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HARMONIC GAUSS SECTIONS, OBJECT INCLUSION MAPS AND YANG-MILLS CONNECTIONS

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Introduction

Harmonic Gauss sections are critical points of the *vertical energy* through arbitrary vertical variations with respect to some fixed tangential splitting of a target Grassmann manifold bundle (definition in [11], [12]). On the other hand, *Yang-Mills connections* are critical points of the square norm of curvature form through arbitrary variations in the connection forms. There is a relationship between these variational solutions which is previously observed by C. M. Wood in [1],

Let G be a Lie group which admits a bi-invariant Riemannian metric and P be a right principal G-bundle over a Riemannian manifold M. The horizontally lifted metric on P by a right connection form ω makes the fibration $\pi_P: P \to M$ into a Riemannian submersion with totally geodesic fibers [10]. The Yang-Mills equation for a single connection form ω is translated into the harmonic section equation for the Gauss section $\gamma_P = [P \ni u \mapsto \mathrm{Ker}\,(d\pi_P)_u \subset T_u P]$ on P with the ω-horizontally lifted metric [12]. In fact, C. M. Wood obtained:

THEOREM. *Let ω be a right connection form on P. Then*
(i) *ω is flat (resp. Yang-Mills) if and only if γ_P is a horizontal (resp. harmonic) section with respect to ${}^P\nabla$,*
(ii) *ω is parallel if γ_P is a covariantly horizontal section with respect to ${}^P\nabla$, where ${}^P\nabla$ is the Riemannian connection of P.*

In this paper we study several characterizations of Yang-Mills connections in terms of harmonic Gauss sections, each of which is a generalization of the theorem of C. M. Wood. Our results extend the lists of similarities between theories of harmonic maps and Yang-Mills connections by J. P. Bourguignon [1].

Let Q be a left principal G-bundle over a Riemannian manifold N and η be a left connection form on Q, where the left G-action in the definition of *left* $\cdots$ is the reciprocal of the right G-action in that of *right* $\cdots$. The *joint space* $P \cdot Q$ (definition in § 1) also has the horizontally lifted metric by the *joint form* $\omega \diamond \eta$ (definition in § 1) which makes the fibration $\pi_{P \cdot Q}: P \cdot Q \to M \times N$ into

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a Riemannian submersion with totally geodesic fibers.

A pair of the Yang-Mills equations for ω and η is translated into the harmonic section equation for the Gauss section $\gamma_{P\cdot Q}=[P\cdot Q\ni u\cdot v\mapsto \mathrm{Ker}\,(d\pi_{P\cdot Q})_{u\cdot v}\subset T_{u\cdot v}(P\cdot Q)]$ on $P\cdot Q$ with the $\omega\diamond\eta$-horizontally lifted metric:

THEOREM A. *Let ω, η be a right connection form on P and a left connection form on Q, respectively. Then*

(i) *ω and η are both flat (resp. Yang-Mills) if and only if*
$\gamma_{P\cdot Q}$ is a horizontal (resp. harmonic) section with respect to ${}^{P\cdot Q}\nabla$,

(ii) *ω and η are both parallel if*
$\gamma_{P\cdot Q}$ is a covariantly horizontal section with respect to ${}^{P\cdot Q}\nabla$,
where ${}^{P\cdot Q}\nabla$ is the Riemannian connection of $P\cdot Q$.

In the case of $N=M$, $Q=P^{-1}$ and $\eta=\omega^{-1}$, where P^{-1} (resp. ω^{-1}) is the *inverse* of P (resp. ω) [5], we have immediately the following:

COROLLARY B. *Let ω be a right connection form on P. Then*

(i) *ω is flat (resp. Yang-Mills) if and only if*
$\gamma_{P\cdot P^{-1}}$ is a horizontal (resp. harmonic) section with respect to ${}^{P\cdot P^{-1}}\nabla$.

(ii) *ω is parallel if*
$\gamma_{P\cdot P^{-1}}$ is a covariantly horizontal section with respect to ${}^{P\cdot P^{-1}}\nabla$.

By pulling-back $\gamma_{P\cdot P^{-1}}$ along the *object inclusion map* ε(definition in § 1), the Gauss section ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}=[M\ni x\mapsto \mathrm{Ker}\,(d\pi_{P\cdot P^{-1}})_{\varepsilon(x)}\subset T_{\varepsilon(x)}(P\cdot P^{-1})]$ is induced and the above relation (ii) is improved (Theorems 4.7, 4.14, 4.16):

THEOREM C. *Let ω be a right connection form on P. Then ω is flat, parallel or Yang-Mills if and only if*
${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ is a horizontal, covariantly horizontal or harmonic section with respect to ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$, respectively,
where ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$ is the induced connection from ${}^{P\cdot P^{-1}}\nabla$ via ε.

$\varepsilon(M)$ is a totally geodesic submanifold of $P\cdot P^{-1}$ (Proposition 2.16) so that ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$ splits into the Riemannian connection ${}^{M}\nabla$ of M and the normal connection $\nabla^{(-)}$ with respect to ε. On the other hand, ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ can be reduced to ${}^{\varepsilon}\gamma^{(-)}_{P\cdot P^{-1}}=[M\ni x\mapsto \mathrm{Ker}\,(d\pi_{P\cdot P^{-1}})_{\varepsilon(x)}\subset \boldsymbol{E}^{(-)}_{\varepsilon(x)}]$, where $\boldsymbol{E}^{(-)}_{\varepsilon(x)}$ is the -1-eigenspace of the differential of the inversion of $P\cdot P^{-1}$, which is also the orthogonal complement of $(d\varepsilon)_x T_x M$ in $T_{\varepsilon(x)}(P\cdot P^{-1})$.

THEOREM D (Theorem 5.12). *Let ω be a right connection form on P. Then ω is flat, parallel or Yang-Mills if and only if*
${}^{\varepsilon}\gamma^{(-)}_{P\cdot P^{-1}}$ is a horizontal, covariantly horizontal or harmonic section
with respect to $\nabla^{(-)}$, respectively.

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§ 1. Joints of Principal Bundles and Connection Forms

Let G be a Lie group and $P \xrightarrow{\pi_P} M$ (resp. $Q \xrightarrow{\pi_Q} N$) be a right (resp. left) principal G-bundle over a C^∞-manifold M (resp. N). G acts on $P\times Q$ as follows; ${}^{P\times Q}R: P\times Q\times G \to P\times Q\,;\ ((u, v), g)\mapsto(ug, g^{-1}v)$ (resp. ${}^{P\times Q}L: G\times P\times Q \to P\times Q\,;\ (g, (u, v))\mapsto(ug^{-1}, gv)$.) We denote the quotient topological space of the right (resp. left) G-space $(P\times Q, {}^{P\times Q}R)$ (resp. $(P\times Q, {}^{P\times Q}L)$) by $P\cdot Q$ and the quotient map by $j_{P\cdot Q}: P\times Q\to P\cdot Q\,;\ (u, v)\mapsto u\cdot v$. The map $\pi_{P\cdot Q}: P\cdot Q\to M\times N$; $u\cdot v\mapsto(\pi_P(u), \pi_Q(v))$ is well-defined. By [3, Chapter 5, Section 2, Proposition 1], $P\cdot Q$ is given a C^∞-manifold structure (which is called the *joint space* of P and Q).

$(P\times Q, {}^{P\times Q}R) \xrightarrow{j_{P\cdot Q}} P\cdot Q$ (resp. $(P\times Q, {}^{P\times Q}L) \xrightarrow{j_{P\cdot Q}} P\cdot Q$) is a right (resp. left) principal G-bundle. $P\cdot Q \xrightarrow{\pi_{P\cdot Q}} M\times N$ is a C^∞-fiber bundle. The canonical projection $p_P: (P\times Q, {}^{P\times Q}R)\to P;\ (u, v)\mapsto u$ (resp. $p_Q: (P\times Q, {}^{P\times Q}L)\to Q\,;\ (u, v)\mapsto v$) is a right (resp. left) principal G-bundle homomorphism. Throughout the following sections, $(P\times Q, {}^{P\times Q}R)$ will be simply abbreviated to $P\times Q$. Let $(G, \mathrm{Ad}_G, \mathfrak{g})$ be the adjoint representation of G. $\operatorname{Ker} d\pi_P$ (resp. $\operatorname{Ker} d\pi_Q$) is isomorphic to the trivial vector bundle $P\times\mathfrak{g}$ (resp. $\mathfrak{g}\times Q$). Similarly we observe the following:

PROPOSITION 1.1.

$$\operatorname{Ker} dj_{P\cdot Q} \longrightarrow P\times Q\times\mathfrak{g}\,;\ A^\sharp_{(u,v)}=(A^\sharp_u, -A^\flat_v)\mapsto((u, v), A) \text{ and}$$

$$\operatorname{Ker} d\pi_{P\cdot Q} \longrightarrow (P\times Q)_{\mathrm{Ad}_G}\mathfrak{g}\,;\ (dj_{P\cdot Q})_{(u,v)}(A^\sharp_u, A^\flat_v)\mapsto[(u, v), A]$$

are both vector bundle isomorphisms,
where $A^\sharp_u=d/dt|_{t=0}u(\exp tA)$, $A^\flat_v=d/dt|_{t=0}(\exp tA)v$ $(A\in\mathfrak{g})$ *e. t. c..*

Let ω (resp. η) be a right (resp. left) connection from on P (resp. Q). Then from Proposition 1.1, $p_P^*\omega$ and $-p_Q^*\eta$ are both right connection forms on $P\times Q$ and therefore a tensorial form of type $(G, \mathrm{Ad}_G, \mathfrak{g})$ (c. f. [6]).

PROPOSITION 1.2. *Let* $\theta\in\boldsymbol{R}$.
(i) $(1-\theta)p_P^*\omega-\theta p_Q^*\eta$ *is a right connection form on* $P\times Q$.
(ii) $(1-\theta)p_P^*\omega+\theta p_Q^*\eta$ *is a tensorial* 1*-form on* $P\times Q$ *of type* $(G, \mathrm{Ad}_G, \mathfrak{g})$ *if and only if* $\theta=1/2$.

We write shortly $\omega\star\eta$ instead of $1/2(p_P^*\omega-p_Q^*\eta)$. On the other hand, from the above proposition, $1/2(p_P^*\omega+p_Q^*\eta)$ can be reduced to a 1-form on $P\cdot Q$ taking values in $(P\times Q)\times_{\mathrm{Ad}_G}\mathfrak{g}$, which is denoted by $\omega\diamond\eta$ (which is called the *joint form* of ω and η).

For any $u\in P$, $v\in Q$, $\mathrm{Ker}(\omega\bigstar\eta)_{(u,v)}=(\mathrm{Ker}\,\omega_u\oplus 0_v)\oplus\{(A_u^\#, A_v^\flat)\,|\,A\in\mathfrak{g}\}\oplus(0_u\oplus\mathrm{Ker}\,\eta_v)$, where $(\mathrm{Ker}\,\omega_u\oplus 0_v)\oplus(0_u\oplus\mathrm{Ker}\,\eta_v)$ is the $\omega\bigstar\eta$-horizontal lift of $\mathrm{Ker}(\omega\diamond\eta)_{u\cdot v}$ and $\{(A_u^\#, A_v^\flat)\,|\,A\in\mathfrak{g}\}$ is the $\omega\bigstar\eta$-horizontal lift of $\mathrm{Ker}(d\pi_{P\cdot Q})_{u\cdot v}$. On the other hand, $\mathrm{Ker}\,\omega\diamond\eta$ is a horizontal distribution relative to $\pi_{P\cdot Q}$:

PROPOSITION 1.3. $T(P\cdot Q)=\mathrm{Ker}\,d\pi_{P\cdot Q}\oplus\mathrm{Ker}\,\omega\diamond\eta$.

Proof. For each $u\in P$, $v\in Q$, the restriction of surjective linear map $(\omega\diamond\eta)_{u\cdot v}\colon T_{u\cdot v}(P\cdot Q)\to((P\times Q)\times_{\mathrm{Ad}_G}\mathfrak{g})_{u\cdot v}$ to $\mathrm{Ker}\,(d\pi_{P\cdot Q})_{u\cdot v}$ coincides with the linear isomorphism in Proposition 1.1.

The above proposition implies that $\omega\diamond\eta$ plays the same role of a connection form. Let ${}^{\omega\bigstar\eta}d$ (resp. ${}^{\omega\bigstar\eta}D$) be the exterior covariant differentiation on $P\cdot Q$ (resp. $P\times Q$) with respect to $\omega\bigstar\eta$.

PROPOSITION 1.4. *For any $u\in P$, $v\in Q$, $X, Y\in\mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v}$,*

$$({}^{\omega\bigstar\eta}d(\omega\diamond\eta))_{u\cdot v}(X, Y)=[(u, v),\ 1/2((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)({}^{\omega\bigstar\eta}H_{u,v}^{(u,v)}X,\ {}^{\omega\bigstar\eta}H_{u,v}^{(u,v)}Y)],$$

where ${}^\omega\Omega$ (resp. ${}^\eta\Omega$) is the curvature form of ω (resp. η) and ${}^{\omega\bigstar\eta}H_{u,v}^{(u,v)}\colon T_{u\cdot v}(P\cdot Q)\to\mathrm{Ker}\,(\omega\bigstar\eta)_{(u,v)}$ is the $\omega\bigstar\eta$-horizontal lifting.

