T. OTSUKI
KODAI MATH. J.
13 (1990), 58—83

CERTAIN PROPERTIES OF $S(x, n)$

BY TOMINOSUKE OTSUKI

Let $S(x, n)$ be the function of x and n defined as follows:

$$S(x, n)=-BS_3(x, n)+(n-x)^{n-1}S_4(x, n), \tag{1}$$

where $B=(n-1)^{n-1}$,

$$\begin{aligned} S_3(x, n)=&(8n^2-5)x^3-2(8n^3+20n^2-15n+20)x^2 \\ &+3(24n^3-68n^2+42n-5)x+4n(4n-1)(4n-3) \end{aligned} \tag{2}$$

and

$$\begin{aligned} S_4(x, n)=&(n-1)(4n^2-10n+5)x^4+(8n^3-52n^2+87n-40)x^3 \\ &+3(12n^3-42n^2+37n-5)x^2+3n(16n^2-32n+9)x+12n^2(2n-1). \end{aligned} \tag{3}$$

The present author proved the following facts:

FACT 1. *$S(x, n)>0$ for $0\leqq x\leqq n$, $x\neq 1$, with $n\geqq 2$*

(Proposition 4 in [1]);

FACT 2. *$S(x, n)$ is decreasing in $0<x<1$ with $n\geqq 2$, and increasing in $1<x<n$ with $2\leqq n\leqq \dfrac{11+\sqrt{77}}{4}=4.9437410\cdots$*

(Proposition 8 in [2]).

The proof of the second part of Fact 2 was too long and worked out elaborately even though it was expected with $n\geqq 2$. We shall give another proof of it with $n\geqq 2$.

MAIN THEOREM. *$S(x, n)$ is increasing in $1<x<n$ with $n\geqq 2$.*

By means of the argument of § 3 in [2], setting $x=1+y$ and $n=1+m$, we have from (1) and (2)

$$\begin{aligned} S_3(1+y)=&(8m^2+16m+3)y^3-(16m^3+64m^2-22m-15)y^2 \\ &+10m(4m^2-14m-7)y+60m^2(2m+1), \end{aligned}$$

Received May 14, 1989

$$S_4(1+y)=m(4m^2-2m-1)y^4+3(8m^3-12m^2+m+1)y^3$$
$$+3(28m^3-38m^2-6m+5)y^2+10m(16m^2-8m-7)y$$
$$+60m^2(2m+1),$$

where $S_3(x)=S_3(x, n)$ and $S_4(x)=S_4(x, n)$ for simplicity, and

$$\frac{1}{B}\left[\frac{\partial}{\partial x}S(x, n)\right]_{x=1+y}=-10m(4m^2-14m-7)$$
$$+2(16m^3+64m^2-22m-15)y-3(8m^2+16m+3)y^2$$
$$+\left(1-\frac{y}{m}\right)^{m-1}\left\{\left(1-\frac{y}{m}\right)S_4'(1+y)-S_4(1+y)\right\}$$

and

$$(m-y)S_4'(1+y)-mS_4(1+y)=10m^2(4m^2-14m-7)$$
$$+2m(4m^3-154m^2+57m+50)y-3(4m^4+54m^3-85m^2$$
$$-10m+10)y^2-(8m^4+44m^3-101m^2+12m+9)y^3$$
$$-m(m+4)(4m^2-2m-1)y^4.$$

Furthermore, setting $y=m(1-t)$, from the above expressions we obtain

(4) $$\sigma(t)=\frac{1}{Bm}\left[\frac{\partial}{\partial x}S(x, n)\right]_{x=1+m(1-t)}$$
$$=L_2(t)-t^{m-1}L_4(t),$$

(5) $$L_2(t)=a_0+2a_1t-3a_2t^2,$$

where

(6) $$\begin{cases} a_0=a_0(m)=8m^3+40m^2+87m+40, \\ a_1=a_1(m)=8m^3-16m^2+31m+15, \\ a_2=a_2(m)=m(8m^2+16m+3) \end{cases}$$

and

(7) $$L_4(t)=b_0-b_1t+3b_2t^2-b_3t^3+b_4t^4,$$

where

(8) $$\begin{cases} b_0=b_0(m)=m(m+1)(4m^4+18m^3+29m^2+20m+5), \\ b_1=b_1(m)=(m+1)^2(16m^4+48m^3+8m^2-67m-40), \\ b_2=b_2(m)=(m+2)(8m^5+20m^4-10m^3-35m^2-3m+5), \\ b_3=b_3(m)=m(m+3)(16m^4+16m^3-40m^2+3m+3), \\ b_4=b_4(m)=m^3(m+4)(4m^2-2m-1). \end{cases}$$

It is sufficient to prove $\sigma(t)>0$ for $0<t<1$ with $m\geqq 1$.

Now, we have

$$L_2(1)=a_0+2a_1-3a_2=-10(4m^2-14m-7),$$

$$L_4(1)=b_0-b_1+3b_2-b_3+b_4=-10(4m^2-14m-7)$$

$$=L_2(1),$$

and hence we can put

$$L_4(t)-L_2(t)=b_4t^4-b_3t^3+3(a_2+b_2)t^2$$

$$-(2a_1+b_1)t+b_0-a_0=(t-1)L_3(t),$$

where

$$L_3(t)=b_4t^3+(b_4-b_3)t^2+(3a_2+3b_2-b_3+b_4)t+a_0-b_0.$$

Then, we have

$$L_3(1)=3b_4-2b_3+3b_2-b_0+3a_2+a_0$$

$$=40m^3-180m^2+70m+70$$

$$=10(m-1)(4m^2-14m-7)=-(m-1)L_2(1),$$

and hence we can put

$$L_3(t)+(m-1)L_2(t)=b_4t^3-\{3(m-1)a_2+b_3-b_4\}t^2$$

$$+\{2(m-1)a_1+3a_2+3b_2-b_3+b_4\}t+ma_0-b_0$$

$$=(t-1)L_2^*(t),$$

where

$$L_2^*(t)=b_4t^2-\{3(m-1)a_2+b_3-2b_4\}t-ma_0+b_0.$$

Next, we have

$$L_2^*(1)=3b_4-b_3+b_0-3(m-1)a_2-ma_0$$

$$=-20m^4+90m^3-35m^2-35m$$

$$=-5m(m-1)(4m^2-14m-7)=\frac{1}{2}m(m-1)L_2(1),$$

and hence we can put

$$2L_2^*(t)-m(m-1)L_2(t)=\{3m(m-1)a_2+2b_4\}t^2$$

$$-2\{3(m-1)a_2+m(m-1)a_1+b_3-2b_4\}t$$

$$-m(m+1)a_0+2b_0=(t-1)L_1^*,$$

where

$$L_1^*(t)=\{3m(m-1)a_2+2b_4\}t+m(m+1)a_0-2b_0$$

and

$$3m(m-1)a_2+2b_4=m^2(8m^4+52m^3+6m^2-47m-9)$$
$$=m^2(m+1)(8m^3+44m^2-38m-9),$$
$$-m(m+1)a_0+2b_0=m(m+1)(8m^4+28m^3+18m^2-47m-30).$$

Now, we set

(9) $$L_1(t)=d_1t-d_0,$$

where

(10) $$\begin{cases} d_0=d_0(m)=8m^4+28m^3+18m^2-47m-30, \\ d_1=d_1(m)=m(8m^3+44m^2-38m-9). \end{cases}$$

From these computations we obtain

$$2L_2^*(t)=m(m-1)L_2(t)-m(m+1)(1-t)L_1(t),$$
$$L_3(t)=-(m-1)L_2(t)$$
$$-(1-t)\left[\frac{m(m-1)}{2}L_2(t)-\frac{m(m+1)}{2}(1-t)L_1(t)\right]$$
$$=-(m-1)\left\{1+\frac{m}{2}(1-t)\right\}L_2(t)+\frac{m(m+1)}{2}(1-t)^2L_1(t),$$

and

$$L_4(t)=L_2(t)-(1-t)L_3(t),$$

i.e.

