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NUMBER OF DEFICIENT VALUES OF A CLASS of MEROMORPHIC FUNCTION

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Abstract

We proved the following Theorem

Theorem. Let $f(z)$ be a meromorphic function of lower order $\mu<\infty$. If $\sum_a \delta(a, f')=2$ then we have

$$P_0+P_1\leqq\mu+1,$$

where P_0, P_1 are the numbers of finite deficient values of $f(z), f'(z)$ respectively.

1. Lemmas.

We need the following four known results.

LEMMA A [1 Theorem 1]. *Let $f(z)$ be a meromorphic function of lower order $\mu<\infty$. Assume that there exists a positive integer P which satifies*

$$P-\frac{1}{2}\leqq\mu<P+\frac{1}{2}.$$

Assume also that for some $A_0>0$ and $0<\varepsilon<1$,

$$K(f)=\overline{\lim_{r\to\infty}}\frac{N(r, f)+N(r, 1/f)}{T(r, f)}<\frac{\varepsilon}{A_0(P+1)},$$

then

1) $P\geqq 1$.
2) *For $r>r_0$ and all $1<\sigma\leqq 36$, we have*

$$\begin{cases} T(\sigma r, f)=\sigma^P T(r, f)(1+\eta(r, \sigma)) \\ |\eta(r, \sigma)|<\varepsilon. \end{cases} \tag{1.1}$$

3) *Let $E(\mu, P)$ denotes the Weierstrass primary factor of genus P and a_ν, b_μ ($\nu=1, 2, \cdots$; $\mu=1, 2, \cdots$) are zeros and poles of $f(z)$ respectively, then we have*

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$$f(z)=z^k e^{\alpha_0 z^P+\alpha_1 z^{P-1}+\cdots+\alpha_P} \frac{\prod_{\nu=1}^{\infty} E\left(\frac{z}{a_\nu}, P\right)}{\prod_{\mu=1}^{\infty} E\left(\frac{z}{b_\mu}, P\right)},$$

where k is an integer. If we set

$$C(r)=\alpha_0+\frac{1}{P}\left\{\sum_{|a_\nu|\leq r} a_\nu^{-P}-\sum_{|b_\mu|\leq r} b_\mu^{-P}\right\},$$

then for $r>r_0$

$$\begin{cases} |C(\sigma r)-C(r)|<\varepsilon|C(r)| \\ T(r, f)=(1+\eta_1(r))\dfrac{|C(r)|}{\pi} r^P, \qquad |\eta_1(r)|<\varepsilon. \end{cases} \tag{1.2}$$

LEMMA B [1 Lemma 5]. *Let the assumptions and notations be the same as in Lemma A, we further assume that*

$$K(f)<\frac{\varepsilon}{B_0(P+1)\log(P+1)+B_1(P+1)\log 1/\delta},$$

where $0<\varepsilon\leqq 1$, $o<\delta<1/e$, and B_1, B_0, A_0 are constants which satisfy $B_1>B_0>A_0$, set $\alpha=\exp\left(\frac{1}{P+1}\right)$ and $C_j=C(\alpha^j)$, where j is an integer satifying $\alpha^j\leqq r<\alpha^{j+1}$, then for $z\in\Gamma_j-E_j(|z|=r)$, $j\geqq j_0$,

$$|\log|f(z)|-Re\{C_j z^P\}|<4\varepsilon|C_j|r^P, \tag{1.3}$$

where $\Gamma_j=z$, $\alpha^j<|z|\leqq\alpha^{j+(3/2)}\}$, E_j is the set of a finite number of circles which contain the zeros and poles of $f(z)$ and the sum of radius of those circles is no more than $4e\delta\alpha^{j+2}$.

LEMMA C [4 Theorem 4.1]. *Let $f(z)$ be a meromorphic fuction. Then for $r>r_0$*

$$T(r, f)<C\{T(2r, f')+\log r\}, \tag{1.4}$$

where C is a constant which is only dependent on $f(0)$.