Proof. By the structure equations for ω and η (c. f. [6]),

$$\begin{aligned}{}^{P\times Q}d\Big(\frac{1}{2}(p_P^*\omega+p_Q^*\eta)\Big)&=\frac{1}{2}((p_P^*)^P d\omega+(p_Q^*)^Q d\eta)\\&=\frac{1}{2}\Big\{p_P^*\Big({}^\omega\Omega+\frac{1}{2}[\omega\wedge\omega]\Big)+p_Q^*\Big({}^\eta\Omega-\frac{1}{2}[\eta\wedge\eta]\Big)\Big\}\\&=\frac{1}{2}((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)-\frac{1}{4}([p_P^*\omega\wedge p_P^*\omega]-[p_Q^*\eta\wedge p_Q^*\eta])\\&=\frac{1}{2}((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)-\Big[\frac{1}{2}(p_P^*\omega-p_Q^*\eta)\wedge\frac{1}{2}(p_P^*\omega+p_Q^*\eta)\Big].\end{aligned}$$

$1/2((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)$ is a tensorial 2-form on $P\times Q$ so that ${}^{\omega\bigstar\eta}D(1/2(p_P^*\omega+p_Q^*\eta))=1/2((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)$.

COROLLARY 1.5. *ω and η are both flat if and only if $\mathrm{Ker}\,\omega\diamond\eta$ is an involutive distribution on $P\cdot Q$.*

In the case of $N=M$, $Q=P^{-1}=P\times_\lambda G$ which is called the *inverse* of P in [5] defined by the left G-action $\lambda\colon G\times G\to G$; $(g_1, g_2)\mapsto g_2g_1^{-1}$, for the diffeomorphism $i\colon P\to P^{-1}$; $u\mapsto u^{-1}=[u, e]$, we denote $-(i^{-1})^*\omega$ by ω^{-1} which is a left connection form on P^{-1}, where e is the identity element of G. Note that ${}^{\omega^{-1}}\Omega=-(i^{-1})^{*\omega}\Omega$.

COROLLARY 1.6. *ω is flat if and only if* $\mathrm{Ker}\,\omega\diamondsuit\omega^{-1}$ *is an involutive distribution on* $P\cdot P^{-1}$.

$P\cdot P^{-1}$ is the *Lie groupoid associated to* $P\overset{\pi_P}{\to}M$ and $\pi_{P\cdot P^{-1}}: P\cdot P^{-1}\to M\times M$ is the *anchor* of $P\cdot P^{-1}$ [8, Chapter 1, Example 1.10]. Let $\Delta_M: M\to M\times M$ be the diagonal map of M and $\Delta_M^{-1}(P\cdot P^{-1})$ be the pull-back fiber bundle over M via Δ_M which is the *inner subgroupoid* of $P\cdot P^{-1}$.

$\Delta_M^{-1}(P\cdot P^{-1})$ is isomorphic to the automorphism bundle $P\times_{\mathrm{Ad}}G$. In fact, because $\Delta_M^{-1}(P\cdot P^{-1})$ can be identified with $\{(x, u\cdot v^{-1})\in M\times P\cdot P^{-1}\mid \pi_P(u)=x=\pi_P(v)\}$, therefore, the following map is well-defined and gives a fiber bundle isomorphism; $\Delta_M^{-1}(P\cdot P^{-1})\to P\times_{\mathrm{Ad}}G$; $(x, u\cdot v^{-1})\mapsto[u, g]$ $(v=ug, g\in G)$.

Let $\tilde{\Delta}_M: P\times_{\mathrm{Ad}}G\to P\cdot P^{-1}$ be the induced fiber bundle homomorphism from Δ_M and $e: M\to P\times_{\mathrm{Ad}}G$ be the canonical section which assigns x to the identity element e_x of the fiber over x, then their composite $\varepsilon=\tilde{\Delta}_M\circ e: M\to P\cdot P^{-1}$ is a C^∞-map, which is called the *object inclusion map* of $P\cdot P^{-1}$ and $\pi_{P\cdot P^{-1}}\circ\varepsilon=\Delta_M$.

Let $\varepsilon^{-1}(P\times P^{-1})$ be the pull-back principal G-bundle over M via $\varepsilon\cdot\varepsilon^{-1}(P\times P^{-1})$ is isomorphic to P. Actually we have a principal bundle homomorphism $\tilde{\varepsilon}: P\to P\times P^{-1}$; $u\mapsto(u, u^{-1})$ such that $j_{P\cdot P^{-1}}\circ\tilde{\varepsilon}=\varepsilon\circ\pi_P$ and $\tilde{\varepsilon}^*(\omega\bigstar\omega^{-1})=\omega$. Because for $x\in M$, $u, v\in P$, $\varepsilon(x)=u\cdot v^{-1}$ if and only if $u=v\in\pi_P^{-1}(x)$, then $\varepsilon^{-1}(P\times P^{-1})$ can be identified with $\{(x, (u, u^{-1}))\in M\times P\times P^{-1}\mid u\in\pi_P^{-1}(x)\}$ so that the map $\varepsilon^{-1}(P\times P^{-1})\to P$; $(x, (u, u^{-1}))\mapsto u$ is well-defined and gives a right principal G-bundle isomorphism.

PROPOSITION 1.7. $\varepsilon^{-1}(P\times P^{-1})\times_{\mathrm{Ad}_G}\mathfrak{g}$ *is isomorphic to the adjoint bundle* $P\times_{\mathrm{Ad}_G}\mathfrak{g}$.

The notion of general groupoid includes its inversion. In the case of $P\cdot P^{-1}$, its inversion is $\iota: P\cdot P^{-1}\to P\cdot P^{-1}$; $u\cdot v^{-1}\mapsto v\cdot u^{-1}$, which is obviously well-defined and $\iota\circ\iota=\mathrm{id}_{P\cdot P^{-1}}$. ε is also given by $\varepsilon(x)=u\cdot u^{-1}$ $(u\in\pi_P^{-1}(x))$ so that $\iota\circ\varepsilon=\varepsilon$, therefore, the differential $d\iota: T(P\cdot P^{-1})\to T(P\cdot P^{-1})$ induces a vector bundle involutive automorphism of $\varepsilon^{-1}T(P\cdot P^{-1})$ and a splitting $(\varepsilon^{-1}T(P\cdot P^{-1}))_x=T_{\varepsilon(x)}(P\cdot P^{-1})=\boldsymbol{E}^{(+)}_{\varepsilon(x)}\oplus\boldsymbol{E}^{(-)}_{\varepsilon(x)}$, for each $x\in M$, where $\boldsymbol{E}^{(\pm)}_{\varepsilon(x)}$ is the ±1-eigenspace of $(d\iota)_{\varepsilon(x)}$. $\varepsilon^{-1}\boldsymbol{E}^{(\pm)}=\coprod_{x\in M}\boldsymbol{E}^{(\pm)}_{\varepsilon(x)}$ is a vector subbundle of $\varepsilon^{-1}T(P\cdot P^{-1})$ and $\varepsilon^{-1}T(P\cdot P^{-1})=\varepsilon^{-1}\boldsymbol{E}^{(+)}\oplus\varepsilon^{-1}\boldsymbol{E}^{(-)}$.

Let ${}^{\omega}H^u_x$ (resp. ${}^{\omega^{-1}}H^{u^{-1}}_x$): $T_xM\to\mathrm{Ker}\,\omega_u$ (resp. $\mathrm{Ker}\,\omega_u^{-1}$) be the ω-(resp. ω^{-1}-) horizontal lifting. Note that for $u\in\pi_P^{-1}(x)$, $g\in G$, $X\in T_xM$,

$$(dj_{P\cdot P^{-1}})_{(ug,\, g^{-1}u^{-1})}({}^{\omega}H^{ug}_xX,\ \pm{}^{\omega^{-1}}H^{g^{-1}u^{-1}}_xX)$$
$$=(dj_{P\cdot P^{-1}})_{(u,\, u^{-1})}({}^{\omega}H^u_xX,\ \pm{}^{\omega^{-1}}H^u_xX).$$

We write shortly ${}^{\omega\diamondsuit\omega^{-1}}X^{(\pm)}$ instead of $(dj_{P\cdot P^{-1}})_{(u,\, u^{-1})}({}^{\omega}H^u_xX,\ \pm{}^{\omega^{-1}}H^{u^{-1}}_xX)$.

PROPOSITION 1.8. *For any* $x\in M$,
(i) $(d\varepsilon)_x(X)={}^{\omega\diamond\omega^{-1}}X^{(+)}$ *for all* $X\in T_xM$, *therefore,*

$$(d\varepsilon)_x(T_xM)=\boldsymbol{E}^{(+)}_{\varepsilon(x)}=\{{}^{\omega\diamond\omega^{-1}}X^{(+)}\,|\,X\in T_xM\},$$

(ii) $\boldsymbol{E}^{(-)}_{\varepsilon(x)}=\mathrm{Ker}\,(d\pi_{P\cdot P^{-1}})_{\varepsilon(x)}\oplus\{{}^{\omega\diamond\omega^{-1}}X^{(-)}\,|\,X\in T_xM\}$,
(iii) $\mathrm{Ker}\,(\omega\diamond\omega^{-1})_{\varepsilon(x)}=(d\varepsilon)_x(T_xM)\oplus\{{}^{\omega\diamond\omega^{-1}}X^{(-)}\,|\,X\in T_xM\}$.

COROLLARY 1.9. *For any* ω, $\varepsilon^*(\omega\diamond\omega^{-1})=0$, *therefore,* ε *is an* $\omega\diamond\omega^{-1}$*-horizontal lift of* Δ_M.

§2. Horizontally Lifted Metrics by the Joint Forms

In this section, we assume that a Lie group G admits a bi-invariant Riemannian metric $\langle\,,\rangle$. Let M (resp. N) be a Riemannian manifold with a Riemannian metric ${}^Mg$ (resp. ${}^Ng$). For a right (resp. left) principal G-bundle $P\xrightarrow{\pi_P}M$ (resp. $Q\xrightarrow{\pi_Q}N$) with a right (resp. left) connection form ω (resp. η), the ω-(resp. η-)horizontally lifted metric ${}^P_\omega g$ (resp. ${}^Q_\eta g$) is defined as follows:

$${}^P_\omega g=(\pi_P^*)^Mg+\langle\omega,\,\omega\rangle\quad(\text{resp. } {}^Q_\eta g=(\pi_Q^*)^Ng+\langle\eta,\,\eta\rangle).$$

$\pi_P:(P,\,{}^P_\omega g)\to(M,\,{}^Mg)$ (resp. $\pi_Q:(Q,\,{}^Q_\eta g)\to(N,\,{}^Ng)$) is a Riemannian submersion with totally geodesic fibers since the ω-(resp. η-) parallel translations between fibers are isometries in the above metric [10, Theorem 3.5]. We denote the canonical projection by p_M (resp. p_N): $M\times N\to M$ (resp. N); $(x,\,y)\mapsto\omega$ (resp. y). The joint form $\omega\diamond\eta$ lifts the Riemannian product metric ${}^{M\times N}g=(p_M^*)^Mg+(p_N^*)^Ng$ to the metric ${}^{P\cdot Q}_{\omega\diamond\eta}g$ on $P\cdot Q$ defined by ${}^{P\cdot Q}_{\omega\diamond\eta}g=(\pi_{P\cdot Q}^*)^{M\times N}g+2\langle\omega\diamond\eta,\,\omega\diamond\eta\rangle$. From [10, Theorem 3.5], we have $\pi_P^Q:(P\cdot Q,\,{}^{P\cdot Q}_{\omega\diamond\eta}g)\to(M,\,{}^Mg)$; $u\cdot v\mapsto\pi_P(u)$, π_Q^P: $(P\cdot Q,\,{}^{P\cdot Q}_{\omega\diamond\eta}g)\to(N,\,{}^Ng)$; $u\cdot v\mapsto\pi_Q(v)$ and $\pi_{P\cdot Q}:(P\cdot Q,\,{}^{P\cdot Q}_{\omega\diamond\eta}g)\to(M\times N,\,{}^{M\times N}g)$ are Riemannian submersions with totally geodesic fibers.

Let ${}^{P\times Q}g=(p_P^*)^P_\omega g+(p_Q^*)^Q_\eta g$ be the Riemannian product metric on $P\times Q$. Then ${}^{P\times Q}g$ coincides with the $\omega\bigstar\eta$-horizontally lifted metric from ${}^{P\cdot Q}_{\omega\diamond\eta}g$:

PROPOSITION 2.1. ${}^{P\times Q}g=(j_{P\cdot Q}^*)^{P\cdot Q}_{\omega\diamond\eta}g+2\langle\omega\bigstar\eta,\,\omega\bigstar\eta\rangle$.

Throughout the following sections, the $\omega\bigstar\eta$-horizontally lifted vector ${}^{\omega\diamond\eta}H^{(u,v)}_{u\cdot v}U$ will be abbreviated to $\bar{U}$ or $(U)^-$ for $U\in T_{u\cdot v}(P\cdot Q)$. Let ${}^{P\cdot Q}\nabla$ be the Riemannian connection of ${}^{P\cdot Q}_{\omega\diamond\eta}g$. We denote the associated projections in Proposition 1.3 by $\rho:T(P\cdot Q)\to\mathrm{Ker}\,d\pi_{P\cdot Q}$; $X\mapsto X^\top$, $\rho^\perp:T(P\cdot Q)\to\mathrm{Ker}\,\omega\diamond\eta$; $X\mapsto X^\perp$. ${}^{P\cdot Q}\mathcal{A}=[(X,\,Y)\mapsto({}^{P\cdot Q}\nabla_{X^\perp}Y^\top)^\perp+({}^{P\cdot Q}\nabla_{X^\perp}Y^\perp)^\top]$ is called the O'Neill's tensor A on $(P\cdot Q,\,{}^{P\cdot Q}_{\omega\diamond\eta}g)$ [9, Lemma 2], and we have immediately the following:

PROPOSITION 2.2. *For any* $u\in P$, $v\in Q$, $X,\,Y\in T_{u\cdot v}(P\cdot Q)$,

$$((\omega\diamond\eta)\circ{}^{P\cdot Q}\mathcal{A})_{u\cdot v}(X,\,Y)=\left[(u,\,v),\,-\frac{1}{2}((p_P^*)^\omega\Omega+(p_Q^*)^\eta\Omega)(\bar{X},\,\bar{Y})\right].$$

Let ${}^{\omega\diamond\eta}\nabla$ be the covariant differentiation on $P\cdot Q$ with respect to $\omega\star\eta$ in $(P\times Q)\times_{\mathrm{Ad}_G}\mathfrak{g}$ and ${}^{\omega\star\eta}_{P\cdot Q}\nabla$ be the induced differentiation on $P\cdot Q$ with respect to ${}^{\omega\star\eta}\nabla$ and ${}^{P\cdot Q}\nabla$ in $T^*(P\cdot Q)\otimes(P\times Q)\times_{\mathrm{Ad}_G}\mathfrak{g}$. The Riemannian connection of ${}^P_\omega g$, ${}^Q_\eta g$ or ${}^{P\times Q}g$ is denoted by ${}^P\nabla$, ${}^Q\nabla$ or ${}^{P\times Q}\nabla$, respectively.