(11) $$L_4(t)=\left\{1+(m-1)(1-t)+\frac{m(m-1)}{2}(1-t)^2\right\}L_2(t)$$
$$-\frac{m(m+1)}{2}(1-t)^3L_1(t).$$

Thus we obtain an important expression:

(12) $$\sigma(t)=\varphi_1(t)L_1(t)+\varphi_2(t)L_2(t),$$

where

(13) $$\begin{cases} \varphi_1(t)=\dfrac{m(m+1)}{2}t^{m-1}(1-t)^3, \\ \varphi_2(t)=1-t^{m-1}\left\{1+(m-1)(1-t)+\dfrac{m(m-1)}{2}(1-t)^2\right\}. \end{cases}$$

LEMMA 1. i) *$d_1(m)$ is ↗ in $1<m<\infty$ and $\geqq 5$ for $m\geqq 1$,*
ii) *$d_0(m)$ is ↗ in $1<m<\infty$ and*

$$\begin{cases} d_0(m)<0 & \textit{for } 1\leqq m<1.17\cdots, \\ \quad >0 & \textit{for } 1.17\cdots<m<\infty, \end{cases}$$

iii) *$d_1(m)-d_0(m)$ is $\searrow\nearrow$ in $1<m<\infty$ and >0 for $m\geqq 1$.*

Proof. i) Since we have

$$8m^3+44m^2-38m-9\geqq 14m-9\geqq 5 \qquad \text{for } m\geqq 1$$

and

$$24m^2+88m-38\geqq 74 \qquad \text{for } m\geqq 1,$$

we obtain the claim i).

ii)

$$d_0'(m)=32m^3+84m^2+36m-47$$

$$\geqq 152m-47\geqq 105 \qquad \text{for } m\geqq 1,$$

and we see $d_0(m)$ is $\nearrow$ in $1<m<\infty$. Since we have $d_0(1.17)=-0.513538\cdots$ and $d_0(1.18)=1.118318\cdots$, we obtain the claim ii).

iii) We have

$$L_1(1)=d_1-d_0=16m^3-56m^2+38m+30,$$

and $24m^2-56m+19=-4.04$ at $m=1.8$ and $=3$ at $m=2$. Thus, we see that $d_1(m)-d_0(m)$ is $\searrow\nearrow$ in $1<m<\infty$, which is equal to 10.272 at $m=1.8$ and 10 at $m=2$. For $1.8\leqq m\leqq 2$, we have

$$d_1(m)-d_0(m)>10.272-2\times 4.04\times 0.2=8.656.$$

Therefore, it must be

$$d_1(m)-d_0(m)>8.656 \qquad \text{for } m\geqq 1.$$

Q. E. D.

LEMMA 2. *Regarding $L_2(t)$, we have the following:*

i) *a_0, a_1 and a_2 are positive for $m\geqq 1$;*

ii) *$1/3<a_1/3a_2<1/2$ for $1\leqq m<5/4$ and $0<a_1/3a_2\leqq 1/3$ for $m\geqq 5/4$;*

iii) *for the root $\beta(m)$ of the quadratic equation $L_2(t)=0$:*

$$\beta(m)=\frac{a_1+\sqrt{a_1a_1+3a_0a_2}}{3a_2}, \tag{14}$$

$$\beta(m)>1 \qquad \textit{for } 1\leqq m<\frac{7+\sqrt{77}}{4} \quad \textit{and}$$

$$\frac{3}{4}<\beta(m)<1 \qquad \textit{for } m>\frac{7+\sqrt{77}}{4}=3.9437410\cdots;$$

iv) $L_2(t)>0$ *for* $0\leqq t\leqq 1$ *with* $1\leqq m<\frac{7+\sqrt{77}}{4}$.

Proof. i) is evident. On ii) we have

$$\frac{a_1}{3a_2}=\frac{8m^3-16m^2+31m+15}{3m(8m^2+16m+3)}\longrightarrow\frac{1}{3}\qquad\text{as } m\to\infty .$$

The condition $a_1/3a_2\leqq 1/3$ is equivalent to

$$32m^2-28m-15=(4m-5)(8m+3)\geqq 0,\qquad \text{i.e. } m\geqq\frac{5}{4}.$$

Next, the condition $a_1/3a_2\leqq 1/2$ is equivalent to

$$8m^3+80m^2-53m-30>0,$$

whose left hand side $\geqq 5$ for $m\geqq 1$. These imply the claim ii).

iii) Since we have $L_2(1)=-10(4m^2-14m-7)$ and

$$L_2\left(\frac{3}{4}\right)=a_0+\frac{3}{2}a_1-\frac{27}{16}a_2$$
$$=\frac{1}{16}(104m^3-176m^2+2055m+1000)>0\qquad\text{for } m\geqq 1,$$

we obtain the claim iii) immediately.

iv) is evident from iii). Q.E.D.

LEMMA 3. i) $\varphi_1(t)>0$ *for* $0<t<1$ *with* $m\geqq 1$, *and*

$$\varphi_1'(t)=\frac{m(m+1)}{2}t^{m-2}(1-t)^2\{m-1-(m+2)t\}. \tag{15}$$

ii) $0<\varphi_2(t)<1$ *for* $0<t<1$ *with* $m>1$, *and*

$$\varphi_2'(t)=-\frac{(m+1)m(m-1)}{2}t^{m-2}(1-t)^2, \tag{16}$$

$$\varphi_2''(t)=-\frac{(m+1)m(m-1)}{2}t^{m-3}(1-t)(m-2-mt). \tag{17}$$

iii) $\varphi_1(1)=\varphi_2(1)=0$.

Proof. We obtain easily (15), (16) and (17) from (13). The rest things are evident from them. Q.E.D.

LEMMA 4. *We have*

$$\alpha(m)=d_0(m)/d_1(m)<\beta(m)\qquad \textit{for } m\geqq 1.$$

Proof. Since $\beta(m)>3/4$ for $m\geqq 1$ by iii) of Lemma 2 and $\alpha(m)<0$ for $1\leqq$

$m<1.17\cdots$ by ii) of Lemma 1, we shall prove the inequality for $m>1.17\cdots$. Since $a_2(m)>0$ for $m\geqq 1$, it is sufficient to show

$$a_0d_1^2+2a_1d_1d_0-3a_2d_0^2>0 .$$

By (6) and (10) we have

$$\begin{aligned}a_0d_1-a_1d_0&=m(8m^3+40m^2+87m+40)(8m^3+44m^2-38m-9)\\&\quad-(8m^3-16m^2+31m+15)(8m^4+28m^3+18m^2-47m-30)\\&=m(64m^6+672m^5+2152m^4+2556m^3-1906m^2\\&\quad-2303m-360)-(64m^7+96m^6-56m^5+324m^4\\&\quad+1490m^3-707m^2-1635m-450)\\&=3(192m^6+736m^5+744m^4-1132m^3-532m^2+425m+150),\end{aligned}$$

$$\begin{aligned}(a_1d_1-a_2d_0)/m&=(8m^3-16m^2+31m+15)(8m^3+44m^2-38m-9)\\&\quad-(8m^2+16m+3)(8m^4+28m^3+18m^2-47m-30)\\&=64m^6+224m^5-760m^4+2020m^3-374m^2\\&\quad-849m-135)-(64m^6+352m^5+616m^4-4m^3\\&\quad-938m^2-621m-90)\\&=-128m^5-1376m^4+2024m^3+564m^2-228m-45,\end{aligned}$$

and hence

$$\begin{aligned}&(a_0d_1^2+2a_1d_1d_0-3a_2d_0^2)/3m\\&=(8m^3+44m^2-38m-9)(192m^6+736m^5+744m^4\\&\quad-1132m^3-532m^2+425m+150)\\&\quad-(8m^4+28m^3+18m^2-47m-30)(128m^5+1376m^4\\&\quad-2024m^3-564m^2+228m+45)\\&=1536m^9+14336m^8+31040m^7-6016m^6-88960m^5\\&\quad+16312m^4+50304m^3-4762m^2-9525m-1350\\&\quad-1024m^9-14592m^8-24640m^7+42432m^6+118912m^5\\&\quad-50440m^4-92592m^3-7014m^2+8955m+1350,\end{aligned}$$

i. e.

