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PERMUTABILITY OF ENTIRE FUNCTIONS

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1. Introduction. Two entire functions $f(z)$ and $g(z)$ are said to be permutable if they satisfy the relation

$$f(g(z))=g(f(z)) \tag{1}$$

in the complex plane. As a continuing study of Zheng and Zhou's earlier paper [8] on permutability of entire functions which are entire solutions of certain differential equations, in this note, we shall exhibit some extensions of results in [8] and [2]. And with the aid of theory of differential equation, we will discuss the following question which was posed by H. Urabe [5]:

Question A: What can be said about $f(z)$ and $g(z)$, if both f and g are entire and satisfy the relation

$$f(f(z))=g(g(z))? \tag{2}$$

Let P denote a differential polynomial in $w, w', \cdots, w^{(n)}$ $(n\geqq 1)$ with polynomials as coefficients, i.e.

$$P=P(z, w, w', \cdots, w^{(n)})=\sum_{\lambda\in I} c_\lambda(z)w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$$

where $c_\lambda(z)$'s are polynomials and I is a finite set of multi-indices $\lambda=(i_0, \cdots, i_n)$ for which $c_\lambda(z)\neq 0$ and $i_0, i_1, \cdots, i_n$ are non-negative integers and $D(P)=\max_I\{i_0+i_1+\cdots+i_n\}>0$.

By using the same method as in the proof of Theorem 1 in [8], we immediately get the following.

THEOREM 1. *Suppose that $f(z)$ and $g(z)$ are permutable transcendental entire functions of finite order and $f(z)$ is of positive lower order. If $f(z)$ satisfies the equation $P=0$, then there exist polynomials $a_j(z)$ $(j\in J=\{(j_0, \cdots, j_n);\ 0\leqq j_k\leqq D(P),\ k=0, 1, \cdots, n\}$ not all identically zero such that*

$$\sum_{j\in J} a_j(z)g^{j_0}(g')^{j_1}\cdots(g^{(n)})^{j_n}=0, \tag{3}$$

with $\max_J\{j_0+j_1+\cdots+j_n\}\leqq(n+1)D(P)$.

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In order to prove our main result in the sequel, we need the following result about linear differential equation of the second order. The result itself is interesting and useful.

THEOREM 2. *Let $P_0(z)(\not\equiv 0)$, $P_1(z)$, and $P_2(z)$ be three polynomials with degree m_0, m_1 and m_2 respectively and Q a polynomial. Then any entire solution $g(z)$ of the linear differential equation*

$$P_0(z)g''(z)+P_1(z)g'(z)+P_2(z)g(z)=Q(z) \tag{4}$$

is of perfectly regular growth and

(i) *when* $2m_1 \leqq m_0+m_2$, $\rho(g)=1+(m_2-m_0)/2$;

(ii) *when*, $2m_1>m_0+m_2$, $\rho(g)=1+(m_1-m_0)$;

where $\rho(g)$ denotes the order of $g(z)$.

This can be verified directly by virtue of the Wiman-Valiron theory (see [6], Chapter 4).

Now our main result can be stated as follows:

THEOREM 3. *Let $f(z)=\sin z+Q(z)$, where $Q(z)$ is a polynomial and $g(z)$ a non-linear entire function of finite order, which is permutable with $f(z)$. Then*

(i) *when* $\deg Q=0$, $g=f$ *or* $g=-\sin z+\kappa\pi$, $f=\sin z+\kappa\pi$,

(ii) *when* $\deg Q=1$, $g=f+2\kappa\pi$ *or* $-f+2\kappa\pi$ *for some integer* κ,

(iii) *when* $\deg Q>1$ *and* $Q(z)\not\equiv -Q(-z)$, *we have* $g=f$,

(iv) *when* $\deg Q>1$ *and* $Q(z)\equiv -Q(-z)$, *we have* $g=f$ *or* $-f$.

2. *Proof of Theorem* 3. We first relate a lemma which is useful in itself, and is an easy consequence of a result of A. Mokhon'ko [3].

LEMMA. *Let $f(z)$ be a transcendental entire function and $a_k(z)$ $(k=0, 1, \cdots, n)$ be $n+1$ meromorphic functions with*

$$T(r, a_k(z))=o(T(r, f)).$$

Then the identity

$$a_n(z)f^n(z)+\cdots+a_1(z)f(z)\equiv a_0(z)$$

holds if and only if $a_n(z)\equiv\cdots\equiv a_0(z)\equiv 0$.