COROLLARY 2.3. *For any* $u\in P$, $v\in Q$, $X, Y, Z\in T_{u\cdot v}(P\cdot Q)$,

$$ {}^{\omega\star\eta}_{P\cdot Q}\nabla((\omega\diamond\eta)\circ{}^{P\cdot Q}\mathcal{A}))_{u\cdot v}(X, Y, Z)=\left[(u, v), -\frac{1}{2}((p_P^*)^P\nabla^\omega\Omega+(p_Q^*)^Q\nabla^\eta\Omega)(\bar{X}, \bar{Y}, \bar{Z})\right]. $$

Proposition 1.4 implies the following lemma:

LEMMA 2.4.

$$ ({}^{\omega\star\eta}d(\omega\diamond\eta))_{u\cdot v}(X, Y)=0 $$

for all $u\in P$, $v\in Q$, $X\in T_{u\cdot v}(P\cdot Q)$, $Y\in\mathrm{Ker}\,(d\pi_{P\cdot Q})_{u\cdot v}$.

LEMMA 2.5. *For any* $X, Y\in\mathcal{X}(P\cdot Q)$,

$$ (\omega\diamond\eta)({}^{P\cdot Q}\nabla_{X^\perp}Y^\top)={}^{\omega\star\eta}\nabla_{X^\perp}((\omega\diamond\eta)(Y^\top)). $$

Proof. Since each fiber of $P\cdot Q$ is totally geodesic and the decomposition in Proposition 1.3 is orthogonal with respect to the lifted metric,

$$ (\omega\diamond\eta)({}^{P\cdot Q}\nabla_{Y^\top}X^\perp)=(\omega\diamond\eta)(({}^{P\cdot Q}\nabla_{Y^\top}X^\perp)^\perp)=0 $$

therefore $(\omega\diamond\eta)({}^{P\cdot Q}\nabla_{X^\perp}Y^\top)=(\omega\diamond\eta)([X^\perp, Y^\top])$. By Lemma 2.4,

$$ \begin{aligned} 0&=({}^{\omega\star\eta}d(\omega\diamond\eta))(X^\perp, Y^\top)\\ &=\frac{1}{2}\{{}^{\omega\star\eta}\nabla_{X^\perp}((\omega\diamond\eta)(Y^\top))-{}^{\omega\star\eta}\nabla_{Y^\top}((\omega\diamond\eta)(X^\perp))-(\omega\diamond\eta)([X^\perp, Y^\top])\}\\ &=\frac{1}{2}\{{}^{\omega\star\eta}\nabla_{X^\perp}((\omega\diamond\eta)(Y^\top))-(\omega\diamond\eta)([X^\perp, Y^\top])\}. \end{aligned} $$

Hence $(\omega\diamond\eta)({}^{P\cdot Q}\nabla_{X^\perp}Y^\top)={}^{\omega\star\eta}\nabla_{X^\perp}((\omega\diamond\eta)(Y^\top))$.

LEMMA 2.6. *For any* $u\in P$, $v\in Q$, $X, Y, Z\in\mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v}$,

$$ ((\omega\diamond\eta)\circ({}^{P\cdot Q}\nabla{}^{P\cdot Q}\mathcal{A}))_{u\cdot v}(X, Y, Z)=({}^{\omega\star\eta}_{P\cdot Q}\nabla((\omega\diamond\eta)\circ{}^{P\cdot Q}\mathcal{A}))_{u\cdot v}(X, Y, Z). $$

From Corollary 2.3 and Lemma 2.6, we have

PROPOSITION 2.7. *For any* $u\in P$, $v\in Q$, $X, Y, Z\in\mathrm{Ker}\,(\omega\diamond\eta)_{uv}$,

$$ ((\omega\diamond\eta)\circ{}^{P\cdot Q}\nabla{}^{P\cdot Q}\mathcal{A})_{u\cdot v}(X, Y, Z)=\left[(u, v), -\frac{1}{2}((p_P^*)^P\nabla^\omega\Omega+(p_Q^*)^Q\nabla^\eta\Omega)(\bar{X}, \bar{Y}, \bar{Z})\right]. $$

ω (resp. η) is called a *parallel connection* if ${}^P\nabla^\omega\Omega(\mathrm{Ker}\,\omega, \mathrm{Ker}\,\omega, \mathrm{Ker}\,\omega)\equiv 0$

(resp. ${}^{Q}\nabla^{\eta}\Omega(\mathrm{Ker}\,\eta, \mathrm{Ker}\,\eta, \mathrm{Ker}\,\eta)\equiv 0$). For instance, in the canonical fibration over a Riemannian symmetric space, the canonical invariant connection with respect to its symmetric pair is a parallel connection [4].

COROLLARY 2.8. *ω and η are both parallel if and only if*

$$((\omega\diamondsuit\eta)\circ{}^{P\cdot Q}\nabla^{P\cdot Q}\mathcal{A})_{u\cdot v}(X, Y, Z)=0 \quad \textit{for all } u\in P,\ v\in Q,\ X, Y, Z\in\mathrm{Ker}\,(\omega\diamondsuit\eta)_{u\cdot v}.$$

Let ${}^{\omega}D$ (resp. ${}^{\eta}D$) be the exterior covariant differentiation on P (resp. Q) with respect to ω (resp. η) and ${}^{\omega}D^*$ (resp. ${}^{\eta}D^*$) be the exterior covariant co-differentiation on P (resp. Q) of ${}^{\omega}D$ (resp. ${}^{\eta}D$).

PROPOSITION 2.9. *For any $u\in P$, $v\in Q$, $Z\in\mathrm{Ker}\,(\omega\diamondsuit\eta)_{u\cdot v}$,*

$$\mathrm{Trace}\,((\omega\diamondsuit\eta)\circ{}^{P\cdot Q}\nabla^{P\cdot Q}\mathcal{A})_{u\cdot v}(\rho^{\perp}, \rho^{\perp}, Z)=\left[(u, v),\ \frac{1}{2}((p_P^*)^{\omega}D^{*\omega}\Omega+(p_Q^*)^{\eta}D^{*\eta}\Omega)(\bar{Z})\right].$$

ω (resp. η) is called a *Yang-Mills connection* if ω (resp. η) satisfies the *Yang-Mills equation*: ${}^{\omega}D^{*\omega}\Omega\equiv 0$ (resp. ${}^{\eta}D^{*\eta}\Omega\equiv 0$) (c. f. [2]). For instance, parallel connections are Yang-Mills connections.

COROLLARY 2.10. *ω and η are both Yang-Mills if and only if*

$$\mathrm{Trace}\,((\omega\diamondsuit\eta)\circ{}^{P\cdot Q}\nabla^{P\cdot Q}\mathcal{A})_{u\cdot v}(\rho^{\perp}, \rho^{\perp}, Z)=0 \quad \textit{for all } u\in P,\ v\in Q,\ Z\in\mathrm{Ker}\,(\omega\diamondsuit\eta)_{u\cdot v}.$$

Notice that we can describe a pair of the Yang-Mills equations for ω and η in terms of the Ricci curvature tensor ${}^{P\cdot Q}Ric$ of $(P\cdot Q,\ {}^{P\cdot Q}_{\omega\diamondsuit\eta}g)$.

PROPOSITION 2.11. *For any $u\in P$, $v\in Q$, $Z\in\mathrm{Ker}\,(\omega\diamondsuit\eta)_{u\cdot v}$,*

$$((\omega\diamondsuit\eta)\circ{}^{P\cdot Q}Ric)_{u\cdot v}(Z)=\left[(u, v),\ \frac{1}{2}((p_P^*)^{\omega}D^{*\omega}\Omega+(p_Q^*)^{\eta}D^{*\eta}\Omega)(\bar{Z})\right].$$

A pair of the Yang-Mills equations for ω and η is a necessary condition for the Einstein equation with respect to $(P\cdot Q,\ {}^{P\cdot Q}_{\omega\diamondsuit\eta}g)$ as an analog to the relation between the Yang-Mills equation for ω and the Einstein with respect to $(P, {}^{\omega}_{P}g)$ [12, Corollary 2.19].

COROLLARY 2.12.
(i) *ω and η are both Yang-Mills if and only if*

$${}^{P\cdot Q}Ric(\mathrm{Ker}\,d\pi_{P\cdot Q}, \mathrm{Ker}\,\omega\diamondsuit\eta)=0\,.$$

(ii) *If $(P\cdot Q,\ {}^{P\cdot Q}_{\omega\diamondsuit\eta}g)$ is an Einstein space, then ω and η are both Yang-Mills.*

In the case of $N=M$, $Q=P^{-1}$ and $\eta=\omega^{-1}$, the inversion $i:(P, {}^{\omega}_{P}g)\to(P^{-1}, {}^{\omega^{-1}}_{P^{-1}}g)$ is an isometry. Therefore we have immediately the followings:

COROLLARY 2.13. *ω is parallel if and only if*

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})_{u\cdot v^{-1}}(X, Y, Z)=0$$

for all $u, v\in P$, $X, Y, Z\in \operatorname{Ker}(\omega\diamond\omega^{-1})_{u\cdot v^{-1}}$.

COROLLARY 2.14. *ω is Yang-Mills if and only if*

$$\operatorname{Trace}((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})_{u\cdot v^{-1}}(\rho^{\perp}, \rho^{\perp}, Z)=0$$

for all $u, v\in P$, $Z\in \operatorname{Ker}(\omega\diamond\omega^{-1})_{u\cdot v^{-1}}$.

COROLLARY 2.15.
(i) *ω is Yang-Mills if and only if* ${}^{P\cdot P^{-1}}Ric(\operatorname{Ker} d\pi_{P\cdot P^{-1}}, \operatorname{Ker}\omega\diamond\omega^{-1})=0$.
(ii) *If* $(P\cdot P^{-1}, {}^{P\cdot P^{-1}}_{\omega\diamond\omega^{-1}}g)$ *is an Einstein space, then ω is Yang-Mills.*

The diagonal map $\Delta_M:(M, 2^M g)\to(M\times M, {}^{M\times M}g)$ is a totally geodesic isometric embedding so that Corollary 1.19 implies the following:

PROPOSITION 2.16. *$\varepsilon:(M, 2^M g)\to(P\cdot P^{-1}, {}^{P\cdot P^{-1}}_{\omega\diamond\omega^{-1}}g)$ is a totally geodesic isometric embedding for any connection form ω on P.*

COROLLARY 2.17.
(i) $\varepsilon^{*P\cdot P^{-1}}\mathcal{A}=0$, *equivalently, for any* $x\in M$, $X, Y\in T_xM$, $u\in\pi_P^{-1}(x)$,

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(+)}, {}^{\omega\diamond\omega^{-1}}Y^{(+)})=0\,.$$

(ii) *For any* $x\in M$, $X, Y\in T_xM$, $u\in\pi_P^{-1}(x)$,

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(-)}, {}^{\omega\diamond\omega^{-1}}Y^{(-)})=0\,.$$

Under the identification $\varepsilon^{-1}(P\times P^{-1})\cong P$, we observe the following:

PROPOSITION 2.18. *For any* $x\in M$, $X, Y\in T_xM$, $u\in\pi_P^{-1}(x)$,

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(+)}, {}^{\omega\diamond\omega^{-1}}Y^{(-)})=[u, -{}^{\omega}\Omega_u({}^{\omega}H_x^uX, {}^{\omega}H_x^uY)]\,.$$

COROLLARY 2.19. *ω is flat if and only if*

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(+)}, {}^{\omega\diamond\omega^{-1}}Y^{(-)})=0$$

for all $x\in M$, $X, Y\in T_xM$, $u\in\pi_P^{-1}(x)$, *equivalently,*

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\mathcal{A})((d\varepsilon)_xT_xM, \operatorname{Ker}(\omega\diamond\omega^{-1})_{\varepsilon(x)})=0, \quad \text{for all } x\in M.$$

By using the fact that the inversion $\imath$ is an isometry, we get

LEMMA 2.20. *For any* $x\in M$, $X, Y, Z\in T_xM$, $u\in\pi_P^{-1}(x)$,

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(+)}, {}^{\omega\diamond\omega^{-1}}Y^{(+)}, {}^{\omega\diamond\omega^{-1}}Z^{(+)})=0\,,$$

$$((\omega\diamond\omega^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}X^{(-)}, {}^{\omega\diamond\omega^{-1}}Y^{(-)}, {}^{\omega\diamond\omega^{-1}}Z^{(+)})=0\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(-)})=0\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(-)})=0\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(-)})$$
$$=[u,\ -({}^{P}\nabla^{\omega}\Omega)_u({}^{\omega}H^u_xX,\ {}^{\omega}H^u_xY,\ {}^{\omega}H^u_xZ)]\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(-)})$$
$$=[u,\ -({}^{P}\nabla^{\omega}\Omega)_u({}^{\omega}H^u_xX,\ {}^{\omega}H^u_xY,\ {}^{\omega}H^u_xZ)]\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(+)})$$
$$=[u,\ ({}^{P}\nabla^{\omega}\Omega)_u({}^{\omega}H^u_xX,\ {}^{\omega}H^u_xY,\ {}^{\omega}H^u_xZ)]\,,$$

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}X^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Y^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(+)})$$
$$=[u,\ -({}^{P}\nabla^{\omega}\Omega)_u({}^{\omega}H^u_xX,\ {}^{\omega}H^u_xY,\ {}^{\omega}H^u_xZ)]\,.$$

Proposition 1.16 and the above lemma imply the followings:

PROPOSITION 2.21. *For any $x\in M$, $u\in\pi_P^{-1}(x)$, the following three conditions are equivalent;*

(i) $((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})((d\varepsilon)_xT_xM,\ (d\varepsilon)_xT_xM,\ \mathrm{Ker}\,(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)})=0$,

(ii) $((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})(\mathrm{Ker}(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)},\ \mathrm{Ker}(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)},\ \mathrm{Ker}(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)})$
$=0$,

(iii) $({}^{P}\nabla^{\omega}\Omega)_u(\mathrm{Ker}\,\boldsymbol{\omega}_u,\ \mathrm{Ker}\,\boldsymbol{\omega}_u,\ \mathrm{Ker}\,\boldsymbol{\omega}_u)=0$.