LEMMA D [3, Proof of Theorem]. *Let assumptions be the same as the Theorem, then*

$$K(f'')=\overline{\lim_{r\to\infty}}\frac{N(r, f'')+N(r, 1/f'')}{T(r, f'')}=0.$$

2. Main Lemmas.

LEMMA 1. *Let the assumptions be the same as the Theorem, then for an arbitrary set of finite deficient values $\{a_1, a_2, \cdots, a_\tau\}$ and $\{a_{\tau+1}, \cdots a_{\tau'}\}$ of $f(z)$ and $f'(z)$, there exists a set of positive numbers $\{\rho_j\}_{j=1}^{\infty}$, $\alpha^j\leqq\rho_j<\alpha^{j+1}$ and sets $\{E_{1j}, E_{2j}, \cdots, E_{\tau j}, \cdots, E_{\tau' j}\}$ which are on $|z|=\rho_j$ so that* meas $E_{l}^{*}=$meas$\{\phi\,; 0$

$\leqq\phi<2\pi,\ \rho_j e^{i\phi}\in E_{lj}\}>\eta>0$ $(l=1, 2, \cdots, \tau')$, *and for* $z\in E_{lj}$ *we have*

$$\log|f(z)-a_l|<-kT(\rho_j, f''), \qquad l=1, 2, \cdots, \tau. \tag{2.1}$$

$$\log|f'(z)-a_l|<-kT(\rho_j, f''), \qquad l=\tau+1, \tau+2, \cdots, \tau'. \tag{2.2}$$

and

$$\log|f'(z)|<-kT(\rho_j, f''), \qquad l=1, 2, \cdots, \tau. \tag{2.3}$$

$$\log|f''(z)|<-kT(\rho_j, f''), \qquad l=1, 2, \cdots, \tau'. \tag{2.4}$$

where $\alpha=\exp(1/\mu+1)$, k, η *are absolute constants.*

In order to prove Lemma 1, we must prove following Lemma.

LEMMA 2. *Let* $g(z)$ *be meromorphic in* $|z|<R$ $(0<R<\infty)$, *then for* $z=re^{i\theta}$ *and* ρ, $r<\rho\leqq R$,

$$\log^+\frac{|g'(z)|}{|g(z)|}\leqq 5\log 2+\log^+\rho+3\log^+\frac{1}{\rho-r}+\log^+\mathfrak{R}+\log^+\frac{1}{\delta(z)}$$
$$+\log^+\frac{1}{\gamma}+\log^+T(\rho, g)+\log^+\log^+\frac{1}{|g(0)|}. \tag{2.5}$$

$$\log^+\frac{|g''(z)|}{|g(z)|}\leqq C\Big\{1+\log^+\rho+\log^+\frac{1}{\rho-\gamma}+\log^+\mathfrak{R}+\log^+\frac{1}{\delta(z)}$$
$$+\log^+\frac{1}{r}+\log^+T(\rho, g)+\log^+\log^+\frac{1}{|g(0)|}\Big\}, \tag{2.6}$$

where $\mathfrak{R}=n(\rho, g)+n(\rho, 1/g)$, $\delta(z)=\min\{|z-a|\}$, *here* a *takes all zeros and poles of* $g(z)$ *in* $|z|\leqq\rho$, *where* C *is an absolute constant.*

Proof. For (2.5), see [4 Lemma 6.2]. Using Poisson-Jensen formula, we have for $|z|\leqq r<R$

$$\log g(z)=\frac{1}{2\pi}\int_0^{2\pi}\log|g(\rho e^{i\phi})|\frac{\rho e^{i\phi}+z}{\rho e^{i\phi}-z}d\phi$$
$$-\sum_j\log\frac{\rho^2-\bar{a}_j z}{\rho(z-a_j)}+\sum_k\log\frac{\rho^2-b_k z}{\rho(z-b_k)}+ic,$$

where a_j, b_k are zeros and poles of $g(z)$ in $|z|\leqq\rho$ respectively.

