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# EULER CHARACTERISTICS OF SOME SIX-DIMENSIONAL RIEMANNIAN MANIFOLDS

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## 1. Introduction.

Let $(M, g)$ be a compact and orientable Riemannian manifold of even dimension. With respect to the Euler characteristic $\chi(M)$ of $M$ and sectional curvature of $(M, g)$, it is an open question whether positivity of sectional curvature implies positivity of $\chi(M)$ or not. In the case of $m=\dim M=4$, J. Milnor (S. S. Chern [3]) proved that positivity of sectional curvature implies $\chi(M)>0$. So the question remains open for $m\geqq 6$. For simplicity, by $K(X, Y)>0$ we mean that the sectional curvature for arbitrary 2-plane $\{X, Y\}$ at an arbitrary point of $M$ is positive. Examples of known results are as follows:

(1) If $(M, g)$ is homogeneous and $K(X, Y)>0$, then $\chi(M)>0$ (A. Weinstein [11], p. 150).

(2) If $(M, g)$ admits a non-trivial Killing vector field, $m=6$, and $K(X, Y)>0$, then $\chi(M)>0$ (A. Weinstein [11], p. 150).

(3) If $(M, g)$ is isometrically immersed in $E^{m+2}$ and $K(X, Y)>0$, then $M$ is a homology sphere, in particular, $\chi(M)=2$ (A. Weinstein [10], D. Meyer [8]).

(4) If $(M, g)$ is conformally flat and $K(X, Y)>0$, then $\chi(M)>0$ (S. I. Goldberg [5], p. 227).

(5) In a 6-dimensional vector space with an inner product, one can define a curvature-like tensor $R$ so that the formal sectional curvature is positive and the formal Gauss-Bonnet integrand is negative (cf. R. Geroch [4], P. Klembeck [6], J. P. Bourguignon and H. Karcher [1]).

Assumptions in (1), (2) and (4) are not open in $C^\infty$-category in the sense that if one deforms the metric $g$ slightly in $C^3$-topology then theorems are not applicable. So extending (4) for $m=6$ we show that if the conformal curvature tensor $C$ of $(M, g)$, $m=6$, is not so much deviated from zero and if the Ricci curvature is not so much deviated from the average (i. e., the scalar curvature divided by 6) then $\chi(M)>0$ holds.

To state our theorem we fix notations. By $R$, $\rho$, and $S$ we denote the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature of $(M, g)$, respectively. By $\rho_{(1)}\leqq\rho_{(2)}\leqq\cdots\leqq\rho_{(m)}$ we denote the eigenvalues of the Ricci

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tensor with respect to $g$. $\rho_{(i)}$'s are continuous functions on $M$. Next we define a function $F(\alpha, \beta)$ by

$$(1.1) \qquad 100F(\alpha, \beta)=-540(\alpha-1)\beta^2+9(\alpha^3-5\alpha^2+10\alpha-6),$$

and we put (for details, see §2)

$$(1.2) \qquad W(C)=2|\nabla C|^2-(16/3)|C^*|^2-6(R, C, C)$$
$$+16(\rho; C, C)-(9/5)S|C|^2-3(\rho; \rho; C).$$

THEOREM A. *Assume that the scalar curvature $S$ of a compact and orientable 6-dimensional Riemannian manifold $(M, g)$ is positive and*

(i) $\rho_{(6)}\leqq\beta S$,
(ii) $\rho_{(1)}+\rho_{(2)}>(\alpha/5)S$,
(iii) $W(C)\leqq F(\alpha, \beta)S^3$

*hold on $M$ for some real numbers $\alpha\geqq1$ and $\beta$. Then $\chi(M)>0$.*

As a special case, if $\beta=1/\sqrt{24}=0.204\cdots$, then $F(\alpha, 1/\sqrt{24})$ is positive for $\alpha: 1<\alpha<1.292\cdots$. And for $\alpha=1.14$, $F(1.14, 1/\sqrt{24})=0.003\cdots$. Therefore we obtain the following.

COROLLARY B. *Assume that $S>0$ in a compact and orientable 6-dimensional Riemannian manifold $(M, g)$ and that*

(i)′ $\rho_{(6)}\leqq0.204S=1.224(S/6)$,
(ii)′ $\rho_{(1)}+\rho_{(2)}>0.228S=0.684(2S/6)$,
(iii)′ $W(C)\leqq0.003S^3$.

*Then $\chi(M)>0$.*

If $\alpha=1$, then $F(1, \beta)=0$. Therefore we get

COROLLARY C. *Assume that $S>0$ in a compact and orientable 6-dimensional Riemannian manifold $(M, g)$ and*

(ii)″ $\rho_{(1)}+\rho_{(2)}>(1/5)S$,
(iii)″ $W(C)\leqq0$.

