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KODAI MATH. J.
15 (1992), 324—340

PROJECTIVE SPACES IN A WIDER SENSE, I

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Introduction.

The purpose of this paper is to generalize the notion of projective spaces in compact symmetric spaces. For pairs $\{o, p\}$ of antipodal points in a compact symmetric space M, we attach to each p a pair of totally geodesic submanifolds (M_+^o, M_-^o) (simply denoted by (M_+, M_-)) in M. The general theory of (M_+, M_-) has been developed by B. Y. Chen and T. Nagano and now it plays an important role as a new method in the global study of compact symmetric spaces (cf. [5], [6]). We generalize projective spaces in terms of (M_+, M_-).

The aim of our study was to find a geometry for exceptional Lie groups. The compact Lie group $F_{4(-52)}$ is realized as the isometry group of the Cayley projective plane and the non-compact Lie group $E_{6(-26)}$ is the projective transformation group. For the compact exceptional Lie groups E_6, E_7 and E_8, we would like to find good symmetric spaces which play the same role as the Cayley plane does. The study originates from H. Freudenthal [7] and B. A. Rozenfeld [10].

First we intended to solve a problem proposed by H. Freudenthal (p. 175, [7]). Roughly speaking, it asks us whether the adjoint compact symmetric spaces of type $E\mathrm{III}$, $E\mathrm{VI}$ and $E\mathrm{VIII}$ (in the sense of E. Cartan) can be regarded as generalized projective planes. This problem was solved affimatively in [2], [3] and [4].

We know a unified construction of real simple Lie algebras (cf. [1]). In order to study the above problem, we constructed the usual projective planes explicitly by making use of the unifield algebras. Then we encountered the symmetric spaces of type $E\mathrm{III}$, $E\mathrm{VI}$ and $E\mathrm{VIII}$, and moreover we obtained the real and the complex Grassmann manifolds $G^R(4, 4n)^*$ and $G^C(2, 2n)$ (cf. Example 1.2). We found some common structures existing in these spaces (cf. Definition 1.1) and we called the symmetric spaces with such structures the projective spaces in a wider sense (cf. [3], [4]).

In this paper especially the projective planes in the wider sense are studied. For these planes we first establish a duality between points and lines (cf. Corollary 1.8) and also give the intersection number of two lines. We list the classification of the planes at the end of this paper.

Received September 26, 1991, revised March 23, 1992.

We summarize the notations used below. Let $\boldsymbol{R}P_n$, $\boldsymbol{C}P_n$ and $\boldsymbol{Q}P_n$ denote the real, the complex and the quaternion projective spaces respectively and let $\mathfrak{C}P_2$ be the Cayley projective plane. Let $G^R(n, m)=SO(n+m)/SO(n)\times SO(m)$, $G^C(n, m)=SU(n+m)/S(U_n\times U_m)$ and $G^H(n, m)=Sp(n+m)/Sp(n)\times Sp(m)$. For a compact symmetric space M, we denote by M^* the adjoint space of M (cf. [8]). It is also called the bottom space of M by Chen and Nagano.

1. Projective planes in a wider sense.

Let M be a compact symmetric space. For each p in M, the involutive isometry s_p is called the symmetry at p. Let G denote the closure of the group generated by all symmetries of M with respect to the compact-open topology. For some point o in M, let K_o be the isotropy subgroup in G at o (or we simply denote it by K). Then K is compact and $M=G/K$.

For $g\in G$, the mapping $\mathrm{Ad}(g): h\to ghg^{-1}$ is an isomorphism of G into itself. Let e be the identity element of G and T_eG the tangent space to G at e. We put $g_*=d\,\mathrm{Ad}(g)$ (the differential of $\mathrm{Ad}(g)$ on T_eG. Then $g\exp(X)(o)=\exp(g_*(X))(u)$ holds for $X\in T_eG$ and $u=g(o)\in M$. Hence, when we regard each tangent space T_vM (to M at v) as $T_vM\subset T_eG$, $g_*(T_oM)=T_uM$ holds.

Let p be an antipodal point of o on a closed smooth geodesic. Then the orbit $K(p)$ of p becomes a connected, compact, complete totally geodesic submanifold. We call $K(p)$ a polar and denote it by $M_+{}^o(p)$. If it is one point, it is called a pole. There exists a unique complete, connected totally geodesic submanifold $M_-{}^o(p)$ whose tangent space is the normal space of $T_oM_+{}^o(p)$. We call $M_-{}^o(p)$ the orthogonal complement of $M_+{}^o(p)$.

DEFINITION 1.1. Let M be a compact, connected symmetric space. M is called an *n-dimensional projective space in the wider sense* ($n\geqq 2$) if it satisfies the following conditions:

(1) M is the bottom space,

(2) M has a sequence of totally geodesic submanifolds $\{M_i\}$ $(i=1, \cdots, n)$ such that

(2-1) $M=M_n$, $M_i\supset M_{i-1}$ and each M_{i-1} is a polar in M_i,

(2-2) the orthogonal complement of M_{i-1} in M_i $(i=2, \cdots, n)$ is conjugate to M_1 under the isometry group G.

We call the polar which is conjugate to M_1 a *line* and, if $M=M_2$, M is called a *projective plane in the wider sense*. The incidence relation as a projective geometry is introduced into M by the inclusion relation of sets.

EXAMPLE 1.2. The examples of n-dimensional projective spaces in the wider sense are $\boldsymbol{R}P_n$, $\boldsymbol{C}P_n$, $\boldsymbol{Q}P_n$, $G^C(2, 2n)$ and $G^R(4, 4n)^*$. For instance, in the case of $M=G^C(2, 2n)$ we have $M_i=G^C(2, 2i)$ $(1\leqq i\leqq n)$. Since $M_1=G^C(2, 2)$ there, the orthogonal complement of M_i in M_{i+1} is conjugate to $G^C(2, 2)$. The projective spaces in the wider sense have been classified. If $n \geqq 3$, they are $G^{R}(m, nm)^{*}$, $G^{C}(m, nm)$ and $G^{H}(m, nm)$ where m is an arbitrary natural number. Note that $\boldsymbol{R}P_n = G^{R}(1, n)^{*}$, $\boldsymbol{C}P_n = G^{C}(1, n)$ and $\boldsymbol{Q}P_n = G^{H}(1, n)$.

The classification in the case of $n=2$ is listed in the last section. The result was given essentially by Chen and Nagano [6]. According to the list we notice that the space of type EVI has two kinds of structures of projective planes in the wider sense. There is an important sequence of planes:

$$\boldsymbol{R}P_2 \subset \boldsymbol{C}P_2 \subset \boldsymbol{Q}P_2 \subset \mathfrak{C}P_2 \subset E\mathrm{III} \subset E\mathrm{VI} \subset E\mathrm{VIII}.$$

This sequence was the starting point of our study.