Differentiating this twice we have

$$\frac{g''(z)}{g(z)}-\left[\frac{g'(z)}{g(z)}\right]^2=\frac{1}{2\pi}\int_0^{2\pi}\log|g(\rho e^{i\phi})|\frac{4\rho e^{i\phi}}{(\rho e^{i\phi}-z)^3}d\phi$$
$$+\sum_j\left[\frac{\bar{a}_j^2}{(\rho^2-\bar{a}_j z)^2}-\frac{1}{(a_j-z)^2}\right]-\sum_k\left[\frac{\bar{b}_k^2}{(\rho^2-\bar{b}_k z)^2}-\frac{1}{(b_k-z)^2}\right].$$

Since $|z-a_j|\geqq\delta(z)$, $|z-b_k|\geqq\delta(z)$, we have

$$\left|\frac{g''(z)}{g(z)}\right| \leqq \left|\frac{g'(z)}{g(z)}\right|^2 + \frac{4\rho}{(\rho-r)^3}\left\{m(\rho, g)+m\left(\rho, \frac{1}{g}\right)\right\} + \mathfrak{K}\left[\frac{1}{(\rho-r)^2}+\frac{1}{\delta(z)^2}\right],$$

Using (2.5), we prove (2.6).

3. Proof of Lemma 1.

Since Lemma A and D imply $P=\mu$ (2.3), we have

$$T(\sigma r, f'')=\sigma^{\mu} T(r, f'')(1+o(1)), \qquad 1<\sigma\leqq 36.$$

$$T(r, f'')=(1+o(1))\frac{C(r)r^{\mu}}{\pi}$$

$$|C(\sigma r)-C(r)|=o\{|C(r)|\}. \tag{3.1}$$

Let $a_{l\nu}=\{l=1, 2, \cdots, \tau;\ \nu=1, 2, \cdots, n_l(\alpha^{j+(3/2)}, f=a_l)\}$ be zeros of $f(z)-a_l$ in $|z|\leqq\alpha^{j+(3/2)}$, $a_{l\nu}\{l=\tau+1, \cdots, \tau';\ \nu=1, 2, \cdots, n_l=n(\alpha^{j+(3/2)}, f'=a_l)\}$ be zeros of $f'(z)-a_l$ in $|z|\leqq\alpha^{j+(3/2)}$. Using Broutroux-Cartan's Theorem, the inequalities we have

$$\prod_{\nu=1}^{n_l}|z-a_{l\nu}|>\left(\frac{H}{\tau}\right)^{n_l}, \qquad l=1, 2, \cdots, \tau. \tag{3.2}$$

$$\prod_{\nu=1}^{n_l}|z-a_{l\nu}|>\left(\frac{H}{\tau'-\tau}\right)^{n_l} \qquad l=\tau+1, \tau+2, \cdots, \tau'. \tag{3.3}$$

hold outside a finite number of small circles (r_j^1), (r_j^2) respectively, and the sums of radii are no more than $\frac{2eH}{\tau}$, $\frac{2eH}{\tau'-\tau}$ respectively. Set $(r_j^3)=\left\{\bigcup_{l=1}^{\tau'}\bigcup_{\nu=1}^{n_l}\left(|z-a_{l\nu}|<\frac{2eH}{\mathfrak{K}}\right)\right\}$ were $\mathfrak{K}=\sum_{l=1}^{\tau'} n_l$. Set $\mathfrak{K}'=n(\alpha^{j+(3/2)}, f)$ and $(r_j^4)=\bigcup_{\nu=1}^{\mathfrak{K}'}\left\{|z-b_\nu|<\frac{2eH}{\mathfrak{K}'}\right\}$, where $\{b_\nu\}$ is poles of $f(z)$ in $|z|\leqq\alpha^{j+(3/2)}$. Take $H=\frac{\alpha^j}{54e}(\alpha-1)$ and in Lemma B, $\delta<\frac{\alpha-1}{36e\alpha^2}$, and set $(r_j)=\bigcup_{i=1}^{4}(r_j^i)\cup E_j$. The sum of diameters of (r_j) is no more than $\frac{5}{9}(\alpha-1)\alpha^j$, but the width of (α^j, α^{j+1}) is $(\alpha-1)\alpha^j$, hence there exists ρ_j, $\rho_j\in(\alpha^j, \alpha^{j+1})$ so that $(|z|=\rho_j)\cap(r_j)=\phi$. Set

$$\delta_0=\min\{\delta(a_1, f), \cdots, \delta(a_\tau, f);\ \delta(a_{\tau+1}, f'), \cdots, \delta(a_{\tau'}, f')\}. \tag{3.4}$$

