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TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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Abstract

A classification Theorem for totally umbilical CR-submanifolds of a Kaehler manifold is proved.

1. Introduction.

CR-submanifolds of a Kaehler manifold [1] being generalization of holomorphic and totally real submanifolds of a Kaehler manifold, has recently become subject of sufficient interest. Totally umbilical CR-submanifolds of a Kaehler manifold have been studied by A. Bejancu [3], Blair and Chen [4]. The purpose of this paper is to classify all totally umbilical CR-submanifolds of a Kaehler manifold. In fact we prove the following theorem.

THEOREM. *Let M, ($\dim M \geqq 5$) be a complete simply connected totally umbilical CR-submanifold of a Kaheler manifold $\bar{M}$. Then M is one of the following:*

(i) *Locally the Riemannian product of a holomorphic submanifold and a totally real submanifold of $\bar{M}$*
(ii) *totally real submanifold*
(iii) *isometric to an ordinary sphere*
(iv) *homothetic to a Sasakian manifold.*

The cases (iii) *and* (iv) *occur when* $\dim M$ *is odd.*

2. Preliminaries.

Let $\bar{M}$ be an m-dimensional Kaehler manifold with almost complex structure J. Then the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies [11].

$$\bar{R}(JX, JY)Z=\bar{R}(X, Y)Z,\ \bar{R}(X, Y)JZ=J\bar{R}(X, Y)Z. \tag{2.1}$$

An n-dimensional submanifold M of $\bar{M}$ is said to be a CR-submanifold if on M there exist two orthogonal complementary distributions D and $D^{\perp}$ such that $JD=D$ and $JD^{\perp}\subset\nu$, where ν is the normal bundle of M [1]. If $D=\{o\}$, (resp.

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$D^{\perp}=\{o\}$), then M is said to be totally real (resp. holomorphic) submanifold. It follows that $\dim D$=even and that the normal bundle ν splits as $\nu=JD^{\perp}\oplus\mu$, where μ is invariant sub-bundle of ν under J. The Riemannian connection $\bar{\nabla}$ on $\bar{M}$ induces the connections ∇ on M and the normal connection $\nabla^{\perp}$ in ν obeying the Gauss and Weingarten farmulae

$$\bar{\nabla}_X Y=\nabla_X Y+h(X, Y), \tag{2.2}$$

$$\bar{\nabla}_X N=-A_N X+\nabla_X^{\perp} N, \tag{2.3}$$

where X, Y are vector fields on M, $N\in\nu$ and h, A_N are called the second fundamental forms related as

$$g(h(X, Y), N)=g(A_N X, Y). \tag{2.4}$$

The CR-submanifold M is said to be totally umbilical if

$$h(X, Y)=g(X, Y)H,$$

where $H=\frac{1}{n}(\text{trace } h)$, called the mean curvature vector. For totally umbilical CR-submanifold M, the equations (2.2) and (2.3) take the form

$$\bar{\nabla}_X Y=\nabla_X Y+g(X, Y)H \tag{2.5}$$

$$\bar{\nabla}_X N=-g(H, N)X+\nabla_X^{\perp} N. \tag{2.6}$$

The equation of Codazzi for totally umbilical CR-submanifold M is given by

$$\bar{R}(X, Y; Z, N)=g(Y, Z)g(\nabla_X^{\perp} H, N)-g(X, Z)g(\nabla_Y^{\perp} H, N), \tag{2.7}$$

where $\bar{R}(X, Y; Z, N)=g(\bar{R}(X, Y)Z, N)$ and X, Y, Z are vector fields on M and $N\in\nu$.

By an extrinsic sphere we mean a submanifold of an arbitrary Riemannian manifold which is totally umbilic and has nonzero parallel mean curvature vector [10]. We need the following Theorem of Yamaguchi, Nemoto and Kawabata [13].

"A complete connected and simply connected extrinsic sphere M^n in a Kaehler manifold $\bar{M}^{2m}$ is one of the following:

1. M^n is isometric to an ordinary sphere
2. M^n is homothetic to a Sasakian manifold
3. M^n is totally real submanifold and the f-structure is not parallel in the normal bundle."