(18) $$a_0d_1^2+2a_1d_1d_0-3a_2d_0^2=6m^2\times P(m),$$

where

(19) $$P(m)=256m^8-128m^7+3200m^6+18208m^5 +14976m^4-17064m^3-21144m^2-5888m-285\,.$$

Since we have

$$\begin{aligned}P'(m)&=2048m^7-896m^6+19200m^5+91040m^4\\&\quad+59904m^3-51192m^2-42288m-5888\\&\geqq 20352m^5+91040m^4+59904m^3-51192m^2-42288m-5888\\&\geqq 171296m^3-51192m^2-42288m-5888\\&\geqq 77816m-5888>0 \qquad \text{for } m\geqq 1,\end{aligned}$$

$P(m)$ is increasing in $1<m<\infty$. We have

$$P(1)=-7869 \quad\text{and}\quad P(1.1)=2160.6218\,,$$

which implies $P(m)>0$ for $m>1.1$. From these facts we see that

$$\alpha(m)<\beta(m) \qquad \text{for } m\geqq 1.1.$$

Q. E. D.

Proof of Main Theorem.

When $1\leqq m\leqq 1.17\cdots$ (the root of $d_0(m)=0$),

$$L_1(t)>0 \quad\text{and}\quad L_2(t)>0 \qquad \text{for } 0<t<1$$

by Lemma 1 and Lemma 2. By Lemma 3 and (12) we obtain

$$\sigma(t)>0 \qquad \text{for } 0<t<1\,.$$

When $m>1.17\cdots$, we have $0<\alpha(m)<1$ by Lemma 1. We shall show $\sigma(t)>0$ for $0\leqq t\leqq\alpha(m)$. If it does not hold, then there exist ξ_1, ξ_2 such that $0<\xi_1\leqq\xi_2<\alpha(m)$,

$$\sigma(\xi_1)=0,\quad \sigma'(\xi_1)\leqq 0 \quad\text{and}\quad \sigma(\xi_2)=0,\quad \sigma'(\xi_2)\geqq 0\,.$$

We have $L_2(t)>0$ for $0\leqq t\leqq\alpha(m)$ by Lemma 2 and

$$\begin{aligned}\Big(\varphi_2+\varphi_1\frac{L_1}{L_2}\Big)'&=\frac{1}{L_2^2}[\varphi_2'L_2^2+\varphi_1'L_1L_2+\varphi_1(L_1'L_2-L_1L_2')]\\&=\frac{1}{L_2^2}\Big[-\frac{(m+1)m(m-1)}{2}t^{m-2}(1-t)^2L_2^2\\&\qquad+\frac{(m+1)m}{2}\{m-1-(m+2)t\}L_1L_2\\&\qquad+\frac{(m+1)m}{2}t^{m-1}(1-t)^3(L_1'L_2-L_1L_2')\Big]\end{aligned}$$

$$(20)\qquad =\frac{(m+1)m}{2L_2^2}t^{m-2}(1-t)^2[-(m-1)L_2^2$$
$$+\{m-1-(m+2)t\}L_1L_2+t(1-t)(L_1'L_2-L_1L_2')]$$

by Lemma 3. Now, we compute the expression in the above brackets. We obtain by (5), (6) and (9), (10)

$$-(m-1)L_2^2+\{m-1-(m+2)t\}L_1L_2+t(1-t)(L_1'L_2-L_1L_2')$$
$$=L_2[-(m-1)L_2+\{m-1-(m+2)t\}L_1+t(1-t)L_1']-t(1-t)L_1L_2'$$
$$=L_2[-(m-1)(a_0+2a_1t-3a_2t^2)+\{m-1-(m+2)t\}(d_1t-d_0)$$
$$+t(1-t)d_1]-2t(1-t)(d_1t-d_0)(a_1-3a_2t)$$
$$=-(a_0+2a_1t-3a_2t^2)[(m-1)(a_0+d_0)$$
$$-\{(m+2)d_0+md_1-2(m-1)a_1\}t+\{(m+3)d_1-3(m-1)a_2\}t^2]$$
$$+2t(1-t)\{d_0a_1-(d_1a_1+3d_0a_2)t+3d_1a_2t^2\},$$

and

$$-(m-1)a_0(a_0+d_0)$$
$$=-2(m-1)a_0(4m^4+18m^3+29m^2+20m+5)=-2c_0\,,$$
$$a_0\{(m+2)d_0+md_1-2(m-1)a_1\}-2(m-1)a_1(a_0+d_0)+2d_0a_1$$
$$=2d_0\{(m+1)a_0-(m-2)a_1\}+m(d_1-d_0)a_0-4(m-1)a_0a_1$$
$$=6(192m^7+864m^6+1376m^5+540m^4-1307m^3$$
$$-2183m^2-1340m-300)=6c_1,$$
$$-a_0\{(m+3)d_1-3(m-1)a_2\}+2a_1\{(m+2)d_0+md_1-2(m-1)a_1\}$$
$$+3(m-1)a_2(a_0+d_0)-2(d_1a_1+3d_0a_2)-2d_0a_1$$
$$=\{-(m+3)a_0+4ma_1+3(m-3)a_2\}d_1$$
$$-\{2(m+1)a_1+3(m-3)a_2\}(d_1-d_0)+2(m-1)(3a_0a_2-2a_1^2)$$
$$=6m(64m^7+64m^6-632m^5-1172m^4-142m^3+783m^2+475m+15)=6c_2,$$
$$-2a_1\{(m+3)d_1-3(m-1)a_2\}$$
$$-3a_2\{(m+2)d_0+md_1-2(m-1)a_1\}+6d_1a_2+2(d_1a_0+3d_0a_2)$$
$$=-2\{(m+2)a_1+3(m-1)a_2\}d_1+3ma_2(d_1-d_0)+12(m-1)a_1a_2$$
$$=-2m^2(256m^6+1024m^5+192m^4-1696m^3+204m^2-385m-159)=-2c_3,$$

$$3a_2\{(m+3)d_1-3(m-1)a_2\}-6d_1a_2$$
$$=6m^3(4m^3+14m^2-9m-4)(8m^2+16m+3)$$
$$=6m^3(m+4)(4m^2-2m-1)(8m^2+16m+3)=6c_4 .$$

Thus, we obtain the formula:

$$\left(\varphi_2(t)+\varphi_1(t)\frac{L_1(t)}{L_2(t)}\right)'=\frac{(m+1)m}{L_2(t)^2}t^{m-2}M_4(t), \tag{21}$$

where

$$M_4(t)=-c_0+3c_1t+3c_2t^2-c_3t^3+3c_4t^4 , \tag{22}$$

$$\begin{cases} c_0=c_0(m)=(m-1)(8m^3+40m^2+87m+40)(4m^4+18m^3+29m^2+20m+5), \\ c_1=c_1(m)=192m^7+864m^6+1376m^5+540m^4-1307m^3-2183m^2-1340m-300, \\ c_2=c_2(m)=m(64m^7+64m^6-632m^5-1172m^4-142m^3+783m^2+475m+15, \\ c_3=c_3(m)=m^2(256m^6+1024m^5+192m^4-1696m^3+204m^2-385m-159), \\ c_4=c_4(m)=m^3(m+4)(8m^2+16m+3)(4m^2-2m-1). \end{cases} \tag{23}$$

We obtained the same auxiliary function $M_4(t)=M_4(t, m)$ in § 3 in [2], which is derived from (4). We can use $\sigma(t)/L_2(t)$ in place of $\sigma(t)$ for the interval $0\leqq t\leqq \alpha(m)$, and so we may put

$$M_4(\xi_1)\leqq 0 \quad \text{and} \quad M_4(\xi_2)\geqq 0 , \tag{24}$$

which contradicts the following lemma.