Using Poisson-Jensen's formula, we have for $z=\rho_j e^{i\phi}$, from (3.2)

$$\log\frac{1}{|f(z)-a_l|}\leqq\frac{\alpha^{j+(3/2)}+\alpha^{j+1}}{\alpha^{j+(3/2)}-\alpha^{j+1}}\int_0^{2\pi}\log^+\frac{1}{|f(\alpha^{j+(3/2)}e^{i\theta})-a_l|}d\theta$$

$$+\sum_{\nu=1}^{n_l}\log\left|\frac{(\alpha^{i+3/2})^2-\bar{a}_{l\nu}z}{\alpha^{j+(3/2)}(z-a_{l\nu})}\right|$$

$$\leqq Cm\left(\alpha^{j+3/2}, \frac{1}{f-a_l}\right)+\sum_{\nu=1}^{n_l} \log \frac{2\alpha^{j+3/2}}{|z-a_{l\nu}|}$$

$$\leqq Cm\left(\alpha^{j+3/2}, \frac{1}{f-a_l}\right)+n(\alpha^{j+3/2}, f=a_l) \log \frac{2\tau\alpha^{j+3/2}}{H}$$

$$\leqq CT\left(\alpha^{j+2}, \frac{1}{f-a_l}\right)\leqq CT(\alpha^{j+2}, f). \tag{3.5}$$

By Lemma C,

$$T(\alpha^{j+2}, f)\leqq C\{T(2\alpha^{j+2}, f')\}\leqq C\{T(4\alpha^{j+2}, f'')\}. \tag{3.6}$$

Using (3.1) we have

$$T(4\alpha^{j+2}, f'')<2(4\alpha^2)^{\mu}T(\alpha^j, f'')<2(4\alpha^2)^{\mu}T(\rho_j, f''). \tag{3.7}$$

Since $\sum\limits_a \delta(a, f')=2$, the order of $f'(z)$ equals lower order μ (2.3) and the order of $f(z)$ equals $\mu<\infty$. So that

$$T(r, f'')=m(r, f'')+N(r, f'')$$

$$\leqq m\left(r, \frac{f''}{f}\right)+m(r, f)+3N(r, f)\leqq 4T(r, f). \tag{3.8}$$

Using (3.5)-(3.8), we have

$$\log \frac{1}{|f(\rho_j e^{i\phi})-a_l|}\leqq CT(\rho_j, f) \tag{3.9}$$

where C is constant that is not depedent on j, l.

For $l=1, 2, \cdots, \tau$, set $E^*_{lj}=E\left\{\phi\,; o\leqq\phi<2\pi, \log\frac{1}{|f(\rho_j e^{i\phi})-a_l|}>\frac{1}{2}m\left(\rho_j, \frac{1}{f-a_l}\right)\right\}$ by (3.9)

$$m\left(\rho_j, \frac{1}{f-a_l}\right)=\frac{1}{2\pi}\left[\int_{E^*_{lj}}+\int_{CE^*_{lj}}\right]\log^+\frac{1}{|f(\rho_j e^{i\phi})-a_l|}d\phi$$

$$\leqq\frac{C}{2\pi}T(\rho_j, f)\operatorname{meas} E^*_{lj}+\frac{1}{2}m\left(\rho_j, \frac{1}{f-a_l}\right).$$

so that

$$\operatorname{meas} E^*_{lj}\leqq\frac{\pi}{C}\frac{m\left(\rho_j, \frac{1}{f-a_l}\right)}{T(\rho_j, f)}\geqq\frac{\pi}{2C}\delta(a_l, f)\geqq\frac{\pi}{2C}\delta_0, \qquad (j\geqq j_0). \tag{3.10}$$

Hence for $z\in E_{lj}=\{z\,; z=\rho_j e^{i\phi}, \phi\in E^*_{lj}\}$, using (3.8) we have

$$\log\frac{1}{|f(\rho_j e^{i\phi})-a_l|}>\frac{1}{2}m\left(\rho_j, \frac{1}{f-a_l}\right)>\frac{\delta_0}{4}T(\rho_j, f)>\frac{\delta_0}{16}T(\rho_j, f''). \tag{3.11}$$

