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CENTRAL SECTIONS OF CENTRALLY SYMMETRIC CONVEX BODIES

Dedicated to Professor T. Otsuki on his 70th birthday

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§ 0. Introduction.

Let K and K' be two centrally symmetric convex bodies in the 3-dimensional Euclidean space E^3 with their centers at the origin O. The following problem is still open:

Suppose that for each plane L through O in E^3

$$\text{Area}(K\cap L)<\text{Area}(K'\cap L)$$

holds. Then does the inequality $\text{Vol}(K)<\text{Vol}(K')$ *follow*?

This problem has a natural meaning for any dimension $m\geqq 3$ by taking a hyperplane through the origin O of E^m as L. Let K and K' be centrally symmetric convex bodies in E^m with their centers at O. Then the following are known.

(i) Equality of (m$-$1)-dimensional volumes $\text{Vol}(K\cap L)=\text{Vol}(K'\cap L)$ for each L implies that K and K' are congruent; in particular $\text{Vol}(K)=\text{Vol}(K')$. This is shown by the generalized Funk's spherical integration theorem, which says that two even functions f_1 and f_2 on the $(m-1)$-dimensional unit sphere $S^{m-1}(1)$ are identical, if the integrals of f_1 and f_2 on each totally geodesic $(m-2)$-sphere are identical (cf. P. Funk [7], T. Bonnesen and W. Fenchel [2], p. 136-138, A.L. Besse [1], p. 103-104, p. 124-125 for $m=3$. Generalization to general m is not difficult.).

(ii) If K is an ellipsoid in E^m and $\text{Vol}(K\cap L)<\text{Vol}(K'\cap L)$ holds for each L, then $\text{Vol}(K)<\text{Vol}(K')$ follows (H. Busemann [3]). However, if K' is an ellipsoid then the question has not been answered yet.

(iii) By probabilistic arguments, D.G. Larman and C.A. Rogers [9] established the existence of a centrally symmetric convex body K in E^m, for $m\geqq 12$, such that for each hyperplane L, $\text{Vol}(K\cap L)<\text{Vol}(B^m\cap L)$ holds, nevertheless $\text{Vol}(K)>\text{Vol}(B^m)$, where B^m denotes the m-dimensional unit ball.

For a general survey on this problem and related subjects see an article

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by D. G. Larman [8] in the Proceedings of the International Congress of Mathematicians in Helsinki, 1978.

Now we return to the 3-dimensional case, which seems to be most important at present.

By $B^3(R)$ we denote the ball of radius R with center O in E^3, and by $S^2(1)$ we denote the unit sphere in E^3.

Let ε be a positive number and N be a natural number. Then $2N$ points $\pm p_1, \pm p_2, \cdots, \pm p_N$ on $S^2(1)$ are called *ε-properly distributed* on $S^2(1)$, if for any two different elements $x, y \in \{\pm p_1, \pm p_2, \cdots, \pm p_N\}$ two geodesic ε-disks on $S^2(1)$ centered at x and y are disjoint. By $\Theta = \{\pm p_1, \pm p_2, \cdots, \pm p_N\}$ we denote an ε-proper distribution of $2N$ points on $S^2(1)$.

By $K(\varepsilon, N, \Theta)$ we denote a centrally symmetric convex body obtained from $B^3(1)$ by removing $2N$ spherical caps of $B^3(1)$ of angular radius ε corresponding to Θ. $K(\varepsilon, N, \Theta)$ is a natural object as a centrally symmetric convex body which enables us to calculate various quantities and was studied in [9] for general dimension m.

For each ε-proper distribution Θ of $2N$ points on $S^2(1)$, if one varies planes L through O in E^3, then the mean value of $\mathrm{Area}(K(\varepsilon, N, \Theta) \cap L)$ is independent of Θ, and so we denote it by $M(\varepsilon, N)$.

Let $R(\varepsilon, N)$ be a real number determined by

$$\mathrm{Vol}(B^3(R(\varepsilon, N))) = \mathrm{Vol}(K(\varepsilon, N, \Theta)).$$

Then $R(\varepsilon, N) < 1$. If one could define Θ such that

$$\mathrm{Area}(K(\varepsilon, N, \Theta) \cap L) < \pi R(\varepsilon, N)^2 \tag{0.1}$$

holds for each L, then replacing $R(\varepsilon, N)$ by a slightly smaller R', one would get a counter example $K(\varepsilon, N, \Theta)$:

$$\mathrm{Area}(K(\varepsilon, N, \Theta) \cap L) < \mathrm{Area}(B^3(R') \cap L),$$

$$\mathrm{Vol}(K(\varepsilon, N, \Theta)) > \mathrm{Vol}(B^3(R')).$$

As Proposition 3.6 we prove the following.

THEOREM A. *$M(\varepsilon, N) < \pi R(\varepsilon, N)^2$ holds.*

This means that the mean value of the left hand side of (0.1) is always smaller than the right hand side. Therefore, at a glance, it seems to be possible to construct counter-examples to the question by distributing $2N$ points "homogeneously".

The purpose of this paper is to give some evidence that $\pi R(\varepsilon, N)^2 - M(\varepsilon, N)$ is too small to give Θ satisfying (0.1).

If N is not so large and ε is so small, then one may find L which does not meet any removed caps of $K(\varepsilon, N, \Theta)$.

If N is not so large and ε is so big as possible, then the variation of

Area($K(\varepsilon, N, \Theta)\cap L$) with respect to L is so big. In §9 we show two examples related to an octahedron and icosahedron, and one additional example which is not centrally symmetric.

To study the cases where $100\leqq N<\infty$, we define an ideal homogeneous model Θ_0' called the H-model of ε-proper distribution of $2N$ points on $S^2(1)$ in §4. Θ_0' is not concrete, but it is an abstract model which is nearly homogeneous and which allows us to calculate necessary quantities for $\varepsilon\to 0$, $N\to\infty$.

THEOREM B. *Let $N\geqq 100$. For the H-model Θ_0', there exists some plane L through O such that*

$$\mathrm{Area}(K(\varepsilon, N, \Theta_0')\cap L)>\pi R(\varepsilon, N)^2.$$

§1. Volumes of spherical caps.

Let $B^m(1)$ be the unit ball with center O in the m-dimensional Euclidean space E^m. For a positive number ε and a point p in the boundary of $B^m(1)$, ε-spherical cap $C^m(p, \varepsilon)$ of $B^m(1)$ is defined by

$$C^m(p, \varepsilon)=\{x\in B^m(1)\,;\,(x, p)>\cos\varepsilon\},$$

where (x, p) denotes the inner product of x and p, as position vectors. Then the volume of $C^m(p, \varepsilon)$ is given by (cf. [9], p. 166).

$$\mathrm{Vol}(C^m(p, \varepsilon))=\frac{\pi^{(m-1)/2}}{\Gamma((m+1)/2)}\int_0^{\varepsilon}\sin^m\theta\, d\theta.$$

LEMMA 1.1. *For $m=2$ and 3 we get*

$$\mathrm{Area}(C^2(p, \varepsilon))=\varepsilon-\frac{1}{2}\sin 2\varepsilon, \tag{1.1}$$

$$\mathrm{Vol}(C^3(p, \varepsilon))=\frac{\pi}{3}(\cos^3\varepsilon-3\cos\varepsilon+2). \tag{1.2}$$

§2. Mean value of Area $(K(\varepsilon, N, \Theta)\cap L)$.