COROLLARY 2.22. *ω is parallel if and only if*

$$((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})((d\varepsilon)_xT_xM,\ (d\varepsilon)_xT_xM,\ \mathrm{Ker}\,(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)})=0$$

for all $x\in M$.

Let $\{E_i\}_{i=1}^m$ be an orthonormal basis for $(T_xM, 2^Mg)$, then $\{{}^{\omega\Diamond\omega^{-1}}E_i^{(+)}, {}^{\omega\Diamond\omega^{-1}}E_i^{(-)}\}_{i=1}^m$ is an orthonormal basis for $(\mathrm{Ker}\,(\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})_{\varepsilon(x)}, {}^{P\cdot P^{-1}}_{\omega\Diamond\omega^{-1}}g)$ where $m=\dim M$. The following lemma is obtained from Lemma 2.20:

LEMMA 2.23. *For any $x\in M$, $Z\in T_xM$, $u\in\pi_P^{-1}(x)$,*

$$\sum_{i=1}^m((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}E_i^{(+)},\ {}^{\omega\Diamond\omega^{-1}}E_i^{(+)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(+)})=0\,,$$

$$\sum_{i=1}^m((\boldsymbol{\omega}\Diamond\boldsymbol{\omega}^{-1})\circ{}^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\Diamond\omega^{-1}}E_i^{(-)},\ {}^{\omega\Diamond\omega^{-1}}E_i^{(-)},\ {}^{\omega\Diamond\omega^{-1}}Z^{(+)})=0\,,$$

$$\sum_{i=1}^{m}((\boldsymbol{\omega}\diamondsuit\boldsymbol{\omega}^{-1})\circ^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}E_i^{(+)},\ {}^{\omega\diamond\omega^{-1}}E_i^{(+)},\ {}^{\omega\diamond\omega^{-1}}Z^{(-)})$$

$$=\left[u,\ \frac{1}{2}({}^{\omega}D^{*\omega}\Omega)_u({}^{\omega}H_x^u Z)\right],$$

$$\sum_{i=1}^{m}((\boldsymbol{\omega}\diamondsuit\boldsymbol{\omega}^{-1})\circ^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})({}^{\omega\diamond\omega^{-1}}E_i^{(-)},\ {}^{\omega\diamond\omega^{-1}}E_i^{(-)},\ {}^{\omega\diamond\omega^{-1}}Z^{(-)})$$

$$=\left[u,\ \frac{1}{2}({}^{\omega}D^{*\omega}\Omega)_u({}^{\omega}H_x^u Z)\right].$$

Proposition 1.9 and the above lemma imply the following:

PROPOSITION 2.24. *$\boldsymbol{\omega}$ is Yang-Mills if and only if*

$$\sum_{i=1}^{m}((\boldsymbol{\omega}\diamondsuit\boldsymbol{\omega}^{-1})\circ^{P\cdot P^{-1}}\nabla^{P\cdot P^{-1}}\mathcal{A})((d\varepsilon)_x E_i,\ (d\varepsilon)_x E_i,\ \mathrm{Ker}\,(\boldsymbol{\omega}\diamondsuit\boldsymbol{\omega}^{-1})_{\varepsilon(x)})=0$$

for all $x\in M$.

§ 3. Gauss Sections on the Joint Spaces

Let $G_r(T(P\cdot Q))\xrightarrow{\pi}P\cdot Q$ be the Grassmann bundle associated to $T(P\cdot Q)$ which typical fiber is the real Grassmann manifold of the r-dimensional planes, where $r=\dim G$. When we choose the Riemannian metric ${}^{P\cdot Q}_{\omega\diamond\eta}g$, $G_r(T(P\cdot Q))$ can be identified with the $O(r)\times O(m+n)$-quotient space $O(T(P\cdot Q),\ {}^{P\cdot Q}_{\omega\diamond\eta}g)/O(r)\times O(m+n)$ of the orthonormal frame bundle $O(T(P\cdot Q),\ {}^{P\cdot Q}_{\omega\diamond\eta}g)$ of $T(P\cdot Q)$ with respect to ${}^{P\cdot Q}_{\omega\diamond\eta}g$, where $m=\dim M$, $n=\dim N$. The quotient map of $O(T(P\cdot Q),\ {}^{P\cdot Q}_{\omega\diamond\eta}g)$ onto $G_r(T(P\cdot Q))$ is denoted by ζ, which is a right principal $O(r)\times O(m+n)$-fibration. Since the structure group of $\pi^{-1}T(P\cdot Q)$ is reduced to $O(r)\times O(m+n)$, the $O(r)\times O(m+n)$-submodule splitting $\boldsymbol{R}^{r+m+n}=(\boldsymbol{R}^r,\ 0_{m+n})\oplus(0_r,\ \boldsymbol{R}^{m+n})$ induces the vector subbundle splitting $\pi^{-1}T(P\cdot Q)=K\oplus K^{\perp}$ where

$$K=[O(T(P\cdot Q),\ {}^{P\cdot Q}_{\omega\diamond\eta}g)\xrightarrow{\zeta}G_r(T(P\cdot Q))]\times_{\sigma}(\boldsymbol{R}^r,\ 0_{m+n}),$$

$$K^{\perp}=[O(T(P\cdot Q),\ {}^{P\cdot Q}_{\omega\diamond\eta}g)\xrightarrow{\zeta}G_r(T(P\cdot Q))]\times_{\sigma}(0_r,\ \boldsymbol{R}^{m+n})\ \text{and}$$

$\sigma: O(r)\times O(m+n)\subset GL(\boldsymbol{R}^{r+m+n})$ is the natural linear representation of $O(r)\times O(m+n)$.

On the other hand, the Riemannian connection ${}^{P\cdot Q}\nabla$ of ${}^{P\cdot Q}_{\omega\diamond\eta}g$ induces the splitting $T(G_r(T(P\cdot Q)))=\mathrm{Ker}\,d\pi\oplus(\mathrm{Ker}\,d\pi)^{\perp}$ so that the differential $d\gamma$ of $\gamma=\gamma_{P\cdot Q}$ splits into $(d\gamma)^V$ and $(d\gamma)^H$ where the former is the *vertical differential* of γ with respect to ${}^{P\cdot Q}\nabla$ [11], [12]. γ is called a *horizontal section* with respect to ${}^{P\cdot Q}\nabla$ if $(d\gamma)^V\equiv0$. The *vertical energy density* of γ is the C^{∞}-function $\boldsymbol{e}^V(\gamma): P\cdot Q\to\boldsymbol{R}$ defined by $\boldsymbol{e}^V(\gamma)(u\cdot v)=\|(d\gamma)^V\|_{u\cdot v}^2$ $(u\in P,\ v\in Q)$.

Let $\mathcal{K}$ be the vector subbundle $\{(\kappa,\ -\kappa^{\dagger})\,|\,\kappa\in\mathrm{Hom}(K,\ K^{\perp})\}$ of $\mathrm{Hom}(K,\ K^{\perp})\oplus\mathrm{Hom}(K^{\perp},\ K)$ where $\kappa^{\dagger}$ is the adjoint of κ. C. M. Wood has introduced in [11] an isomorphism $\boldsymbol{I}=(\kappa, -\kappa^{\dagger})$: Ker $d\pi\cong\mathcal{K}$; $(d\zeta)_E A_E^{\#}\mapsto E\circ A_{\mathfrak{k}}\circ E^{-1}$ ($E\in O(T(P\cdot Q), {}^{P\cdot Q}_{\omega\diamond\eta}g)$, $A\in\mathfrak{o}(r+m+n)$) where $A_{\mathfrak{k}}$ is the $\mathfrak{k}$-component of A and $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{o}(r)\times\mathfrak{o}(m+n)$ in $\mathfrak{o}(r+m+n)$ with respect to the Killing-Cartan form of $O(r+m+n)$ which is denoted by $1/2\langle\!\langle\,,\rangle\!\rangle$. κ: Ker $d\pi\to\mathrm{Hom}(K, K^{\perp})$ and $\kappa^{\dagger}$: Ker $d\pi\to\mathrm{Hom}(K^{\perp}, K)$ are vector bundle isomorphisms. The metric is taken to be that derived from $1/2\langle\!\langle\,,\rangle\!\rangle$ under the $O(r)\times O(m+n)$-quotient map $O(r+m+n)\to G_r(\boldsymbol{R}^{r+m+n})$. Thus $\boldsymbol{I}$ is 2-homothetic and $\kappa^{\dagger}$ is an isometry [12]. Note that $\gamma^{-1}K\cong\mathrm{Ker}\, d\pi_{P\cdot Q}$, $\gamma^{-1}K^{\perp}\cong\mathrm{Ker}\,\omega\diamond\eta$,

$$\gamma^{-1}\,\mathrm{Hom}(K, K^{\perp})\cong\mathrm{Hom}(\mathrm{Ker}\, d\pi_{P\cdot Q}, \mathrm{Ker}\,\omega\diamond\eta)\quad\text{and}$$

$$\gamma^{-1}\,\mathrm{Hom}(K^{\perp}, K)\cong\mathrm{Hom}(\mathrm{Ker}\,\omega\diamond\eta, \mathrm{Ker}\, d\pi_{P\cdot Q})\,.$$

We denote also the induced vector bundle isomorphisms via γ by $\boldsymbol{I}: \gamma^{-1}\mathrm{Ker}\, d\pi\to\gamma^{-1}\mathcal{K}$, $\kappa: \gamma^{-1}\mathrm{Ker}\, d\pi\to\mathrm{Hom}(\mathrm{Ker}\, d\pi_{P\cdot Q}, \mathrm{Ker}\,\omega\diamond\eta)$ and $\kappa^{\dagger}: \gamma^{-1}\mathrm{Ker}\, d\pi\to\mathrm{Hom}(\mathrm{Ker}\,\omega\diamond\eta, \mathrm{Ker}\, d\pi_{P\cdot Q})$ for convenience of notation. $(d\gamma)^V$ is evaluated in $\mathrm{Hom}(\mathrm{Ker}\,\omega\diamond\eta, \mathrm{Ker}\, d\pi_{P\cdot Q})$ as the O'Neill's tensor ${}^{P\cdot Q}\mathcal{A}$:

PROPOSITION 3.1.
(i) $\boldsymbol{I}((d\gamma)^V Y)={}^{P\cdot Q}\mathcal{A}_Y=[W\mapsto{}^{P\cdot Q}\mathcal{A}(Y, W)]$, *for any* $u\in P$, $v\in Q$, $Y\in T_{u\cdot v}(P\cdot Q)$.
(ii) $2\boldsymbol{e}^V(\gamma)(u\cdot v)=\|{}^{P\cdot Q}\mathcal{A}\|^2_{u\cdot v}$, *for any* $u\in P$, $v\in Q$.

Proof. (i) By the Gauss's and the Weingarten's formulas,

$$({}^{P\cdot Q}\nabla_Y W^{\top})^{\perp}=({}^{P\cdot Q}\nabla_{Y^{\perp}} W^{\top})^{\perp},\quad ({}^{P\cdot Q}\nabla_Y W^{\perp})^{\top}=({}^{P\cdot Q}\nabla_{Y^{\perp}} W^{\perp})^{\top}$$

since each fiber is totally geodesic. From [12, Corollary 1.9],

$$\begin{aligned}\boldsymbol{I}((d\gamma)^V Y)&=[W\mapsto({}^{P\cdot Q}\nabla_Y W^{\perp})^{\top}+({}^{P\cdot Q}\nabla_Y W^{\top})^{\perp}]\\&=[W\mapsto({}^{P\cdot Q}\nabla_{Y^{\perp}} W^{\perp})^{\top}+({}^{P\cdot Q}\nabla_{Y^{\perp}} W^{\top})^{\perp}]=[W\mapsto{}^{P\cdot Q}\mathcal{A}(Y, W)]\,.\end{aligned}$$

(ii) It follows from (i).

COROLLARY 3.2.
(i) $\kappa^{\dagger}((d\gamma)^V Y)=\rho\circ{}^{P\cdot Q}\mathcal{A}_{Y^{\perp}}\circ\rho^{\perp}$, *for any* $u\in P$, $v\in Q$, $Y\in T_{u\cdot v}(P\cdot Q)$.
(ii) $\boldsymbol{e}^V(\gamma)(u\cdot v)=\|\rho\circ{}^{P\cdot Q}\mathcal{A}_{\rho^{\perp}}\circ\rho^{\perp}\|^2_{u\cdot v}$, *for any* $u\in P$, $v\in Q$.