By Lemma 2, set $g(z)=f(z)-a_l$, $\rho=\alpha^{j+(3/2)}$, $r=\rho_j$, for $z\in E_{lj}$ $(l=1, 2, \cdots, \tau)$

$$\log^+\frac{|f'(z)|}{|f(z)-a_l|}\leqq C\Big\{1+\log^+\alpha^{j+3/2}+3\log^+\frac{1}{\alpha^{j+3/2}-\rho_j}$$

$$+\log^+(n_l+\mathfrak{K}')+\log^+\frac{1}{\delta(z)}+\log^+T(\alpha^{j+3/2}, f-a_l)$$

$$+\log^+\log^+\frac{1}{|f(0)-a_l|}\Big\} \tag{3.12}$$

From (3.6)-(3.8), $l=1, 2, \cdots, \tau$, we have

$$n_l=n\Big(\alpha^{j+3/2}, \frac{1}{f-a_l}\Big)\leqq\frac{2}{\log\alpha}N\Big(\alpha^{j+2}, \frac{1}{f-a_l}\Big)\leqq CT(\rho_j, f).$$

$$\mathfrak{K}'=n(\alpha^{j+3/2}, f-a_l)=n(\alpha^{j+3/2}, f)\leqq CT(\rho_j, f).$$

and for $l=\tau+1, \cdots, \tau'$, by the same reasoning we have

$$n_l=n\Big(\alpha^{j+3/2}, \frac{1}{f'-a}\Big)<CT(\rho_j, f).$$

so

$$\mathfrak{K}=\sum_{l=1}^{\tau'}n_l<CT(\rho_j, f), \qquad n_l+\mathfrak{K}'\leqq CT(\rho_j, f). \tag{3.13}$$

From definition of (r_j^3), (r_j^4), using (3.13) we have

$$\delta(z)\geqq\frac{2eH}{\mathfrak{K}+\mathfrak{K}'}>\frac{C\alpha^j}{T(\rho_j, f)}. \tag{3.14}$$

Combing (3.12)-(3.14), for $z\in E_{lj}$, we have

$$\log\frac{|f'(z)|}{|f(z)-a_l|}<C\{1+\log\rho_j+\log T(\rho_j, f)\}.$$

Hence from (3.11), (3.6), (3.7) for $z\in E_{lj}$, $l=1, 2, \cdots, \tau$, $r>r_0$ we have

$$\begin{aligned}\log|f'(z)|&\leqq\log^+\frac{|f'(z)|}{|f(z)-a_l|}+\log|f(z)-a_l|\\ &=\log^+\frac{|f'(z)|}{|f(z)-a_l|}-\log\frac{1}{|f(z)-a_l|}\\ &\leqq C\{1+\log\rho_j+\log T(\rho_j, f)\}-\frac{\delta_0}{16}T(\rho_j, f'')\\ &\leqq C\{1+\log\rho_j+\log T(\rho_j, f'')\}-\frac{\delta_0}{16}T(\rho_j, f'')\\ &<-\frac{\delta_0}{64}T(\rho_j, f'').\end{aligned} \tag{3.15}$$

$$\log|f''(z)|<-\frac{\delta_0}{64}T(\rho_j, f''). \tag{3.16}$$

For $\tau<l\leqq\tau'$, set $E_{lj}^{*}=\left\{\phi\,;0\leqq\phi<2\pi,\ \log\frac{1}{|f'(\rho_j e^{i\phi})-a_l|}>\frac{1}{2}m\left(\rho_j, \frac{1}{f-a_l}\right)\right\}$ and $E_{lj}=\{z\,;z=\rho_j e^{i\phi},\ \phi\in E_{lj}^{*}\}$, with the same reasoning we can prove $l=\tau+1, \cdots, \tau'$, $z\in E_{lj}$

$$\log\frac{1}{|f'(\rho_j e^{i\phi})-a_l|}>\frac{\delta_0}{16}T(\rho_j, f''), \tag{3,17}$$

$$\log|f''(z)|<-\frac{\delta_0}{64}T(\rho_j, f''). \tag{3.18}$$

Combining (3.11), (3.15)-(3.18), we can prove Lemma 1.

