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COHOMOLOGY OF A HOPF ALGEBRA OVER Z_2

Dedicated to Professor Kenichi Shiraiwa on his 60-th birthday

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Introduction

Let $A=Z_2[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^4)$ be a truncated polynomial algebra having a structure of a Hopf algebra over Z_2, the prime field of characteristic 2, with comultiplication

$$\begin{cases} \phi(x_1)=x_1\otimes 1+1\otimes x_1 \\ \phi(x_2)=x_2\otimes 1+1\otimes x_2 \\ \phi(x_3)=x_3\otimes 1+1\otimes x_3+x_1\otimes x_2 \end{cases}$$

This comultiplication comes from the multiplication for matrices $\left\{\begin{pmatrix} 1 & r_1 & r_3 \\ 0 & 1 & r_2 \\ 0 & 0 & 1 \end{pmatrix}\right\}$, and the Hopf algebra A is related to the Frobenius kernel U_2 of the maximal nilpotent subgroup scheme U of GL_3, defined over an algebraically closed field k of characteristic 2 (cf. W. Waterhouse [5]).

M. Tezuka constructed a DGA-algebra Λ over Z_2 such that $Z_2\to A\otimes\Lambda$ is an a cyclic A-comodule resolution of Z_2 (unpublished) after N. Shimada and A. Iwai [4], and A. Kono, M. Mimura and N. Shimada [3].

In this note we shall verify his result and calculate the cohomology $\mathrm{Ext}_{A^*}(Z_2, Z_2)$ of the dual Hopf algebra A^* which is known to be isomorphic to the cohomology ring $\mathrm{Cotor}^A(Z_2, Z_2)=H^*(\Lambda)$ of the complex (Λ, d).

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1. Notations and Preliminaries

Let $L=Z_2\{x_1, x_1^2, x_2, x_2^2, x_1x_2, x_3, x_3^2\}$ be the linear subspace of A spanned by indicated elements, and sL denote a graded vector space over Z_2 such that

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$$(sL)_n=\begin{cases} L & \text{if } n=1 \\ 0 & \text{otherwise.}\end{cases}$$

Then the suspension map $s: L\to sL$ is a vector space isomorphism. The images of the elements in L by the suspension s are denoted by

$$\begin{array}{cccccccc} L & x_1 & x_1^2 & x_2 & x_2^2 & x_1x_2 & x_3 & x_3^2 \\ \downarrow & & & & & & & \\ sL & a_0 & a_1 & b_0 & b_1 & \alpha & c_0 & c_1 \end{array}$$

Let $\theta: A \xrightarrow{\pi} L \xrightarrow{s} sL$ be the composite map of the canonical projection π and the suspension s. Then θ is a linear map of degree 1.

Then we have $A\cong \mathrm{Ker}\,\theta\oplus L$, where

$$\mathrm{Ker}\,\theta\cong \boldsymbol{Z}_2\{1,\ x_1x_3,\ x_2x_3,\ x_1^3,\ x_2^3,\ x_3^3,\ x_1x_2x_3,\ x_1x_2^2,\ x_1^2x_2,$$
$$x_1x_3^2,\ x_1^2x_3,\ x_2x_3^2,\ x_2^2x_3,\ x_1^3x_2,\ x_1^2x_2^2,\ x_2x_3^3,\ \cdots\}$$

Denote by $T(sL)$ the tensor algebra on the graded vector space sL over $\boldsymbol{Z}_2$. Then

$$T(sL)\cong \boldsymbol{Z}_2\oplus sL\oplus sL\otimes sL\oplus sL\otimes sL\otimes sL\oplus \cdots .$$

2. Construction of a DGA-algebra Λ

We define a map $\theta\cup\theta: A\to T(sL)$ by the following composition;

$$\theta\cup\theta: A \xrightarrow{\phi} A\otimes A \xrightarrow{\theta\otimes\theta} sL\otimes sL \hookrightarrow T(sL)$$

where all the tensor products are over $\boldsymbol{Z}_2$.

Define a DGA-algebra over $\boldsymbol{Z}_2$

$$\Lambda=T(sL)/I$$

to be the quotient algebra of $T(sL)$ by the two-sided ideal I generated by $\theta\cup\theta(\mathrm{Ker}\,\theta)$, the $\theta\cup\theta$ image of $\mathrm{Ker}\,\theta$.

The augmentation $\varepsilon: \Lambda\to \boldsymbol{Z}_2$ is naturally induced from that of $T(sL)$.

The differential $\bar{d}: \Lambda\to\Lambda$ of degree 1 is defined as follows. Consider first the natural section $\iota: sL\xrightarrow{s^{-1}} L\hookrightarrow A$ such that $\theta\cdot\iota=1_{sL}$. Define $\tilde{d}: sL\to sL\otimes sL$ by $\tilde{d}=(\theta\cup\theta)\cdot\iota$, and extend this onto $T(sL)$ as a derivation which is denoted by $\tilde{d}$ also. We can verify the following

LEMMA 2.1. (1) $\tilde{d}(I)\subset I$,
(2) $\tilde{d}\cdot\tilde{d}\equiv 0 \pmod{I}$.