In this section we give the expression of the mean value $M(\varepsilon, N)$ of Area($K(\varepsilon, N, \Theta)\cap L$) for an ε-proper distribution Θ of $2N$ points on $S^2(1)$.

Define a point A in E^3 by $A=(0, 0, 1)$, where coordinates of a point or components of a vector are ones with respect to the standard basis of E^3. Let $K_0(\varepsilon)$ denote the unit ball removed one spherical cap $C^3(A, \varepsilon)$; $K_0(\varepsilon)=B^3(1)-C^3(A, \varepsilon)$. Let g be a great circle on the unit sphere $S^2(1)$ in E^3. Suppose that g meets the geodesic circle on $S^2(1)$ of radius ε centered at A at two points V and Z. Let M be the middle point of the (shorter) geodesic segments VZ of g. The length of the geodesic segment MV is denoted by $\tilde{\varepsilon}$ and the distance on $S^2(1)$ between A and M is denoted by t. Then we get

$$\cos\varepsilon=\cos\varepsilon^{\sim}\cos t. \tag{2.1}$$

The set of all planes through O is identified with a 2-dimensional real projective space RP^2 by considering to each plane L its normal line through O. We identify RP^2 with $S^2_*(1)$, which denotes the closed upper hemisphere removed one half of the equator. RP^2 is also identified with the set of all great circles on $S^2(1)$ by identifying L with $L\cap S^2(1)=g$. For $x\in S^2_*(1)$, $g(x)$ or $L(x)$ means the great circle on $S^2(1)$ or plane through O corresponding to x with respect to the above identification.

By $P(\varepsilon)$ we denote the mean value of $\pi-\mathrm{Area}(K_0(\varepsilon)\cap L)$ with respect to $\{L\}=RP^2$. Then, the mean value $M(\varepsilon, N)$ of $\mathrm{Area}(K(\varepsilon, N, \Theta)\cap L)$ with respect to $\{L\}$ is given by $\pi-2N\cdot P(\varepsilon)$.

LEMMA 2.1. *Let $\varepsilon^{\sim}=\varepsilon^{\sim}(\varepsilon, t)$ be a function defined by* (2.1). *Then $P(\varepsilon)$ is given by*

$$P(\varepsilon)=\int_0^{\varepsilon}\left(\varepsilon^{\sim}-\frac{1}{2}\sin 2\varepsilon^{\sim}\right)\cos t\,dt. \tag{2.2}$$

Proof. Let (s, θ) be a polar coordinate system of $S^2_*(1)$ centered at A. (For a point x in $S^2_*(1)$, $s=s(x)$ is the distance between x and A, and θ is zero for the geodesic segment AX where $X=(1, 0, 0)$.) Then the volume element of $S^2_*(1)$ is given by $\sin s\,ds\,d\theta$.

For $x\in S^2_*(1)$ such that $\pi/2-\varepsilon\leqq s(x)\leqq\pi/2$, $g(x)$ meets the spherical cap $C^3(A, \varepsilon)$ of $B^3(1)$. For $x\in S^2_*(1)$ the distance $t=t(x)$ between A and $g(x)$ is equal to $\pi/2-s(x)$. So, $\varepsilon^{\sim}(\varepsilon, t)$ is determined and

$$\mathrm{Area}(B^2(1))-\mathrm{Area}(K_0(\varepsilon)\cap L(x))=\mathrm{Area}(C^2(M, \varepsilon^{\sim})),$$

where M is the point of $g(x)$ nearest to A. By Lemma 1.1 we get

$$P(\varepsilon)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\varepsilon}\left(\varepsilon^{\sim}-\frac{1}{2}\sin 2\varepsilon^{\sim}\right)\cos t\,dt\,d\theta,$$

where we have used $\mathrm{Vol}(RP^2)=2\pi$. Thus, proof is completed.

Later we need the following relations among $\varepsilon^{\sim}$, ε and t. By $[\varepsilon^k]$ we denote the higher order ($\geqq k$) terms with respect to ε, $\varepsilon^{\sim}$ and t. This is reasonable, because $\varepsilon^{\sim}\leqq\varepsilon$ and $t\leqq\varepsilon$.

LEMMA 2.2. *$\varepsilon^{\sim}$ is expanded as follows*:

$$\varepsilon^{\sim 2}=(\varepsilon^2-t^2)\left(1+\frac{1}{3}t^2\right)+[\varepsilon^6]. \tag{2.3}$$

$$\varepsilon^{\sim}=\sqrt{\varepsilon^2-t^2}\left(1+\frac{1}{6}t^2\right)+[\varepsilon^5]. \tag{2.4}$$

Proof. Expanding $\cos\varepsilon$, $\cos\varepsilon^{\sim}$ and $\cos t$ in each variable and using (2.1) we obtain

$$\varepsilon^{\sim 2}=\varepsilon^2-t^2+\frac{1}{2}t^2\varepsilon^{\sim 2}+\frac{1}{12}(\varepsilon^{\sim 4}+t^4-\varepsilon^4)+[\varepsilon^6],$$

and hence we get (2.3). (2.4) follows from (2.3).

§ 3. $R(\varepsilon, N)$ and $P(\varepsilon)$.

Let $K(\varepsilon, N, \Theta)$ be a centrally symmetric convex body obtained from $B^3(1)$ by removing $2N$ spherical caps as before. By Lemma 1.1 and

$$\mathrm{Vol}(K(\varepsilon, N, \Theta))=\frac{4\pi}{3}-2N\cdot\mathrm{Vol}(C^3(A, \varepsilon))$$

we see that $R(\varepsilon, N)$ satisfying

$$\mathrm{Vol}(K(\varepsilon, N, \Theta))=\mathrm{Vol}(B^3(R(\varepsilon, N))) \tag{3.1}$$

is given by

$$R(\varepsilon, N)^3=1-\frac{N}{2}(\cos^3\varepsilon-3\cos\varepsilon+2). \tag{3.2}$$

Then $\mathrm{Area}(K(\varepsilon, N, \Theta)\cap L)<\pi R(\varepsilon, N)^2$ is equivalent to

$$\mathrm{Area}(K(\varepsilon, N, \Theta)\cap L)<\mathrm{Area}(B^3(R(\varepsilon, N))\cap L). \tag{3.3}$$

We define $A(\varepsilon, N)$ by

$$A(\varepsilon, N)=\pi(1-R(\varepsilon, N)^2). \tag{3.4}$$

$A(\varepsilon, N)$ is the lower bound of the sum of areas of spherical caps removed in $K(\varepsilon, N, \Theta)\cap L$ for $K(\varepsilon, N, \Theta)$ to satisfy (3.1) and (3.3).

For some pairs (ε, N) we calculate values of $P(\varepsilon)$ and $A(\varepsilon, N)$ showing the inequality $2N\cdot P(\varepsilon)>A(\varepsilon, N)$. The difference $2N\cdot P(\varepsilon)-A(\varepsilon, N)$ may be helpful to understand the situation.