4. Proof of Theorem.

Set $C(\alpha^j)=|C(\alpha^j)|e^{i\omega_j}$. We divide $|z|=\rho_j$ into 2μ arcs $\alpha_1, \cdots, \alpha_\mu\,;\ \beta_1, \cdots, \beta_\mu$, so that $\cos(\mu\phi+\omega_j)\geqq 0$ on α_k $(k=1,2,\cdots,\mu)$; $\cos(\mu\phi+\omega_j)\leqq 0$ on β_k $(k=1,2,\cdots,\mu)$. Also set

$$\alpha_k(\phi)=\{\phi\,;0\leqq\phi<2\pi,\ \rho_j e^{i\phi}\in\alpha_k\}=\left\{\phi_{jk},\ \phi_{jk}+\frac{\pi}{\mu}\right\},$$

$$\beta_k(\phi)=\{\phi\,;0\leqq\phi<2\pi,\ \rho_j e^{i\phi}\in\beta_k\}=\left\{\Psi_{jk},\ \Psi_{jk}+\frac{\pi}{\mu}\right\}.$$

And $\phi_0<\frac{1}{10\mu}\eta$, we set

$$\tilde{\alpha}_k(\phi)=\left\{\phi_{jk}+\phi_0,\ \phi_{jk}+\frac{\pi}{\mu}-\phi_0\right\};\quad \tilde{\beta}_k(\phi)=\left\{\Psi_{jk}+\phi_0,\ \Psi_{jk}+\frac{\pi}{\mu}-\phi_0\right\}.$$

By Lemma B, for $z\in\tilde{\alpha}_k=\{z\,;z=\rho_j e^{i\phi},\ \phi\in\tilde{\alpha}_k(\phi)\}$,

$$\log|f''(\rho_j e^{i\phi})|\geqq Re\{C_j z^\mu\}-4\varepsilon|C_j|\rho_j^\mu,$$

where $\varepsilon<\frac{\sin\mu\phi_0}{10}$, for $z\in\tilde{\alpha}_k$

$$\begin{aligned}\log|f''(\rho_j e^{i\phi})| &\geqq |C_j|\rho_j^\mu\cos(\mu\phi+\omega_j)-4\varepsilon\rho_j^\mu|C_j|\\ &\geqq |C_j|\rho_j^\mu\sin\mu\phi_0-4\varepsilon\rho_j^\mu|C_j|\\ &>\frac{1}{2}|C_j|\rho_j^\mu\sin\mu\phi_0(>o).\end{aligned} \tag{4.1}$$

Assume that $\{a_1, a_2, \cdots, a_\tau\}$, $\{a_{\tau+1}, \cdots, a_{\tau'}\}$ are arbitrarily taken as finite deficients of $f(z)$ and $f'(z)$ respectively and $\{\rho_j\}$, E_{lj} $(l=1, 2, \cdots, \tau')$ are the same as Lemma 1. Using Lemma 1 and (4.1) we have $\left(\bigcup_{l=1}^{\tau'}E_{lj}\right)\cap\left(\bigcup_{k=1}^{\mu}\tilde{\alpha}_k\right)=\phi$. Because from Lemma 1 meas $E_{lj}^{*}>\eta$ $(l=1, 2, \cdots, \tau')$ and $2\mu\phi_0<\eta/5$, so we have a $\tilde{\beta}_{l'}$, $E_{lj}\cap\tilde{\beta}_{l'}\neq\phi$ $(l=1, 2, \cdots, \tau, \tau+1, \cdots, \tau'\,;\ l'\in(1, 2, \cdots \mu))$. From Lemma B, $z\in\tilde{\beta}_l$,

$$\log|f''(z)| \leqq Re\{C_j z^{\mu}\} + 4\varepsilon |C_j| \rho_j^{\mu}$$
$$= |C_j| \rho_j^{\mu} \cos(\mu\phi + \omega_j) + 4\varepsilon |C_j| \rho_j^{\mu}$$
$$\leqq -\rho_j^{\mu} |C_j| \sin(\mu\phi_0) + 4\varepsilon |C_j| \rho_j^{\mu} \leqq -\frac{1}{2} \sin(\mu\phi_0) |C_j| \rho_j^{\mu}. \quad (4.2)$$