PROPOSITION 3.3.
(i) *For any* $u\in P$, $v\in Q$, $Y\in T_{u\cdot v}(P\cdot Q)$, $W\in\mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v}$,

$$(\omega\diamond\eta)(\kappa^{\dagger}((d\gamma)^V Y)W)=\left[(u, v), -\frac{1}{2}((p_P^*)^{\omega}\Omega+(p_Q^*)^{\eta}\Omega)(\bar{Y}, \bar{W})\right].$$

(ii) $\boldsymbol{e}^V(\gamma)(u\cdot v)=1/2(\|{}^{\omega}\Omega\|^2_u+\|{}^{\eta}\Omega\|^2_v)$, *for any* $u\in P$, $v\in Q$.

Proof. (i) It follows from Proposition 2.2 and Corollary 3.2. (ii) Let $\{{}^M E_i\}_{i=1}^m$, $\{{}^N E_j\}_{j=1}^n$ be orthonormal bases for $(T_x M, {}^M g)$, $(T_y N, {}^N g)$, respectively, we set

$$^{P\cdot Q}E_i=(dj_{P\cdot Q})_{(u,v)}({}^{\omega}H_x^{uM}E_i,\ 0_v),\qquad {}^{P\cdot Q}E_{m+j}=(dj_{P\cdot Q})_{(u,v)}(0_u,\ {}^{\eta}H_y^{vN}E_j).$$

$\{^{P\cdot Q}E_\alpha\}_{\alpha=1}^{m+n}$ is an orthonormal basis for $(\mathrm{Ker}(\omega\diamond\eta)_{u\cdot v},\ {}^{P\cdot Q}g)$. $(d\gamma)_{u\cdot v}^V\mathrm{Ker}(d\pi_{P\cdot Q})_{u\cdot v}=0$ so that

$$e^V(\gamma)(u\cdot v)=\sum_{\alpha=1}^{m+n}\|(d\gamma)^{V\,P\cdot Q}E_\alpha\|_{u\cdot v}^2=\sum_{\alpha=1}^{m+n}\|\kappa^\dagger((d\gamma)^{V\,P\cdot Q}E_\alpha)\|_{u\cdot v}^2$$

$$=\sum_{\alpha,\beta=1}^{m+n}\|\kappa^\dagger((d\gamma)^{V\,P\cdot Q}E_\alpha)^{P\cdot Q}E_\beta\|_{u\cdot v}^2=2\sum_{\alpha,\beta=1}^{m+n}\|(\omega\diamond\eta)(\kappa^\dagger((d\gamma)^{V\,P\cdot Q}E_\alpha)^{P\cdot Q}E_\beta)\|_{u\cdot v}^2$$

$$=\frac{1}{2}\sum_{\alpha,\beta=1}^{m+n}\|((p_P^*)^{\omega}\Omega+(p_Q^*)^{\eta}\Omega)((^{P\cdot Q}E_\alpha)^-,\ (^{P\cdot Q}E_\beta)^-)\|_{u\cdot v}^2=\frac{1}{2}(\|^{\omega}\Omega\|_u^2+\|^{\eta}\Omega\|_v^2).$$

THEOREM 3.4. *ω and η are both flat if and only if $\gamma_{P\cdot Q}$ is a horizontal section with respect to ${}^{P\cdot Q}\nabla$.*

COROLLARY 3.5. *ω is flat if and only if $\gamma_{P\cdot P^{-1}}$ is a horizontal section with respect to ${}^{P\cdot P^{-1}}\nabla$.*

PROPOSITION 3.6.
(i) *For any $x\in M$, $u\in\pi_P^{-1}(x)$, $X, Y\in T_xM$,*

$$(\omega\diamond\omega^{-1})(\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(+)})^{\omega\diamond\omega^{-1}}Y^{(+)})=0,$$

$$(\omega\diamond\omega^{-1})(\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(+)})^{\omega\diamond\omega^{-1}}Y^{(-)})=[u,\ -{}^{\omega}\Omega({}^{\omega}H_x^uX,\ {}^{\omega}H_x^uY)],$$

$$(\omega\diamond\omega^{-1})(\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(-)})^{\omega\diamond\omega^{-1}}Y^{(+)})=[u,\ {}^{\omega}\Omega({}^{\omega}H_x^uX,\ {}^{\omega}H_x^uY)],$$

$$(\omega\diamond\omega^{-1})(\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(-)})^{\omega\diamond\omega^{-1}}Y^{(-)})=0.$$

(ii) $\|(d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(+)}\|_{\varepsilon(x)}^2=\|(d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(-)}\|_{\varepsilon(x)}^2$ *for any $x\in M$, $X\in T_xM$.*

(iii) $e^V(\gamma_{P\cdot P^{-1}})(\varepsilon(x))=2\sum_m^{i=1}\|(d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}E_i^{(+)}\|_{\varepsilon(x)}^2=\|^{\omega}\Omega\|_u^2$ *for any $x\in M$, $u\in\pi_P^{-1}(x)$.*

Proof. (i) It follows from Corollary 2.17. (ii) From (i), we get

$$\|(d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(+)}\|_{\varepsilon(x)}^2=\sum_{i=1}^{m}\|\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(+)})^{\omega\diamond\omega^{-1}}E_i^{(-)}\|_{\varepsilon(x)}^2$$

$$=\frac{1}{2}\sum_{i=1}^{m}\|-{}^{\omega}\Omega({}^{\omega}H_x^uX,\ {}^{\omega}H_x^uE_i)\|_{\varepsilon(x)}^2=\frac{1}{2}\sum_{i=1}^{m}\|{}^{\omega}\Omega({}^{\omega}H_x^uX,\ {}^{\omega}H_x^uE_i)\|_{\varepsilon(x)}^2$$

$$=\sum_{i=1}^{m}\|\kappa^\dagger((d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(-)})^{\omega\diamond\omega^{-1}}E_i^{(+)}\|_{\varepsilon(x)}^2=\|(d\gamma_{P\cdot P^{-1}})^{V\,\omega\diamond\omega^{-1}}X^{(-)}\|_{\varepsilon(x)}^2.$$

(iii) It follows from (ii).

Let ∇^V be the induced connection in $\operatorname{Ker} d\pi$ from ${}^{P\cdot Q}\nabla$ and ${}^{\pi}({}^{P\cdot Q}\nabla)$ be the induced connection in $\pi^{-1}T(P\cdot Q)$ from ${}^{P\cdot Q}\nabla$ and ${}^{\mathcal{K}}\nabla$ be the induced connection in $\mathcal{K}$ from ${}^{\pi}({}^{P\cdot Q}\nabla)$. Notice that $\boldsymbol{I}: (\operatorname{Ker} d\pi, \nabla^V) \to (\mathcal{K}, {}^{\mathcal{K}}\nabla)$ is *not* connection-preserving but $\boldsymbol{I}: (\gamma^{-1}\operatorname{Ker} d\pi, {}^{\gamma}\nabla^V) \to (\gamma^{-1}\mathcal{K}, {}^{\gamma^{-1}\mathcal{K}}\nabla)$ is connection-preserving, where ${}^{\gamma}\nabla^V$ (resp. ${}^{\gamma^{-1}\mathcal{K}}\nabla$) is the induced connection via γ in $\gamma^{-1}\operatorname{Ker} d\pi$ (resp. $\gamma^{-1}\mathcal{K}$) from ∇^V (resp. ${}^{\mathcal{K}}\nabla$) [12, Theorem 1.5, Corollary 1.6].

PROPOSITION 3.7. *For any* $X, Y \in \mathfrak{X}(P\cdot Q)$,
(i) $\boldsymbol{I}({}^{\gamma}\nabla_X^V((d\gamma)^V Y)) = {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y$,
(ii) $\kappa^{\dagger}({}^{\gamma}\nabla_X^V((d\gamma)^V Y)) = \rho \circ {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y \circ \rho^{\perp}$.

LEMMA 3.8. *For any* $X, Y, W \in \mathfrak{X}(P\cdot Q)$,

$$(\rho \circ ({}^{P\cdot Q}\nabla_X^{P\cdot Q}\mathcal{A})_Y \circ \rho^{\perp})(W) = (\rho \circ {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y \circ \rho^{\perp})(W) - {}^{P\cdot Q}\mathcal{A}(({}^{P\cdot Q}\nabla_X Y)^{\perp}, W^{\perp}).$$

Let ${}^{\gamma}\nabla^V(d\gamma)^V$ be the *vertical second fundamental form* of γ with respect to ${}^{P\cdot Q}\nabla$ (definition in [11]). γ will be called a *covariantly horizontal section* with respect to ${}^{P\cdot Q}\nabla$ if ${}^{\gamma}\nabla^V(d\gamma)^V \equiv 0$.

PROPOSITION 3.9. *For any* $u \in P$, $v \in Q$, $X, Y, W \in T_{u\cdot v}(P\cdot Q)$,
(i) $\kappa^{\dagger}({}^{\gamma}\nabla^V(d\gamma)^V(X, Y))(W) = ({}^{P\cdot Q}\nabla^{P\cdot Q}\mathcal{A}(X, Y, W^{\perp}))^{\top}$,
(ii) $\kappa^{\dagger}({}^{\gamma}\nabla^V(d\gamma)^V(X^{\top}, Y^{\top}))(W) = 0$.

Proof. (i) From Proposition 3.7 and Lemma 3.8,

$$\begin{aligned}
\kappa^{\dagger}({}^{\gamma}\nabla^V(d\gamma)^V(X, Y))(W) &= \kappa^{\dagger}({}^{\gamma}\nabla_X^V((d\gamma)^V Y))(W) - \kappa^{\dagger}((d\gamma)^V({}^{P\cdot Q}\nabla_X Y))(W) \\
&= (\rho \circ {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y \circ \rho^{\perp})(W) - (\rho \circ {}^{P\cdot Q}\mathcal{A}_{{}^{P\cdot Q}\nabla_X Y} \circ \rho^{\perp})(W) \\
&= (\rho \circ {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y \circ \rho^{\perp})(W) - ({}^{P\cdot Q}\mathcal{A}({}^{P\cdot Q}\nabla_X Y, W^{\perp}))^{\top} \\
&= (\rho \circ {}^{\gamma^{-1}\mathcal{K}}\nabla_X^{P\cdot Q}\mathcal{A}_Y \circ \rho^{\perp})(W) - {}^{P\cdot Q}\mathcal{A}(({}^{P\cdot Q}\nabla_X Y)^{\perp}, W^{\perp}) \\
&= (\rho \circ ({}^{P\cdot Q}\nabla_X^{P\cdot Q}\mathcal{A})_Y \circ \rho^{\perp})(W).
\end{aligned}$$

(ii) It follows from (i) and [9, Lemma 4].
Notice that ${}^{\gamma}\nabla^V(d\gamma)^V(X^{\top}, Y^{\perp})$ and ${}^{\gamma}\nabla^V(d\gamma)^V(X^{\perp}, Y^{\top})$ de *not* vanish, in general.

COROLLARY 3.10. *For any* $u \in P$, $v \in Q$, $X, Y \in T_{u\cdot v}(P\cdot Q)$, $W \in \operatorname{Ker}(\omega \diamond \eta)_{u\cdot v}$,

$$\begin{aligned}
&(\omega \diamond \eta)_{u\cdot v}(\kappa^{\dagger}({}^{\gamma}\nabla^V(d\gamma)^V(X, Y))(W)) \\
&\quad = \left[(u, v), -\frac{1}{2}((p_P^*)^P\nabla^{\omega}\Omega + (p_Q^*)^Q\nabla^{\eta}\Omega)(\bar{X}, \bar{Y}, \bar{W})\right].
\end{aligned}$$

Proof. It follows from Propositions 2.7 and 3.9.

THEOREM 3.11.
(i) *ω and η are both parallel if and only if*

$$^{\gamma}\nabla(d\gamma)^{V}(\mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v},\ \mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v})=0, \quad \textit{for all } u\in P,\ v\in Q.$$

(ii) *If* $\gamma_{P\cdot Q}$ *is a covariantly horizontal section with respect to* $^{P\cdot Q}\nabla$, *then* ω *and* η *are both parallel.*

COROLLARY 3.12.
(i) ω *is parallel if and only if*

$$^{\gamma_{P\cdot P^{-1}}}\nabla^{V}(d\gamma_{P\cdot P^{-1}})^{V}(\mathrm{Ker}\,(\omega\diamond\omega^{-1})_{u\cdot v^{-1}},\ \mathrm{Ker}\,(\omega\diamond\omega^{-1})_{u\cdot v^{-1}})=0 \quad \textit{for all } u,\ v\in P.$$

(ii) *If* $\gamma_{P\cdot P^{-1}}$ *is a covariantly horizontal section with respect to* $^{P\cdot P^{-1}}\nabla$, *then* ω *is parallel.*

Let $\tau^{V}(\gamma)=\mathrm{Trace}^{\gamma}\nabla^{V}(d\gamma)^{V}$ be the *vertical tension field* of γ with respect to $^{P\cdot Q}\nabla$. γ is called a *harmonic section* (or *vertical harmonic map*) with respect to $^{P\cdot Q}\nabla$ if γ satisfies the *harmonic section equation*: $\tau^{V}(\gamma)\equiv 0$.