*Then $\chi(M)>0$.*

Let $\{e_i\}$ be eigenvectors (at a point of $M$) of the Ricci tensor with respect to $g$ corresponding to $\{\rho_{(i)}\}$. If $(M, g)$ is conformally flat, then the sectional curvature $K(e_i, e_j)$ is given by (cf. [5], p. 229)

$$4K(e_i, e_j)=\rho_{(i)}+\rho_{(j)}-(1/5)S.$$

Hence, positivity of sectional curvature implies $\rho_{(i)}+\rho_{(j)}>(1/5)S$ for $m=6$. Thus, Corollary C is a generalization of (4) for $m=6$.

## 2. Expressions of $\chi(M)$.

In this section by $(M, g)$ we denote a compact and orientable 6-dimensional Riemannian manifold (except Lemma 2.1). By $R$, $\rho$, $S$, and $C$ we denote the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, and the Weyl conformal curvature tensor of $(M, g)$, respectively. For tensor fields $P$, $Q$, $T$, $U$, $V$, and $W$ of respective type we use the following notations:

$$(P, Q)=P_{ijkl}Q^{ijkl}, \qquad |P|^2=(P, P),$$

$$(P, Q, T)=P^{ij}{}_{kl}Q^{kl}{}_{rs}T^{rs}{}_{ij},$$

$$(U; Q, T)=U^{rs}Q_{rjkl}T_s{}^{jkl},$$

$$(U; V; T)=U^{ik}V^{jl}T_{ijkl},$$

$$(UVW)=U^i_jV^j_kW^k_i,$$

$$(P:P:P)=P^{ijkl}P_i{}^r{}_k{}^sP_{jslr}.$$

Then $\chi(M)$ is given by (cf. T. Sakai [9], p. 602)

$$384\pi^2\chi(M)=\int_M[S^3-12S|\rho|^2+3S|R|^2+16(\rho\rho\rho) +24(\rho;\rho;R)-24(\rho;R,R)+2(R,R,R)-8(R:R:R)]. \tag{2.1}$$

$(R:R:R)$ may be replaced by (cf. [9], p. 592)

$$-8\int_M(R:R:R)=\int_M[-2|\nabla R|^2+8|\nabla\rho|^2-2|\nabla S|^2+8(\rho\rho\rho) -8(\rho;\rho;R)-4(\rho;R,R)+4(R,R,R)]. \tag{2.2}$$

The following relations are verified by direct calculations:

$$|C|^2=|R|^2-|\rho|^2+(1/10)S^2, \tag{2.3}$$

$$(\rho;\rho;C)=(\rho;\rho;R)+(1/2)(\rho\rho\rho)-(11/20)S|\rho|^2+(1/20)S^3, \tag{2.4}$$

$$(\rho;C,C)=(\rho;R,R)-(\rho;\rho;R)-(3/4)(\rho\rho\rho)+(3/8)S|\rho|^2-(1/40)S^3, \tag{2.5}$$

$$(R,C,C)=(R,R,R)-2(\rho;R,R)+(1/5)S|R|^2+(1/2)(\rho;\rho;R) +(1/2)(\rho\rho\rho)-(1/5)S|\rho|^2+(1/100)S^3 \tag{2.6}$$

$$(R:R:R)=(C:C:C)-(3/2)(\rho;C,C)+(3/20)S|C|^2 -(3/8)(\rho\rho\rho)+(3/16)S|\rho|^2-(7/400)S^3, \tag{2.7}$$

$$|\nabla R|^2=|\nabla C|^2+|\nabla\rho|^2-(1/10)|\nabla S|^2. \tag{2.8}$$

We define $C^*$ by

$$C^*{}_{jkl}=\nabla_s C^s{}_{jkl}\,. \tag{2.9}$$

LEMMA 2.1. *On a compact and orientable m-dimesional Riemannian manifold, the following relation holds*:

$$((m-2)^2/2(m-3)^2)\int_M |C^*|^2=\int_M [|\nabla\rho|^2+(\rho\rho\rho) -(\rho\,;\,\rho\,;\,R)-(m/4(m-1))|\nabla S|^2]\,. \tag{2.10}$$

*Proof.* The next relation is a classical one:

$$-((m-2)/(m-3))\nabla^l C_{ijkl}=\nabla_i\rho_{jk}-\nabla_j\rho_{ik}-(1/2(m-1))(\nabla_i S\, g_{jk}-\nabla_j S\, g_{ik})\,. \tag{2.11}$$

Transvecting the last equation with $\nabla^i\rho^{jk}$ and integrating the result, we get

$$-((m-2)/(m-3))\int_M [\nabla^l C_{ijkl}\nabla^i\rho^{jk}]=\int_M [|\nabla\rho|^2+(\rho\rho\rho) -(\rho\,;\,\rho\,;\,R)-(m/4(m-1))|\nabla S|^2]\,, \tag{2.12}$$

where we have applied the Green-Stokes' theorem and the Ricci identity for $\nabla\nabla\rho$. On the other hand we get

$$\begin{aligned}2\nabla^l C_{ijkl}\nabla^i\rho^{jk}&=\nabla^l C_{ijkl}(\nabla^i\rho^{jk}-\nabla^j\rho^{ik})\\&=-((m-2)/(m-3))\nabla^l C_{ijkl}\nabla_s C^{ijks}\,,\end{aligned}$$

where we have used (2.11). From the last equation and (2.12) we get (2.10).
Now we return to the case of $m=6$.