Table 3.1. For pairs (ε, N) such that $N\varepsilon^2 \doteqdot 1$:

ε	N	$2N\cdot P(\varepsilon)$	$A(\varepsilon, N)$	$2N\cdot P(\varepsilon)-A(\varepsilon, N)$
0.1	100	$7.8409\cdots 10^{-3}$	$7.8327\cdots 10^{-3}$	$8.2\cdot 10^{-6}$
0.075	177	$4.3944\cdots 10^{-3}$	$4.3918\cdots 10^{-3}$	$2.6\cdot 10^{-6}$
0.05	400	$1.9626\cdots 10^{-3}$	$1.9621\cdots 10^{-3}$	$5.1\cdot 10^{-7}$
0.025	1600	$4.9082\cdots 10^{-4}$	$4.9079\cdots 10^{-4}$	$3.2\cdot 10^{-8}$

Table 3.2. For pairs (ε, N) such that $\varepsilon=0.05$:

N	$2N\cdot P(\varepsilon)$	$A(\varepsilon, N)$	$2N\cdot P(\varepsilon)-A(\varepsilon, N)$
100	$4.9066\cdots 10^{-4}$	$4.9048\cdots 10^{-4}$	$1.9\cdot 10^{-7}$
200	$9.8133\cdots 10^{-4}$	$9.8100\cdots 10^{-4}$	$3.3\cdot 10^{-7}$
300	$1.4720\cdots 10^{-3}$	$1.4715\cdots 10^{-3}$	$4.4\cdot 10^{-7}$
500	$2.4533\cdots 10^{-3}$	$2.4528\cdots 10^{-3}$	$5.4\cdot 10^{-7}$

In the table 3.1, $N\varepsilon^2 \doteqdot 1$ corresponds to the fact that the sum of areas of $2N$ geodesic ε-disks in $S^2(1)$ is about one half of the total area of $S^2(1)$. In the table 3.2, we notice that the number N is limited by (3.8) below.

LEMMA 3.3. *For $\varepsilon<0.136$, $P(\varepsilon)$ is estimated by*

$$\frac{\pi}{8}\varepsilon^4-\frac{\pi}{40}\varepsilon^6<P(\varepsilon)<\frac{\pi}{8}\varepsilon^4-\frac{\pi}{60}\varepsilon^6. \tag{3.5}$$

Proof. Expanding $\sin 2\varepsilon^{\sim}$ and using (2.3) and (2.4) we obtain

$$\begin{aligned}\varepsilon^{\sim}-\frac{1}{2}\sin 2\varepsilon^{\sim}&=\frac{2}{3}\varepsilon^{\sim}\Big(\varepsilon^{\sim 2}-\frac{1}{5}\varepsilon^{\sim 4}+[\varepsilon^6]\Big)\\ &=\frac{2}{3}\sqrt{\varepsilon^2-t^2}\Big(\varepsilon^2-t^2-\frac{1}{5}\varepsilon^4+\frac{9}{10}\varepsilon^2t^2-\frac{7}{10}t^4\Big)+[\varepsilon^7].\end{aligned}$$

Expanding $\cos t$ we get

$$\begin{aligned}P(\varepsilon)&=\int_0^\varepsilon\Big(\varepsilon^{\sim}-\frac{1}{2}\sin 2\varepsilon^{\sim}\Big)\cos t\,dt\\ &=\frac{2}{3}\Big(\varepsilon^2-\frac{1}{5}\varepsilon^4\Big)\int_0^\varepsilon\sqrt{\varepsilon^2-t^2}\,dt-\frac{2}{3}\Big(1-\frac{2}{5}\varepsilon^2\Big)\int_0^\varepsilon\sqrt{\varepsilon^2-t^2}\,t^2\,dt\\ &\quad-\frac{2}{15}\int_0^\varepsilon\sqrt{\varepsilon^2-t^2}\,t^4\,dt+[\varepsilon^8].\end{aligned}$$

On the other hand, for each even integer k, we have

$$\int_0^\varepsilon\sqrt{\varepsilon^2-t^2}\,t^k\,dt=\frac{\pi(k-1)!!}{2(k+2)!!}\varepsilon^{k+2} \tag{3.6}$$

and so we get

$$P(\varepsilon)=\frac{\pi}{8}\varepsilon^4-\frac{\pi}{48}\varepsilon^6+[\varepsilon^8]. \tag{3.7}$$

For $\varepsilon<0.136$ by numerical calculation (by computer) we can verify (3.5). For example, if $\varepsilon=0.136$, then

$$\pi/8-\pi\varepsilon^2/40=0.3912\cdots,$$

$$P(\varepsilon)/\varepsilon^4=0.3914\cdots,$$

$$\pi/8-\pi\varepsilon^2/60=0.3917\cdots.$$

The meaning of the value 0.136 is explained later in § 5.

LEMMA 3.4. *For* $\varepsilon<\pi/2$, $N\varepsilon^2$ *is estimated by*

$$N\varepsilon^2<2+\frac{1}{5}\varepsilon^2. \tag{3.8}$$

Proof. The area of a geodesic disk of radius ε on $S^2(1)$ is $2\pi(1-\cos\varepsilon)$. So the total area of $2N$ geodesic disks is $4\pi N(1-\cos\varepsilon)$, which is smaller than the area 4π of $S^2(1)$. Expanding $\cos\varepsilon$ we get the inequality.

LEMMA 3.5. *For* $\varepsilon<0.136$, $A(\varepsilon, N)$ *is estimated by*

$$\pi N\left(\frac{1}{4}\varepsilon^4-\frac{1}{12}\varepsilon^6\right)<A(\varepsilon, N)<\pi N\left(\frac{1}{4}\varepsilon^4-\frac{1}{20}\varepsilon^6\right). \tag{3.9}$$

Proof. Since

$$\cos^3\varepsilon-3\cos\varepsilon+2=\frac{3}{4}\varepsilon^4-\frac{1}{4}\varepsilon^6+\frac{13}{320}\varepsilon^8+[\varepsilon^{10}],$$

by (3.2) we obtain the expansion of $R(\varepsilon, N)^3$ and hence

$$R(\varepsilon, N)=1-\frac{N}{8}\varepsilon^4+\frac{N}{24}\varepsilon^6-\left(\frac{13N}{1920}+\frac{N^2}{64}\right)\varepsilon^8$$
$$+N[\varepsilon^{10}]+N^2[\varepsilon^{10}]+\sum_{h=3}^{\infty}N^h[\varepsilon^{4h}].$$

Furthermore we get

$$R(\varepsilon, N)^2=1-\frac{N}{4}\varepsilon^4+\frac{N}{12}\varepsilon^6-\left(\frac{13N}{960}+\frac{N^2}{64}\right)\varepsilon^8$$
$$+N[\varepsilon^{10}]+N^2[\varepsilon^{10}]+\sum_{h=3}^{\infty}N^h[\varepsilon^{4h}].$$

Since $N\varepsilon^2$ is bounded, we can put $N^h[\varepsilon^{4h}]=N[\varepsilon^{2h+2}]$ and so

$$R(\varepsilon, N)^2=1-\frac{N}{4}\varepsilon^4+\frac{N}{12}\varepsilon^6-\frac{N^2}{64}\varepsilon^8+N[\varepsilon^8].$$

Therefore

$$A(\varepsilon, N)=\pi N\left(\frac{1}{4}\varepsilon^4-\frac{1}{12}\varepsilon^6+\frac{N}{64}\varepsilon^8+[\varepsilon^8]\right). \tag{3.10}$$

We use (3.8) to obtain the upper estimate of $A(\varepsilon, N)$ and we replace $N\varepsilon^8/64$ in (3.10) by $\varepsilon^6/32$. Then, (3.9) is verified by numerical calculation. For each value of ε the range of N is limited by (3.8).

