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THE VALUE DISTRIBUTION OF ENTIRE FUNCTIONS OF FINITE ORDER

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1. Introduction.

In [8] Tsuzki proved the following

THEOREM A. *Let $f(z)$ be an entire function of order less than one and let $\{w_n\}$ be an unbounded sequence. Assume that there exists a real number β such that $0<\beta<\pi/2$ and all the roots of equations*

$$f(z)=w_n \qquad (n=1, 2, \cdots) \tag{1}$$

belong to the sector $\{z\,;\,|\arg z-\pi|\leqq\beta\}$. Then $f(z)$ is a linear function.

In [4], [5], [1] Kimura, Kobayashi, Baker and Liverpool improved the above result respectively. In this paper we generalize Theorem A to the following

THEOREM 1. *Let $f(z)$ be an entire function and let $\{w_n\}$ be an unbounded sequence. Suppose that for some positive integer m.*

$$\varliminf_{r\to\infty}\frac{T(r, f)}{r^m}=0\,. \tag{2}$$

Assume that there exists some $\varepsilon>0$ such that all the roots of equations (1) *belong to the following set*

$$\bigcup_{k=0}^{m-1}\left\{z\,;\,\frac{2k}{m}\pi+\varepsilon<\arg z<\frac{2k+1}{m}\pi-\varepsilon\right\}. \tag{3}$$

Then $f(z)$ is a polynomial.

The direction $\arg z=\theta$ is said to be a limiting direction of the complex set E, if θ is a cluster point of the set $\{\arg z\,;\,z\in E\}$. As the corollary of Theorem 1 we have

COROLLARY 1. *Let $f(z)$ be an entire function of finite order and let $\{w_n\}$*

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be an unbounded sequence. Assume that $\bigcup_{n=1}^{\infty}\{z\,;\,f(z)=w_n\}$ *has only* $k(<\infty)$ *distinct limiting directions, then* $f(z)$ *is a polynomial of degree at most* k.

In [7] Ozawa proposed the following conjecture:

Let $f(z)$ be an entire function, $\{w_n\}$ be an unbounded sequence and $L_1, L_2, \cdots, L_p$ be p distinct straightlines any two of which are not parallel with each other. Assume that all the roots of equations (1) lie on $L_1, L_2, \cdots, L_p$. Then $f(z)$ is a polynomial of degree at most $2p$.

By Corollary 1 we deduce the following

COROLLARY 2. *Ozawa's conjecture is true.*

A meromorphic function $F(z)$ is said to have a factorization with left factor f and right factor g, if it is expressible in the form $f(g(z))$, where f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be pseudoprime if every factorization of the above form implies that either f is rational or g is a polynomial. If $F(z)$ is pseudoprime when only entire factors are considered in the factorization of the above form, it is called E-pseudoprime. In this paper we prove the following

THEOREM 2. *Let* $F(z)$ *be a meromorphic function of order less than* m *(a positive integer). Assume that there exist two complex number* A_1, A_2 *(finite or infinite) such that all the roots of equation* $F(z)=A_j$ $(j=1, 2)$ *belong to the following set*

$$T_j=\bigcup_{k=0}^{m-1}\left\{z\,;\,\frac{2k}{m}\pi+\varepsilon<\arg z-\alpha_j<\frac{2k+1}{m}\pi-\varepsilon\right\} \tag{4}$$

for some $\varepsilon>0$ *and two real numbers* α_j $(j=1, 2)$. *Then* $F(z)$ *is pseudoprime.*

In [2] Baker proved the following

THEOREM B. *Let* $F(z)$ *be an entire function of finite order and let there exist a complex number* A *such that the set of the roots of* $F(z)=A$ *has only one limiting direction. Then* $F(z)$ *is E-pseudoprime.*