LEMMA 1.3. *If M is a projective plane in the wider sense, it satisfies the following properties*

(1) *for any $p, q \in M$, $s_p = s_q$ is equivalent to $p = q$,*

(2) *there exists a polar $M_+^o(p)$ such that $s_o s_p = s_q$ holds for some $q \in M_+^o(p)$.*

Proof. We can prove (1) and (2) by Chen's results in [6]. Namely Theorem 4.1 says that $s_p = s_q$ holds in M if and only if $p=q$ or q is a pole of p. And Theorem 5.1 asserts that if M has a pole it is a double covering space of some symmetric space (that is, M is not the bottom space). (2) is also obtained by Theorem 4.4. It shows that (2) holds if and only if $M_+^o(p)$ and $M_-^o(p)$ are conjugate. □

Remark 1.4. Let M be a compact irreducible symmetric space. Then, if M satisfies the above two properties, it becomes the bottom space. We can see this fact from the explicit classification of the spaces which satisfy (1) and (2). Hence we can use the properties (1) and (2) as the definition of projective planes in the wider sense.

Let M be a projective plane in the wider sense and $L(M)$ the set of all lines. Let $L(o)$ be the polar $M_+^o(p)$ which is conjugate to M_1. We define a map L from M to $L(M)$ by $o \to L(o)$. Since $\operatorname{rank} M_-^o(p) = \operatorname{rank} M$, all lines have the same rank as symmetric spaces.

LEMMA 1.5. *If $M_+^u(v) = M_+^o(p)$ for $u, v \in M$, then $u = o$ holds.*

Proof. Since $p \in M_+^o(p)$ (hence also $p \in M_+^u(v)$), the symmetries s_u and s_o leave p fixed. Then the tangent space T_pM to M at p has two direct sum decompositions $T_pM = T_pM_+^u(p) \oplus T_pM_-^u(p)$ and $T_pM = T_pM_+^o(p) \oplus T_pM_-^o(p)$ where the subspaces are the (± 1)-eigenspaces of the differentials $(s_u)_*$ and $(s_o)_*$ of s_u and s_o respectively. Since $M_+^u(p) = M_+^o(p)$, we obtain $T_pM_+^u(p) = T_pM_+^o(p)$ and, hence, $T_pM_-^u(p) = T_pM_-^o(p)$. These implies $(s_u)_* = (s_o)_*$ in T_pM. By Lemma 11.2 (p. 62 [8]), we get $s_u = s_o$. Therefore we have $u = o$ for M is the bottom space. □

LEMMA 1.6. *The map L is well-defined and bijective.*

Proof. Let A be a maximal flat torus in M which passes through o and p. Assume that $M_+^o(v)$, $v \in A$, is conjugate to $M_+^o(p)$ under G. Then there exists $g \in G$ such that $gM_+^o(v)=M_+^o(p)$. Since $gM_+^o(v)=M_+^{g(o)}(g(v))$, we obtain $g(o)=o$ by Lemma 1.5. This means $g \in K_o$ and $M_+^o(v)=gM_+^o(v)=M_+^o(p)$. Thus L is well-defined.

The surjectivity is known by the definition of L. The injectivity is an easy consequence of Lemma 1.5. $\square$

We can introduce a differentiable structure into $L(M)$ since L is bijective. Hence $L(M)$ can be regarded as a symmetric space. It has the same structure as M. Let π be a map, from $M \cup L(M)$ onto itself, which maps a point (resp. a line) to some line (resp. to some point). If π satisfies the two properties

(1) $\pi^2=$identity map,

(2) $p \in \pi(q) \Leftrightarrow \pi(p) \ni q$, for $p, q \in M$,

then π is called a polarity. This gives a duality between points and lines in M.

PROPOSITION 1.7. *The map L induces a polarity in M.*

Proof. Since L is bijective by Lemma 1.6, we can define a polarity π by $\pi(p)=L(p)$ and $\pi(L(p))=p$. And π satisfies the condition (2) by Theorem 4.4 [6]. $\square$

COROLLARY 1.8. *A duality between points and lines holds in each projective plane in the wider sense.*

2. The intersection of two lines.

Let M be a projective plane in the wider sense. Throughout this section M will be semi-simple as a symmetric space. Our aim is to determine the intersection N of any two lines in M (See Theorem 2.11). We will see that N is a finite set in general and the cardinal number $\#N$ of N is constant for M. For example, $\#N=1$, 1, 1, 1, 1, 3 and 135 according to $\boldsymbol{RP}_2$, $\boldsymbol{CP}_2$, $\boldsymbol{QP}_2$, $\mathfrak{C}P_2$, $E\mathrm{III}$, $E\mathrm{VI}$ and $E\mathrm{VIII}$. These numbers are listed later as $\#(M_+)$. By the duality of M (Corollary 1.8), the set of all lines, which pass through two points, has the same structure as N. Hence, in the case of $E\mathrm{III}$, there exists in general only one line which passes through two points.

We have a Cartan decomposition $T_eG=T_eK \oplus \mathfrak{M}$ with respect to the differential $(s_o)_*$ $(=d\,\mathrm{Ad}(s_o))$ of $\mathrm{Ad}(s_o)$ where $M=G/K$ $(K=K_o)$. Since we identify $\mathfrak{M}$ and T_oM, any geodesic of M, which passes o and has a tangent vector $X \in T_oM$, can be given by $\gamma(t)=\exp(tX)(o)$ where $t \in \boldsymbol{R}$ and $\exp(tX) \in G$.

LEMMA 2.1. *If $p \in M$ satisfies $s_p s_q = s_q s_p$ for any $q \in \gamma(t)$, then* $\exp(tX)$ *leave*

p fixed as a transformation of M.

Proof. Put $a=a_t=\exp(tX)$ and $q=a_t(o)$ for $t\in \boldsymbol{R}$. Let $p\in M$ satisfy the above condition. Then we obtain, from $s_q=as_oa^{-1}$, that

$$s_ps_q=s_qs_p \Longleftrightarrow s_pas_oa^{-1}=as_oa^{-1}s_p \Longleftrightarrow a^{-1}s_pa=s_oa^{-1}s_pas_o \Longleftrightarrow s_u=s_v,$$

where $u=a^{-1}(p)$ and $v=s_oa^{-1}(p)$. Since M is the bottom space, $s_u=s_v$ implies $u=v$. On the other hand, $s_oa^{-1}s_o=a$ holds because $(s_o)_*X=-X$. From this we have $v=s_oa^{-1}(p)=as_o(p)$ and hence $a^{-1}(p)=u=v=as_o(p)$. Especially if we put $t=0$, we have $p=s_o(p)$. And $a^{-1}(p)=a(p)$, i.e., $a^2(p)=p$ holds for any $a(=a_t)$. Since $t\in \boldsymbol{R}$ is arbitrary, we have $a(p)=p$. □

COROLLARY 2.2. *Let A be a maximal flat torus in M which passes through o. If a symmetry s_p, $p\in M$, commutes with any symmetry s_q of A, then all isometries* $\exp(X)$, *$X\in T_oA$, leave p fixed.*

Let o, $p\in M$. We denote by K_p the isotropy subgroup of G at p and put $U_p=K_o\cap K_p$. Let $U_p{}^0$ be the identity component of U_p. And we denote the corresponding Lie algebra by $\mathfrak{U}_p$, and put $\mathfrak{G}=T_eG$ and $\mathfrak{K}_o=T_eK_o$.