For example, if $\varepsilon=0.136$ (in this case $1\leqq N\leqq 108$) and if $N=100$, then

$$\pi N(1/4-\varepsilon^2/12)=78.05\cdots,$$

$$A(\varepsilon, N)/\varepsilon^4=78.22\cdots,$$

$$\pi N(1/4-\varepsilon^2/20)=78.24\cdots.$$

Now we prove the following.

PROPOSITION 3.6. *$2N\cdot P(\varepsilon)>A(\varepsilon, N)$ holds for each pair (ε, N) such that $2N$ points can be ε-properly distributed on $S^2(1)$.*

Proof. For $\varepsilon<0.136$ we see that $2N\cdot P(\varepsilon)>A(\varepsilon, N)$ by the first inequality of (3.5) and the second inequality of (3.9).

For $\varepsilon\geqq 0.136$ we can verify $2N\cdot P(\varepsilon)>A(\varepsilon, N)$ by numerical calculation. For each value of ε, the range of N is limitted by (3.8). If ε gets larger, then the maximum of N gets smaller.

§4. *H*-model.

From now on we show some evidence that

$$\pi R(\varepsilon, N)^2-M(\varepsilon, N)=2N\cdot P(\varepsilon)-A(\varepsilon, N)$$

is too small to construct concrete examples $K(\varepsilon, N, \Theta)$ satisfying (3.1) and (3.3).

We define $\Theta_0'=\{\pm q_1, \pm q_2, \cdots, \pm q_N\}$ somewhat abstractly. First we define q_1, q_2 and q_3 in the following setting.

⟨4-1⟩ Setting.

(i) $\{q_1, q_2, q_3\}$ makes an equilateral (geodesic) triangle on $S^2(1)$.

(ii) The center of the triangle $q_1q_2q_3$ is $A=(0, 0, 1)$.

(iii) For each i $(i=1, 2, 3)$, q_i represents a hexagon H_i on $S^2(1)$ and q_i is the center of H_i.

(iv) The area of H_i is equal to $4\pi/2N$.

(v) H_1, H_2 and H_3 are placed naturally so that two edges of each H_i coincide with respective one edge of the other hexagons.

⟨4-2⟩ Definition of hexagon H $(=H_1)=ABCDEF$.

(i) A, D and the center Q $(=q_1)$ of H are in the (x^2, x^3)-plane.

(ii) The lengths of the geodesic segments AB, BQ, QC, CD, DE, EQ, QF, FA are all equal to α.

(iii) $\angle BAQ=\angle FAQ=\angle CDQ=\angle EDQ=\pi/3$.

Coordinate expressions of these points are as follows:

$$A=(0, 0, 1),$$

$$B=((\sqrt{3}/2)\sin\alpha, (1/2)\sin\alpha, \cos\alpha),$$

$$C=(c^1,\ c^2,\ c^3)/2(3\cos^2\alpha+1),$$

$$c^1=\sqrt{3}(3\cos^2\alpha+1)\sin\alpha,$$

$$c^2=(13\cos^2\alpha-1)\sin\alpha,$$

$$c^3=14\cos^3\alpha-6\cos\alpha,$$

$$D=(0,\ d^2,\ d^3)/(3\cos^2\alpha+1)^2,$$

$$d^2=8(5\cos^2\alpha-1)\cos\alpha\sin\alpha,$$

$$d^3=41\cos^4\alpha-26\cos^2\alpha+1,$$

$$Q=(0,\ 4\cos\alpha\sin\alpha,\ 5\cos^2\alpha-1)/(3\cos^2\alpha+1).$$

⟨4-3⟩ Area of H.

We denote the lengths of geodesic segments AQ and BC by λ and μ, respectively. Then we get

$$\cos\lambda=(A,\ Q)=(5\cos^2\alpha-1)/(3\cos^2\alpha+1),$$

$$\cos\mu=(B,\ C)=(\cos^2\alpha+1)/2.$$

With respect to the triangles ABQ and BCQ we get the classical relations:

$$\sin\alpha\sin\angle ABQ=\sin\lambda\sin(\pi/3) \tag{4.1}$$

$$\sin\alpha\sin(\pi/3)=\sin\mu\sin\angle QBC \tag{4.2}$$

and we can calculate $\angle ABQ$ and $\angle QBC$. Further, we obtain

$$\mathrm{Area}(H)=4(\angle ABQ+\angle QBC)-8\pi/3. \tag{4.3}$$

For a given value of α, we can calculate $\angle ABQ$ and $\angle QBC$ by (4.1) and (4.2). Next by (4.3) we obtain the area Area(H) corresponding to α. Conversely, for a given natural number N we can find the (approximated) value of α so that Area$(H)=4\pi/2N$.

Example. (i) For $N=100$, $\alpha=0.1551\cdots$.
(ii) For $N=400$, we get the following values:

$\mathrm{Area}(H)=2\pi/N=0.0157\cdots$	$\alpha=0.0777\cdots$
$\lambda=0.0778\cdots$	$\mu=0.0776\cdots$
$\angle ABQ=1.0498\cdots$	$\angle QBC=1.0485\cdots$

LEMMA 4.1. Area$(H)=2\pi/N$ *is expanded as follows*:

$$\frac{2\pi}{N}=\frac{3\sqrt{3}}{2}\alpha^2+\frac{5\sqrt{3}}{16}\alpha^4+[\alpha^6].$$

Proof. Expanding $\cos\lambda$ and $\cos\mu$ with respect to α, we get

$$\cos\lambda=1-\frac{1}{2}\alpha^2-\frac{5}{24}\alpha^4-\frac{77}{1440}\alpha^6+[\alpha^8],$$

$$\cos\mu=1-\frac{1}{2}\alpha^2+\frac{1}{6}\alpha^4-\frac{1}{45}\alpha^6+[\alpha^8],$$

and using relations (4.1) and (4.2) we obtain

$$\sin<ABQ=\frac{\sqrt{3}}{2}\Big(1+\frac{1}{4}\alpha^2-\frac{1}{48}\alpha^4+[\alpha^6]\Big),$$

$$\sin<QBC=\frac{\sqrt{3}}{2}\Big(1+\frac{1}{8}\alpha^2-\frac{7}{384}\alpha^4+[\alpha^6]\Big).$$

$\cos<ABQ$ and $\cos<QBC$ are obtained from these. We put $4Z=\mathrm{Area}(H)$. Expanding $\sin(<ABQ+<QBC)=\sin(Z+2\pi/3)$, we obtain

$$\frac{\sqrt{3}}{4}\Big(\frac{3}{4}\alpha^2+\frac{111}{192}\alpha^4+[\alpha^6]\Big)=\frac{1}{2}Z+\frac{\sqrt{3}}{4}Z^2+[Z^3],$$

from which we obtain the relation in Lemma 4.1.

By Lemma 4.1 we get

$$\alpha^2N=\frac{4\sqrt{3}}{9}\pi-\frac{5\sqrt{3}}{54}\pi\alpha^2+[\alpha^4], \tag{4.4}$$

and by numerical calculation we can verify $\alpha^2N>4\sqrt{3}\pi/9-\pi\alpha^2/5$ for $\alpha<0.156$. Then $\alpha^2N>2.4$ for $\alpha<0.156$, and hence we get the following.