PROPOSITION 3.13. *For any* $u\in P$, $v\in Q$, $W\in\mathrm{Ker}\,(\omega\diamond\eta)_{u\cdot v}$,

$$(\omega\diamond\eta)_{u\cdot v}(\kappa^{\dagger}(\tau^{V}(\gamma))(W)=\Big[(u,\ v),\ \frac{1}{2}((p_P^*)^{\omega}D^{*\omega}\Omega+(p_Q^*)^{\eta}D^{*\eta}\Omega)(\overline{W}))\Big].$$

Proof. From Proposition 3.9 (ii),

$$\begin{aligned}\kappa^{\dagger}(\tau^{V}(\gamma))(W)&=\mathrm{Trace}\,\kappa^{\dagger}(^{\gamma}\nabla^{V}(d\gamma)^{V}(\rho,\ \rho))(W)+\mathrm{Trace}\,\kappa^{\dagger}(^{\gamma}\nabla^{V}(d\gamma)^{V}(\rho^{\perp},\ \rho^{\perp}))(W)\\ &=\mathrm{Trace}\,\kappa^{\dagger}(^{\gamma}\nabla^{V}(d\gamma)^{V}(\rho^{\perp},\ \rho^{\perp}))(W)\end{aligned}$$

so that

$$\begin{aligned}(\omega\diamond\eta)(\kappa^{\dagger}(\tau^{V}(\gamma))(W))&=\mathrm{Trace}\Big[(u,\ v),\ -\frac{1}{2}((p_P^*)^{P}\nabla^{\omega}\Omega+(p_Q^*)^{Q}\nabla^{\eta}\Omega)(\bar{\rho}^{\perp},\ \bar{\rho}^{\perp},\ \overline{W})\Big]\\ &=\Big[(u,\ v),\ \frac{1}{2}((p_P^*)^{\omega}D^{*\omega}\Omega+(p_Q^*)^{\eta}D^{*\eta}\Omega)(\overline{W})\Big].\end{aligned}$$

THEOREM 3.14. ω *and* η *are both Yang-Mills if and only if* $\gamma_{P\cdot Q}$ *is a harmonic section with respect to* $^{P\cdot Q}\nabla$.

COROLLARY 3.15. ω *is Yang-Mills if and only if* $\gamma_{P\cdot P^{-1}}$ *is a harmonic section with respect to* $^{P\cdot P^{-1}}\nabla$.

§ 4. Gauss Sections along Object Inclusion Map

In this section, we prepare a general argument on the pull-back Gauss sections via a C^{∞}-map and an application to the object inclusion map $\varepsilon: M\to P\cdot P^{-1}$.

Let M and L be C^{∞}-manifolds and $\varphi: M\to L$ be a C^{∞}-map. For a real C^{∞}-vector bundle $\mathcal{E}\xrightarrow{\pi_{\mathcal{E}}} L$ of rank $r+s+t(<+\infty)$, we denote the induced vector bundle from $\mathcal{E}$ via φ by $\varphi^{-1}\mathcal{E}\xrightarrow{\varphi_{\pi\mathcal{E}}} M$. The Grassmann bundle of the r-dimensional planes associated to $\mathcal{E}$ is denoted by $G_r(\mathcal{E})\xrightarrow{\pi_{G_r(\mathcal{E})}} L$. Let $\mathcal{F}$ be a vector subbundle with rank r of $\mathcal{E}$ and $\gamma_{\mathcal{F}}=[L\ni l\mapsto\mathcal{F}_l\subset\mathcal{E}_l]$ be the corresponding Gauss section into $G_r(\mathcal{E})$.

The induced fiber bundle $\varphi^{-1}G_r(\mathcal{E})\xrightarrow{\varphi_{\pi G_r(\mathcal{E})}} M$ is naturally identified with $G_r(\varphi^{-1}\mathcal{E})\xrightarrow{\pi_{G_r(\varphi^{-1}\mathcal{E})}} M$. Let $\bar{\varphi}: G_r(\varphi^{-1}\mathcal{E})\to G_r(\mathcal{E})$ be the induced fiber bundle homomorphism with $\pi_{\mathcal{E}}\circ\bar{\varphi}=\varphi\circ{}^{\varphi}\pi_{\mathcal{E}}\cdot\varphi^{-1}\mathcal{F}$ is naturally identified with a vector subbundle of $\varphi^{-1}\mathcal{E}$ so that $\gamma_{\varphi^{-1}\mathcal{F}}$ maps M into $G_r(\varphi^{-1}\mathcal{E})$ and $\bar{\varphi}\circ\gamma_{\varphi^{-1}\mathcal{F}}=\gamma_{\mathcal{F}}\circ\varphi$.

Let $(\mathcal{E}, h_{\mathcal{E}})\xrightarrow{\pi_{\mathcal{E}}} L$ be a real C^∞-vector bundle with a fiber metric $h_{\mathcal{E}}$ and $O(\mathcal{E}, h_{\mathcal{E}})\xrightarrow{\pi_{O(\mathcal{E},h_{\mathcal{E}})}} L$ be the orthonormal frame bundle of $(\mathcal{E}, h_{\mathcal{E}})$. For the induced vector bundle $(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})\xrightarrow{\varphi_{\pi\mathcal{E}}} M$, the orthonormal frame bundle $O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})$ is naturally identified with $\varphi^{-1}O(\mathcal{E}, h_{\mathcal{E}})\xrightarrow{\varphi_{\pi O(\mathcal{E},h)}} M$. Let ${}^{\mathcal{E}}\zeta$ (resp. ${}^{\varphi^{-1}\mathcal{E}}\zeta$) be the quotient map $O(\mathcal{E}, h_{\mathcal{E}})\to G_r(\mathcal{E})$ (resp. $O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})\to G_r(\varphi^{-1}\mathcal{E})$), which is a right principal $O(r)\times O(s+t)$-fibration. The pull-back vector bundle $(\pi_{G_r(\mathcal{E})})^{-1}\mathcal{E}\to G_r(\mathcal{E})$ (resp. $(\pi_{G_r(\varphi^{-1}\mathcal{E})})^{-1}(\varphi^{-1}\mathcal{E})\to G_r(\varphi^{-1}\mathcal{E})$) splits via $\pi_{G_r(\mathcal{E})}$ (resp. $\pi_{G_r(\varphi^{-1}\mathcal{E})}$) into ${}^{\mathcal{E}}K\oplus{}^{\mathcal{E}}K^{\perp}$ (resp. ${}^{\varphi^{-1}\mathcal{E}}K\oplus{}^{\varphi^{-1}\mathcal{E}}K^{\perp}$) where

$$ {}^{\mathcal{E}}K=[O(\mathcal{E}, h_{\mathcal{E}})\xrightarrow{{}^{\mathcal{E}}\zeta} G_r(\mathcal{E})]\times_{\sigma}(\boldsymbol{R}^r, 0_{s+t}),\quad {}^{\mathcal{E}}K^{\perp}=[O(\mathcal{E}, h_{\mathcal{E}})\xrightarrow{{}^{\mathcal{E}}\zeta} G_r(\mathcal{E})]\times_{\sigma}(0_r, \boldsymbol{R}^{s+t}) $$

(resp. ${}^{\varphi^{-1}\mathcal{E}}K=[O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})\xrightarrow{{}^{\varphi^{-1}\mathcal{E}}\zeta} G_r(\varphi^{-1}\mathcal{E})]\times_{\sigma}(\boldsymbol{R}^r, 0_{s+t})$,

$$ {}^{\varphi^{-1}\mathcal{E}}K^{\perp}=[O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})\xrightarrow{{}^{\varphi^{-1}\mathcal{E}}\zeta} G_r(\varphi^{-1}\mathcal{E})]\times_{\sigma}(0_r, \boldsymbol{R}^{s+t})) $$

and $\sigma: O(r)\times O(s+t)\subset GL(\boldsymbol{R}^{r+s+t})$ is the natural linear representation of $O(r)\times O(s+t)$. There are natural vector bundle isomorphisms; $\bar{\varphi}^{-1}({}^{\mathcal{E}}K)\cong{}^{\varphi^{-1}\mathcal{E}}K$, $\bar{\varphi}^{-1}({}^{\mathcal{E}}K^{\perp})\cong{}^{\varphi^{-1}\mathcal{E}}K^{\perp}$, $\bar{\varphi}^{-1}\,\mathrm{End}\,((\pi_{G_r(\mathcal{E})})^{-1}\mathcal{E})\cong\mathrm{End}((\pi_{G_r(\varphi^{-1}\mathcal{E})})^{-1}(\varphi^{-1}\mathcal{E}))$.

As in § 3, we set ${}^{\mathcal{E}}\mathcal{K}=\{(\kappa, -\kappa^{\dagger})\,|\,\kappa\in\mathrm{Hom}({}^{\mathcal{E}}K, {}^{\mathcal{E}}K^{\perp})\}$,

$$ {}^{\varphi^{-1}\mathcal{E}}\mathcal{K}=\{(\kappa, -\kappa^{\dagger})\,|\,\kappa\in\mathrm{Hom}({}^{\varphi^{-1}\mathcal{E}}K, {}^{\varphi^{-1}\mathcal{E}}K^{\perp})\}. $$

Under the natural identification $\bar{\varphi}^{-1}({}^{\mathcal{E}}\mathcal{K})\cong{}^{\varphi^{-1}\mathcal{E}}\mathcal{K}$, there is no confusion when we write $\bar{\varphi}: {}^{\varphi^{-1}\mathcal{E}}\mathcal{K}\to{}^{\mathcal{E}}\mathcal{K}$. ${}^{\mathcal{E}}\boldsymbol{I}$ and ${}^{\varphi^{-1}\mathcal{E}}\boldsymbol{I}$ are defined by

$$ {}^{\mathcal{E}}\boldsymbol{I}:\ \mathrm{Ker}\, d\pi_{G_p(\mathcal{E})}\cong{}^{\mathcal{E}}\mathcal{K}\,;\ (d^{\mathcal{E}}\zeta)_E A_E^{\#}\mapsto E\circ A_{\mathfrak{k}}\circ E^{-1}, $$

$$ {}^{\varphi^{-1}\mathcal{E}}\boldsymbol{I}:\ \mathrm{Ker}\, d\pi_{G_r(\varphi^{-1}\mathcal{E})}\cong{}^{\varphi^{-1}\mathcal{E}}\mathcal{K}\,;\ (d^{\varphi^{-1}\mathcal{E}}\zeta)_{\tilde{E}} A_{\tilde{E}}^{\#}\mapsto\tilde{E}\circ A_{\mathfrak{k}}\circ\tilde{E}^{-1} $$

$(E\in O(\mathcal{E}, h_{\mathcal{E}}),\ \tilde{E}\in O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}}),\ A\in\mathfrak{o}(r+s+t))$, respectively, where $A_{\mathfrak{k}}$ is the $\mathfrak{k}$-component of A and $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{o}(r)\times\mathfrak{o}(s+t)$ in $\mathfrak{o}(r+s+t)$ with respect to the Killing-Cartan form of $O(r+s+t)$. C^∞-fiber bundle homomorphism $\bar{\varphi}: G_r(\varphi^{-1}\mathcal{E})\to G_r(\mathcal{E})$ maps each fiber of $G_r(\varphi^{-1}\mathcal{E})$ onto that of $G_r(\mathcal{E})$

so that the differential $d\bar{\varphi}: TG_r(\varphi^{-1}\mathcal{E}) \to TG_r(\mathcal{E})$ maps $\operatorname{Ker} d\pi_{r(\varphi^{-1}\mathcal{E})}$ to $\operatorname{Ker} d\pi_{G_r(\mathcal{E})}$. Let $(d\bar{\varphi})^{\mathcal{V}}: \operatorname{Ker} d\pi_{G_r(\varphi^{-1}\mathcal{E})} \to \operatorname{Ker} d\pi_{G_r(\mathcal{E})}$ be the restriction of $d\bar{\varphi}$ to $\operatorname{Ker} d\pi_{G_r(\varphi^{-1}\mathcal{E})}$. Then ${}^{\mathcal{E}}\boldsymbol{I} \circ (d\bar{\varphi})^{\mathcal{V}} = \bar{\varphi}_\kappa \circ {}^{\varphi^{-1}\mathcal{E}}\boldsymbol{I}$.

The induced linear homomorphisms between C^∞-sections are denoted by ${}^{\mathcal{E}}\boldsymbol{I}$, ${}^{\varphi^{-1}\mathcal{E}}\boldsymbol{I}$, $\bar{\varphi}_\kappa$ and $(d\bar{\varphi})^{\mathcal{V}}$.

Let $(\mathcal{E}, h_{\mathcal{E}}, {}^{\mathcal{E}}\nabla, {}^{\mathcal{E}\nabla}\omega)$ be a system of a C^∞-vector bundle with a fiber metric $h_{\mathcal{E}}$, a covariant differentiation ${}^{\mathcal{E}}\nabla$ compatible to h and the connection form ${}^{\mathcal{E}\nabla}\omega$ of ${}^{\mathcal{E}}\nabla$. The induced system via φ is denoted by $(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}}, {}^{\varphi}({}^{\mathcal{E}}\nabla), {}^{\varphi({}^{\mathcal{E}}\nabla)}\omega)$. Notice that

$$(d\bar{\varphi})_{\varphi^{-1}\mathcal{E}\zeta(\tilde{E})}(\operatorname{Ker} {}^{\varphi({}^{\mathcal{E}}\nabla)}\omega_{\varphi^{-1}\mathcal{E}\zeta(\tilde{E})}) \subset \operatorname{Ker} {}^{\mathcal{E}\nabla}\omega_{\bar{\varphi}(\varphi^{-1}\mathcal{E}\zeta(\tilde{E}))} \quad \text{for any } \tilde{E} \in O(\varphi^{-1}\mathcal{E}, {}^{\varphi}h_{\mathcal{E}})$$

$$\text{and} \quad (d\bar{\varphi})^{\mathcal{V}} \circ (d\gamma_{\varphi^{-1}\mathcal{F}})^V = (d\gamma_{\mathcal{F}})^V \circ d\varphi \,.$$

PROPOSITION 4.1.

$$(d\gamma_{\mathcal{F}})^V_{\varphi(x)}((d\varphi)_x T_x M) = 0 \qquad \textit{for all } x \in M$$

if and only if $\gamma_{\varphi^{-1}\mathcal{F}}$ *is a horizontal section with respect to* ${}^{\varphi}({}^{\mathcal{E}}\nabla)$.