LEMMA 2.3. *Let p, q be points of $L(o)$ with $s_os_p=s_q$ and let A be a maximal flat torus in $L(q)$ which passes through o and p. If a symmetry $s_r, r\in L(o)$, commutes with any s_u, $u\in A$, then there exists $k\in$*$\exp(\mathfrak{K}_o)$ *such that $r=k(q)$ and $k(A)=A$.*

Proof. Assume that s_r, $r\in L(o)$, satisfies the above conditiom. Since $L(o)=K_o(p)$ and q, $r\in L(o)$, there exists $g\in\exp(\mathfrak{K}_o)$ such that $r=g(q)$. Then also $\mathfrak{K}_r=g_*(\mathfrak{K}_q)$ holds in $\mathfrak{G}$, where g_* is the differential of $\mathrm{Ad}(g)$. First we show $g_*(\mathfrak{A})\cup\mathfrak{A}\subset\mathfrak{M}\cap\mathfrak{K}_r$ in $\mathfrak{G}$ where $\mathfrak{G}=\mathfrak{K}_o\oplus\mathfrak{M}$ and $\mathfrak{A}=T_oA$. We know $\exp(\mathfrak{A})(q)=q$ from $A\subset L(q)$ and Corollary 2.2. This implies $\mathfrak{A}\subset\mathfrak{K}_q$ and hence $g_*(\mathfrak{A})\subset\mathfrak{K}_r$. By Corollary 2.2 one also has $\mathfrak{A}\subset\mathfrak{K}_r$ because s_r commutes with all symmetries s_u, $u\in A$. Since $A\subset M$, we get $\mathfrak{A}\subset\mathfrak{M}$. And $g_*(\mathfrak{A})\subset\mathfrak{M}$ can be obtained from the following argument. Since $g(o)=o$, $s_og=gs_o$ holds. For any $Z\in\mathfrak{A}$, one has $(s_o)_*g_*(Z)=g_*(s_o)_*(Z)=-g_*(Z)$. Therefore $g_*(Z)\in\mathfrak{M}$.

We take $X\in g_*(\mathfrak{A})$ and $Y\in\mathfrak{A}$ such that these centralizers become $g_*(\mathfrak{A})$ and $\mathfrak{A}$ respectively (cf. p. 248 [8]). On the identity component $U_r{}^0$ we define a differentiable function $F\colon U_r{}^0\to\boldsymbol{R}$, by $F(k)=B(X, k_*Y)$ for $k\in U_r{}^0$ where B is the Killing form of $\mathfrak{G}$. Since $U_r{}^0$ is compact, we can assume that F takes an extremal value at $k=h$. Then, it holds that, for any $Z\in\mathfrak{U}_r$,

$$\begin{aligned}0&=\left\{\frac{d}{dt}B(X, (\exp(tZ))_*h_*(Y))\right\}_{t=0}\\&=B(X, [Z, h_*(Y)])\\&=-B([X, h_*(Y)], Z).\end{aligned}$$

Since $\mathfrak{A}\subset\mathfrak{M}\cap\mathfrak{K}_r$, $h_*(Y)\in\mathfrak{M}\cap\mathfrak{K}_r$ holds. This implies $[X, h_*(Y)]\in\mathfrak{K}_o\cap\mathfrak{K}_r(=\mathfrak{U}_r)$ because $X\in g_*(\mathfrak{A})$ and $g_*(\mathfrak{A})\subset\mathfrak{M}\cap\mathfrak{K}_r$. So, we obtain $B(Z, Z)=0$ for $Z=[X, h_*(Y)]$ in the above equation. This means $Z=[X, h_*(Y)]=0$ because M is semi-simple and so the Killing form B is negative definite. From the definition of X and Y, this gives $h(A)=g(A)$. If we put $k=h^{-1}g$, k belongs to the identity component of K_o and it satisfies the above properties: in fact, $k(A)=A$, $k(o)=o$ and $k(q)=h^{-1}g(q)=h^{-1}(r)=r$. □

Let p, q be points of $L(o)$ with $s_o s_p=s_q$. Let A be a maximal flat torus in $L(q)$ which passes o and p. Define two subsets S_o and S in M by $S_o=L(o)\cap A$ and $S=\bigcap_{r\in A} L(r)$. Since $A\subset L(q)$, we have $q\in S$ by the duality. By Lemma 1.3 and the transitivity of points in $L(o)$, for any $u\in L(o)$ there exists $v\in L(o)$ such that $s_o s_u=s_v$. Hence we can define a map $\phi: S\to S_o$ by $\phi(u)=v$ because $S\subset L(o)$.

LEMMA 2.4. *Let $u\in M$ and A' be a maximal flat torus which passes through o and u. Then $(s_u)_*(Z)=-Z$ holds for $Z\in T_oA'$ where we regard T_oA' as a subspace of T_oG.*

Proof. Let A' be a maximal flat torus which satisfies the above condition. Take $X\in T_oA'$ such that $u=\exp(X)(o)$. Put $a_t=\exp(tX)$ for $t\in \boldsymbol{R}$ and put $a=a_1$. Then for $Z\in T_oA'$ we have

$$s_u \exp(tZ) s_u = s_{a(o)} \exp(tZ) s_{a(o)} = a s_o a^{-1} \exp(tZ) a s_o a^{-1} = \exp(-tZ)$$

because $a\exp(tZ)=\exp(tZ)a$ and $s_o\exp(tZ)s_o=\exp(-tZ)$ hold. Hence $(s_u)_*(Z)=-Z$ holds for $Z\in T_oA'$. □

LEMMA 2.5. *The map ϕ is well-defined and bijective.*

Proof. If $s_o s_u=s_v$ and $s_o s_u=s_w$ hold, $s_v=s_w$ gives $v=w$ because M is the bottom space. Next we show that $\phi(S)\subset S_o$. From Lemma 2.3, for any $u\in S$ there exists $k\in\exp(\mathfrak{K}_o)$ such that $u=k(q)$ and $k(A)=A$. Hence it holds that, by $k^{-1}s_o k=s_o$ and $s_o s_p=s_q$,

$$s_o s_u = s_o s_{k(q)} = s_o k s_q k^{-1} = k s_o s_q k^{-1} = k s_p k^{-1} = s_{k(p)}.$$

This implies $\phi(u)=k(p)$. And we have also $k(p)\in L(o)\cap A$ because $p\in L(o)\cap A$ and $k(L(o)\cap A)=L(o)\cap A$.