LEMMA 4.2. *For $\alpha<0.156$, αN is estimated by*

$$\alpha N>\frac{12}{5\alpha}. \tag{4.5}$$

⟨4-4⟩ Mean values.

Let Ω be the domain in $S^2(1)$ defined by three hexagons H_1, H_2, and H_3. Since q_1 $(=Q)$, q_2 and q_3 are defined as centers of three hexagons, to define Θ_0' as a standard model we suppose that $2N-6$ points $\{\pm q_4, \cdots, \pm q_N\}$ are distributed in $S^2(1)-\Omega\cup(-\Omega)$ (abstractly and) nearly homogeneously.

Let $\{L_A\}$ be the set of all planes which contain the line AO. Planes L_A are parametrized by angles θ from the first axis; $0\leqq\theta<\pi$. We want to calculate the rotational mean value at A, that is, the mean value $M_0'(\varepsilon, N; A)$ of $\mathrm{Area}(K(\varepsilon, N, \Theta_0')\cap L_A)$ with respect to planes $\{L_A\}$.

Let Θ_0 be an abstract ε-proper distribution of $2N$ points on $S^2(1)$, which is nearly homogeneously distributed. Then the mean value $M_0(\varepsilon, N; A)$ of $\mathrm{Area}(K(\varepsilon, N, \Theta_0)\cap L_A)$ with respect to $\{L_A\}$ is equal to $M(\varepsilon, N)=\pi-2N\cdot P(\varepsilon)$. Here we divide $2N\cdot P(\varepsilon)$ into two factors:

$$2N\cdot P(\varepsilon)=2P(\varepsilon,\ N,\ \Omega)+P(\varepsilon,\ N,\ S^2(1)-\Omega\cup(-\Omega)),$$

where $P(\varepsilon, N, \Omega)$ is defined as follows: Let $\theta\in[\pi/6, \pi/2]$ and let $\{\rho_\theta(s)\}$ be the geodesic emanating from A such that the angle between $d\rho_\theta(0)/ds$ and the x^1-axis is θ. By $l(\theta)$ we denote the length of the geodesic segment $\{\rho_\theta(s)\}\cap\Omega$. Then the effect of the mean value of sum of areas of removed caps restricted to $\{\rho_\theta(s)\}\cap\Omega$ is $2N\cdot P(\varepsilon)\cdot l(\theta)/2\pi$, and

$$P(\varepsilon,\ N,\ \Omega)=\frac{6}{\pi}\cdot 2N\cdot P(\varepsilon)\cdot\frac{1}{2\pi}\int_{\pi/6}^{\pi/2} l(\theta)d\theta. \tag{4.6}$$

We denote the mean value of $\pi-\mathrm{Area}(K(\varepsilon, N, \Theta')\cap L_A)$ with respect to $\{L_A\}$ by $2P(q_1q_2q_3)$, where $\Theta'=\{\pm q_1, \pm q_2, \pm q_3\}$. To define Θ_0' we replace $P(\varepsilon, N, \Omega)$ by $P(q_1q_2q_3)$.

DEFINITION. H-model Θ_0' of ε-proper distribution of $2N$ points on $S^2(1)$ is $\{\pm q_1, \pm q_2, \pm q_3, \cdots, \pm q_N\}$ such that $M_0'(\varepsilon, N; A)$ is calculated by

$$\pi-M_0'(\varepsilon,\ N;\ A)=2N\cdot P(\varepsilon)-2P(\varepsilon,\ N,\ \Omega)+2P(q_1q_2q_3). \tag{4.7}$$

Here we notice that the condition (i) of Setting ⟨4-1⟩ is related to the case where N is not small. Since we are studying the case where $N\geqq 100$, this may be natural.

§ 5. The range of ε with respect to α.

For a given value of N, α and H are determined. Let M' be the middle point of the geodesic segment AB. Let ν be the distance between M' and Q. Then the range of ε is estimated by $0<\varepsilon<\nu$. The coordinates of M' are given by

$$(\sqrt{3}\sin\alpha,\ \sin\alpha,\ 2(\cos\alpha+1))/[8(\cos\alpha+1)]^{1/2}.$$

Therefore

$$\begin{aligned}\cos\nu&=(M',\ Q)\\ &=(3\cos^3\alpha+5\cos^2\alpha+\cos\alpha-1)/(3\cos^2\alpha+1)(2\cos\alpha+2)^{1/2}\\ &=1-\frac{3}{8}\alpha^2-\frac{17}{128}\alpha^4+[\alpha^6].\end{aligned}$$

Furthermore we obtain

$$\nu=\frac{\sqrt{3}}{2}\alpha+\frac{5\sqrt{3}}{48}\alpha^3+[\alpha^5].$$

Consequently

$$\left(\frac{\nu}{\sin\alpha}\right)^2=\frac{3}{4}+\frac{9}{16}\alpha^2+[\alpha^4], \tag{5.1}$$

and hence

$$\left[1-\left(\frac{\nu}{\sin\alpha}\right)^2\right]^{-1/2}=2+\frac{9}{4}\alpha^2+[\alpha^4]. \tag{5.2}$$

By numerical calculation we get the following.

LEMMA 5.1. *For* $\alpha<0.156$

$$\left(\frac{\varepsilon}{\sin\alpha}\right)^2<\frac{3}{4}+\frac{3}{5}\alpha^2, \tag{5.3}$$

$$\left[1-\left(\frac{\varepsilon}{\sin\alpha}\right)^2\right]^{-1/2}<2+3\alpha^2. \tag{5.4}$$

For $N=100$, we get $\alpha=0.1551\cdots$ and $\nu=0.1350\cdots$. Since we are studying the case where $N\geqq 100$, the ranges of α and ε may be set as follows:

$$0<\alpha<0.156, \qquad 0<\varepsilon<0.136$$

§6. $P(\varepsilon, N, \Omega)$.

By $S=ABCD$ we denote the quadrangle on $S^2(1)$ defined by A, B, C and D. Let $S^*=A^*B^*C^*D^*$ be the quadrangle on the tangent space $T_AS^2(1)$ to $S^2(1)$ at A satisfying the following conditions.

(i) $|A^*B^*|=|B^*C^*|=|C^*D^*|=\alpha$, $|A^*D^*|=2\alpha$.

(ii) By the exponential map φ at A, $\varphi(A^*)=A$, $\varphi(B^*)=B$, and $\varphi(A^*D^*)$ is contained in the geodesic segment AD.

Then $\angle D^*A^*B^*=\pi/3$ follows.

LEMMA 6.1. $\varphi^{-1}(S)$ *contains* S^*.

Proof. We define C_* and D_* by $C_*=\varphi^{-1}(C)$ and $D_*=\varphi^{-1}(D)$. $\angle C_*A^*D^*$ is calculated by the coordinates of C;

$$\cos\angle C_*A^*D^*=(13\cos^2\alpha-1)/2(49\cos^4\alpha-2\cos^2\alpha+1)^{1/2}.$$

Then $\cos^2\angle C_*A^*D^*-3/4<0$ is equivalent to

$$(11\cos^2\alpha+1)(\cos^2\alpha-1)<0,$$

and so we see that $\angle C_*A^*D^*>\angle C^*A^*D^*=\pi/6$.