LEMMA 2.2.

$$-8\int_M (R:R:R)=\int_M [-2|\nabla C|^2+(16/3)|C^*|^2+4(R,\,C,\,C)+4(\rho\,;\,C,\,C) -(4/5)S|C|^2+3(\rho\rho\rho)-(3/2)S|\rho|^2+(7/50)S^3] \tag{2.13}$$

*Proof.* By (2.10) with $m=6$ we get

$$(8/9)\int_M |C^*|^2=\int_M [|\nabla\rho|^2+(\rho\rho\rho)-(\rho\,;\,\rho\,;\,R)-(6/20)|\nabla S|^2]\,. \tag{2.14}$$

By (2.2)$\sim$(2.8), and (2.14), we get (2.13).
Now we get the following.

$$\begin{aligned}384\pi^2\chi(M)=\int_M[&-8(C:C:C)+2(R,\,C,\,C)-8(\rho\,;\,C,\,C)\\&+(7/5)S|C|^2+3(\rho\,;\,\rho\,;\,C)\\&+(3/2)(\rho\rho\rho)-(27/20)S|\rho|^2+(21/100)S^3]\end{aligned} \tag{2.15}$$

$$=\int_M [-2|\nabla C|^2+(16/3)|C^*|^2+6(R,\,C,\,C) \tag{2.16}$$

$$-16(\rho\,;C,\,C)+(9/5)S|C|^2+3(\rho\,;\rho\,;C)$$
$$+(3/2)(\rho\rho\rho)-(27/20)S|\rho|^2+(21/100)S^3]\,.$$

(2.15) follows from (2.1) and (2.3)$\sim$(2.7). (2.16) follows from (2.7), (2.13) and (2.15).

(2.15) is purely algebraic. However, the term $(C:C:C)$ is not so familiar. (2.16) is what we removed $(C:C:C)$ from (2.15) by using classical quantities.

## 3. Proof of Theorem A.

As before $\rho_{(1)}\leqq\rho_{(2)}\leqq\cdots\leqq\rho_{(6)}$ denote the eigenvalues of the Ricci tensor with respect to $g$.

LEMMA 3.1.

$$4^{-3}\Sigma_{(i\cdots b)}[\rho_{(i)}+\rho_{(j)}-(\alpha/5)S][\rho_{(k)}+\rho_{(l)}-(\alpha/5)S][\rho_{(a)}+\rho_{(b)}-(\alpha/5)S]$$
$$=(3/2)(\rho\rho\rho)-(9/20)(5-2\alpha)S|\rho|^2+(3/100)(25-30\alpha+15\alpha^2-3\alpha^3)S^3\,, \tag{3.1}$$

*where $\Sigma_{(i\cdots b)}$ denotes the summation over all permutations $(i,\,j,\,k,\,l,\,a,\,b)$ of* (1, 2, 3, 4, 5, 6).

*Proof.* First we notice that

$$(\rho\rho\rho)=\Sigma_i\rho_{(i)}^3\,,$$
$$S|\rho|^2=\Sigma_i\rho_{(i)}^3+\Sigma_{(i\neq j)}\rho_{(i)}^2\rho_{(j)}\,,$$
$$S^3=\Sigma_i\rho_{(i)}^3+3\Sigma_{(i\neq j)}\rho_{(i)}^2\rho_{(j)}+6\Sigma_{(i<j<k)}\rho_{(i)}\rho_{(j)}\rho_{(k)}\,.$$

Next we expand the left hand side of (3.1). Then we get (3.1).

*Proof of Theorem A.* $\rho_{(6)}\leqq\beta S$ implies $|\rho|^2\leqq 6\beta^2S^2$. For simplicity we denote the left hand side of (3.1) by $Z(\alpha)$. Then we obtain the following.

$$\begin{aligned}&(3/2)(\rho\rho\rho)-(27/20)S|\rho|^2+(21/100)S^3\\&=Z(\alpha)-(9/10)(\alpha-1)S|\rho|^2+(9/100)(\alpha^3-5\alpha^2+10\alpha-6)S^3\\&\geqq Z(\alpha)+F(\alpha,\,\beta)S^3.\end{aligned}$$

The assumption (ii) of Theorem A implies $Z(\alpha)>0$. Therefore, by (iii) and (2.16) we get $\chi(M)>0$.

*Remark.* The assumption (iii) ((iii)′, (iii)″, resp.) in Theorem A (Corollaries, resp.) may be replaced by

(iii)* $$W^*(C)\leqq F(\alpha,\,\beta)S^3\,,$$

where

$$W^*(C)=8(C:C:C)-2(R,\ C,\ C)+8(\rho\ ;\ C,\ C)-(7/5)S|C|^2-3(\rho\ ;\ \rho\ ;\ C)\,.$$

*Remark.* In Corollary C, if we replace (ii)′ by

$$\rho_{(1)}+\rho_{(2)}\geqq(1/5)S\,,$$

then the conclusion is $\chi(M)\geqq 0$.

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