Next we show the injectivity. If $\phi(u)=\phi(r)$ for some u, $r\in S$, $s_o s_u=s_o s_r$ holds. Hence one has $s_u=s_r$ and so $u=r$ because M is the bottom space. We show the surjectivity. By Lemma 1.3, for any $u\in S_o$, there exists $v\in L(o)$ such that $s_o s_u=s_v$. Then, since $s_o s_v=s_u$ holds, we may show $v\in S$. Let $r\in A$. Assume $r=\exp(X)(o)$ with $X\in T_oA$. Since o, $u\in A$ and s_u leave o fixed, we have $(s_u)_*=-1$ on T_oA by Lemma 2.4. Hence it holds that

$$s_v(r) = s_o s_u(r) = s_o s_u \exp(X)(o) = \exp((s_o)_*(s_u)_* X)(o) = \exp(X)(o) = r.$$

This gives $s_r=s_v s_r s_v$, that is, $s_r s_v=s_v s_r$. Since $r\in A$ is arbitrary and $v\in L(o)$, by Lemma 2.3 there exists $k\in\exp(\mathfrak{K}_o)$ such that $v=k(q)$ and $k(A)=A$. Therefore we have $v\in S$ because $q\in S$ and $k(S)=S$. □

LEMMA 2.6. *Let $k\in K_o$ and $u, v\in S$. Then $k(u)=v$ is equivalent to $k\phi(u)=\phi(v)$.*

Proof. Since $s_o s_u=s_{\phi(u)}$, we have, by $s_o k=k s_o$,

$$s_o s_{k(u)}=s_o k s_u k^{-1}=k s_o s_u k^{-1}=k s_{\phi(u)} k^{-1}=s_{k\phi(u)}.$$

Hence, $k(u)=v$ is equivalent to $k\phi(u)=\phi(v)$. □

COROLLARY 2.7. *For $u\in S$, $U_u=U_{\phi(u)}$ holds.*

We take three points $\{o, p, q\}$ and a maximal flat torus A as in Lemma 2.3. The aim is to study the set of lines which pass through any two points $u, v\in M$. Without loss of generality, we may assume $v=o$ and $u\in A$ by the transitivity. For any $u\in A$, define two subsets by,

$$N(u)=\{v\in M\,|\,o,\ u\in L(v)\},$$

$$N_o(u)=\{k(v)\,|\,\text{any } v\in S_o \text{ and any } k\in U_u{}^0\}.$$

We will see later that these sets are isometric (Proposition 2.10) and that S_o and $U_u{}^0$ (i.e., $N_o(u)$) can be determined explicitly. Note that $N_o(u)$ is not necessarily connected.

PROPOSITION 2.8. *Let A' be another maximal flat torus in M which passes through two points $o, u\in A$. Then there exists $k\in U_u{}^0$ such that $k(A')=A$.*

Proof. We have a direct sum decomposition $T_eG=T_eK\oplus\mathfrak{M}$ where $M=G/K$ and $\mathfrak{M}=T_oM$. Put $\mathfrak{A}=T_oA$ and $\mathfrak{A}'=T_oA'$. Take $X\in\mathfrak{A}$ and $Y\in\mathfrak{A}'$ such that these centralizers become $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. Next we define a differentiable function $F: U_u{}^0\to\boldsymbol{R}$ by $F(k)=B(X, k_*Y)$, $k\in U_u{}^0$, where B is the Killing form of T_eG. Since $U_u{}^0$ is a compact group, we may assume that F takes an extremal value at $k=h$. Then it holds that, for $Z\in\mathfrak{U}_u$ $(=T_eU_u{}^0)$,

$$\begin{aligned}0&=\left\{\frac{d}{dt}B(X, (\exp(tZ))_*h_*(Y))\right\}_{t=0}\\&=B(X, [Z, h_*(Y)])\\&=-B([X, h_*(Y)], Z).\end{aligned}$$

On the other hand, X and $h_*(Y)$ are tangent vectors to A and to $h(A')$ at o respectively. Since A and $h(A')$ passes through o and u, $(s_o)_*$ and $(s_u)_*$ act as -1 for both X and $h_*(Y)$ by Lemma 2.4. This gives $[X, h_*(Y)]\in\mathfrak{U}_u$. Hence, if we put $Z=[X, h_*(Y)]$ in the above equation, we get $Z=0$ because B is nondegenerate. This shows $h(A')=A$. □

LEMMA 2.9. *$N(u)=\{k(v)\mid$ any $v\in S$ and any $k\in U_u{}^0\}$ holds for each $u\in A$.*

Proof. Take $v\in N(u)$. Then we have o, $u\in L(v)$ by the definition. Since $L(v)$ and M have the same rank, there exists a maximal flat torus A' of M such that $A'\subset L(v)$ and it passes through o and u. By Lemma 2.8, we can take $k\in U_u{}^0$ such that $k(A')=A$. Since $kL(v)=L(k(v))$, $A\subset L(k(v))$ holds. By the duality, we get $k(v)\in S$. Therefore we obtain $v=k^{-1}k(v)\in k^{-1}S$ for $k^{-1}\in U_u{}^0$. Conversely, we take $v\in S$ and $k\in U_u{}^0$. Then $L(k(v))\supset k(A)$ holds by $k(v)\in kS$ and the duality. Hence we have $L(k(v))\ni o$, u since $k(A)\ni o$, u. □

We define a map $\Phi: N(u)\to N_o(u)$ by $\Phi(k(v))=k(\phi(v))$ where we use the expression in Lemma 2.9 for $N(u)$.

PROPOSITION 2.10. *Φ is an isometry from $N(u)$ to $N_o(u)$.*

Proof. First we show that Φ is well-defined and injective by the following arguments: for v, $w\in S$ and k, $h\in U_u{}^0$, it holds that

$$\begin{aligned} k(v)=h(w) &\iff v=k^{-1}h(w) \\ &\iff \phi(v)=k^{-1}h(\phi(w)) \qquad \text{(by Lemma 2.6)} \\ &\iff k(\phi(v))=h(\phi(w)) \\ &\iff \Phi(k(v))=\Phi(h(w)). \end{aligned}$$

The surjectivity of Φ can be given by that of ϕ.

Let C be a connected componect of $N(u)$. Then, by Lemma 2.9, C must meet S because $U_u{}^0$ is connected. So we may assume $v\in C\cap S$. Then $U_v\cap U_u{}^0 =U_{\phi(v)}\cap U_u{}^0$ holds in G by Corollary 2.7. Hence C and $\Phi(C)$ become totally geodesic submanifolds with the type $U_u{}^0/U_v\cap U_u{}^0$. And, since both have the induced metric from $U_u{}^0/U_v\cap U_u{}^0$, they are isometric. □

THEOREM 2.11. (1) *The set of all lines which pass through o, $u\in A$ becomes a totally geodesic submanifold in M. It is isometric to $N_o(u)$.*

(2) (*the dual of* (1)): *The intersection of two lines $L(o)$ and $L(u)$, o, $u\in A$, becomes a totally geodesic submanifold in $L(M)$. It is isometric to $N_o(u)$.*

Proof. We obtain (1) by Proposition 2.10 and (2) by the duality in Corollary 1.8. □

3. The determination of the intersection number of two lines.

In this section we keep the notation in §2 unless otherwise stated. Let p, q be points of $L(o)$ with $s_o s_p=s_q$. Let A be a maximal flat torus in $L(q)$ which passes through o and p. The structure of the set of all lines, passing through o, $u\in A$, can be determined by $N_o(u)$ (cf. Theorem 2.11). Therefore we must analyze the set $S_o(=L(o)\cap A)$ and the isotropy group $U_u{}^0$.