Next we show that the orthogonal projection of C_* to the line A^*C^* lies in the extension of A^*C^*. That is,

$$|A^*C_*|\cos(\angle C_*A^*D^*-\pi/6)>\sqrt{3}\,\alpha. \tag{6.1}$$

By the expression of $\cos\angle C_*A^*D^*$ we get

$$\cos(\angle C_*A^*D^*-\pi/6)=4\sqrt{3}\cos^2\alpha/(49\cos^4\alpha-2\cos^2\alpha+1)^{1/2}.$$

Then (6.1) is equivalent to

$$|A^*C_*|>(49\cos^4\alpha-2\cos^2\alpha+1)^{1/2}\alpha/4\cos^2\alpha. \tag{6.2}$$

Since $|A^*C_*|=|AC|$, and $\cos|AC|$ is known by the coordinates of C, (6.2) is equivalent to

$$(7\cos^3\alpha-3\cos\alpha)/(3\cos^2\alpha+1)<\cos\beta, \tag{6.3}$$

where

$$\beta=(49\cos^4\alpha-2\cos^2\alpha+1)^{1/2}\alpha/4\cos^2\alpha.$$

We expand the both sides of (6.3) and get

$$(7\cos^3\alpha-3\cos^2\alpha)/(3\cos^2\alpha+1)=1-\frac{3}{2}\alpha^2+\frac{1}{8}\alpha^4+[\alpha^6]$$

$$<1-\frac{3}{2}\alpha^2+\frac{1}{4}\alpha^4 \qquad \text{for} \quad \alpha<0.156$$

$$\cos\beta=1-\frac{3}{2}\alpha^2+\frac{3}{8}\alpha^4+[\alpha^6]$$

$$>1-\frac{3}{2}\alpha^2+\frac{1}{4}\alpha^4 \qquad \text{for} \quad \alpha<0.156.$$

Therefore we get (6.1). Since $\varphi^{-1}(BC)$ and $\varphi^{-1}(CD)$ are convex in $T_AS^2(1)$, we see that $\varphi^{-1}(S)$ contains S^*. (q. e. d.)

By Lemma 6.1 we obtain

$$\int_{\pi/6}^{\pi/2} l(\theta)d\theta>\int_{\pi/6}^{\pi/3}\frac{\sqrt{3}\alpha}{2\cos\theta}d\theta+\int_{\pi/3}^{\pi/2}\frac{\sqrt{3}\alpha}{\cos(\theta-\pi/3)}d\theta.$$

Since

$$\int\frac{1}{\cos\theta}d\theta=\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right),$$

we obtain the following.

$$P(\varepsilon, N, \Omega)>\frac{3\sqrt{3}}{\pi^2}N\cdot P(\varepsilon)\cdot\alpha\left(\log\tan\frac{5\pi}{12}+\log\tan\frac{\pi}{3}\right).$$

Therefore we get

LEMMA 6.2. *$P(\varepsilon, N, \Omega)$ is estimated by*

$$2P(\varepsilon, N, \Omega)>\frac{3\sqrt{3}}{\pi^2}\log(3+2\sqrt{3})\cdot 2N\cdot P(\varepsilon)\cdot\alpha>\frac{98\alpha}{100}\cdot 2N\cdot P(\varepsilon). \tag{6.4}$$

§ 7. Mean value $U(b, \varepsilon)$.

Let $X=(1, 0, 0)$ and $Y=(0, 1, 0)$. Let $T=(\sin b, 0, \cos b)$ where $0<b\leqq\pi/2$. Let L_A be a plane in $\{L_A\}$ and let $g=g(L_A)=g(\theta)$ be the corresponding great circle on $S^2(1)$, $0\leqq\theta<\pi/2$. The point of intersection of g and the equator is $(\cos\theta, \sin\theta, 0)$.

The distance $w=w(b, \theta)$ between T and g is given by

$$\sin w=\sin b\sin\theta. \tag{7.1}$$

Let $\varepsilon<b$ and let θ_0 be the value of parameter of L_A for which g is tangent to the geodesic circle $C[T, \varepsilon]$ of radius ε centered at T in $S^2(1)$. By putting $w=\varepsilon$ in (7.1) we see that θ_0 is determined by

$$\sin\theta_0=\frac{\sin\varepsilon}{\sin b}. \tag{7.2}$$

For $\theta\in[0, \theta_0]$, we denote the points of intersection of $g(\theta)$ and $C[T, \varepsilon]$ by V and Z. The half of the distance between V and Z is denoted by $\varepsilon^*=\varepsilon^*(\varepsilon, b, \theta)$. Then

$$\cos\varepsilon=\cos\varepsilon^*\cos w. \tag{7.3}$$

With respect to only one spherical cap $C^3(T, \varepsilon)$, the mean value $U(b, \varepsilon)$ of areas of removed caps with respect to $\{L_A\}$ is calculated by

$$U(b, \varepsilon)=\frac{2}{\pi}\int_0^{\theta_0}\left(\varepsilon^*-\frac{1}{2}\sin 2\varepsilon^*\right)d\theta. \tag{7.4}$$

If one changes the variables, then (7.4) is rewritten as

$$U(b, \varepsilon)=\frac{2}{\pi}\int_0^{\varepsilon}\left(\varepsilon^*-\frac{1}{2}\sin 2\varepsilon^*\right)\frac{\cos w}{\sqrt{\sin^2 b-\sin^2 w}}dw. \tag{7.5}$$

Next we obtain an estimate of $U(\alpha, \varepsilon)$.

LEMMA 7.1.

$$\begin{aligned}\int_0^{\varepsilon}\frac{\sqrt{\varepsilon^2-w^2}}{\sqrt{\sin^2 b-w^2}}w^k dw \\ =\frac{1}{\sin b}\left[\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!2^l}\left(\frac{\varepsilon}{\sin b}\right)^{2l}\frac{\pi(k+2l-1)!!}{2(k+2l+2)!!}\right]\varepsilon^{k+2}.\end{aligned} \tag{7.6}$$

Proof. By

$$\frac{1}{\sqrt{1-x}}=\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!2^l}x^l$$

and (3.6) we obtain

$$\begin{aligned}\int_0^\varepsilon \frac{\sqrt{\varepsilon^2-w^2}}{\sqrt{\sin^2 b-w^2}} w^k dw &= \int_0^\varepsilon \frac{\sqrt{\varepsilon^2-w^2}}{\sin b}\left[\sum_{l=0}^\infty \frac{(2l-1)!!}{l!2^l}\left(\frac{w}{\sin b}\right)^{2l}\right] w^k dw \\ &= \sum_{l=0}^\infty \frac{(2l-1)!!}{l!2^l \sin^{2l+1} b}\int_0^\varepsilon \sqrt{\varepsilon^2-w^2}\, w^{k+2l} dw \\ &= \sum_{l=0}^\infty \frac{(2l-1)!!}{l!2^l \sin^{2l+1} b}\cdot\frac{\pi(k+2l-1)!!}{2(k+2l+2)!!}\varepsilon^{k+2l+2},\end{aligned}$$

from which we obtain (7.6).