Proof. As $\theta\cdot\iota=1_{sL}$, we have $\theta\cdot(1-\iota\theta)=0$, and hence $(1-\iota\theta)$-image$\subset$Ker θ, it follows $(\theta\cup\theta)((1-\iota\theta)$-image$)\subset I$. Then we have

$$\bar{d}\theta=((\theta\cup\theta)\cdot\iota)\theta=\{(\theta\cup\theta)\cdot(1-\iota\theta)\}-\theta\cup\theta\equiv\theta\cup\theta \pmod{I},$$

and hence

$$\bar{d}\cdot(\theta\cup\theta)=\bar{d}\theta\cup\theta+\theta\cup\bar{d}\theta\equiv(\theta\cup\theta)\cup\theta+\theta\cup(\theta\cup\theta)=0 \pmod{I}.$$

Therefore we have proved $\bar{d}(I)=\bar{d}(\theta\cup\theta(\text{Ker }\theta))\subset I$, and $\bar{d}\cdot\bar{d}\equiv 0 \pmod{I}$ at a time. q. e. d.

Thus the derivation $\bar{d}$ on $T(sL)$ induces naturally a differential operator $\bar{d}$ on Λ.

To investigate the products in Λ, we will list the basis elements of $\theta\cup\theta(\text{Ker }\theta)=I_{(2)}$, the part of tensor degree 2 of the ideal I.

(2.2)
$$\begin{aligned}
&\theta\cup\theta(x_1^3)=[a_0, a_1]=a_0\cdot a_1+a_1\cdot a_0,\\
&\theta\cup\theta(x_2^3)=[b_0, b_1],\\
&\theta\cup\theta(x_3^3)=[c_0, c_1],\\
&\theta\cup\theta(x_1x_3)=[a_0, c_0]+a_1\cdot b_0+a_0\cdot\alpha,\\
&\theta\cup\theta(x_2x_3)=[b_0, c_0]+a_0\cdot b_1+\alpha\cdot b_0,\\
&\theta\cup\theta(x_1x_2x_3)=[\alpha, c_0]+a_1\cdot b_1+\alpha^2,\\
&\theta\cup\theta(x_1^2x_2)=[a_1, b_0],\\
&\theta\cup\theta(x_1^2x_3)=[a_1, c_0],\\
&\theta\cup\theta(x_1x_2^2)=[a_0, b_1],\\
&\theta\cup\theta(x_1x_3^2)=[a_0, c_1],\\
&\theta\cup\theta(x_2x_3^2)=[b_0, c_1],\\
&\theta\cup\theta(x_2^2x_3)=[b_1, c_0],\\
&\theta\cup\theta(x_1^2x_2^2)=[a_1, b_1],\\
&\theta\cup\theta(x_1^2x_3^2)=[a_1, c_1],\\
&\theta\cup\theta(x_2^2x_3^2)=[b_1, c_1],\\
&\theta\cup\theta(x_1^3x_2)=[\alpha, a_1],\\
&\theta\cup\theta(x_1x_2^3)=[\alpha, b_1],
\end{aligned}$$

$$\theta\cup\theta(x_1x_2x_3^2)=[\alpha, c_1],$$

$$\theta\cup\theta(\text{any other monomial})=0.$$

Consequently we have seen that a_1, b_1 and c_1 commute with all elements in Λ.

3. Twisted tesor product $A\otimes\Lambda$

In the preceding section, we defined the differential algebra $(\Lambda, \bar{d})$ over $\boldsymbol{Z}_2$. Let $\bar{\theta}$ be the composite map $A\xrightarrow{\theta} sL\hookrightarrow T(sL)\xrightarrow{\text{projection}}\Lambda$, then we can get the relation $\bar{d}\cdot\bar{\theta}+\bar{\theta}\cup\bar{\theta}=0$ as in the proof of Lemma 2.1. So we can construct the twisted tensor product $A\otimes_{\bar{\theta}}\Lambda$ with respect to $\bar{\theta}$ (cf. E. H. Brown [2]). That is, $A\otimes_{\bar{\theta}}\Lambda$ is an A-comodule with the differential operator

$$d(x\otimes\lambda)=x\otimes\bar{d}\lambda+(1\otimes\theta\otimes 1)(\psi(x)\otimes\lambda). \tag{3.1}$$

By the definition, this complex $A\otimes_{\bar{\theta}}\Lambda$ is isomorphic to $A\otimes(T(s\bar{A})/I)$ where $T(s\bar{A})$ is the cobar construction (cf. J. F. Adams [1]), and we denote $A\otimes_{\bar{\theta}}\Lambda$ by $(A\otimes\Lambda, d)$.