We have a direct sum decomposition $T_eG=T_eK\oplus\mathfrak{M}$ with respect to the involutive automorphism $g\to s_o g s_o$ of G. Put $\mathfrak{G}=T_eG$ and $\mathfrak{K}=T_eK$. Since A is a maximal flat torus also in M, one obtains an eigenspace decomposition of $\mathfrak{G}$ with respect to $\mathfrak{A}$ where $\mathfrak{A}=T_oA$ ($\mathfrak{M}=T_oM$). We review some facts on this decomposition after O. Loos [9] (p. 58-p. 62).

Set $Q(A)=s_o s_A$. Then $Q(A)$ is a flat torus in G. When we consider the adjoint representation $\operatorname{Ad}Q(A)$ of $Q(A)$ on complexification $\mathfrak{G}_C$ of $\mathfrak{G}$, we have an eigenspace decomposition

$$\mathfrak{G}_C=(\mathfrak{G}_C)^A\oplus\Sigma\mathfrak{G}_\chi\,,$$

where $(\mathfrak{G}_C)^A$ is the set of fixed points of $\operatorname{Ad}Q(A)$ on $\mathfrak{G}_C$ and $\{\chi\}$ are the mutually different non-trivial characters of the representation with the corresponding eigenspaces

$$\mathfrak{G}_\chi=\{Z\in\mathfrak{G}_C\mid \operatorname{Ad}Q(u)(Z)=\chi(u)Z\quad\text{for all } u\in A\}\,.$$

Each χ corresponds to a linear form λ_χ on $\mathfrak{A}$ by

$$\chi(\exp(X))=\exp(2\lambda(X))\qquad\text{for}\quad X\in\mathfrak{A}\,,$$

where we denote λ_χ by λ for simplicity. We call λ a root relative to $\mathfrak{A}$ and denote the set of roots by Δ.

Set, for $\lambda\in\Delta$,

$$\mathfrak{G}_\lambda=\{Z\in\mathfrak{G}_C\mid [H,Z]=\lambda(H)Z\quad\text{for all } H\in\mathfrak{A}\}\,.$$

It follows that $\mathfrak{G}_\chi=\mathfrak{G}_\lambda$ if $\lambda=\lambda_\chi\in\Delta$ and $\mathfrak{G}_\lambda=0$ if $\lambda\neq0$ and $\lambda\notin\Delta$. Here $(\mathfrak{G}^{\mathfrak{A}})_C=(\mathfrak{G}_C)^{\mathfrak{A}}=(\mathfrak{G}_C)^A$ where $\mathfrak{G}^{\mathfrak{A}}=\{X\in\mathfrak{G}\mid[X,\mathfrak{A}]=0\}$. Then we obtain

$$\mathfrak{G}_C=(\mathfrak{K}^{\mathfrak{A}})_C\oplus\mathfrak{A}_C\oplus\Sigma\mathfrak{G}_\lambda\,,\tag{1}$$

where Σ is the sum over $\lambda\in\Delta$ and $\mathfrak{K}^{\mathfrak{A}}=\{X\in\mathfrak{K}\mid[X,\mathfrak{A}]=0\}$. Put $\mathfrak{K}_\lambda=\mathfrak{K}\cap(\mathfrak{G}_\lambda\oplus G_{-\lambda})$ and $\mathfrak{M}_\lambda=\mathfrak{M}\cap(\mathfrak{G}_\lambda\oplus\mathfrak{G}_{-\lambda})$. Then it holds

$$\mathfrak{K}=\mathfrak{K}^{\mathfrak{A}}\oplus\Sigma\mathfrak{K}_\lambda\quad\text{and}\quad\mathfrak{M}=\mathfrak{A}\oplus\Sigma\mathfrak{M}_\lambda\,,\tag{2}$$

where Σ runs over positive roots.

Define a set U_λ by

$$U_\lambda=\operatorname{kernel}(\chi)=\{u=\exp(H)\in A\mid\chi(u)=\exp(2\lambda(H))=1\}\,.$$

Then we have

$$\mathfrak{U}_u=\mathfrak{K}^{\mathfrak{A}}\oplus\Sigma\mathfrak{K}_\lambda\,,\tag{3}$$

where $\mathfrak{U}_u$ is the Lie algebra of U_u (the isotropy group of o and $u\in A$) and Σ

runs over λ such that $u \in U_\lambda$. (3) gives the isotropy group $\exp(\mathfrak{U}_u)$ at $u \in A$ explicitly.

For o, $u \in A$, set

$$\Delta_o = \{\lambda \in \Delta \mid S_o \subset U_\lambda\} \quad \text{and} \quad \Xi_u = \{\lambda \in \Delta \mid u \in U_\lambda\}.$$

When $\Xi_u \subset \Delta_o$ (resp. $\Xi_u \cap (\Delta - \Delta_o) \neq \emptyset$), we say that two points o and u are *in the general position* (resp. *in the singular position*). Then, from the duality, we also say that two lines $L(o)$ and $L(u)$ are in the general position (resp. in the singular position).

EXAMPLE 3.1. Let M be an usual projective plane (being not in the wider sense). Since M is of rank one, S_o consists of one point and $\Delta = \Delta_o$ holds. Hence $\Xi_u \subset \Delta_o$. This means that two points are always in the general position. We usualy say that there exists only one line which passes through any two points.

LEMMA 3.2. *S_o is a finite set.*

Proof. We know from Lemma 3.15 in [6] that, in an abelian Lie group, two antipodal points of the identity element are always antipodal to each other. When we regard the base point o as the identity element, we may regard the maximal flat torus A in M as an abelian Lie group. Hence any two u, $v \in S_o$ are antipodal to each other because $S_o = L(o) \cap A$. This means that s_u leaves v fixed and u is an isolated point in S_o. Since S_o is a compact discrete set, it is a finite set. □

LEMMA 3.3. *If o, $u \in A$ are in the general position, $N_o(u)$ is a finite set.*

Proof. Assume that o, $u \in A$ are in the general position. Then $\Xi_u \subset \Delta_o$ holds. The above identity (3) means that $\exp(\mathfrak{U}_u)$ leaves all elements in S_o fixed. Therefore $N_o(u) = S_o$. By Lemma 3.2 we have that $N_o(u)$ is a finite set. □

Let C be a component of the set of regular elements in the maximal flat torus A (resp. in T_oA) (cf. p. 68 [9]). We call the closure $\bar{C}$ of C a (closed) cell and, if $o \in \bar{C}$ (resp. $0 \in \bar{C}$), we call it a fundamental cell. From now on we study the number of all cells in A. We will use the following notation:

D: a fundamental cell in T_oA,

$v_1, v_2, \cdots, v_l$: the vertexes of D, where l is the rank of M and the suffixes $\{1, \cdots, l\}$ correspond to that of the fundamental roots $\{\lambda_1, \cdots, \lambda_l\}$ respectively,

v_0: the origin of D,

c_i: the number of all points in D which are conjugate to v_i under the affine Weyl group of T_oA,

n_i: the number of all points in A which are conjugate to $\exp(v_i)(o)$ under the affine Weyl group of T_oA,

r_i: the number of all cells which have v_i as a vertex when we regard T_oA as the tangent space of some maximal torus in the universal covering space of M.

However, if $\exp(v_i)(o)=\exp(v_0)(o)=o$ in M, let c_i, n_i and r_i denote the numbers for $v_i/2$.