Since ε^*, ε and w satisfy the relations satisfied by $\varepsilon^\sim$, ε and t, we have the corresponding equalities as in Lemma 2.2. So as in the proof of Lemma 3.3 we obtain

$$\text{(7.7)}\quad \left(\varepsilon^*-\frac{1}{2}\sin 2\varepsilon^*\right)\cos w = \frac{2}{3}\sqrt{\varepsilon^2-w^2}\left[\varepsilon^2-\frac{1}{5}\varepsilon^4-\left(1-\frac{2}{5}\varepsilon^2\right)w^2-\frac{1}{5}w^4\right]+[\varepsilon^7].$$

By (7.5) with $b=\alpha$ we get

$$U(\alpha, \varepsilon) < \frac{2}{\pi}\int_0^\varepsilon \left(\varepsilon^*-\frac{1}{2}\sin 2\varepsilon^*\right)\frac{\cos w}{\sqrt{\sin^2\alpha-w^2}}dw.$$

Applying (7.6) and (7.7) to the last inequality we obtain

$$\begin{aligned}U(\alpha, \varepsilon) &< \frac{2}{3}\left(1-\frac{1}{5}\varepsilon^2\right)\frac{\varepsilon^4}{\sin\alpha}\left[\frac{1}{2}+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\frac{1}{4!!}+\frac{3}{2\cdot 4}\left(\frac{\varepsilon}{\sin\alpha}\right)^4\frac{3!!}{6!!}+\cdots\right] \\ &\quad -\frac{2}{3}\left(1-\frac{2}{5}\varepsilon^2\right)\frac{\varepsilon^4}{\sin\alpha}\left[\frac{1}{4!!}+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\frac{3!!}{6!!}+\frac{3}{2\cdot 4}\left(\frac{\varepsilon}{\sin\alpha}\right)^4\frac{5!!}{8!!}+\cdots\right] \\ &\quad -\frac{2\varepsilon^6}{15\sin\alpha}\left[\frac{3!!}{6!!}+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\frac{5!!}{8!!}+\frac{3}{2\cdot 4}\left(\frac{\varepsilon}{\sin\alpha}\right)^4\frac{7!!}{10!!}+\cdots\right] \\ &\quad +\arcsin\left(\frac{\varepsilon}{\sin\alpha}\right)[\varepsilon^7] \\ &= \frac{2\varepsilon^4}{3\sin\alpha}\left[\frac{3}{8}+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\cdot\frac{1}{16}+\frac{3}{8}\left(\frac{\varepsilon}{\sin\alpha}\right)^4\cdot\frac{3}{128}+\cdots\right] \\ &\quad -\frac{2\varepsilon^6}{15\sin\alpha}\left[\frac{5}{16}+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\cdot\frac{5}{128}+\frac{3}{8}\left(\frac{\varepsilon}{\sin\alpha}\right)^4\cdot\frac{3}{256}+\cdots\right] \\ &\quad +\arcsin\left(\frac{\varepsilon}{\sin\alpha}\right)[\varepsilon^7].\end{aligned}$$

Since

$$\left\{\frac{(2l-1)!!}{(2l+2)!!}-\frac{(2l+1)!!}{(2l+4)!!}\right\}=\left\{\frac{3}{8}, \frac{1}{16}, \frac{3}{128}, \cdots\right\}$$

is decreasing with respect to l, and

$$\left\{\frac{15(2l-1)!!}{(2l+6)!!}\right\}=\left\{\frac{5}{16},\ \frac{5}{128},\ \frac{3}{256},\ \cdots\right\}$$

is composed of positive numbers, we obtain

$$U(\alpha,\ \varepsilon)<\frac{2\varepsilon^4}{3\sin\alpha}\left[\left(\frac{3}{8}-\frac{3}{128}\right)+\frac{1}{2}\left(\frac{\varepsilon}{\sin\alpha}\right)^2\left(\frac{1}{16}-\frac{3}{128}\right)\right.$$

$$\left.+\frac{3}{128}\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!2^l}\left(\frac{\varepsilon}{\sin\alpha}\right)^{2l}\right]-\frac{2\varepsilon^6}{15\sin\alpha}\cdot\frac{5}{16}+[\varepsilon^8]$$

$$=\frac{2\varepsilon^4}{3\sin\alpha}\left[\frac{45}{128}+\frac{5}{256}\left(\frac{\varepsilon}{\sin\alpha}\right)^2+\frac{3}{128\sqrt{1-(\varepsilon/\sin\alpha)^2}}\right]$$

$$-\frac{\varepsilon^6}{24\sin\alpha}+[\varepsilon^8].$$

For $\alpha<0.156$ by Lemma 5.1 we obtain

$$U(\alpha,\ \varepsilon)<\frac{\varepsilon^4}{\sin\alpha}\left(\frac{141}{512}+\frac{7}{128}\alpha^2\right)-\frac{\varepsilon^6}{24\sin\alpha}+[\varepsilon^8].$$

For $\alpha<0.156$, $\alpha/\sin\alpha$ is increasing, and so

$$\frac{1}{\sin\alpha}<\frac{0.156}{\alpha\sin 0.156}<\frac{1.004\cdots}{\alpha}.$$

Therefore we obtain

$$U(\alpha,\ \varepsilon)<\frac{28}{100\alpha}\varepsilon^4-\frac{1}{24\alpha}\varepsilon^6+[\varepsilon^8],$$

and hence by numerical calculation we get

$$U(\alpha,\ \varepsilon)<\frac{28}{100\alpha}\varepsilon^4-\frac{1}{24\alpha}\varepsilon^6.$$

Since $\alpha<|AQ|$ we see that

$$P(q_1q_2q_3)<3U(\alpha,\ \varepsilon),$$

and hence we obtain

LEMMA 7.2. *For $\alpha<0.156$, $2P(q_1q_2q_3)$ is estimated by*

$$2P(q_1q_2q_3)<\frac{168}{100\alpha}\varepsilon^4-\frac{1}{4\alpha}\varepsilon^6. \tag{7.8}$$

§8. Proof of Theorem B.

PROPOSITION 8.1. *For $N\geqq 100$, $A(\varepsilon,\ N)>\pi-M_0'(\varepsilon,\ N;\ A)$ holds.*

Proof. By $N\geqq 100$ we obtain $\alpha<0.156$ and $\varepsilon<0.136$. Applying estimates (3.5), (3.9) and (6.4) and (7.8) to (4.7), we obtain

$$A(\varepsilon, N)-\pi+M_0'(\varepsilon, N; A)=A(\varepsilon, N)-2N\cdot P(\varepsilon)+2P(\varepsilon, N, \Omega)-2P(q_1q_2q_3)$$

$$>\pi N\Big(\frac{1}{4}\varepsilon^4-\frac{1}{12}\varepsilon^6\Big)-\Big(1-\frac{98\alpha}{100}\Big)2N\Big(\frac{\pi}{8}\varepsilon^4-\frac{\pi}{60}\varepsilon^6\Big)-\frac{168}{100\alpha}\varepsilon^4+\frac{1}{4\alpha}\varepsilon^6$$

$$=\frac{98}{100}\alpha N\Big(\frac{\pi}{4}\varepsilon^4-\frac{\pi}{30}\varepsilon^6\Big)-\frac{1}{20}\pi N\varepsilon^6-\frac{168}{100\alpha}\varepsilon^4+\frac{1}{4\alpha}\varepsilon^6.$$