A a corollary of Theorem 2 we improve Theorem B to the following

COROLLARY 3. *Let* $F(z)$ *be a meromorphic function of finite order and let there exist two distinct complex numbers* A_1, A_2 *(finite or infinite) such that the set of the roots of* $F(z)=A_j$ $(j=1, 2)$ *has only finitely many limiting directions. Then* $F(z)$ *is pseudoprime.*

Let $f(z)$ be an entire function and $f_1(z)=f(z)$, $f_2(z)=f(f(z)), \cdots, f_n(z), \cdots$ be its sequence of interates. Regarding the Fatou set $F(f)$ of those points of the complex plane where $\{f_n(z)\}$ does not form a normal family, Baker proved in [2] the following

THEOREM C. *Let $f(z)$ be a transcendental entire function and let the set*

$$F(f)-\{z\,;\,|\arg z|<\delta\}$$

be bounded for every $\delta>0$, then $f(z)$ is of infinite order.

In this paper we improve Theorem C to the following

THEOREM 3. *Let $f(z)$ be a transcendental entire function and let θ_j $(j=1, 2, \cdots, m)$ be m real numbers. Assume that the set*

$$F(f)-\bigcup_{j=1}^{m}\{z\,;\,|\arg z-\theta_j|<\delta\}$$

is bounded for every $\delta>0$, then $f(z)$ is of infinite order.

By Theorem 3 we easily obtain the following

COROLLARY 4. *Let $f(z)$ be a transcendental entire function of finite order, then $F(f)$ cannot be contained in any finitely many strip regions.*

2. Some lemmas.

To prove our theorems, we need the following lemmas.

LEMMA 1. *Let $f(z)$ be an entire function with the zeros $\{z_j\}$ and $0<|z_1|\leqq |z_2|\leqq\cdots\leqq|z_j|\leqq\cdots$. Then for any positive integer n we have*

$$\left(\frac{f'(z)}{f(z)}\right)^{(n-1)}=(n-1)!\left[-\sum_{|z_j|\leqq r}\frac{1}{(z_j-z)^n}+O\left(\frac{T(er, f)}{r^n}\right)\right]\quad(r\to\infty). \tag{5}$$

Proof. Let $|z|<r$, by Poisson-Jensen formula we have

$$\frac{f'(z)}{f(z)}=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\frac{2re^{i\theta}}{(re^{i\theta}-z)^2}d\theta+\sum_{|z_j|\leqq r}\left\{\frac{1}{z-z_j}+\frac{\bar{z}_j}{r^2-\bar{z}_jz}\right\}.$$

Differentiating this $n-1$ times we obtain

$$\left(\frac{f'(z)}{f(z)}\right)^{(n-1)}=\frac{n!}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\frac{2re^{i\theta}}{(re^{i\theta}-z)^{n+1}}d\theta \tag{6}$$

$$+(n-1)!\sum_{|z_j|\leqq r}\left\{\frac{(-1)^{n+1}}{(z-z_j)^n}+\frac{\bar{z}_j^n}{(r^2-\bar{z}_jz)^n}\right\}.$$

We also have

(7) $$\left|\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\frac{2re^{i\theta}}{(re^{i\theta}-z)^{n+1}}d\theta\right|\leqq O\left(\frac{T(er,f)}{r^n}\right) \quad (r\to\infty),$$

(8) $$\left|\sum_{|z_j|\leqq r}\frac{\bar{z}_j^n}{(r^2-\bar{z}_j z)^n}\right|\leqq O\left(\frac{n(r,f=0)}{r^n}\right)\leqq O\left(\frac{T(er,f)}{r^n}\right) \quad (r\to\infty).$$

By (6), (7) and (8) we deduce (5), Lemma 1 is thus proved.

LEMMA 2. *Let $\theta_j\in[0, 2\pi)$ $(j=1, 2, \cdots, p)$ be p distinct real numbers. then for any constant $M>0$ there exists some integer $m>M$ such that $\cos m\theta_j>\sqrt{3}/2$ $(j=1, 2, \cdots, p)$.*

This lemma is Lemma 1.1 of paper [6].