3. Proof of the Theorem.

Let M be totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$. Then using (2.5), (2.6) and $J\bar{\nabla}_X W=\bar{\nabla}_X JW$ for $X, W\in D^{\perp}$, we get

$$J\nabla_X W+g(X, W)JH=-g(JW, H)X+\nabla_X^{\perp} JW. \tag{3.1}$$

Taking inner product with X we get

$$g(H, JW)\|X\|^2=g(X, W)g(H, JX). \tag{3.2}$$

Interchanging the role of X and W in above equation we get

$$g(H, JX)\|W\|^2=g(X, W)g(H, JW).$$

Using (3.2) in above equation we have

$$g(H, JW)=\frac{g(X, W)^2}{\|X\|^2\|W\|^2}g(H, JW). \tag{3.3}$$

The possible solutions of equation (3.3) are:

(a) $H=0$ or (b) $H\perp JW$, or (c) $X\|W$.

Thus we have one of the following:

(a) M is totally geodesic, (b) $H\in\mu$ (c) $\dim D^\perp=1$.

Combining (a) with a result in [4] we get part (i) of the Theorem.

Next suppose that $H\neq 0$ and $H\in\mu$. We observe that for $N\in JD^\perp$ and $X\in D$, $\bar\nabla_X JN=J\bar\nabla_X N$ gives $\nabla_X JN=J\nabla_X^\perp N$. This implies that for $N\in JD^\perp$ and $X\in D$, $\nabla_X^\perp N\in JD^\perp$. Also $g(N, H)=0$ for $N\in JD^\perp$ implies $g(\nabla_X^\perp N, H)=-g(N, \nabla_X^\perp H)$, this together with $\nabla_X^\perp N\in JD^\perp$ gives $g(N, \nabla_X^\perp H)=o$. Hence for $X\in D$, we get $\nabla_X^\perp H\in\mu$. Now for $X\in D$, we have from $\bar\nabla_X JH=J\bar\nabla_X H$, with the help of (2.6), that

$$\nabla_X^\perp JH=-g(H, H)JX+J\nabla_X^\perp H. \tag{3.4}$$

Since $\nabla_X^\perp H\in\mu$, from (3.4) it follows that $JX=o$ for all $X\in D$. Hence $D=\{o\}$, this proves part (ii) of the theorem.

Lastly suppose $H\neq o$, $H\notin\mu$ and that $\dim D^\perp=1$. Since $\dim M\geqq 5$, we can choose vectors $X, Y\in D$ such that $g(X, Y)=g(X, JY)=0$. Now from (2.7) it follows that $\bar R(JX, Y; JY; N)=0$, $N\in\nu$. Using (2.1) we get $\bar R(JY, X; JY, N)=0$. This, with the help of (2.6) gives

$$g(\nabla_X^\perp H, N)=0\forall N\in\nu.$$

This proves that $\nabla_X^\perp H=0$ for $X\in D$. Next we let $X\in D^\perp$. Then there exists a normal N' such that $JX=N'$. Now for $N\in\mu$ we have $\bar R(X, Y; JY, JN)=0$, $Y\in D$. Using (2.1) in this we get $\bar R(X, Y; Y, N)=0$ and this together with (2.7) gives $g(\nabla_X^\perp H, N)=0$, from which it follows that $\nabla_X^\perp H\in JD^\perp$. Now again from (2.7) and (2.1) we have $\bar R(X, Y; Y, X)=\bar R(X, Y; JY, N')=0$, $N'=JX\in JD^\perp$. Using linearity of $\bar R$ in $\bar R(X, Y; Y, X)=0$, we get $\bar R(X, Y; JY, X)=0$. This gives $\bar R(X, Y; Y, N')=0$. From this using (2.7) we get $g(\nabla_X^\perp H, N')=0$. From this it follows that $\nabla_X^\perp H\in\mu$. Thus we have proved for $X\in D^\perp$, $\nabla_X^\perp H\in JD^\perp\cap\mu=\{o\}$, *i.e.* $\nabla_X^\perp H=0$. Hence $\nabla_X^\perp H=0$ for all vector fields X on M *i.e.* M is an extrinsic sphere. Then parts (iii) and (iv) of the Theorem follow from theorem of Yamaguchi, Nemoto and Kawabata in § 2.