LEMMA 5. *For m such that $d_0(m)>0$, we have*

$$M_4(t, m)<0 \qquad \textit{for } 0\leqq t\leqq \alpha(m)=d_0(m)/d_1(m) .$$

Proof. Since $c_0>0$ for $m>1$, we have $M_4(0, m)<0$. From (20) and (21) we have at $t=\alpha=\alpha(m)$

$$2M_4(\alpha)=-(m-1)L_2^2+\alpha(1-\alpha)d_1L_2$$
$$=-L_2\{(m-1)L_2-\alpha(1-\alpha)d_1\}$$
$$=-\frac{L_2}{d_1^2}\{(m-1)(a_0d_1^2+2a_1d_1d_0-3a_2d_0^2)-d_0d_1(d_1-d_0)\}$$

and by (18), (19)

$$(m-1)(a_0d_1^2+2a_1d_1d_0-3a_2d_0^2)-d_0d_1(d_1-d_0)$$
$$=6m^2(m-1)P(m)-2m(8m^4+28m^3+18m^2-47m-30)$$
$$\times(8m^3+44m^2-38m-9)(8m^3-28m^2+19m+15)$$
$$=4m\times Q(m) ,$$

where

$$Q(m)=128m^{10}-1984m^9+8160m^8+34448m^7-16456m^6 \tag{25}$$
$$-95892m^5+28102m^4+65128m^3-4944m^2-13860m-2025\,.$$

We show that $Q(m)>0$ for $m\geqq 1$. Since we have

$$\frac{1}{8\times 5!}Q^{(5)}(m)=4032m^5-31248m^4+57120m^3$$
$$+90426m^2-12342m-11986.5\,,$$

and $4032m^3-31248m^2+57120m+90426>76918$ for $m\geqq 1$, because $3\times 4032m^2-2\times 31248m+57120$ is equal to 6720, -2634.24, 672 at $m=1$, 3.9, 4, respectively and $31248/3\times 4032=2.58\cdots$, and $4032m^3-31248m^2+57120m+90426$ is equal to 120330, 77086.128, 76986 at $m=1$, 3.9, 4, respectively and $>76986-672\times 0.1=76918.8$ for $3.9<m<4$, therefore

$$\frac{1}{8\times 5!}Q^{(5)}(m)>76918m^2-12342m-11986.5\geqq 76918.8 \qquad \text{for } m\geqq 1\,.$$

Hence we have

$$\frac{1}{4!}Q^{(4)}(m)=26880m^6-249984m^5+571200m^4+1205680m^3$$
$$-246840m^2-479460m+28102\geqq 855578 \qquad \text{for } m\geqq 1$$

and $Q^{(3)}(m)$ is increasing in $1<m<\infty$. Furthermore

$$\frac{1}{3!}Q^{(3)}(m)=15360m^7-166656m^6+456960m^5+1205680m^4-329120m^3$$
$$-958920m^2+112408m+65128\geqq 400840 \qquad \text{for } m\geqq 1\,,$$

and

$$\frac{1}{2}Q''(m)=5760m^8-71424m^7+228480m^6$$
$$+723408m^5-246840m^4-958920m^3+168612m^2$$
$$+195384m-4944\geqq 39516 \qquad \text{for } m\geqq 1\,,$$

and $Q'(m)$ is increasing in $1<m<\infty$. Then, we have

$$Q'(m)=1280m^9-17856m^8+65280m^7+241136m^6-98736m^5$$
$$-479460m^4+112408m^3+195384m^2-9888m-13860\,,$$

which is equal to -4312, -763.213215, 1002.36431 at $m=1$, 1.03, 1.04, respectively, from which we see that $Q(m)$ is $\searrow$ in $1<m<1.03$ and $\nearrow$ in $1.04<m<\infty$. We have $Q(1.03)=722.7389906$ and $Q(1.04)=723.669575$ and

$$Q(m)>722.7389906-763.213215\times 0.01=715.1068585 \qquad \text{for } 1.03<m<1.04 .$$

Thus, we obtain $Q(m)>0$ for $m>1$ and so

$$M_4(\alpha(m))<0 \qquad \text{for } m\geqq 1 . \tag{26}$$

Next, we investigate the sign of $M_4'(\alpha(m))$. From (20) we obtain

$$2M_4'(\alpha)=L_2[-2(m-1)L_2'+\{m-(m+4)\alpha\}d_1],$$

$$-2(m-1)L_2'(\alpha)+\{m-(m+4)\}d_1$$
$$=-2(m-1)(2a_1-6a_2\alpha)+\{m-(m+4)\alpha\}d_1$$
$$=-4(m-1)a_1+12(m-1)a_2\frac{d_0}{d_1}+md_1-(m+4)d_0 ,$$

$$-4(m-1)a_1+md_1-(m+4)d_0$$
$$=-4(8m^4-24m^3+47m^2-16m-15)+(8m^5+44m^4-38m^3-9m^2)$$
$$-8m^5-60m^4-130m^3-25m^2+218m+120$$
$$=-48m^4-72m^3-222m^2+282m+180 ,$$

and

$$M_4'(\alpha)d_1=6(m-1)a_2d_0-(24m^4+36m^3+111m^2-141m-90)d_1$$
$$=6m(m-1)(64m^6+352m^5+616m^4-4m^3-938m^2 \tag{27}$$
$$-621m-90)-m(192m^7+1344m^6+1560m^5+2172m^4$$
$$-11466m^3+399m^2+4689m+810)=3m\times R(m) ,$$

where

$$R(m)=64m^7+128m^6+8m^5-1964m^4+1954m^3+501m^2-501m-90 . \tag{28}$$

Since we have

$$\frac{1}{4!}R^{(4)}(m)=2240m^3+1920m^2+40m-1964>0 \qquad \text{for } m\geqq 1 ,$$

$$\frac{1}{3!}R^{(3)}(m)=2240m^4+2560m^3+80m^2-7856m+1954 ,$$

which is equal to -1022, 1169.62863 at $m=1$, 1.17, respectively, and so >0 for $m\geqq 1.17$,

$$\frac{1}{2}R''(m)=1344m^5+1920m^4+80m^3-11784m^2+5862m+501 ,$$

which is equal to -2098.93495, -129.678661, 49.28175698 at $m=1.17$, 1.37, 1.38, respectively, and so

$$R'(m)=448m^6+768m^5+40m^4-7856m^3+5862m^2+1002m-501$$

is $\searrow\nearrow$ in $1.17<m<\infty$, is equal to -978.490785, -99.0477806, 48.90886028 at $m=1.17$, 1.61, 1.62, respectively. Therefore $R(m)$ is $\searrow\nearrow$ in $1.17<m<\infty$, and

$$R(m)=\begin{cases} -3.1536809 & \text{at } m=1.17\,, \\ -9.99221242 & \qquad 1.83\,, \\ 50.72843899 & \qquad 1.84\,, \end{cases}$$

and hence we obtain $R(m)<0$ for $1.17\leqq m<1.83\cdots$ and $R(m)>0$ for $m>1.83\cdots$, i.e.

$$(29) \qquad M_4'(\alpha(m)) \begin{cases} <0 & \text{for } 1.17<m<1.83\cdots\,, \\ >0 & \text{for } 1.83\cdots<m<\infty\,, \end{cases}$$

because $d_1(m)>0$ for $m\geqq 1$ by Lemma 1.