PROPOSITION 3.4. *For* $i\in\{1, \cdots, l\}$, $r_i n_i/c_i$ *is equal to the number of all cells in* A.

Proof. Take any $i\in\{1, \cdots, l\}$. In $T_oA(\subset T_oM)$ there are n_i points conjugate to v_i and there are r_i cells around v_i. Since we count these $r_i n_i$ cells c_i times repeatedly, $r_i n_i/c_i$ becomes the number of all cells in A. $\square$

If we regard the fundamental cell D and the extended Dynkin diagram of M as those of some compact, simply connected, semi-simple Lie group $G(M)$ respectively, then the normalizer $K_i(M)$ of $\exp(v_i)(\in G(M))$ can be obtained from the diagram by the same method as Borel-Siebenthal's one. However, if $\exp(v_i)(o)=\exp(v_0)(o)=o$ in M, let $K_i(M)$ denote the normalizer of $\exp(v_i/2)$ $(\in G(M))$. Let $W(K_i)$ be the Weyl group corresponding to $K_i(M)$. Then the order $\#W(K_i)$ of $W(K_i)$ is equal to r_i. But, if the diagram is of the following type, we must calculate r_i directly because the corresponding $G(M)$ does not exist:

$$\circledcirc\Longrightarrow\bigcirc\!-\!\!-\!\bigcirc\!-\!\!-\!\bigcirc\!-\!\!-\!\bigcirc\!-\cdots-\!\bigcirc\!-\!\!-\!\bigcirc\!-\!\!-\!\bigcirc\Longrightarrow\bigcirc\,,$$

where $\circledcirc$ means $(-1)\times$(the highest root). For example, we see this type when $M=SO(2n)/U(n)$ (n is odd), $SU(n+m)/S(U_n\times U_m)$ $(n\neq m)$ or $Sp(n+m)/Sp(n)\times Sp(m)$.

EXAMPLE 3.5. We consider $G\,\mathrm{I}=G_2/SO(4)$ as M since it becomes a projective plane in the wider sense. Then $M_+=M_-=S^2\cdot S^2$ (semi-direct product of two spheres). As a symmetric space, M is irreducible and of rank two. Let A be a maximal torus in M which passes through o and T_oA its tangent space at o. Let Δ be the set of roots of M with respect to A. Take a fundamental root system $\{\lambda_1, \lambda_2\}$ such that the highest root μ is equal to $2\lambda_1+3\lambda_2$. Then the extended Dynkin diagram is

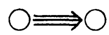
$$\overset{-\mu}{\circledcirc}\!-\!\!-\!\overset{\lambda_1}{\bigcirc}\Rrightarrow\overset{\lambda_2}{\bigcirc}\ .$$

And a fundamental cell is given by

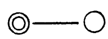
$$D=\{x\in T_oA \mid \lambda_1(x\boldsymbol{i})\geqq 0,\ \lambda_2(x\boldsymbol{i})\geqq 0 \text{ and } \mu(x\boldsymbol{i})\leqq\pi\}\,.$$

Let v_1, v_2 be the vertexes of D corresponding to λ_1 and λ_2 respectively.

We regard the extended Dynkin diagram of the symmetric space M as that of the simple Lie group G_2 (in this case $G(M)=G_2$ holds by chance). We denote by $K_\iota(M)$ (or simply by K_i) the normalizer of $\exp(v_\iota)$ in G_2 ($i=1, 2$). In this case $K_1=SO(4)$ and $K_2=SU(3)$. The diagrams, the types and the number of elements of the Weyl groups for $\{K_i\}$ are given as follows:

 type G_2 $\quad r_0=\#(W(G_2))=12 \quad c_0=1$,

 type $A_1\times A_1$ $\quad r_1=\#(W(K_1))=4 \quad c_1=1$,

 type A_2 $\quad r_2=\#(W(K_2))=6 \quad c_2=1$,

Then we have

$$\begin{aligned}\text{the number of all cells in } A &= (\#(W(G_2))\times 1)/c_0=12\\ &=(\#(W(K_1))\times n_1)/c_1=4n_1\\ &=(\#(W(K_2))\times n_2)/c_2=6n_2\,,\end{aligned}$$

where $W(G_2)$ denotes the Weyl group of G_2. From these equations we obtain $n_1=3$ and $n_2=2$. This means that the cardinal number of $L(o)\cap A$ is 3 $(=n_1)$ because the orbit of $\exp(v_1)(o)$ becomes $L(o)$. Hence we can say that there exist three lines which pass through any two points in the general position.

For this model it holds that $n_1=\#(W(G_2))/\#(W(K_1))$. Therefore n_1 is also equal to the Euler number of $G\,\mathrm{I}$.

We state here two facts about the Euler number $\chi(G)$ of compact, semi-simple Lie groups G. But we don't use them in our discussion.

First we consider a compact, semi-simple symmetric space M which is not necessarily a projective plane in the wider sense. Denote by A a maximal flat torus in M which passes through o. Let M_+ be a polar of o. Assume that M_+ is the orbit of $\exp(v_i)(o)$ (or $\exp(v_\iota/2)(o)$). Then we have the following theorem where n is the rank of M and Σ means the sum over all polars M_+ of o.

THEOREM 3.6. $2^n=1+\Sigma c_\iota\#(W(G(M)))/c_0\#(W(K_i(M)))$.

This theorem can be obtained from two identities $2^n=\Sigma n_\iota$ (cf. Corollary 6.6 [5]) and $n_i=c_\iota\#(W(G(M)))/c_0\#(W(K_i(M)))$ (cf. Proposition 3.4). Notice that $n_0=1$ always.

When M is a compact, semi-simple Lie group G, we set $G_+=M_+$ and $e=o$ (the identity element). Then $G_+=G/K_\iota$ holds for some i and the isotropy group K_ι has the maximal rank.

THEOREM 3.7. $\chi(G_+)=c_\iota\#(W(G))/c_0\#(W(K_i))$.

If G is simply connected, we know that $c_0=c_i=1$ and K_ι is connected. Then the identity becomes the well-known one. Moreover, we obtain a Chen-Nagano's identity in Theorem 3.4 [6] from Theorem 3.6, 3.7. But their identity holds for all compact Lie groups.

COROLLARY 3.8. $$2^n=1+\Sigma\chi(G_+).$$

From the above arguments, we know that it is very important to determine the numbers $\{c_i\}$. And so we have calculated them for all compact irreducible symmetric spaces and for all orbits of $\exp(v_i)(o)$ (resp. $\exp(v_i/2)(o)$). In the table at the end of this paper, we list $\{c_i\}$ for projective planes in the wider sense.