LEMMA 3. *If the conditions of Ozawa's conjecture are satisfied, then the order of $f(z)$ is finite.*

This lemma is a special case of Theorem 2 of paper [3].

3. Proof of the theorems.

Proof of Theorem 1. Let ω be an m-th root of unity. Set

$$B_{m-j}(z)=(-1)^j\sum_{1\leqq k_1<\cdots<k_j\leqq m}f(\omega^{k_1}z)f(\omega^{k_2}z)\cdots f(\omega^{k_j}z),$$

$A_{m-j}(z)=B_{m-j}(z^{1/m})$ is obviously an entire function and it is easily seen that

$$f^m(z)+B_{m-1}(z)f^{m-1}(z)+\cdots+B_1(z)f(z)+B_0(z)=0.$$

Thus the entire algebroid function $g(z)=f(z^{1/m})$ satisfies the following equation

$$g^m+A_{m-1}(z)g^{m-1}+\cdots+A_1(z)g+A_0(z)=0.$$

Set

(9) $$\varphi_n(z)=w_n^m+w_n^{m-1}A_{m-1}(z)+\cdots+w_nA_1(z)+A_0(z).$$

By (3) it is obvious that the zeros $\{a_{nj}\}$ of $\varphi_n(z)$ (which are the zeros of $g(z)-w_n$) all lie in the half plane $\operatorname{Im} z>0$.

Because $\{w_n\}$ is unbounded, without loss of generality we may assume that $w_n\to\infty$ as $n\to\infty$ (otherwise consider its some suitable subsequence). Let n be sufficiently large. It follows from (9) that

$$\log\varphi_n(z)=m\log w_n+\log\left(1+\frac{A_{m-1}(z)}{w_n}+\cdots+\frac{A_0(z)}{w_n^m}\right)$$
$$=m\log w_n+\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\left(\frac{A_{m-1}(z)}{w_n}+\cdots+\frac{A_0(z)}{w_n^m}\right)^j$$

$$=m \log w_n+\frac{A_{m-1}(z)}{w_n}+O\left(\frac{1}{w_n^2}\right) \qquad (n\to\infty).$$

By this we obtain that

$$\lim_{n\to\infty} w_n[\log \varphi_n(z)]^{(q)}=A_{m-1}{}^{(q)}(z) \qquad (q=1, 2, \cdots). \tag{10}$$

From (2) and (9) it follows that

$$\underline{\lim}_{r\to\infty} \frac{T(r, \varphi_n)}{r}=0. \tag{11}$$

By Lemma 1 and (11) we obtain that there exists a sequence $r_k\to\infty$ such that

$$(\log \varphi_n(z))'=-\lim_{k\to\infty} \sum_{|a_{nj}|\leqq r_k} \frac{1}{a_{nj}-z}. \tag{12}$$

Taking $z_0\in\{z\,;\operatorname{Im} z<0\}$ such that $\varphi_n(z_0)\neq 0$, by (12) we deduce that

$$\lim_{k\to\infty} \sum_{|a_{nj}|\leqq r_k} \frac{\operatorname{Im}(a_{nj}-z_0)}{|a_{nj}-z_0|^2}$$

is a finite number. Since $a_{nj}\in\{z\,;\operatorname{Im}(z)>0\}$, we have $\operatorname{Im}(a_{nj}-z_0)>|\operatorname{Im}(z_0)|>0$. From this we know that the following series is convergent

$$\sum_{j=1}^{\infty} \frac{1}{|a_{nj}-z_0|^2}.$$

It tells us that the order of $N(r, g(z+z_0)=w_n)$ is not larger than 2 for every w_n. By the second fundamental theorem of algebroid functions we obtain that the order of $g(z)$ is not larger than 2. This implies that the order of $\varphi_n(z)$ is not larger than 2. By Lemma 1 we have

$$(\log \varphi_n(z))^{(q)}=-(q-1)!\sum_{j=1}^{\infty} \frac{1}{(a_{nj}-z)^q} \qquad (q\geqq 3). \tag{13}$$