Now we return to the Proof Theorem 3. Obviously $f(z)$ satisfies the following two differential equations

$$f''(z)+f(z)=Q+Q''=A(z) \quad \text{(say)} \tag{5}$$

$$(f(z)-Q(z))^2+(f'(z)-Q'(z))^2=1. \tag{6}$$

Succesively differentiating both sides of the equality $f(g(z))=g(f(z))$ twice, we have

$$f'(g)g'=g'(f)f',$$

$$f''(g)g'^2+f'(g)g''=g''(f)f'^2+g'(f)f''.$$

Substituting the above equalities into (5), (6) respectively, we get

$$(f'/g')^2g''(f)+(f''/g'^2-g''f'/g'^3)g'(f)+g(f)=A(g), \tag{7}$$

$$(g(f)-Q(g))^2+[(f'/g')g'(f)-Q'(g)]^2=1. \tag{8}$$

On the other hand, it follows from (5), Theorem 1 that there exist polynomials $P_i(z)$ ($i=0, 1, 2, 3$) not all identically zero such that

$$P_0g''+P_1g'+P_2g=P_3. \tag{9}$$

We may assume that $P_0(z)\not\equiv 0$. Indeed, when $P_0(z)\equiv 0$, by a result of Yanagihara [7, Theorem 1], $P_1(z)\not\equiv 0$ unless $g(z)$ is linear. Thus by differentiating both sides of (9), we get a second order differential equation of g with $P_1(z)$ as the coefficient of g''. By (6) and the same method as in the proof of Theorem 1, there exist polynomials $Q_i(z)$ ($i=0, 1, 2, 3, 4$) not all identically zero such that

$$Q_0g^2-2Q_1g+Q_2g'^2-2Q_3g'=Q_4. \tag{10}$$

Now we show that Q_0 in (10) is not identically zero. If $Q_0(z)\equiv 0$, then (10) becomes

$$-2Q_1g+Q_2g'^2-2Q_3g'=Q_4. \tag{11}$$

Applying the Wiman-Valiron theory to (9), we can verify that $g(z)$ has a positive lower order, moreover by a result of Zheng and Zhou [8, Theorem 2], we have that $\rho(g)=\rho(f)=1$. It follows from this, Theorem 1, and equation (11) that there exist polynomials A_1, A_2, A_3 and A_4 not all identically zero such that

$$2\imath A_1f+4A_2f'^2-2A_3f'=A_4. \tag{12}$$

That is

$$A_1(e^{\imath z}-e^{-\imath z})+A_2(e^{\imath z}+e^{-\imath z})^2+(4A_2Q'-A_3)(e^{\imath z}+e^{-\imath z})=B,$$

where B is a polynomial. By Borel's theorem (cf. Gross [1], p. 108), we have $A_2\equiv 0$, further, $A_1\equiv A_3\equiv 0$. This is a contradiction, and hence $Q_0(z)\not\equiv 0$.

Thus from (9) and (10), we have

$$P_0(f)g''(f)+P_1(f)g'(f)+P_2(f)g(f)=P_3(f), \tag{13}$$

$$Q_0(f)g^2(f)-2Q_1(f)g(f)+Q_2(f)g'^2(f)-2Q_3(f)g'(f)=Q_4(f). \tag{14}$$

Eliminating the term $g''(f)$ from (7) and (13) and term $g^2(f)$ from (8) and (14), we get

$$(P_0(f)(f''/g'^2-g''f'/g'^3)-P_1(f)(f'/g')^2)g'(f)+(P_0(f)-P_2(f)(f'/g)^2)g(f)$$
$$=P_0(f)A(g)-P_3(f)(f'/g')^2 \tag{15}$$

and

$$2(Q_1(f)-Q_0(f)Q(g))g(f)+(Q_0(f)(f'/g')^2-Q_2(f))g'^2(f)$$
$$+2(Q_3(f)-Q_0(f)Q'^2(g)(f'/g'))g'(f)=Q_0(f)C(g)-Q_4(f) \qquad (16)$$

where $C(z)=1-Q_2(z)-Q'^2(z)$.

We shall treat two cases, separately.

Case (I) $P_0(f)\not\equiv P_2(f)(f'/g')^2$. In the case, by finding the expression of $g(f)$ from (15) and then substituting it into (16), and by Lemma, we have

$$Q_0(f)(f'/g')^2=Q_2(f) \quad \text{i.e.}$$
$$g'^2=(Q_0(f)/Q_2(f))f'^2=R(f)f'^2, \qquad (17)$$

where $R(z)$ is a rational function. If there exists a finite complex number b such that $R(b)$ is infinite, then one can find a simple zero z_0 of $f(z)-b$ $(=\sin z+Q(z)-b)$ so that z_0 is a pole of the right of (17), but g' is entire. This is a contradiction. Thus it follows that $R(z)$ is a polynomial, and $q_0=\deg Q_0\geqq \deg Q_2=q_2$. By Wiman-Valiron method, we easily derive the fact:

Any entire function $g(z)$ satisfying the differential equation (10) is certainly of perfectly regular growth and has order $1+(q_0-q_2)/2$, when $q_0+2\geqq q_2$.

Since g has order 1, we have $q_0=q_2$, which implies that $R(z)$ is a constant. Therefore from (17) we have $g'=cf'$, and hence $g=cf+d$ for some constants $c(\neq 0)$ and d.