EXAMPLE 3.9. Let M be a compact, irreducible symmetric space with the Dynkin diagram of type C_3. Then M has two locally isometric spaces. One is the bottom space and the other is the simply connected space. The examples of such M are $G^C(3, 3)^*$, $G^C(3, 3)$, $C\,\mathrm{I}(3)^*$, $C\,\mathrm{I}(3)$, $G^H(3, 3)^*$, $G^H(3, 3)$, $D\mathrm{III}(3)^*$, $D\mathrm{III}(3)$, $Sp(3)^*$, $Sp(3)$, $E\mathrm{VII}^*$ and $E\mathrm{VII}$. Then the extended Dynkin diagram and the highest root of M are always given by

$$\overset{-\mu}{\circledcirc}\Longrightarrow\overset{\lambda_1}{\bigcirc}\!-\!\!\!-\!\overset{\lambda_2}{\bigcirc}\Longleftarrow\overset{\lambda_3}{\bigcirc} \qquad \text{and} \qquad \mu=2\lambda_1+2\lambda_2+\lambda_3.$$

(1) Let M be the bottom space. Let each vertex v_i of D correspond to the simple root λ_i. Now $o=\exp(v_0)(o)=\exp(v_3)(o)$ holds. Hence we have $c_0=2$. Since v_1 and v_2 are conjugate, we get $c_1=2$ (or $c_2=2$). The point conjugate to $v_3/2$ does not exist in D except itself. This means $c_3=1$. Note that we use the notations $\{c_3, n_3, r_3\}$ for $v_3/2$. When we regard the above extended Dynkin diagram of M as that of $Sp(3)$ (i.e., $G(M)=Sp(3)$), the numbers $\{r_i\}$ can be given by

(v_0)		$\bigcirc\!-\!\!\!-\!\bigcirc\Longleftarrow\bigcirc$	$r_0=\#(W(C_3))=2^3 3!$	$c_0=2$,
(v_1)	$\circledcirc$	$\bigcirc\Longleftarrow\bigcirc$	$r_1=\#(W(A_1\times C_2))=2^3 2!$	$c_1=2$,
$(v_3/2)$		$\bigcirc\!-\!\!\!-\!\bigcirc\quad\bigcirc$	$r_3=\#(W(A_2\times D_1))=3!$	$c_3=1$.

Then we have three orbits of o, $\exp(v_1)(o)$ and $\exp(v_3/2)(o)$. By Proposition 3.4, we obtain

$$\#(W(C_3))/c_0=\#(W(A_1\times C_2))\times n_1/c_1=\#(W(A_2\times D_1))\times n_3/c_3.$$

It follows that $n_1=3$ and $n_3=4$. Certainly $2^n=1+n_1+n_3$ holds. (cf. Theorem 3.6).

(2) Let M be simply connected. Then $\exp(v_1)(o)$ and $\exp(v_2)(o)$ are not conjugate but they become two polars with the same type. The orbit of $\exp(v_3/2)(o)$ is not a polar since $\exp(v_3)(o)$ is a pole (=a polar consisting of one point) of o. Hence there exist three polars which consist of the orbits of $\exp(v_i)(o)$ $(i=1, 2, 3)$. So we can caluculate $\{r_i\}$ as follows. In this case, we

use the notions $\{c_3, n_3, r_3\}$ for v_3;

(v_0)	$\bigcirc - \bigcirc \Leftarrow \bigcirc$	$r_0=\#(W(C_3))=2^3 3!$	$c_0=1$,
(v_1)	$\circledcirc \qquad \bigcirc \Leftarrow \bigcirc$	$r_1=\#(W(A_1\times C_2))=2^3 2!$	$c_1=1$,
(v_2)	$\circledcirc \Rightarrow \bigcirc \qquad \bigcirc$	$r_2=\#(W(C_2\times A_1))=2^3 2!$	$c_2=1$,
(v_3)	$\circledcirc \Rightarrow \bigcirc - \bigcirc$	$r_3=\#(W(C_3))=2^3 3!$	$c_3=1$.

The symmetry of the fundamental cell D disappears since all $c_i=1$. By Proposition 3.4, we have

$$\#(W(C_3))/c_0=\#(W(A_1\times C_2))\times n_1/c_1=\#(W(C_2\times A_1))\times n_2/c_2=\#(W(C_3))\times n_3/c_3.$$

These identities give $n_1=3$, $n_2=3$ and $n_3=1$. Also $2^n=1+n_1+n_2+n_3$ holds.

EXAMPLE 3.10. (cf. [2]). Let $M=E_6/(Spin(10)\times T)/Z_4$ ($=E\mathrm{III}$ simply) where T is the one dimensional torus and Z_4 is the cyclic group of order 4. The rank of M is two. Let A be a maximal flat torus in M and D a fundamental cell of T_oA. Take a fundamental root system $\{\lambda_1, \lambda_2\}$ such that the highest root μ is equal to $2\lambda_1+2\lambda_2$. Then we have the following:

a set of positive roots:

$$\Delta^+=\{\lambda_1, \lambda_2, 2\lambda_1, 2\lambda_1+\lambda_2, \lambda_1+\lambda_2, 2\lambda_1+2\lambda_2\},$$

the multiplicity of positive roots: 8, 6, 1, 1, 6, 8,

the extended Dynkin diagram: $\underset{-\mu}{\circledcirc} \Rightarrow \underset{\lambda_2}{\bigcirc} \Rightarrow \underset{\lambda_1}{\bigcirc}$

the type of orbits M_+ of $\exp(v_i)(o)$ as symmetric spaces:

$$(v_1)\ G^{OR}(2, 8) \quad \text{and} \quad (v_2)\ D\mathrm{III}(5),$$

the type of the orthogonal complement M_- to M_+:

$$(v_1)\ G^{OR}(2, 8) \quad \text{and} \quad (v_2)\ S^2\times G^C(1, 5),$$

the cardinal number $\#(M_+\cap A)$: (v_1) 1 and (v_2) 5.

Hence

$$(M, M_+)=(E\mathrm{III}, G^{OR}(2, 8)).$$

is a projective plane in the wider sense. In this plane $L(o)\cap A=\{\exp(v_1)(o)\}$ holds, that is, S_o consists of one point. Let R_u be the set of positive roots which satisfy $u\in U_\lambda$. If $u\in A\cap L(o)$, $R_u=\{\lambda_2, 2\lambda_1, 2\lambda_1+\lambda_2, 2\lambda_1+2\lambda_2\}$. If $u\in A\cap D\mathrm{III}(5)$, $R_u=\{\lambda_1+\lambda_2, 2\lambda_1+2\lambda_2\}$ or $R_u=\{\lambda_1, 2\lambda_1\}$ (but these sets are conjugate to each other). We know that $u\in A\cap D\mathrm{III}(5)$ if and only if u is on a closed geodesic in A with the minimal length. Then moreover $u\in A$ satisfies the condition $\Xi_u\cap(\Delta-\Delta_0)\neq\emptyset$. Thus o and u are in the singular position. After all, we obtain that

(1) for two points in the general position, there exists only one line which passes through them,

(1)* two lines in the general position intersect at only one point.

(2) for two points in the singular position, the set of all lines passing through them becomes $\boldsymbol{C}P_4$ as a symmetric space.

(2)* the intersection of two lines in the singular position becomes $\boldsymbol{C}P_4$.

DEFINITION 3.11. Let $p, q \in M$. We consider the two following statements (a) and (b);

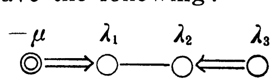
(a): p and q are in the singular position in the sense of symmetric spaces (cf. p. 295 [8]).

(b): p and q are in the singular position in the sense of projective planes in the wider sense.