Next, we have from (10)

$$1-\alpha(m)=\frac{d_1(m)-d_0(m)}{d_1(m)}=\frac{2}{m}h(m),$$

where

$$(30) \qquad h(m):=\frac{8m^3-28m^2+19m+15}{8m^3+44m^2-38m-9}$$

which is positive for $m\geqq 1$ by Lemma 1, $\searrow$ in $1<m<2.1\cdots$ and $\nearrow$ in $2.1\cdots<m<\infty$, since we have

$$\begin{aligned}&(24m^2-56m+19)(8m^3+44m^2-38m-9)\\ &\quad -(24m^2+88m-38)(8m^3-28m^2+19m+15)\\ &=3(192m^4-304m^3-116m^2-272m+133)\end{aligned}$$

and $4\times 192m^3-3\times 304m^2-2\times 116m-272$ is $\nearrow$ in $1<m<\infty$, equal to -648, -80, 167.808 at $m=1$, 1.5, 1.6, respectively, and so $192m^4-304m^3-116m^2-272m+133$ is $\searrow\nearrow$ in $1<m<\infty$, equal to -367, -31.0688, 233.8832 at $m=1$, 2.1, 2.2, respectively, and hence <0 for $1\leqq m<2.1\cdots$ and >0 for $m>2.1\cdots$.

Now, for convenience of calculation, setting $t=1-u$, we have

$$\begin{aligned}\frac{1}{6}M_4''(t)&=6c_4t^2-c_3t+c_2\\ &=6c_4u^2-(12c_4-c_3)u+6c_4-c_3+c_2\end{aligned}$$

and

(31) $$\frac{1}{6m}M_4''(t)=m^2u^2(192m^5+1056m^4+984m^3$$
$$-804m^2-546m-72)-mu(128m^6+1088m^5+1776m^4$$
$$+88m^3-1296m^2+241m+159)+96m^6+160m^5$$
$$-280m^4-892m^3+1096m^2+634m+15,$$

and

(32) $$\frac{1}{3}M_4'(t)=4c_4t^3-c_3t^2+2c_2t+c_1$$
$$=-4c_4u^3+(12c_4-c_3)u^2-2(6c_4-c_3+c_2)u+4c_4-c_3+2c_2+c_1$$
$$=-m^3u^3(128m^5+704m^4+656m^3-536m^2-364m-48)$$
$$+m^2u^2(128m^6+1088m^5+1776m^4+88m^3-1296m^2+241m+159)$$
$$-mu(192m^6+320m^5-560m^4-1784m^3+2192m^2+1268m+30)$$
$$+64m^6+192m^5-312m^4+596m^3-1074m^2-1310m-300$$

(see Lemma 3.3 in [2]).

Let $\gamma=\gamma(m)$ be the root of $6c_4t^2-c_3t+c_2=0$ as

(33) $$\gamma(m)=(c_3+\sqrt{c_3c_3-24c_2c_4})/12c_4,$$

for which $0<\gamma(m)<1$ for $m\geqq 1$ by Lemma 3.3 in [2]. For $m\geqq 6$, we have

$$h(m)\geqq h(6)=\frac{849}{3075}=\frac{283}{1025}=0.276097\cdots$$

by (30) and so

$$1-\alpha(m)=\frac{2}{m}h(m)>\frac{0.55}{m},$$

and

$$\frac{1}{6m}M_4''\left(1-\frac{0.55}{m}\right)=25.6m^6-380.32m^5-937.36m^4$$
$$-642.74m^3+1565.59m^2+336.285m-94.23$$
$$\leqq -2180.56m^4-642.74m^3+1565.59m^2+336.285m-94.23$$
$$\leqq -80791.01m^2+336.285m-94.23<0 \quad \text{for } 6\leqq m\leqq 10,$$

from which we obtain

(34) $$\alpha<1-\frac{0.55}{m}<\gamma \qquad \text{for } 6\leqq m\leqq 10.$$

Next, for $m\geqq 10$, we have

$$h(m)\geqq h(10)=\frac{5405}{12011}=0.4500041\cdots$$

and

$$1-\alpha(m)>\frac{0.9}{m}$$

and

$$\frac{1}{6m}M_4''\left(1-\frac{0.9}{m}\right)=-19.2m^6-663.68m^5-1023.04m^4-174.16m^3$$
$$+1611.16m^2-25.16m-186<0 \qquad \text{for } m\geqq 10\,,$$

from which we obtain

(35) $$\alpha<1-\frac{0.9}{m}<\gamma \qquad \text{for } m\geqq 10\,.$$

From (34), (35) and (29) we obtain

$$M_4'(t)>0 \qquad \text{for } 0\leqq t\leqq\alpha(m) \quad \text{with } m\geqq 6$$

and so by (26) and $M_4(0)<0$

(36) $$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq\alpha(m) \quad \text{with } m\geqq 6\,.$$

Next we consider the interval: $1.17\cdots<m<6$. From (32) we have

$$\frac{1}{3}M_4'(1)=4c_4-c_3+2c_2+c_1$$
$$=2(32m^6+96m^5-156m^4+298m^3-537m^2-655m-150)\,,$$

which is negative for $1\leqq m<1.81\cdots$ and positive for $m>1.81\cdots$ by (3.42) in [2].

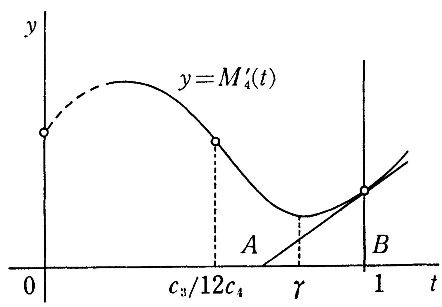

Fig. 1.

Considering the graph of the cubic polynomial of t: $y=M_4'(t)$ (Fig. 1), on the segment AB we have

$$AB=\frac{M_4'(1)}{M_4''(1)}=\frac{4c_4-c_3+2c_2+c_1}{2(6c_4-c_3+c_2)}$$

$$=\frac{32m^6+96m^5-156m^4+298m^3-537m^2-655m-150}{m(96m^6+160m^5-280m^4-892m^3+1096m^2+634m+15)}.$$

Since $6c_4-c_3+c_2>0$ for $m\geqq 1$ by (3.35) in [2], the condition:

$$AB>\frac{1}{3m}$$

is equivalent to

$$T(m):=128m^5-188m^4+1786m^3-2707m^2-2599m-465>0.$$

Since we have

$$\frac{1}{2}T''(m)=1280m^3-1128m^2+5358m-2707$$

$$\geqq 5510m-2707\geqq 2803 \qquad \text{for } m\geqq 1,$$

$$T'(m)=640m^4-752m^3+5358m^2-5414m-2599,$$

which is equal to -2767, -406.42, 718.216 at $m=1$, 1.3, 1.4, respectively, $T(m)$ is $\searrow\nearrow$ in $1<m<\infty$. Furthermore we have

$$T(m)=\begin{cases} -4045 & \text{at } m=\ 1, \\ -45.9522282 & \qquad\ 2.08, \\ 100.8901599 & \qquad\ 2.09, \end{cases}$$

and so as a sensed segment we obtain

(37) $$AB<\frac{1}{3m} \qquad \text{for } 1\leqq m<2.08\cdots,$$

$$AB>\frac{1}{3m} \qquad \text{for } 2.08\cdots<m<\infty.$$

Since we have from (31)

$$\frac{1}{2m}M_4''\left(1-\frac{1}{3m}\right)=160m^6-544m^5-2264m^4$$
$$-2436m^3+4316m^2+1479m-138:=W(m)$$

and

$$\frac{1}{4!}W^{(4)}(m)=2400m^2-2720m-2264,$$

which is $\nearrow$ in $1<m<\infty$ and equal to -2584, 1896 at $m=1$, 2, respectively, and so

$$\frac{1}{3!}W^{(3)}(m)=3200m^3-5440m^2-9056m-2436$$

is $\searrow\nearrow$ in $1<m<\infty$, equal to -13732, -16708, 7836 at $m=1, 2, 3$, respectively, and so