Case (II) $P_0(f)\equiv P_2(f)(f'/g')^2$. By the same method as in Case (I), we can prove that $P_2(z)/P_0(z)$ is a polynomial and $m_2=\deg P_2\geqq \deg P_0=m_0$. We shall prove that $m_2=m_0$. Suppose $m_2>m_0$, since g has order 1, by Theorem 2, we know that $m_0=m_1>m_2$, which contradict $m_2>m_0$. Thus it is showed that P_2 is a constant, so that $g=cf+d$ for some constants c and d. Hence we always have $g=cf+d$ for constants $c(\neq 0)$ and d. Thus it follows that

$$g(f)=f(g)=\sin(cf+d)+Q(cf+d), \quad \text{so that}$$
$$g(z)=\sin(cz+d)+Q(cz+d)=cf+d=c\sin z+cQ(z)+d. \qquad (18)$$

By Borel's theorem [cf. Gross 1, p. 108], we immediately have $c=1$ or $c=-1$.

When $c=1$, applying Borel's theorem again to (18), we have $d=2k\pi$ for some integer k. Thus it is showed that $Q(z)+d=Q(z+d)$. If $d\neq 0$, $Q(z)$ is certainly linear. If Q is a constant, $d=0$.

When $c=-1$, we have $d=2k\pi$ for some integer k and $Q(-z+d)=-Q(z)+d$. If $d\neq 0$, Q is linear. If $d=0$, then $Q(-z)=-Q(z)$. If Q is a constant, $Q=d/2=k\pi$, and $g=-\sin z+k\pi$, $f=\sin z+k\pi$.

Thus Theorem 3 follows.

3. In this section, we shall discuss the question (A) for some classes of entire functions that satisfy certain differential equation.

Let $f(z)$ be an entire function of positive lower order and $g(z)$ an entire function of finite order such that $f(f(z))=g(g(z))$. Then by Polya's theorem [4], we have

$$M(M(r, g), g)\geqq M(r, g(g))=M(r, f(f))\geqq M(cM(r/2, f), f)$$

for some positive constant c. It is clear that there exists a positive number K such that

$$\log M(r, g)>K\log M(r/2, f)>KT(r/2, f), \qquad \text{further}$$

$$3T(2r, g)>KT(r/2, f) \text{ i.e. } T(4r, g)>(K/3)T(r, f).$$

By the last inequality above, we can get the analogues of Theorem 1 for two entire functions satisfying (2) and of Theorem 2 in [8]. These two analogues may be stated as follows.

THEOREM 4. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and of positive lower order and let $P=P(z, w, w', \cdots, w^{(n)})$ be as in Theorem 1. Assume that $f(f(z))=g(g(z))$ and $f(z)$ satisfies $P=0$, then there exist polynomials $a_j(z)$ $(j\in J)$ not all identically zero such that* (3) *holds for $g(z)$ with $D(P)=\max_J\{j_0+j_1+\cdots+j_n\}$, where J and $D(P)$ are defined as in Theorem* 1.

THEOREM 5. *Let $f(z)$ be an entire function of finite lower order, $g(z)$ an entire function of finite order, and $f(f(z))=g(g(z))$. Then the order, lower order and type-class of $g(z)$ do not exceed the ones of $f(z)$.*

Furthermore, we relate the following two results without proofs since they are analogues of Theorems 3 and 4 in [8] and may be proved by exchanging f with g in suitable places in their proof.

THEOREM 6. *Let $f(z)=Q+He^P$, with Q and $H(\not\equiv 0)$ being two polynomials and P a non-constant polynomial. Suppose that $g(z)$ is an entire function of finite order such that $f(f(z))=g(g(z))$, then $g=cf+d$, where $c^n=1$, $d=a_{n-1}(c-1)/(na_n)$, $n=$degree of P, a_n and a_{n-1} are the coefficients of the first and second terms of P, respectively.*

THEOREM 7. *Let $f(z)=\sin P(z)$, where $P(z)$ is a non-constant polynomial. Let $g(z)$ be an entire function of finite order and $f(f(z))=g(g(z))$.*

i) *If $f(z)$ is an odd function, then $g=f$, or $g=-f$;*

ii) *If $f(z)$ is not an odd function, then when $\deg P=1$, we have $g=f$ and when $\deg P>1$, we have either $g=f$ or $g=-\sin(P(z)-c)$, where c satisfies $\exp(inc)=(-1)^{n+1}$, $n=\deg P$ and $P(\sin(z+c))=-P(\sin z)+c$.*

Remark. We don't think that we can get some analogues of Theorem 3 under the case when $f(f(z))=g(g(z))$ instead of $f(g(z))=g(f(z))$, but we conjecture that the result stated in Theorem 3 is true even under the condition

$f(f(z))=g(g(z))$.

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