From (3.1) and $\mu = P$, we have

$$|C_j| \rho_j^{\mu} > |C_j| (\alpha^j)^{\mu} > \frac{\pi}{2} T(\alpha^j, f'') > \frac{\pi}{4} \left(\frac{\alpha^j}{\rho_j}\right)^{\mu} T(\rho_j, f'')$$
$$> \frac{\pi}{4} \left(\frac{\alpha^j}{\alpha^{j+1}}\right)^{\mu} T(\rho_j, f'') = \frac{\pi}{4} \alpha^{-\mu} T(\rho_j, f'').$$

Therefore, from (4.2) there exists a positive C' that is independent of j such that for $z \in \tilde{\beta}_l$,

$$\log|f''(z)| \leqq -C' T(\rho_j, f''). \quad (4.3)$$

Supose $z_0 \in E_{lj} \cap \tilde{\beta}_{l'}$ for $z \in \tilde{\beta}_{l'}$, from (4.3), (2.1) we have

$$|f'(z)| \leqq \rho_j \int_{\widehat{z_0 z}} |f''(z)| d\phi + |f'(z_0)|$$
$$\leqq 2\pi \rho_j e^{-C' T(\rho_j, f'')} + e^{-kT(\rho_j, f'')} \leqq e^{-CT(\rho_j, f'')}. \quad (4.4)$$

were $\widehat{z_0 z} \subset \tilde{\beta}_{l'}$, C independent of j. Therfore we have for $z \in \tilde{\beta}_{l'}$, $l = 1, 2, \cdots, \tau$,

$$|f(z) - a_l| \leqq |f(z) - f(z_0)| + |f(z_0) - a_l| \leqq \rho_j \int_{\widehat{z_0 z}} |f'(z)| d\phi + e^{-kT(\rho_j, f'')}$$
$$\leqq 2\pi \rho_j e^{-C' T(\rho_j, f'')} + e^{-kT(\rho_j, f'')} \leqq e^{-CT(\rho_j, f'')}. \quad (4.5)$$

From the same reason for $z \in \tilde{\beta}_{l'}$, $l = \tau + 1, \cdots, \tau'$,

$$|f'(z) - a_l| < e^{-CT(\rho_j, f'')}. \quad (4.6)$$

$a_{l_1} \neq a_{l_2}$ for $1 \leqq l_1 < l_2 \leqq \tau$ implies $\tilde{\beta}_{l_1'} \neq \tilde{\beta}_{l_2'}$. In fact, if $\beta_{l_1'} = \beta_{l_2'}$ then we have from (4.5)

$$|a_{l_1} - a_{l_2}| \leqq |f(z) - a_l| + |f(z) - a_{l_2}| \leqq 2e^{-CT(\rho_j, f'')}.$$

Setting $j \to \infty$, we have $a_{l_1} = a_{l_2}$. This is a contradiction. By the same reasoning $\tilde{\beta}_{l_1} \neq \tilde{\beta}_{l_2}$ $(\tau + 1 \leqq l_1 < l_2 < \tau')$. And from (4.4) and (4.6) $\tilde{\beta}_{l_1} \neq \tilde{\beta}_{l_2}$ $\{l_1 = 1, 2, \cdots, \tau;$ $l_2 = \tau + 1, \cdots, \tau'$; and $a_l \neq 0$, $l \in (\tau + 1, \cdots, \tau')\}$. Because the number of $\tilde{\beta}_l$ is at most μ, so the Theorem has been proved.

From the proof we have following Corollary,

Corollary. Let $f(z)$ be a meromorphic function of lower order $\mu < \infty$, if

$$\sum_a \delta(a, f') = 2,$$

then the number of deficient values of $f'(z)$ $\nu(f') \leqq \mu + 1$.

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