}$$

for $b\geqq b_1$.

(I) For $x\in E_0$, we obtain from (4.13) that

$$xf'(x)=(t-1)f(x)+mt\int_0^x f(x-u)du\,. \tag{4.15}$$

We get by (4.11) that

$$xf(x)=t\int_0^x f(x-u)(1+mu)du>t\int_0^x f(x-u)du \tag{4.16}$$

on E_0. Hence we obtain from (4.15) that

$$xf'(x)<(t-1)f(x)+mxf(x)<(-2^{-1}+m)f(x)<0 \tag{4.17}$$

on E_0. Therefore, we see that

$$f(x)\geqq Ax^{-1/2+m} \tag{4.18}$$

on E_0 with a positive constant A, which implies that

$$f(x)\longrightarrow\infty \tag{4.19}$$

as $x\downarrow 0$.

(II) Let $x\in E_k$ for $k\geqq 1$. We find from (4.13) that

$$\begin{aligned} xf'(x)=&(t-1)f(x)+mt\int_0^1 f(x-u)du-at\sum_{n=0}^{k-1} f(x-b^n)b^{-\lambda n} \\ &+mt\sum_{n=0}^{k-2}\int_{b^n}^{b^{n+1}} f(x-u)b^{-(\lambda+1)(n+1)}du \\ &+mt\int_{b^{k-1}}^x f(x-u)b^{-(\lambda+1)k}du\,. \end{aligned} \tag{4.20}$$

Since $f(x-b^{k-1})\to\infty$ as $x\downarrow b^{k-1}$ by (4.19), we see from (4.20) that

$$f'(x)\longrightarrow-\infty \tag{4.21}$$

as $x\downarrow b^{k-1}$. Suppose that there is x_1 such that $x_1\in E_k$ and $f'(x_1)=0$. Define $x_2=\inf\{x: x\in E_k,\ f'(x)=0\}$. We find from (4.21) that $x_2\in E_k$ and $f'(x)<0$ on (b^{k-1}, x_2). Hence the induction assumption says that

$$f'(x)<0 \qquad \text{on } \bigcup_{n=0}^{k-1} E_n\cup(b^{k-1}, x_2). \tag{4.22}$$

This implies that

(4.23) $$\sum_{n=0}^{k-2}\int_{b^n}^{b^{n+1}} f(x_2-u)b^{-(\lambda+1)(n+1)}du \leqq \sum_{n=0}^{k-2} f(x_2-b^{n+1})b^{-\lambda(n+1)}(1-b^{-1})$$

and

(4.24) $$\int_0^1 f(x_2-u)du < f(x_2-1).$$

On the other hand, we get by (4.11) that

(4.25) $$t\int_{b^{k-1}}^{x_2} f(x_2-u)b^{-(\lambda+1)k}du < t\int_0^{x_2} f(x_2-u)b^{-k}k_0(u)du = b^{-k}x_2 f(x_2).$$

Therefore, we obtain from (4.20), (4.23), (4.24) and (4.25) that

(4.26) $$0 = x_2 f'(x_2) \leqq (t-1+mb^{-k}x_2)f(x_2) - tf(x_2-1)(a-m)$$
$$-t\sum_{n=1}^{k-1} f(x_2-b^n)b^{-\lambda n}\{a-m(1-b^{-1})\} < 0,$$

noting that $t-1-mb^{-k}x_2 < -2^{-1}+m < 0$ and $a-m > 0$ for $b \geqq b_1$. This is a contradiction. Thus the assertion (4.14) is established. It follows that, for every $b \geqq b_0 \vee b_1$, $\{X_b(t)\}$ is not a unimodal Lévy process and $X_b(2^{-1})$ is unimodal. Recalling (4.3), we see from this that, for every $b \geqq b_0 \vee b_1$, there are $t_1 = 2^{-1}$ and $t_2 \neq 2^{-1}$ such that, for every integer n, $X_b(t)$ is unimodal for $t = c^n t_1$ and not unimodal for $t = c^n t_2$. This completes the proof of Proposition 1.4.

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CENTER FOR MATHEMATICAL SCIENCES
THE UNIVERSITY OF AIZU
IKKI-MACHI
AIZU-WAKAMATSU
FUKUSHIMA-KEN, 965
JAPAN