By (10), (12) and (13) we have

$$A'_{m-1}(z_0)=-\lim_{n\to\infty} w_n\left(\lim_{k\to\infty} \sum_{|a_{nj}|\leqq r_k} \frac{1}{a_{nj}-z_0}\right), \tag{14}$$

$$A^{(q)}_{m-1}(z_0)=-(q-1)!\lim_{n\to\infty} w_n \sum_{j=1}^{\infty} \frac{1}{(a_{nj}-z_0)^q} \qquad (q\geqq 3). \tag{15}$$

By (14) we obtain that

$$\lim_{n\to\infty} |w_n| \sum_{j=1}^{\infty} \frac{1}{|a_{nj}-z_0|^2} \leqq \frac{|A'_{m-1}(z_0)|}{|\operatorname{Im} z_0|}. \tag{16}$$

Without loss of generality we may assume that

$$0<|a_{n1}-z_0|\leqq|a_{n2}-z_0|\leqq\cdots\leqq|a_{nj}-z_0|\leqq\cdots.$$

By (14), (15) and (16) we deduce that for $q>2$

(17)
$$|A_{m-1}^{(q)}(z_0)|\leqq(q-1)!\lim_{n\to\infty}|w_n|\sum_{j=1}^{\infty}\frac{1}{|a_{nj}-z_0|^q}$$
$$\leqq(q-1)!\lim_{n\to\infty}\frac{|w_n|}{|a_{n1}-z_0|^{q-2}}\sum_{j=1}^{\infty}\frac{1}{|a_{nj}-z_0|^2}$$
$$\leqq\frac{(q-1)!|A'_{m-1}(z_0)|}{|\mathrm{Im}\,(z_0)|}\lim_{n\to\infty}\frac{1}{|a_{nj}-z_0|^{q-2}}.$$

Since $f(a_{n1}^{1/m})=w_n$ and $w_n\to\infty$ as $n\to\infty$, we have $a_{n1}\to\infty$ as $n\to\infty$. By (17) we deduce that
$$A_{m-1}^{(q)}(z_0)=0 \qquad (q\geqq3).$$

This proves that $A_{m-1}(z)$ is a polynomial of degree at most two. Thus $B_{m-1}(z)$ is a polynomial of degree at most $2m$. Since
$$-B_{m-1}(z)=f(\omega z)+f(\omega^2 z)+\cdots+f(\omega^m z).$$
We easily obtain that $f^{(3m)}(0)=0$.

For any complex number c, set $f_1(z)=f(z+c)$. Since all the roots of equations $f(z)=w_n$ $(n=1, 2, \cdots)$ belong to the set (3), we can easily see that there exists a positive integer N such that all the roots of equations $f_1(z)=w_n$ $(n=1, 2, \cdots)$ belong to the following set
$$\bigcup_{k=0}^{m-1}\left\{z\,;\frac{2k}{m}\pi+\frac{\varepsilon}{2}<\arg z<\frac{2k+1}{m}\pi-\frac{\varepsilon}{2}\right\}$$
for any $n>N$. Since $f_1(z)$ satisfies all the conditions of $f(z)$, by the above discussion we have $f_1^{(3m)}(0)=0$. Hence $f^{(3m)}(c)=0$ for any complex number c. This proves that $f(z)$ is a polynomial. The proof of Theorem 1 is now complete.