$$\frac{1}{2}W''(m)=2400m^4-5440m^3-13584m^2-7308m+4316$$

is $\searrow\nearrow$ in $1<m<\infty$, equal to -19616, -92344, 23980 at $m=1, 3, 4$, respectively, and so

$$W'(m)=960m^5-2720m^4-9056m^3-7308m^2+8632m+1479$$

is $\searrow\nearrow$ in $1<m<\infty$, equal to -8013, -53382.1056, 29939 at $m=1, 4.9, 5$, respectively, $W(m)$ is $\searrow\nearrow$ in $1<m<\infty$. We have

$$W(m)=\begin{cases} 573 & \text{at } m= \quad 1, \\ 18.60313031 & \quad 1.06, \\ -88.7997998 & \quad 1.07, \\ -6896.16946 & \quad 6.03, \\ 11844.21018 & \quad 6.04, \end{cases}$$

and hence

$$W(m)<0 \qquad \text{for } 1.06\cdots<m\leqq 6,$$

which implies

$$1-\frac{1}{3m}<\gamma(m) \qquad \text{for } 1.06\cdots<m\leqq 6. \tag{38}$$

By means of (37) and (38) we have

$$M_4'(\gamma(m))>0 \qquad \text{for } 2.08\cdots<m\leqq 6$$

and $M_4'(0)=3c_1(m)>0$ for $m>1.17\cdots$ by (3.41) in [2], hence we obtain

$$M_4'(t)>0 \qquad \text{for } 0<t<1 \quad \text{with } 2.08\cdots\leqq m\leqq 6,$$

which implies

$$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq \alpha(m), \quad \text{with } 2.08\cdots\leqq m\leqq 6. \tag{39}$$

Finally we investigate the rest part: $1.17\cdots<m<2.08\cdots$, where $1.17\cdots$ and $2.08\cdots$ mean the roots of $\alpha(m)=0$ and $AB=1/3m$ by (37), respectively. We notice that $M_4'(0)=3c_1$ and $c_1=c_1(m)$ is $\nearrow$ in $1<m<\infty$, since we have from (23)

$$c_1'(m)=1344m^6+5184m^5+6880m^4+2160m^3-3921m^2-4366m-1340$$
$$\geqq 11647m^2-4366m-1340\geqq 5941 \qquad \text{for } m\geqq 1,$$

and $c_1(1.17)=-128.1689\cdots$, $c_1(1.18)=70.6684\cdots$, hence $c_1(m)>0$ for $m\geqq 1.18$. $M_4''(0)=6c_2$ is negative for $1\leqq m<3.4\cdots$ and positive for $m>3.4\cdots$, because

$$\frac{1}{4!}\left(\frac{c_2}{m}\right)^{(4)}=2240m^3+960m^2-3160m-1172$$

which is $\nearrow$ in $1<m<\infty$ and equal to -1132, -504.96, 289.12 at $m=1$, 1.1, 1.2, respectively, and so

$$\frac{1}{3!}\left(\frac{c_2}{m}\right)^{(3)}=2240m^4+1280m^3-6320m^2-4688m-142$$

is $\searrow\nearrow$ in $1<m<\infty$ and equal to -7630, -1379.056, 1922.384 at $m=1$, 1.7, 1.8, respectively, and so

$$\frac{1}{2}\left(\frac{c_2}{m}\right)''=1344m^5+960m^4-6320m^3-7032m^2-426m+783$$

is $\searrow\nearrow$ in $1<m<\infty$ and equal to -10691, -922.33408, 10756.76256 at $m=1$, 2.3, 2.4, respectively, and so

$$\left(\frac{c_2}{m}\right)'=448m^6+384m^5-3160m^4-4688m^3-426m^2+1566m+475$$

is $\searrow\nearrow$ in $1<m<\infty$ and equal to -5401, -13647.6426, 8840.971968 at $m=1$, 2.8, 2.9, respectively, and so c_2/m is $\searrow\nearrow$ in $1<m<\infty$, -545, -3653.22333, 26790 at $m=1$, 3.4, 3.5, respectively, which implies the above claim. First, for m such that $\alpha(m)>0$, $c_1(m)<0$ we have the graph of $M_4'(t)$ by (29) as Fig. 2, we have $M_4'(t)<0$ for $0\leqq t\leqq\alpha(m)$ and hence

$$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq\alpha(m)\,.$$

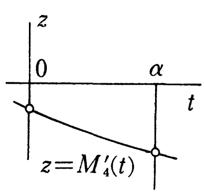

Fig. 2.

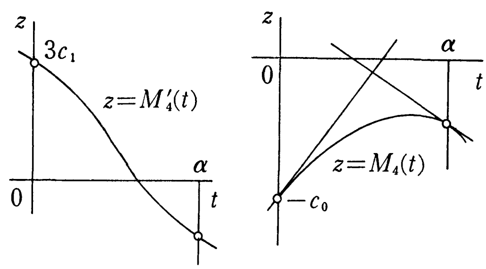

Fig. 3.

Next for m such that $\alpha(m)>0$, $c_1(m)>0$, we consider $M_4''(\alpha(m))$. From (31) and (30), we have

$$M_4''(\alpha(m))=\frac{6m}{(8m^3+44m^2-38m-9)^2}\times\Gamma(m), \tag{40}$$

where

$$\Gamma(m):=4(192m^5+1056m^4+984m^3-804m^2-546m-72)\times(8m^3-28m^2+19m+15)^2 \tag{41}$$

$$-2(128m^6+1088m^5+1776m^4+88m^3-1296m^2+241m+159)$$
$$\times(8m^3-28m^2+19m+15)(8m^3+44m^2-38m-9)$$
$$+(96m^6+160m^5-280m^4-892m^3+1096m^2+634m+15)$$
$$\times(8m^3+44m^2-38m-9)^2,$$

which is ↘ ↗ in $1.17<m<2.09$ and equal to $-12659.00711\cdots$, $-693.19339\cdots$, $300.79369\cdots$ at $m=1.17$, 1.88, 1.89, respectively. Since $\Gamma(m)$ is a polynomial of order 12 in m with integral coefficients of very large, here we used a numerical argument by a computor. Therefore we have

(42) $$M_4''(\alpha(m))<0 \qquad \text{for } 1.17\leqq m<1.88\cdots,$$
$$M_4''(\alpha(m))>0 \qquad \text{for } 1.88\cdots<m<2.09$$

and see that the behavior of $M_4'(t)$ and $M_4(t)$ are as Fig. 3, when $1.17\cdots<m<1.88\cdots$. For such m, we consider the condition:

$$-\frac{M_4(0)}{M_4'(0)}=\frac{c_0}{3c_1}\geqq\alpha=\frac{d_0}{d_1},$$

which is equivalent to

$$c_0d_1-3c_1d_0$$
$$=m(32m^8+272m^7+996m^6+1746m^5+1037m^4-983m^3$$
$$-1865m^2-1035m-200)(8m^3+44m^2-38m-9)$$
$$-3(192m^7+864m^6+1376m^5+540m^4-1307m^3-2183m^2$$
$$-1340m-300)(8m^4+28m^3+18m^2-47m-30)$$
$$=m(256m^{11}+3584m^{10}+18720m^9+47168m^8+44824m^7-37548m^6$$
$$-113292m^5-62319m^4+32577m^3+47315m^2+16915m+1800)$$
$$-3(1536m^{11}+12288m^{10}+38656m^9+49376m^8-16936m^7-134932m^6$$
$$-162030m^5-33985m^4+109291m^3+123070m^2+54300m+9000)=\Pi(m)>0$$

by (10) and (23), where

(43) $$\Pi(m):=256m^{12}-1024m^{11}-18144m^{10}-68800m^9-103304m^8$$
$$+13260m^7+291504m^6+423771m^5+134532m^4$$
$$-280558m^3-352295m^2-161100m-27000,$$

which is ↗ ↘ in $1.17<m<2.09$ by the numerical argument by a computor as for $\Gamma(m)$ and equal to 84962.14696 ⋯, 5745.25969 ⋯, −34308.14753 ⋯ at $m=1.17$, 1.38, 1.39, respectively. Therefore we have $\Pi(m)>0$ for $1.17\leqq m<1.38\cdots$ and <0 for $1.38\cdots<m<2.09$, which implies

$$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq\alpha(m) \quad \text{with } 1.17\cdots<m<1.38\cdots.$$