For the simplicity, we denote $x\otimes 1$ by x $(x\in A)$, $1\otimes\lambda$ by λ $(\lambda\in\Lambda)$, and $d|1\otimes\Lambda$ by d.

From (3.1) we have that

$$\begin{aligned} &dx_1=a_0, \quad dx_1^2=a_1,\\ &dx_2=b_0, \quad dx_2^2=b_1,\\ &dx_1x_2=\alpha+x_1\cdot b_0+x_2\cdot a_0,\\ &dx_3=c_0+x_1\cdot b_0,\\ &dx_3^2=c_1+x_1^2\cdot b_1. \end{aligned} \tag{3.2}$$

We know that the algebra Λ is generated by

$$\{a_0, a_1, b_0, b_1, \alpha, c_0, c_1\},$$

the basis elements of sL, and

$$\begin{aligned} &da_0=\bar{d}a_0=0, \quad da_1=\bar{d}a_1=0,\\ &db_0=\bar{d}b_0=0, \quad db_1=\bar{d}b_1=0,\\ &d\alpha=\bar{d}\alpha=[a_0, b_0]=a_0\cdot b_0+b_0\cdot a_0,\\ &dc_0=\bar{d}c_0=a_0\cdot b_0, \quad dc_1=\bar{d}c_1=a_1\cdot b_1. \end{aligned} \tag{3.3}$$

4. Acyclicity of $A\otimes\Lambda$

We introduce the weight function w in $A\otimes\Lambda$ as follows.

(4.1)

A	x_1	x_1^2	x_2	x_2^2	x_1x_2	x_3	x_3^2	$x_1^ix_2^jx_3^k$
Λ	a_0	a_1	b_0	b_1	α	c_0	c_1	0
w	0	0	0	0	0	1	2	k

Further we put $w(x\otimes\lambda)=w(x)+w(\lambda)$.
Define filtration $F_k=\{x\otimes\lambda\,|\,w(x\otimes\lambda)\leqq k\}$.
Put $E_0(A\otimes\Lambda)=\sum_{k\geqq 0}F_k/F_{k-1}$. Then we have from (3.2) and (3.3) that $d(F_k)\subset F_k$. So d induces differential operator d_0 in $E_0(A\otimes\Lambda)$.

PROPOSITION 4.2. (M. Tezuka) *The twisted tensor product $A\otimes\Lambda$ is an acyclic injective A-comodule resolution of $\boldsymbol{Z}_2$.*

Proof. $E_0(A\otimes\Lambda)$ has the following decomposition.

$$\begin{aligned}E_0(A\otimes\Lambda)\cong\ &\boldsymbol{Z}_2\{x_1, x_2, x_1x_2\}\otimes T(a_0, b_0, \alpha)\\&\otimes(\boldsymbol{Z}_2\{x_1^2\}\otimes\boldsymbol{Z}_2[a_1])\otimes(\boldsymbol{Z}_2\{x_2^2\}\otimes\boldsymbol{Z}_2[b_1])\\&\otimes(\boldsymbol{Z}_2[x_3]/(x_3^2))\otimes\boldsymbol{Z}_2[c_0]\\&\otimes\boldsymbol{Z}_2\{x_3^2\}\otimes\boldsymbol{Z}_2[c_1],\end{aligned}$$

where $\boldsymbol{Z}_2\{x_i, x_j\}$ means the vector space over $\boldsymbol{Z}_2$ generated by x_i and x_j.
By (3.2) and (3.3) we have the following (cf. J. F. Adams [1]);

$$\tilde{H}^*(\boldsymbol{Z}_2\{x_1, x_2, x_1x_2\}\otimes T(a_0, b_0, \alpha), d_0)=0\,,$$

$$\tilde{H}^*(\boldsymbol{Z}_2\{x_1^2\}\otimes\boldsymbol{Z}_2[a_1], d_0)=0\,,$$

$$\tilde{H}^*(\boldsymbol{Z}_2\{x_2^2\}\otimes\boldsymbol{Z}_2[b_1], d_0)=0\,,$$

$$\tilde{H}^*((\boldsymbol{Z}_2[x_3]/(x_3^2))\otimes\boldsymbol{Z}_2[c_0], d_0)=0\,,$$

$$\tilde{H}^*(\boldsymbol{Z}_2\{x_3^2\}\otimes\boldsymbol{Z}_2[c_1], d_0)=0\,,$$

where $\tilde{H}^*=\sum_{i>0}H^i$.
Thus we get the required result. q. e. d.

By definition we have

COROLLARY 4.3. $H^*(\Lambda)=\mathrm{Ker}\ d/\mathrm{Im}\ d\cong\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$.