Generally (b)$\Rightarrow$(a) holds but the converse does not always hold. So, if (a)$\Rightarrow$(b), we call M of type I and, if not so, we call M of type II.

EXAMPLE 3.12. The usual projective planes M (i.e., being not in the wider sense) are of type II since there does not exist two points in the singular position.

Now we consider $(E\mathrm{VII}^*, (T\cdot E\mathrm{IV})/Z_2)$ with the type I. The rank of M is three. Let A be a maximal flat torus in M and let D be a fundamental cell of T_oA. Take a fundamental root system $\{\lambda_1, \lambda_2, \lambda_3\}$ such that the highest root μ is equal to $2\lambda_1+2\lambda_2+\lambda_3$. Then we have the following:

the extended Dynkin diagram: 

the type of orbits M_+ of $\exp(v_i)(o)$ as symmetric spaces:

$$(v_1)\ E\mathrm{III} \quad \text{and} \quad (v_3/2)\ (T\cdot E\mathrm{IV})/Z_2$$

where v_1 and v_2 are conjugate to each other and the orbit of $\exp(v_3/2)(o)$ is a polar of o since $o=\exp(v_3)(o)$.

the type of the orthogonal complement M_- to M_+:

$$(v_1)\ S^2\times G^{\boldsymbol{R}}(2, 10) \quad \text{and} \quad (v_3/2)\ (T\cdot E\mathrm{IV})/Z_2$$

the cardinal number $\#(M_+\cap A)$: (v_1) 3 and $(v_3/2)$ 4

the Cartan matrix C for the set Δ of roots: $C=(a_{ij})$,

$$C=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Define a basis $\{x_i\}$ of T_oA by $\lambda_i(x_j)=a_{ij}$. Then the vertexes $\{v_i\}$ of D are given by

$$v_1=x_1/2+x_2/2+x_3/2,$$
$$v_2=x_1/2+x_2+x_3,$$
$$v_3=x_1/2+x_2+3x_3/2.$$

Since $\#(L(o)\cap A)=4$, we construct in A four conjugate points to $\exp(v_3/2)(o)$ explicitly. Let $i\in\{1, 2, 3, 4\}$. Then $\{x\in T_oA \mid \lambda_i(x)=0\}$ is a wall in T_oA. Define a reflection map W_ι across this wall by $x\to x-\lambda_i(x)x_\iota/\pi\boldsymbol{i}$. We can find four conjugate points $\{\exp(z_\iota)(o)\}$ to $\exp(v_3/2)(o)$ when we operate $\{W_\iota\}$ to $v_3/2$ repeatedly;

$$\begin{aligned} z_1&= x_1/4+x_2/2+3x_3/4 && (z_1=v_3/2),\\ z_2&= x_1/4+x_2/2+x_3/4 && (\text{by } W_3(z_1)=z_2),\\ z_3&= x_1/4+x_3/4 && (\text{by } W_2(z_2)=z_3),\\ z_4&=3x_1/4+x_3/4 && (\text{by } W_1(z_3)=z_4).\end{aligned}$$

Let $\Delta_o=\{\lambda\in\Delta \mid S_o\subset U_\lambda\}$ as before. Then it holds that

$$\begin{aligned}\lambda\in\Delta_o &\Longleftrightarrow \lambda(z_\iota)\in\pi\boldsymbol{iZ} \qquad (\iota=1, 2, 3, 4)\\ &\Longleftrightarrow \lambda(z_1)\in\pi\boldsymbol{iZ} \text{ and } \lambda(x_\iota/2)\in\pi\boldsymbol{iZ} \qquad (\iota=1, 2, 3).\end{aligned}$$

The last condition gives $\Delta_o=\emptyset$. This means that M is of type I because $\lambda\in\Xi_u$ if and only if $\lambda\in\Xi_u\cap(\Delta-\Delta_o)$.

In the following table we list the classification of projective planes (M, M_+) in the wider sense where M's are irreducible compact symmetric spaces. And c_0 (resp. c_ι) denotes the number of all conjugate points to the origin v_0 (resp. v_ι or $v_i/2$) in the fundamental cell. The suffix $i(>0)$ corresponds to the vertex v_i or $v_i/2$ such that M_+ is the orbit of $\exp(v_i)(o)$ or of $\exp(v_\iota/2)(o)$. $\#(M_+)$ denotes the cardinal number of the intersection $L(o)\cap A$.

Classification of projective planes in the wider sense.

M	M_+	(c_0, c_ι)	$\#(M_+)$	Type
(Exceptional spaces)				
E II	$S^2\cdot G^C(3, 3)$	(1, 1)	12	I
E III	$G^{OR}(2, 8)$	(1, 1)	1	II
E V*	A I (8)/Z_4	(2, 1)	36	I
E VI	$G^{OR}(4, 8)$	(1, 1)	3	II
	$S^2\cdot D$ III (6)	(1, 1)	12	I
E VII*	$(T\cdot E\text{IV})/Z_2$	(2, 1)	4	I
E VIII	$G(8, 8)^{\#}$	(1, 1)	135	I
E IX	$S^2\cdot E$ VII	(1, 1)	12	I
F I	$S^2\cdot C$ I (3)	(1, 1)	12	I
F II	S^8	(2, 1)	1	II
G I	$S^2\cdot S^2$	(1, 1)	3	I

(Classical spaces)

AIII	$G(2p, q)$ $(2p\neq q,\ p\leqq q)$	$G(p, p)\times G(p, q-p)$	(1, 1)	${}_{2p}C_p$, $(2p<q)$ ${}_{q}C_p$, $(2p>q)$	II
	$G(2p, 2p)^*$	$G(p, p)\cdot G(p, p)$	(2, 1)	${}_{2p}C_p/2$	II
	$G(p, p)^*$	$U(p)/Z_2$	(2, 1)	2^{p-1}	I $(p\neq 1)$
B I	$G(2p, q)$ $(2p\neq q,\ p\leqq q,\ q: odd)$	$G(p, p)\times G(p, q-p)$	(2, 1)	${}_{2p}C_p$, $(2p<q)$ ${}_{q}C_p$, $(2p>q)$	I
C I	C I$(n)^*$	U I$(n)/Z_2$	(2, 1)	2^{n-1}	I $(n\neq 1)$
C II	$G(2p, q)$ $(2p\neq q,\ p\leqq q)$	$G(p, p)\times G(p, q-p)$	(1, 1)	${}_{2p}C_p$, $(2p<q)$ ${}_{q}C_p$, $(2p>q)$	II
	$G(2p, 2p)^*$	$G(p, p)\cdot G(p, p)$	(2, 1)	${}_{2p}C_p/2$	II
D I	$G(2p, q)$ $(2p\neq q,\ p\leqq q,\ q: even)$	$G(p, p)\times G(p, q-p)$	(2, 1)	${}_{2p}C_p$, $(2p<q)$ ${}_{q}C_p$, $(2p>q)$ (The type of $G(4, 2)$ is II)	I
	$G^{OR}(2p, 2p)^*$	$G(p, p)\cdot G(p, p)$	(4, 1)	${}_{2p}C_p/2$	I
DIII	DIII$(2n)^*$	U II$(2n)/Z_2$	(2, 1)	2^{n-1}	I

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