In the case of $L = P \cdot P^{-1}$, $\mathcal{F} = \operatorname{Ker} d\pi_{P \cdot P^{-1}}$ and $\varphi = \varepsilon$, we have

COROLLARY 4.2.

$$(d\gamma_{P\cdot P^{-1}})^V_{\varepsilon(x)}((d\varepsilon)_x T_x M) = 0 \qquad \textit{for all } x \in M$$

if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ *is a horizontal section with respect to* ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.

From Proposition 3.6 and the above corollary, we have

THEOREM 4.3. ω *is flat if and only if* ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ *is a horizontal section with respect to* ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.

Henceforth let $(M, 2^M g)$ be a Riemannian manifold. Proposition 3.6 implies that:

PROPOSITION 4.4.

$$\boldsymbol{e}^V({}^{\varepsilon}\gamma_{P\cdot P^{-1}})(x) = \frac{1}{2}\boldsymbol{e}^V(\gamma_{P\cdot P^{-1}})(\varepsilon(x)) \qquad \textit{for any } x \in M.$$

Let ${}^L g$ be a Riemmanian metric on L. $G_r(\mathcal{E})$ (resp. $G_r(\varphi^{-1}\mathcal{E})$) has the horizontally lifted metric by ${}^{\mathcal{E}\nabla}\omega$ (resp. ${}^{\varphi({}^{\mathcal{E}}\nabla)}\omega$) and its Riemannian connection ∇ (resp. ${}^{\varphi}\nabla$). Let ∇^V (resp. ${}^{\varphi}\nabla^V$) be the induced connection in $\operatorname{Ker} d\pi_{G_r(\mathcal{E})}$ (resp. $\operatorname{Ker} d\pi_{G_r(\varphi^{-1}\mathcal{E})}$) from ∇ (resp. ${}^{\varphi}\nabla$) and ${}^{\gamma\mathcal{F}}\nabla^V$ (resp. ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla^V$) be the pull-back connection via $\gamma_{\mathcal{F}}$ (resp. $\gamma_{\varphi^{-1}\mathcal{F}}$) from ∇^V (resp. ∇^{φ}).

Note that ${}^{\varphi}({}^{\gamma\mathcal{F}}({}^{\mathcal{E}}\boldsymbol{I})) = {}^{\gamma_{\varphi^{-1}\mathcal{F}}}(\bar{\varphi}({}^{\mathcal{E}}\boldsymbol{I}))$ is connection-preserving [12, Theorem 8(2)]. Let ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{\mathcal{V}}$ be the induced linear isomorphism between C^∞-sections via $\gamma_{\varphi^{-1}\mathcal{F}}$.

LEMMA 4.5. ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}\circ{}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla^{V}={}^{\gamma_{\varphi^{-1}\mathcal{F}}}(\bar{\varphi}(\nabla^{V}))\circ{}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}$.

By a straightforward computation using the above lemma, we have the following formulas analogous to [2, Proposition 2.20]:

PROPOSITION 4.6.

(i) ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}({}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla^{V}(d\gamma_{\varphi^{-1}\mathcal{F}})^{V}(X, Y))$

$$=(d\gamma_{\mathcal{F}})^{V}({}^{\varphi}({}^{L}\nabla)d\varphi)_{x}(X, Y))+{}^{\gamma_{\mathcal{F}}}\nabla^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)_{x}X, (d\varphi)_{x}Y)$$

for any $x\in M$, $X, Y\in T_{x}M$, *where* ${}^{L}\nabla$ *is the Riemmanian connection of* ${}^{L}g$.

(ii) ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}(\tau^{V}(\gamma_{\varphi^{-1}\mathcal{F}})_{x})$

$$=(d\gamma_{\mathcal{F}})_{x}^{V}(\tau(\varphi)_{x})+\sum_{i=1}^{m}{}^{\gamma_{\mathcal{F}}}\nabla^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)_{x}E_{i}, (d\varphi)_{x}E_{i})$$

for any $x\in M$, *where* $\{E_{i}\}_{i=1}^{m}$ *is an orthonormal basis for* $(T_{x}M, 2^{M}g)$.

Proof. (i) Extend X and Y to local vector fields. The above lemma implies that

$$\begin{aligned}
&{}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}({}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla^{V}(d\gamma_{\varphi^{-1}\mathcal{F}})^{V}(X, Y))\\
&={}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}({}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla_{X}^{V}((d\gamma_{\varphi^{-1}\mathcal{F}})^{V}Y))-{}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}((d\gamma_{\varphi^{-1}\mathcal{F}})^{V}({}^{M}\nabla_{X}Y))\\
&={}^{\varphi^{-1}\gamma_{\mathcal{F}}}\nabla_{X}^{V}((d\gamma_{\mathcal{F}})^{V}((d\varphi)Y))-(d\gamma_{\mathcal{F}})^{V}((d\varphi)({}^{M}\nabla_{X}Y))\\
&={}^{\varphi^{-1}\gamma_{\mathcal{F}}}\nabla_{X}^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)Y)+(d\gamma_{\mathcal{F}})^{V}({}^{\varphi}({}^{L}\nabla)_{X}((d\varphi)Y))-(d\gamma_{\mathcal{F}})^{V}((d\varphi)({}^{M}\nabla_{X}Y))\\
&={}^{\gamma_{\mathcal{F}}}\nabla_{(d\varphi)X}^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)Y)+(d\gamma_{\mathcal{F}})^{V}({}^{\varphi}({}^{L}\nabla)_{X}((d\varphi)Y)-(d\varphi)({}^{M}\nabla_{X}Y))\\
&={}^{\gamma_{\mathcal{F}}}\nabla^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)X, (d\varphi)Y)+(d\gamma_{\mathcal{F}})^{V}(({}^{\varphi^{-1}L}\nabla d\varphi)(X, Y)).
\end{aligned}$$

(ii) It follows from (i).

COROLLARY 4.7. *If* φ *is totally geodesic, then*

(i) ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}({}^{\gamma_{\varphi^{-1}\mathcal{F}}}\nabla^{V}(d\gamma_{\varphi^{-1}\mathcal{F}})^{V}(X, Y))={}^{\gamma_{\mathcal{F}}}\nabla^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)_{x}X, (d\varphi)_{x}Y))$

for any $x\in M$, $X, Y\in T_{x}M$,

(ii) ${}^{\gamma_{\varphi^{-1}\mathcal{F}}}(d\bar{\varphi})^{CV}(\tau^{V}(\gamma_{\varphi^{-1}\mathcal{F}})_{x})=\sum_{i=1}^{m}{}^{\gamma_{\mathcal{F}}}\nabla^{V}(d\gamma_{\mathcal{F}})^{V}((d\varphi)_{x}E_{i}, (d\varphi)_{x}E_{i})$

for any $x\in M$.

Even if φ is totally geodesic, vertical harmonicity of $\gamma_{\varphi^{-1}\mathcal{F}}$ generally fails to inherit from that of $\gamma_{\mathcal{F}}$. But an exceptional success lies in the case of $\varphi=\varepsilon$, $\mathcal{F}=\operatorname{Ker} d\pi_{P,P-1}$.

PROPOSITION 4.8.

(i) ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}(d\bar{\varepsilon})^{CV}({}^{\varepsilon}\gamma_{P\cdot P^{-1}}\nabla^{V}(d^{\varepsilon}\gamma_{P\cdot P^{-1}})^{V}(X, Y))={}^{\gamma_{P\cdot P^{-1}}}\nabla^{V}(d\gamma_{P\cdot P^{-1}})^{V}(d\varepsilon)_{x}X, (d\varepsilon)_{x}Y)$ *for any* $x\in M$, $X, Y\in T_{x}M$.

(ii) $${}^{\varepsilon}\gamma_{P\cdot P^{-1}}(d\bar{\varepsilon})^{CV}(\tau^{V}({}^{\varepsilon}\gamma_{P\cdot P^{-1}})_{x})=\sum_{\iota=1}^{m}{}^{\gamma_{P\cdot P^{-1}}}\nabla^{V}(d\gamma_{P\cdot P^{-1}})^{V}((d\varepsilon)_{x}E_{\iota}, (d\varepsilon)_{x}E_{\iota})$$

for any $x\in M$ *where* $\{E_{\iota}\}_{\iota=1}^{m}$ *is an orthonormal basis for* $(T_{x}M, 2^{M}g)$.

COROLLARY 4.9.

$${}^{\gamma_{P\cdot P^{-1}}}\nabla^{V}(d\gamma_{P\cdot P^{-1}})^{V}((d\varepsilon)_{x}T_{x}M, (d\varepsilon)_{x}T_{x}M)=0 \qquad \textit{for all } x\in M$$

if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ *is a covariantly horizontal section with respect to* ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.

THEOREM 4.10. *ω is parallel if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ is a covariantly horizontal section with respect to ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.*

COROLLARY 4.11.

$$\sum_{\iota=1}^{m}{}^{\gamma_{P\cdot P^{-1}}}\nabla^{V}(d\gamma_{P\cdot P^{-1}})^{V}((d\varepsilon)_{x}E_{\iota}, (d\varepsilon)_{x}E_{\iota}))=0 \qquad \textit{for all } x\in M$$

if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ *is a harmonic section with respect to* ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.

THEOREM 4.12. *ω is Yang-Mills if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ is a harmonic section with respect to ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$.*

§5. Reduction of Target Fibers

Let H be a Lie group which admits a bi-invariant metric and H_0 be a closed subgroup of H. We consider a right principal H-bundle $\mathcal{P}$ over a Riemannian manifold M, a principal H_0-subbundle Q, a reduction map of structure group $\bar{i}: Q\to\mathcal{P}$ and a right connection form $\bar{\omega}$ which is reducible with respect to $\bar{i}$. The $\bar{\omega}$- (resp. $\bar{i}^{*}\bar{\omega}$-) horizontally lifted metric on $\mathcal{P}$ (resp. Q) is denoted by ${}^{\mathcal{P}}_{\bar{\omega}}g$ (resp. ${}^{Q}_{\bar{i}^{*}\bar{\omega}}g$).

PROPOSITION 5.1.

(i) $(d\bar{i})_{v}S^{V}=(d\bar{i})_{v}^{V}S$, $(d\bar{i})_{v}S^{H}=(d\bar{i})_{v}^{H}S$ *for any* $v\in Q$, $S\in T_{v}Q$.

(ii) *$\bar{i}: (Q, {}^{Q}_{\bar{i}^{*}\bar{\omega}}g)\to(\mathcal{P}, {}^{\mathcal{P}}_{\bar{\omega}}g)$ is an isometric embedding.*

(iii) $({}^{\bar{i}}\nabla d\bar{i}(S, T^{V}))^{V}=0$, *for all* $v\in Q$, $S, T\in T_{v}Q$.

Proof. (i), (ii) trivial. (iii) Extend S and T to local vector fields. The restriction of $\bar{i}$ to each fiber of Q is a totally geodesic embedding into a fiber of $\mathcal{P}$ so that $({}^{i}\nabla d\bar{i}(S^V, T^V))^V=0$. On the other hand, from (i) and [12, Lemma 1.4],

$$\bar{\omega}({}^{i}\nabla d\bar{i}(S^H, T^V))=\bar{\omega}({}^{\mathcal{P}}\nabla_{(d\bar{i})S^H}(d\bar{i})T^V-(d\bar{i})({}^{Q}\nabla_{S^H}T^V))$$

$$={}^{\mathcal{P}}d_{(d\bar{i})S^H}(\bar{\omega}(d\bar{i})T^V)-\bar{\omega}((d\bar{i}){}^{Q}\nabla_{S^H}T^V)={}^{i}({}^{\mathcal{P}}d)_{S^H}((\bar{i}^*\bar{\omega})T^V)-(\bar{i}^*\omega)({}^{Q}\nabla_{S^H}T^V)$$

$$={}^{Q}d_{S^H}((\bar{i}^*\bar{\omega})T^V)-{}^{Q}d_{S^H}((\bar{i}^*\bar{\omega})T^V)=0.$$

Let H_1, be another closed subgroup of H and $Q/H_0\cap H_1$, $\mathcal{P}/H_1$ be the $H_0\cap H_1$-orbit spase of Q, the H_1-orbit space of $\mathcal{P}$, respectively. The canonical quotient maps are denoted by $\pi_{/H_0\cap H_1}: Q\to Q/H_0\cap H_1$, $\pi_{/H_1}: \mathcal{P}\to\mathcal{P}/H_1$. $\bar{i}$ is right H_0-(therefore $H_0\cap H_1$-) equivariant so that there uniquely exists $i: Q/H_0\cap H_1\to\mathcal{P}/H_1$ such that $i\circ\pi_{/H_0\cap H_1}=\pi_{/H_1}\circ\bar{i}$.

Note that $Q_{/H_0\cap H_1}$, $\mathcal{P}_{/H_1}$ is associated to Q, $\mathcal{P}$, and let ${}^{Q/H_0\cap H_1}_{\bar{i}^*\bar{\omega}}g$, ${}^{\mathcal{P}/H_1}_{\bar{\omega}}g$ be the horizontally lifted metrics on $Q/H_0\cap H_1$, $\mathcal{P}/H_1$ by $\bar{i}^*\bar{\omega}$, $\bar{\omega}$, respectively.

PROPOSITION 5.2.

(i) $(di)_{\pi/H_0\cap H_1(v)}Z^V=(di)^V_{\pi/H_0\cap H_1(v)}Z$, $(di)_{\pi/H_0\cap H_1(v)}Z^H=(di)^H_{\pi/H_0\cap H_1(v)}Z$,

for any $v\in Q$, $Z\in T_{\pi/H_0\cap H_1(v)}(Q/H_0\cap H_1)$.