Here we denoted the differential operator of Λ by d by abuse of notations.

5. Calculation

The purpose of this section is to determine $H^*(\Lambda)$.
First of all we prepare the following

LEMMA 5.1. $dc_0^4=a_1b_1[[a_0, b_0], \alpha]$.

Proof. Using (2.2), we can get

$$[a_0^2, c_0^2]=a_1[a_0, [b_0, \alpha]],$$

$$[b_0^2, c_0^2]=b_1[b_0, [a_0, \alpha]]$$

by a routine but tedious calculation. Substituting these to $dc_0^4=(a_1b_0^2+a_0^2b_1)c_0^2+c_0^2(a_1b_0^2+a_0^2b_1)$, we have the above result. q. e. d.

We have the filtration $F_k=\{\lambda\in\Lambda \mid w(\lambda)\leqq k\}$ using (4.1). To get the result we consider the spectral sequence $\{E_r(\Lambda), d_r\}$ associated with the filtration defined above with $\boldsymbol{Z}_2$ coefficient where d_r is induced from d of Λ.

We know that a_1 and b_1 commute with all elements in Λ by (2.2). So we have

$$F_0=F_0/F_{-1}=T(a_0, b_0, a_1, b_1, \alpha)\cong T(a_0, b_0, \alpha)\otimes\boldsymbol{Z}_2[a_1, b_1].$$

By (2.2) and (4.1), c_0 commutes with all elements in E_0 also. Then we get

$$E_0\cong T(a_0, b_0, \alpha)\otimes\boldsymbol{Z}_2[a_1, b_1]\otimes\boldsymbol{Z}_2[c_0]\otimes\boldsymbol{Z}_2[c_1].$$

By (3.3), a_0, b_0, a_1 and b_1 are permanent cycles, and $d_0\alpha=[a_0, b_0]$, $d_0c_0=0$, and $d_0c_1=0$.

Then we have

$$E_1\cong\boldsymbol{Z}_2[a_0, b_0]\otimes\boldsymbol{Z}_2[a_1, b_1]\otimes\boldsymbol{Z}_2[c_0]\otimes\boldsymbol{Z}_2[c_1],$$

with $d_1c_0=a_0b_0$, $d_1c_0^2=0$, and $d_1c_1=0$.

Subsequently

$$E_2\cong\boldsymbol{Z}_2[a_0, a_1, b_0, b_1]\otimes\boldsymbol{Z}_2[c_1, c_0^2]/(a_0b_0),$$

with $d_2c_1=a_1b_1$, $d_2c_1^2=0$, $d_2c_0^2=a_1b_0^2+a_0^2b_1$, and $d_2c_0^4=0$.

Then we have

$$E_3\cong\boldsymbol{Z}_2[a_0, a_1, b_0, b_1]\otimes\boldsymbol{Z}_2[c_1^2, c_0^4]/(a_0b_0, a_1b_1, a_1b_0^2+a_0^2b_1),$$

with $d_3=0$.

Thus we get $E_4\cong E_3$.

As $dc_1^2=0$, c_1^2 is a permanent cycle. By Lemma 5.1 we have $d_rc_0^4=0$

($r \geqq 3$) which follows that c_0^4 survives forever, and we have

$$E_4 \cong E_5 \cong \cdots \cong E_\infty.$$

As $dc_1 = a_1 b_1$ and $d\alpha^2 = [[a_0, b_0], \alpha]$, Lemma 5.1 shows that $d(c_0^4 + c_1 d\alpha^2) = 0$. Consequently we have obtained the following

THEOREM 5.2. *As an algebra over* $\boldsymbol{Z}_2$

$$\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[u_1, u_2, v_1, v_2, w_1, w_2]/(u_1 v_1, u_2 v_2, u_1^2 v_2 + u_2 v_1^2)$$

where $u_1 = \{a_0\}$, $u_2 = \{a_1\}$, $v_1 = \{b_0\}$, $v_2 = \{b_1\}$, $w_1 = \{c_0^4 + c_1 d\alpha^2\}$, *and* $w_2 = \{c_1^2\}$ *denote the respective cohomology classes of their representative cocycles.*

REFERENCES

[1] J.F. ADAMS, On the cobar construction, "Colloque de Topologie Algébrique", CBRM, Louvain, 1956, 81-87.

[2] E.H. BROWN, Twisted tensor product I, Ann. of Math., **69** (1959), 223-246.

[3] A. KONO, M. MIMURA AND N. SHIMADA, On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7, J. of Pure and Applied Alg., **8** (1976), 267-283.

[4] N. SHIMADA AND A. IWAI, On the cohomology of some Hopf algebras, Nagoya Math. J., **30** (1967), 103-111.

[5] W. WATERHOUSE, "Introduction to Affine Group Schemes", GTM **66**, Springer, N.Y., 1979.

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