Proof of Corollary 1. Set $\arg z=\theta_j$ $(j=1, 2, \cdots, k)$ are the limiting directions of $\bigcup_{n=1}^{\infty}\{z\,;f(z)=w_n\}$. By Lemma 2 there exists a positive integer $m>\rho_f$ (the order of $f(z)$) such that $\cos m\theta_j>\sqrt{3}/2$ $(j=1,2,\cdots, k)$. Hence all $e^{i(\theta_j+\pi/2m)}$ $(j=1, 2, \cdots, k)$ belong to

(18)
$$\bigcup_{k=0}^{m-1}\left\{z\,;\frac{2k}{m}\pi+\frac{\pi}{2m}<\arg z<\frac{2k+1}{m}\pi-\frac{\pi}{2m}\right\}.$$

It is easily seen that $\arg z=\theta_j-\pi/2m$ $(j=1, 2, \cdots, k)$ are the limiting directions of $\bigcup_{n=1}^{\infty}\{z\,;f(e^{-i(\pi/2m)}z)=w_n\}$. From this we know that there exists a positive integer N such that all the roots of equations $f(e^{-i(\pi/2m)}z)=w_n$ belong to the set (18) for any $n>N$. By Theorem 1 we deduce that $f(z)$ is a polynomial. Let $f(z)=a_q z^q+\cdots+a_0$. Then the roots of $f(z)=w_n$ should be distributed asymptotically as q roots of $a_q z^q=w_n$ for sufficiently large n. Hence $q\leqq k$.

The proof of Corollary 1 is now complete.

Proof of Corollary 2. By Lemma 3 we see that $f(z)$ is of finite order. We easily know that the limiting directions of $\bigcup_{n=1}^{\infty}\{z; f(z)=w_n\}$ only may be $\arg z=\theta_1, \theta_2, \cdots, \theta_{2p}$ which are parallel with $L_1, L_2, \cdots, L_p$ respectively. By Corollary 1 we thus complete the proof of Corollary 2.

Proof of Theorem 2. Suppose that $F(z)=f(g(z))$, where f is a transcendental meromorphic function and g is a transcendental entire function. If $f(w)-A_1$ has infinitely many zeros $\{w_n\}$, then all the roots of $g(z)=w_n$ $(n=1, 2, \cdots)$ belong to the set T_1. Because the order of $F(z)$ is less than m, we obviously have

$$\lim_{r\to\infty}\frac{T(r, g)}{r^m}=0 .$$

By Theorem 1, $g(z)$ is a polynomial. This is a contradiction. Hence $f(w)-A_1$ has only finitely many zeros and so does $f(w)-A_2$. Thus

$$\frac{f(w)-A_1}{f(w)-A_2}=R(w)e^{h(w)} ,$$

where $R(w)$ is rational, $h(w)$ is entire and nonconstant. It gives us the following equality

$$\frac{F(z)-A_1}{F(z)-A_2}=R(g(z))e^{h(g(z))} \tag{19}$$

By Pólya's theorem we deduce from (19) that $F(z)$ is of infinite order. This is a contradiction. Hence $F(z)$ is pseudoprime. The proof of Theorem 2 is complete.

Proof of Corollary 3. By the same discussion as in the proof of Corollary 1, we can obtain that $F(z)$ satisfies all the conditions of Theorem 2 for some positive integer m. By Theorem 2 we complete the proof of Corollary 3.

Proof of Theorem 3. Suppose that $f(z)$ is of finite order. We choose a sequence $\{w_n\}\in F(f)$ such that $w_n\to\infty$ as $n\to\infty$. Since $F(f)-\bigcup_{j=1}^{m}\{z; |\arg z-\theta_j|<\delta\}$ is bounded for any $\delta>0$, and $\bigcup_{n=1}^{\infty}\{z; f(z)=w_n\}\subset F(f)$, we know that the number of elements of $\bigcup_{n=1}^{\infty}\{z; f(z)=w_n\}$ which are outside $\bigcup_{j=1}^{m}\{z; |\arg z-\theta_j|<\delta\}$ is at most finite. This implies that the limiting directions of the set $\bigcup_{n=1}^{\infty}\{z; f(z)=w_n\}$ only may be $\arg z=\theta_1, \theta_2, \cdots, \theta_m$. By Corollary 1 we deduce that $f(z)$ is a polynomial. This is a contradiction, Theorem 3 is now proved.

Corollary 4 is obtained by Theorem 3.

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