This theorem thus gives a complete classification of totally umbilical *CR*-submanifolds of a Kaehler manifold.

4. Remark.

In case of complex space form $\overline{M}(c)$ *i.e.* Kaehler manifold of constant holomorphic sectional curvature c, the curvature tensor $\bar{R}$ of $\overline{M}(c)$ is given by

$$\bar{R}(X, Y)Z=\frac{c}{4}g(Y, Z)X-g(X, Z)Y+g(JY, Z)JX \\ -g(JX, Z)JY+2g(X, JY)JZ\,. \tag{4.1}$$

If M is totally umbilical submanifold of $\overline{M}(c)$ and R is curvature tensor of M, then by Gauss equation we have

$$g(R(X, Y)Z, W)=g(\bar{R}(X, Y)Z, W)+\alpha[g(Y, Z)g(X, W)-g(X, Z)g(Y, W)] \tag{4.2}$$

where $\alpha=g(H, H)$.

By [10] or [2] a totally umbilical submanifold of $\overline{M}(c)$ is either holomorphic submanifold or a totally real. Thus we have a corollary in light of equation (4.2).

COROLLARY. *Let M be totally umbilical submanifold of a complex space form $\overline{M}(c)$. Then M is one of the following*

(i) *a complex space form $M(c)$*

(ii) *a totally real submanifold of constant curvature c*

(iii) *a totally real submanifold of constant curvature $c+\alpha$.*

This corollary is essentially theorem due to Chen and Ogiue [10].

REFERENCES

[1] A. BEJANCU, CR-submanifolds of a Kaehler manifold, Proc. Amer. Math. Soc., **69** (1978), 135-142.

[2] A. BEJANCU, CR-submanifolds of a Kaehler manifold-II, Trans. Amer. Math. Soc., **250** (1979), 333-345.

[3] A. BENJANCU, Umbilical CR-submanifolds of a Kaehler manifold, Rend. Mat., **13** (1980), 431-466.

[4] D.E. BLAIR AND B.Y. CHEN, On CR-submanifolds of Hermitian manifolds, Israel J. Math., **34** (1980), 353-363.

[5] B.Y. CHEN, Geometry of submanifolds, Dekker, N.Y. (1973).

[6] B.Y. CHEN, Extrinsic spheres in Kaehler manifolds, Mich. Math. J., **23** (1976), 327-330.

[7] B.Y. CHEN, Geometry of submanifolds and its applications, Sci. Univ. Tokyo, Tokyo (1981).

[8] B. Y. CHEN, On CR-submanifolds of Kaehler manifolds, I, II, J. Diff. Geom., **16** (1981), 305-322, **16** (1981), 493-509.

[9] B. Y. CHEN AND K. OGIUE, On totally real submanifolds, Trans. of Amer. Math. Soc., **193** (1974), 257-266.

[10] B. Y. CHEN AND K. OGIUE, Two Theorems on Kaehler manifolds, Michigan Math. J., **21** (1974), 225-229.

[11] M. OBATA, Riemannian manifolds admitting a solution of a certain system of differential equations, Proc. U. S. Japan Seminar in Differential Geometry (Kyoto, 1965), 101-114.

[12] M. OKUMURA, Certain almost contact hypersurfaces in Kaehler manifold of constant holomorphic sectional curvature, Tohoku Math. J., **16** (1964), 270-284.

[13] S. YAMAGUCHI, H. NEMOTO AND N. KAWABATA, Extrinsic spheres in a Kaehler manifold, Michigan Math. J., **31** (1984), 15-19.

[14] K. YANO, Geometry of complex and almost complex spaces, Pergaman Press, N. Y. 1965.

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