Next, for $1.38\cdots<m<1.83\cdots$ (see (29)) we consider the condition:

$$-\frac{M_4(0)}{M_4'(0)}+\frac{M_4(\alpha)}{M_4'(\alpha)}\geqq\alpha=\frac{d_0}{d_1}, \tag{44}$$

whose left hand becomes by (25), (27)

$$\frac{c_0}{3c_1}-\frac{L_2(\alpha)\times 2mQ(m)}{d_1^2}\Big/\frac{3mR(m)}{d_1}=\frac{c_0}{3c_1}-\frac{2L_2(\alpha)Q(m)}{3d_1R(m)},$$

and hence it is equivalent to

$$-R(c_0d_1-3c_1d_0)+2c_1L_2(\alpha)Q\geqq 0,$$

since $R(m)<0$ for $1.17\leqq m<1.83\cdots$. Using (43) and (18) this condition can be written as

$$\Lambda(m):=12c_1P(m)Q(m)-(8m^3+44m^2-38m-9)^2R(m)\Pi(m)\geqq 0. \tag{45}$$

$\Lambda(m)$ is a polynomial of m of order 25, and we can prove that it is increasing in $1.38\leqq m\leqq 2.08$ and positive there. In fact, for $1.38\leqq m<1.84$, we obtain by (23), (19), (25), (28) and (43)

$$c_1(m)\geqq c_1(1.38)=6901.245307\cdots>6901,$$

$$P(m)\geqq P(1.38)=76169.63772\cdots>76169,$$

$$Q(m)\geqq Q(1.38)=51784.48189\cdots>51784,$$

$$8m^3+44m^2-38m-9\leqq 119.882432<120,$$

$$0>R(m)>R(1.61)+R'(1.61)\times 0.01$$

$$=-529.06177\cdots-99.04778\cdots\times 0.01>-531,$$

$$\Pi(m)\geqq\Pi(1.84)=-19253060.98286>-19253061,$$

and hence

$$\Lambda(m)>12\times 6901\times 76169\times 51784-(120)^2\times 531\times 19253061$$

$$=12(2.721985926\times 10^{13}-1.226805047\times 10^{13})$$

$$=12\times 1.495180879\times 10^{13}$$

$$=179421705500000>0.$$

Therefore the condition (44) is satisfied for $1.38\cdots<m<1.83\cdots$ and we obtain also

$$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq \alpha(m) \quad \text{with } 1.38\cdots<m<1.83\cdots.$$

Next, for $1.83\cdots<m\leqq 1.88\cdots$, we have $M_4'(\alpha(m))>0$ by (29) and $M_4''(\alpha(m))<0$ by (42), and so

$$M_4'(t)>0 \qquad \text{for } 0\leqq t<\alpha(m),$$

which implies also

$$M_4(t)<0 \qquad \text{for } 0\leqq t\leqq \alpha(m) \quad \text{with } 1.83\cdots\leqq m\leqq 1.88\cdots.$$

Last, we consider the rest interval: $1.88\cdots<m<2.08\cdots$. We have there $M_4'(\alpha(m))>0$ and $M_4''(\alpha(m))>0$ and by (30)

$$h(m)<h(1.88)=\frac{4.914176}{128.230976}=0.03832\cdots,$$

$$>h(2.09)=\frac{5.437832}{176.811032}=0.030755\cdots$$

and

$$1-\frac{1}{10m}<1-\frac{0.07664\cdots}{m}<\alpha(m)<1-\frac{0.06151}{m}. \tag{46}$$

We obtain from (31)

$$\begin{aligned}\frac{1}{6m}M_4''\Big(1-\frac{1}{10m}\Big)&=83.2m^6+53.12m^5\\ &\quad -447.04m^4-890.96m^3+1217.56m^2+604.44m-1.62\\ &\leqq 27.40672m^4-890.96m^3+1217.56m^2+604.44m-1.62\\ &\leqq -360.578488m^2+604.44m-1.62\\ &\leqq -75.067559<0 \qquad \text{for } 1.88\leqq m\leqq 2.09.\end{aligned} \tag{47}$$

Therefore $M_4'(t)$ is $\searrow$ in $0<t<1-1/10m$. Then, we obtain from (32)

$$\begin{aligned}\frac{1}{3}M_4'\Big(1-\frac{1}{10m}\Big)&=46.08m^6+170.752m^5-238.944m^4\\ &\quad +774.624m^3-1305.624m^2-1434.026m-301.362\\ &\geqq 244.934912m^4+774.624m^3-1305.624m^2-1434.026m-301.362\\ &\geqq 1016.367073m^2-1434.026m-301.362\\ &\geqq 594.9169028>0 \qquad \text{for } 1.88\leqq m\leqq 2.09.\end{aligned} \tag{48}$$

Since

$$M_4'''(t)>0 \qquad \text{for } t>\frac{c_3}{12c_4}$$

and

$$\frac{c_3}{12c_4} < \frac{2}{3} \qquad \text{for } m \geqq 1,$$

because this is equivalent to

$$\frac{1}{m^2}(8c_4 - c_3) = 384m^5 + 1120m^4 + 624m^3 - 932m^2 + 289m + 159 > 0$$

by (23) and it is evidently satisfied for $m \geqq 1$, $M_4'(t)$ is convex downward in $1 - 1/10m < t < \infty$, since

$$1 - \frac{1}{10m} > \frac{2}{3} \qquad \text{for } m > 1.88.$$

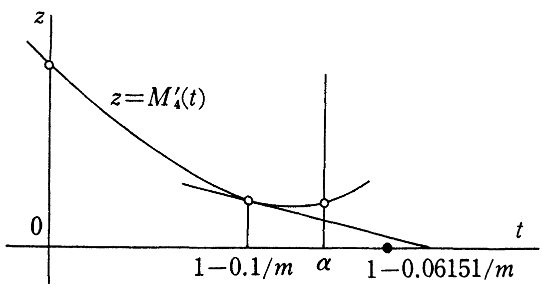

Fig. 4.

Now, using (47) and (48) we obtain

$$-\frac{M_4'\left(1 - \frac{1}{10m}\right)}{M_4''\left(1 - \frac{1}{10m}\right)}$$

$$= \frac{46.08m^6 + 170.752m^5 - 238.944m^4 + 774.624m^3 - 1305.624m^2 - 1434.026m - 301.362}{-2m(83.2m^6 + 53.12m^5 - 447.04m^4 - 890.96m^3 + 1217.56m^2 + 604.44m - 1.62)}$$

and

$$\left(1 - \frac{0.06151}{m}\right) - \left(1 - \frac{1}{10m}\right) = \frac{0.03849}{m}.$$

Hence, for $1.88\cdots < m < 2.08\cdots$ the condition:

(49) $$-M_4'\left(1 - \frac{1}{10m}\right) \Big/ M_4''\left(1 - \frac{1}{10m}\right) > \frac{0.03849}{m}$$

is equivalent to

$$\begin{aligned}&46.08m^6+170.752m^5-238.944m^4+774.624m^3\\&\quad-1305.624m^2-1434.026m-301.362+0.07698\\&\quad\times(83.2m^6+53.12m^5-447.04m^4-890.96m^3\\&\qquad+1217.56m^2+604.44m-1.62)\\&=52.484736m^6+174.8411776m^5-273.3571392m^4+706.0378992m^3\\&\quad-1211.89623m^2-1387.4962m-301.4867076>0,\end{aligned}$$

which is implied from the condition:

$$\Theta(m):=52m^6+174m^5-274m^4+706m^3-1212m^2-1388m-302>0. \tag{50}$$

For $1.88\leqq m\leqq 2.09$ we have

$$\begin{aligned}\Theta'(m)&=312m^5+870m^4-1096m^3+2118m^2-2424m-1388\\&\geqq 1642.3328m^3+2118m^2-2424m-1388>0\end{aligned}$$

and so $\Theta(m)$ is $\nearrow$ in $1.88<m<2.09$ and $\Theta(1.88)=455.46641\cdots$, which implies that $\Theta(m)>0$ for $1.88\leqq m\leqq 2.09$. Thus, we see that the condition (49) is satisfied, and so we obtain

$$M_4'(t)>0 \qquad \text{for } 1-\frac{1}{10m}\leqq t\leqq\alpha(m).$$

Combining this result and the fact $M_4'(t)$ is decreasing in $0<t<1-(1/10m)$ as is shown previously, we see that $M_4(t)$ is increasing in $0<t<\alpha(m)$ and so obtain

$$M_4(t)<0 \qquad \text{for } 0<t<\alpha(m) \quad \text{with } 1.88\cdots<m<2.08\cdots.$$

Thus, we can finish the proof of this lemma. Q.E.D.

LEMMA 6. *For m such that $d_0(m)>0$, i.e. $m>1.17\cdots$, we have*

$$\sigma(t)>0 \qquad \textit{for } \alpha(m)\leqq t<1.$$

Proof. Since we have $L_1(t)>0$, $\varphi_2(t)>0$ for $\alpha(m)<t<1$ by Lemma 1 and Lemma 3, $\sigma(t)$ have the same sign with

$$\frac{\sigma(t)}{\varphi_2(t)}=L_2(t)+\frac{\varphi_1(t)}{\varphi_2(t)}L_1(t).$$

By Lemma 3 and the Cauchy mean value theorem we have

$$\frac{\varphi_1(t)}{\varphi_2(t)}=\frac{\varphi_1(t)-\varphi_1(1)}{\varphi_2(t)-\varphi_2(1)}=\frac{\varphi_1'(t_1)}{\varphi_2'(t_1)}$$

for some $t_1(t<t_1<1)$, and

$$\frac{\varphi_1'(t)}{\varphi_2'(t)}=\frac{m+2}{m-1}t-1<\frac{m+2}{m-1}t_1-1\,.$$

Hence we have for $\alpha(m)<t<1$

$$L_2(t)+\frac{\varphi_1(t)}{\varphi_2(t)}L_1(t)>L_2(t)+\Big(\frac{m+2}{m-1}t-1\Big)L_1(t) \tag{51}$$

$$=\frac{1}{m-1}[(m-1)L_2(t)+\{(m+2)t-(m-1)\}L_1(t)]\,.$$

Then, we have

$$(m-1)L_2(t)+\{(m+2)t-(m-1)\}L_1(t)$$
$$=(m-1)\{a_0+2a_1t-3a_2t^2\}+\{(m+2)t-(m-1)\}(d_1t-d_0)$$
$$=\tilde{a}_0-\tilde{a}_1t+\tilde{a}_2t^2=\widetilde{L}_2(t)\,,$$

where

$$\tilde{a}_0=(m-1)(a_0+d_0)$$
$$=(m-1)(8m^4+36m^3+58m^2+40m+10)$$
$$=2(m-1)(m+1)^2(4m^2+10m+5)\,,$$
$$\tilde{a}_1=-2(m-1)a_1+(m+2)d_0+(m-1)d_1$$
$$=(m-1)(8m^4+28m^3-6m^2-71m-30)$$
$$+(m+2)(8m^4+28m^3+18m^2-47m-30)$$
$$=16m^5+64m^4+40m^3-76m^2-83m-30\,,$$
$$\tilde{a}_2=-3(m-1)a_2+(m+2)d_1$$
$$=-3m(m-1)(8m^2+16m+3)+m(m+2)(8m^3+44m^2-38m-9)$$
$$=m(8m^4+36m^3+26m^2-46m-9)\,,$$

i. e.

$$\widetilde{L}_2(t)=\tilde{a}_0-\tilde{a}_1t+\tilde{a}_2t^2\,,$$

$$\begin{cases}\tilde{a}_0=2(m-1)(m+1)^2(4m^2+10m+5)\,,\\ \tilde{a}_1=16m^5+64m^4+40m^3-76m^2-83m-30\,,\\ \tilde{a}_2=m(8m^4+36m^3+26m^2-46m-9)\,.\end{cases} \tag{52}$$

We see easily $\tilde{a}_0>0$ for $m>1$ and $\tilde{a}_2>0$ for $m\geqq1$.

Now, we compute the discriminant $D(m)$ of the quadratic polynomial $\widetilde{L}_2(t)$ as follows.

$$-D(m)=4\tilde{a}_0\tilde{a}_2-\tilde{a}_1\tilde{a}_1$$
$$=8m(m-1)(m+1)^2(4m^2+10m+5)(8m^4+36m^3+26m^2-46m-9)$$
$$-(16m^5+64m^4+40m^3-76m^2-83m-30)^2,$$

$$(m-1)(m+1)^2(4m^2+10m+5)(8m^4+36m^3+26m^2-46m-9)$$
$$=(m^2-1)(m+1)(32m^6+224m^5+504m^4+256m^3-366m^2-320m-45)$$
$$=(m^2-1)(32m^7+256m^6+728m^5+760m^4-110m^3-686m^2-365m-45)$$
$$=32m^9+256m^8+696m^7+504m^6-838m^5-1446m^4$$
$$-255m^3+641m^2+365m+45,$$

$$(16m^5+64m^4+40m^3-76m^2-83m-30)^2$$
$$=256m^{10}+2048m^9+5376m^8+2688m^7-10784m^6$$
$$-17664m^5-4704m^4+10216m^3+11449m^2+4980m+900,$$

and hence

(53) $$-D(m)=192m^8+1344m^7+4080m^6+6096m^5$$
$$+2664m^4-5088m^3-8529m^2-4620m-900.$$

We see that $-D(m)$ is $\nearrow$ in $1<m<\infty$, since

$$-D'(m)=1536m^7+9408m^6+24480m^5+30480m^4$$
$$+10656m^3-15264m^2-17058m-4620$$
$$\geqq 61296m^2-17058m-4620$$
$$\geqq 39618 \qquad \text{for } m\geqq 1.$$

We have

$$-D(m)=\begin{cases} -4761 & \text{at } m= \quad 1, \\ -532.668055 & \qquad\qquad 1.08, \\ 163.7176446 & \qquad\qquad 1.09, \end{cases}$$

and hence we obtain

(54) $$D(m)>0 \qquad \text{for } 1\leqq m<1.08\cdots,$$
$$D(m)<0 \qquad \text{for } m>1.08\cdots,$$

which implies

(55) $$\tilde{L}_2(t)>0 \qquad \text{for } -\infty<t<\infty \quad \text{with } m>1.08\cdots,$$

Thus, we obtain from (51) and (55)

$$\sigma(t)=\varphi_1(t)L_1(t)+\varphi_2(t)L_2(t)>0 \qquad \text{for } \alpha(m)\leqq t<1 \quad \text{with } m>1.17\cdots.$$

Q. E. D.

Combining Lemma 5 and Lemma 6, we obtain our Main Theorem.

REFERENCES

[1] T. OTSUKI, A certain property of geodesics of the family of Riemanian manifolds $O_n^2(X)$, Kodai Math. J. **12** (1989), 346-391.

[2] T. OTSUKI, Certain inequalities related with a non-linear differential equation (III), TRU Mathematics **24** (1988), 103-134.

DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSIUY OF TOKYO
WAKAMIYA-CHO 26, SHINJUKU-KU
TOKYO, JAPAN 162