By (3.8) and (4.5) we obtain

$$A(\varepsilon, N)-\pi+M_0'(\varepsilon, N; A)>\frac{98}{100}\cdot\frac{12}{5\alpha}\cdot\frac{78}{100}\varepsilon^4-\frac{\pi}{20}\Big(2+\frac{1}{5}\varepsilon^2\Big)\varepsilon^4-\frac{168}{100\alpha}\varepsilon^4$$

$$>\frac{183}{100\alpha}\varepsilon^4-\frac{32}{100}\varepsilon^4-\frac{168}{100\alpha}\varepsilon^4$$

$$=\frac{15-32\alpha}{100\alpha}\varepsilon^4>0. \qquad \text{(q. e. d.)}$$

Proof of Theorem B. By Proposition 8.1 we see that $M_0'(\varepsilon, N; A)>\pi-A(\varepsilon, N)$ holds. Since $M_0'(\varepsilon, N; A)$ is the mean value, we have some plane L through O and A such that

$$\mathrm{Area}(K(\varepsilon, N, \Theta_0')\cap L)>\pi-A(\varepsilon, N). \qquad \text{(q. e. d.)}$$

Let Θ be an ε-proper distribution of $2N$ points on $S^2(1)$, and let q be a point of $S^2(1)$. By $M(q)=M(\varepsilon, N, \Theta; q)$ we denote the rotational mean value at q, *i.e.*, the mean value of $\mathrm{Area}(K(\varepsilon, N, \Theta)\cap L_q)$ with respect to planes $\{L_q\}$ which contain the line qO. If we consider $M(q)$ as a function on $S^2(1)$, $\pi-M(q)$ takes big value at q if relatively many points of Θ are distributed near q, or if q is very near some p_k of Θ. Theorem B implies that even if Θ is nearly homogeneous, the variation of $M(q)$ with respect to q is not so small.

Observations for the case where N is small and Theorem B lead us to the following conjecture.

CONJECTURE. *For an ε-proper distribution Θ of $2N$ points on $S^2(1)$.* $\mathrm{Area}(K(\varepsilon, N, \Theta)\cap L)<\mathrm{Area}(B^3(R)\cap L)$ *for each L may imply*

$$\mathrm{Vol}(K(\varepsilon, N, \Theta))<\mathrm{Vol}(B^3(R)).$$

As a remark we prove the following.

PROPOSITION 8.2. *Let ε and N be given so that $2N$ points can be ε-properly distributed on $S^2(1)$. Then;*

(i) *There exists an ε-proper distribution Θ_* of $2N$ points on $S^2(1)$ such that the maximum value of the rotational mean value function $M_*(q)$ is not greater than the maximum value of $M(q)$ for any other ε-proper distribution Θ of $2N$ points on $S^2(1)$.*

(ii) *There exists an ε-proper distribution Θ^* of $2N$ points on $S^2(1)$ such that* *the maximum value of* Area$(K(\varepsilon, N, \Theta^*)\cap L)$ *with respect to* $\{L\}=RP^2$ *is not greater than the maximum value of* Area$(K(\varepsilon, N, \Theta)\cap L)$ *with respect to* $\{L\}$ *for any other* ε*-proper distribution* Θ *of* $2N$ *points on* $S^2(1)$.

Proof. Let Ψ be a subset of $S^2(1)\times S^2(1)\times\cdots\times S^2(1)$ (N times) composed of elements $(p_1, p_2, \cdots, p_N)$ such that $|p_k(\pm p_l)|\geqq 2\varepsilon$ for $1\leqq k<l\leqq N$. Then Ψ is compact. The rotational mean value function $M(\varepsilon, N, \Theta; q)$ is a continuous function on $\Psi\times S^2(1)$. We define $\Lambda(\varepsilon, N, \Theta)$ by

$$\Lambda(\varepsilon, N, \Theta)=\max_{q\in S^2(1)}\{M(\varepsilon, N, \Theta; q)\}.$$

Then $\Lambda(\varepsilon, N, \Theta)$ is a continuous function on Ψ. Therefore we have some $\Theta_*\in\Psi$, which attains the minimum of $\Lambda(\varepsilon, N, \Theta)$. This proves (i).

(ii) is easily proved.

§9. Appendix.

It is clear that if N is not so large and ε is very small then for any ε-proper distribution of $2N$ points on $S^2(1)$ we can find some L such that $K(\varepsilon, N, \Theta)\cap L=B^3(1)\cap L$.

Let Θ be an ε-proper distribution of $2N$ points on $S^2(1)$. Even if Area$(K(\varepsilon, N, \Theta)\cap L)<\pi$ holds for any L, we see that the variation of Area$(K(\varepsilon, N, \Theta)\cap L)$ is big, if N is not so large. To show this we give two examples corresponding to the closest packings of equal circles on $S^2(1)$.

⟨9-1⟩ Octahedron.

Consider an octahedron inscribed in $S^2(1)$. Vertices define an ε-proper distribution of six points on $S^2(1)$ with $\varepsilon=\pi/4$. Let $\Theta=\{\pm X, \pm Y, \pm A\}$ where $X=(1, 0, 0)$, $Y=(0, 1, 0)$ and $A=(0, 0, 1)$. Let L_0 be the plane passing through Y, $-Y$ and $(\sqrt{2}/2, 0, \sqrt{2}/2)$. Then

$$\text{Area}(K(\pi/4, 3, \Theta)\cap L_0)=2.5707\cdots,$$

$$\pi R(\pi/4, 3)^2=2.3613\cdots.$$

Therefore Area$(K(\pi/4, 3, \Theta)\cap L)<\pi R(\pi/4, 3)^2$ does not hold for L_0.

Notice that Θ corresponds to the closest packing of equal six circles on $S^2(1)$. As for closest packing, see for example [5] and [6] or references there.

⟨9-2⟩ Icosahedron.

Consider an icosahedron inscribed in $S^2(1)$. Vertices define an ε-proper distribution of Θ of eleven points on $S^2(1)$ with $\varepsilon=0.5535\cdots$. Let $p_1\in\Theta$ and let L_0 be the plane orthogonal to $p_1(-p_1)$. Then

$$\text{Area}(K(\varepsilon, 6, \Theta)\cap L_0)=2.9389\cdots,$$

$$\pi R(\varepsilon, 6)^2=2.7281\cdots.$$

Therefore Area$(K(\varepsilon, 6, \Theta)\cap L)<\pi R(\varepsilon, 6)^2$ does not hold for L_0.

Θ corresponds to the closest packing of equal twelve circles on $S^2(1)$.

⟨9-3⟩ Non-symmetric closest packing.

Let $p_1=(\sin b, 0, \cos b)$ with $\sin^2 b=(8-2\sqrt{2})/7$. By $\pi/2$-, π-, and $3\pi/2$-rotation of p_1 around the x^3-axis, we define p_2, p_3, and p_4. By $\pi/4$-rotation of $-p_1$, $-p_2$, $-p_3$, and $-p_4$ around the x^3-axis, we define q_1, q_2, q_3, and q_4. Then $\Sigma=\{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$ defines the closest packing of equal eight circles on $S^2(1)$ with ε such that $\cos^2\varepsilon=(3+\sqrt{2})/7$, i.e., $\varepsilon=0.6532\cdots$. Σ is not centrally symmetric. By $K(\varepsilon, \Sigma)$ we denote the convex body obtained from $B^3(1)$ by removing eight spherical caps of $B^3(1)$ of angular radius ε corresponding to Σ. Let L_0 be the plane passing through A, q_1, and q_3. Then

$$\mathrm{Area}(K(\varepsilon, \Sigma)\cap L_0)=2.8003\cdots,$$

$$\pi R(\varepsilon, 4)^2=2.6234\cdots.$$

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