(ii) $i: (Q/H_0\cap H_1, {}^{Q/H_0\cap H_1}_{\bar{i}^*\bar{\omega}}g)\to(\mathcal{P}/H_1, {}^{\mathcal{P}/H_1}_{\bar{\omega}}g)$ *is an isometric embedding.*

These metrics make $\pi_{/H_0\cap H_1}: (Q, {}^{Q}_{\bar{i}^*\bar{\omega}}g)\to(Q/H_0\cap H_1, {}^{Q/HH_0\cap H_1}_{\bar{i}^*\bar{\omega}}g)$ and $\pi_{/H_1}$: $(\mathcal{P}, {}^{\mathcal{P}}_{\bar{\omega}}g)\to(\mathcal{P}/H_1, {}^{\mathcal{P}/H_1}_{\bar{\omega}}g)$ into Riemannian submersions with totally geodesic fibers. We write the associated orthogonal splittings as follows:

$$TQ=\mathrm{Ker}\, d\pi_{/H_0\cap H_1}\oplus(\mathrm{Ker}\, d\pi_{/H_0\cap H_1})^{\perp}\,;\ S=S_{\mathcal{V}}+S_{\mathcal{H}}\,,$$

$$T\mathcal{P}=\mathrm{Ker}\, d\pi_{/H_1}\oplus(\mathrm{Ker}\, d\pi_{/H_1})^{\perp}\,;\ V=V_{\mathcal{V}}=V_{\mathcal{H}}\,.$$

PROPOSITION 5.3.

$$(d\bar{i})_v S_{\mathcal{V}}=((d\bar{i})_v S)_{\mathcal{V}},\ (d\bar{i})_v S_{\mathcal{H}}=((d\bar{i})_v S)_{\mathcal{H}},\quad \textit{for all } v\in Q,\ S\in T_vQ\,.$$

We denote the horizontal lifts of $Z\in T(Q/H_0\cap H_1)$, $U\in T(\mathcal{P}/H_1)$, by $\mathcal{H}Z$, $\mathcal{H}U$, respectively.

PROPOSITION 5.4.

(i) $\mathcal{H}Z^V=(\mathcal{H}Z)^V$, $\mathcal{H}Z^H=(\mathcal{H}Z)^H$, *for all* $v\in Q$, $Z\in T_{\pi/H_0\cap H_1(v)}(Q/H_0\cap H_1)$.

(ii) $\mathcal{H}U^V=(\mathcal{H}U)^V$, $\mathcal{H}U^H=(\mathcal{H}U)^V$, *for all* $u\in\mathcal{P}$, $U\in T_{\pi/H_1(u)}(\mathcal{P}/H_1)$.

(iii) $\mathcal{H}(di)_{\pi/H_0\cap H_1(v)}W^V=(d\bar{i})_v\mathcal{H}W^V$, $\mathcal{H}(di)_{\pi/H_0\cap H_1(v)}W^H=(d\bar{i})_v\mathcal{H}W^H$,

for all $v\in Q$, $W\in T_{\pi/H_0\cap H_1(v)}(Q/H_0\cap H_1)$.

PROPOSITION 5.5.

$$({}^{i}\nabla d\boldsymbol{i}(Z, W^V))^V=0, \quad \textit{for all } v\in Q,\ Z, W\in T_{\pi/H_0\cap H_1(v)}(Q/H_0\cap H_1).$$

Proof. Extend Z, W to local vector fields. From Propositions 5.1, 5.3, 5.4 and [9, Lemma 1],

$$\begin{aligned}\mathcal{H}({}^{i}\nabla d\boldsymbol{i}(Z, W^V))^V &= (\mathcal{H}^{\mathcal{P}/H_1}\nabla_{(d\boldsymbol{i})Z}(d\boldsymbol{i})W^V)^V-(\mathcal{H}(d\boldsymbol{i})({}^{Q/H_0\cap H_1}\nabla_Z W^V))^V\\ &=(({}^{\mathcal{P}}\nabla_{\mathcal{H}(d\boldsymbol{i})Z}\mathcal{H}(d\boldsymbol{i})W^V)_{\mathcal{H}})^V-((d\boldsymbol{i})\mathcal{H}({}^{Q/H_0\cap H_1}\nabla_Z W^V))^V\\ &=(({}^{\mathcal{P}}\nabla_{(d\bar{\boldsymbol{i}})\mathcal{H}Z}(d\bar{\boldsymbol{i}})\mathcal{H}W^V)_{\mathcal{H}})^V-((d\bar{\boldsymbol{i}})({}^{Q}\nabla_{\mathcal{H}Z}\mathcal{H}W^V)_{\mathcal{H}})^V\\ &=(({}^{\mathcal{P}}\nabla_{(d\bar{\boldsymbol{i}})\mathcal{H}Z}(d\bar{\boldsymbol{i}})\mathcal{H}W^V-(d\bar{\boldsymbol{i}})^{Q}\nabla_{\mathcal{H}Z}\mathcal{H}W^V)^V)_{\mathcal{H}}\\ &=((({}^{\bar{i}}\nabla d\bar{\boldsymbol{i}}(\mathcal{H}Z, \mathcal{H}W^V))^V)_{\mathcal{H}}=0_{\mathcal{H}}=0\,.\end{aligned}$$

Let $\gamma_0: M\to Q/H_0\cap H_1$ be a C^∞-section (if exists). $\gamma_1=\boldsymbol{i}\circ\gamma_0: M\to\mathcal{P}/H_1$ is also a C^∞-section.

PROPOSITION 5.6.

(i) $(d\gamma_1)^V Y=(d\boldsymbol{i})_{\gamma_0(x)}((d\gamma_0)^V Y)$, *for all* $x\in M$, $Y\in T_x M$.

(ii) $\boldsymbol{e}^V(\gamma_1)=\boldsymbol{e}^V(\gamma_0)$.

(iii) ${}^{\gamma_1}\nabla^V(d\gamma_1)^V(X, Y)=(d\boldsymbol{i})_{\gamma_0(x)}({}^{\gamma_0}\nabla^V(d\gamma_0)^V(X, Y))$ *for all* $x\in M$, $X, Y\in T_x M$.

(iv) $\tau^V(\gamma_1)_x=(d\boldsymbol{i})_{\gamma_0(x)}(\tau^V(\gamma_0)_x)$ *for all* $x\in M$.

Proof. (i) It follows from Proposition 5.2 (i). (ii) From (i) and Proposition 5.2 (ii). (iii) Extend X, Y to local vector fields. From (i) and Proposition 5.5,

$$\begin{aligned}{}^{\gamma_1}\nabla^V(d\gamma_1)^V(X, Y) &= ({}^{\mathcal{P}/H_1}\nabla_{(d\gamma_1)X}(d\gamma_1)^V Y-(d\gamma_1)({}^{M}\nabla_X Y))^V\\ &=({}^{\mathcal{P}/H_1}\nabla_{(d\boldsymbol{i})(d\gamma_0)X}(d\boldsymbol{i})(d\gamma_0)^V Y)^V-((d\boldsymbol{i})(d\gamma_0)^V({}^{M}\nabla_X Y))^V\\ &=({}^{i}\nabla_{(d\gamma_0)X}(d\boldsymbol{i})(d\gamma_0)^V Y)^V-((d\boldsymbol{i})({}^{Q/H_0\cap H_1}\nabla_{(d\gamma_0)X}(d\gamma_0)^V Y))^V\\ &\quad+((d\boldsymbol{i})({}^{Q/H_0\cap H_1}\nabla_{(d\gamma_0)X}(d\gamma_0)^V Y))^V-(d\boldsymbol{i})(d\gamma_0)^V({}^{M}\nabla_X Y)\\ &={}^{i}\nabla d\boldsymbol{i}((d\gamma_0 X, (d\gamma_0)^V Y)+d\boldsymbol{i}({}^{\gamma_0}\nabla^V(d\gamma_0)^V(X, Y))=d\boldsymbol{i}({}^{\gamma_0}\nabla^V(d\gamma_0)^V(X, Y)).\end{aligned}$$

(iv) It follows from (iii).

COROLLARY 5.7. *γ_0 is a horizontal, covariantly horizontal or harmonic section if and only if γ_1 is a horizontal, covariantly horizontal or harmonic section, respectively.*

Let $(\mathcal{E}, h_{\mathcal{E}})$, $O(\mathcal{E}, h_{\mathcal{E}})$ be a system of a C^∞-vector bundle of rank $r+s+t$

and the orthonormal frame bundle of $(\mathcal{E}, h_{\mathcal{E}})$, whose structure group is $O(r+s+t)$. For a vector subbundle $\mathcal{S}$ of rank $r+s$, the inclusion is denoted by $i_{\mathcal{S}}^{\mathcal{E}}: \mathcal{S} \to \mathcal{E}$. The orthogonal splitting $\mathcal{E}=\mathcal{S}\oplus\mathcal{S}^{\perp}$ induces the *adapted* (c.f. [7]) orthonormal frame bundle $O(\mathcal{S}\oplus\mathcal{S}^{\perp}, h_{\mathcal{S}}\oplus h_{\mathcal{S}^{\perp}})$, whose structure group is $O(r+s)\times O(t)$, where $h_{\mathcal{S}}$, $h_{\mathcal{S}^{\perp}}$ are the restrictions of $h_{\mathcal{E}}$ to $\mathcal{S}$, $\mathcal{S}^{\perp}$, respectively. The inclusion ${}^{O}i_{\mathcal{S}}^{\mathcal{E}}: O(\mathcal{S}\oplus\mathcal{S}^{\perp}, h_{\mathcal{S}}\oplus h_{\mathcal{S}^{\perp}}) \to O(\mathcal{E}, h_{\mathcal{E}})$ is right $O(r+s)\times O(t)$-equi-variant.

$$O(r+s)\times O(t)\cap O(r)\times O(s+t)=O(r)\times O(s)\times O(t)$$

so that ${}^{O}i_{\mathcal{S}}^{\mathcal{E}}$ is reduced to the inclusion ${}^{Gr}i_{\mathcal{S}}^{\mathcal{E}}: G_r(\mathcal{S}) \to G_r(\mathcal{E})$ which assigns r-plane $\mathcal{F}_x$ in $\mathcal{S}_x$ to $i_{\mathcal{S}}^{\mathcal{E}}\mathcal{F}_x$ in $\mathcal{E}_x$ for $x \in M$. A vector subbundle $\mathcal{F}$ of $\mathcal{S}$ (if exists) defines the Gauss sections:

$$\gamma_{\mathcal{F}}^{\mathcal{S}}: M \to G_r(\mathcal{S})\,;\; x \mapsto \mathcal{F}_x,\quad \gamma_{\mathcal{F}}^{\mathcal{E}}: M \to G_r(\mathcal{E})\,;\; x \mapsto i_{\mathcal{S}}^{\mathcal{E}}\mathcal{F}_x\,.$$

Let ${}^{\mathcal{E}}\nabla$ be a connection in $\mathcal{E}$ compatible with $h_{\mathcal{E}}$, which preserves all C^{∞}-sections of $\mathcal{S}$ (therefore $\mathcal{S}^{\perp}$).

PROPOSITION 5.8. $\boldsymbol{e}^{V}(\gamma_{\mathcal{F}}^{\mathcal{S}})=\boldsymbol{e}^{V}(\gamma_{\mathcal{F}}^{\mathcal{E}})$.

PROPOSITION 5.9.
$\gamma_{\mathcal{F}}^{\mathcal{S}}$ is a horizontal, covariantly horizontal or harmonic section with respect to ${}^{\mathcal{S}}\nabla$ if and only if
$\gamma_{\mathcal{F}}^{\mathcal{E}}$ is a horizontal, covariantly horizontal or harmonic section with respect to ${}^{\mathcal{E}}\nabla$, respectively.

In the case of $r=p$, $s=t=m$, $\mathcal{E}=\varepsilon^{-1}T(P\cdot P^{-1})$, $\mathcal{S}=\varepsilon^{-1}\boldsymbol{E}^{(-)}$, ${}^{\mathcal{E}}\nabla={}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$, $\mathcal{F}=\varepsilon^{-1}\operatorname{Ker} d\pi_{P\cdot P^{-1}}$, $\gamma_{\mathcal{F}}^{\mathcal{S}}={}^{\varepsilon}\gamma_{P\cdot P^{-1}}^{(-)}$ and $\gamma_{\mathcal{F}}^{\mathcal{E}}={}^{\varepsilon}\gamma_{P\cdot P^{-1}}$, from Proposition 2.16 and [7, Chapter VII], we have the followings:

PROPOSITION 5.10. $\boldsymbol{e}^{V}({}^{\varepsilon}\gamma_{P\cdot P^{-1}}^{(-)})=\boldsymbol{e}^{V}({}^{\varepsilon}\gamma_{P\cdot P^{-1}})$.

PROPOSITION 5.11.
${}^{\varepsilon}\gamma_{P\cdot P^{-1}}^{(-)}$ is a horizontal, covariantly horizontal or harmonic section with respect to $\nabla^{(-)}$ if and only if
${}^{\varepsilon}\gamma_{P\cdot P^{-1}}$ is a horizontal, covariantly horizontal or harmonic section with respect to ${}^{\varepsilon}({}^{P\cdot P^{-1}}\nabla)$, respectively.

By combining Theorems 4.3, 4.10, 4.12 and the above proposition and corollary, we obtain the main theorem:

THEOREM 5.12 (THEOREM D).
(i) $\dfrac{1}{2}\|{}^{\omega}\Omega\|_u^2=\boldsymbol{e}^{V}(\gamma^{(-)})(x)$ *for all* $x\in M$, $u\in\pi_P^{-1}(x)$.

(ii) *ω is flat, parallel or Yang-Mills if and only if ${}^{\varepsilon}\gamma_{P\cdot P^{-1}}^{(-)}$ is a horizontal, covariantly horizontal or harmonic section with respect to $\nabla^{(-)}$, respectively.*

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