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A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION BY PRODUCT

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1. Introduction and statement of results.

Let $f(z)$ be an entire function and set

$$m(r, f)=\min_{|z|=r}|f(z)|,$$

$$M(r, f)=\max_{|z|=r}|f(z)|.$$

The relation between $m(r, f)$ and $M(r, f)$ has been very thoroughly explored for functions whose orders lie strictly between 0 and 1. Hayman [7] proved the following result.

THEOREM A. *If $f(z)$ is an entire function such that*

$$m(r, f)\cdot M(r, f)=O(1), \qquad as\ r\to\infty, \tag{1.1}$$

then $f(z)=Ae^{Bz}$, where A, B are constants, or else

$$\lim_{r\to\infty}\frac{\log M(r, f)}{r}=+\infty .$$

He was unable to decide whether (1.1) can hold for functions of order one and of maximal type.

In this connection we prove the following results in this paper.

THEOREM 1. *Suppose that $f(z)$ is an entire function of positive integral order p, and that $f(z)$ has no zeros in a sector $\{z; |\arg z|<\pi-\pi/2p+\eta\}$ $(\eta>0)$ and $\delta(0, f)=1$. If there exists a Jordan curve l joining $z=0$ to $z=\infty$ such that*

$$f(z)\cdot f(\omega z)\cdots f(\omega^{2p-1}z)=O(1) \qquad (z\in l) \tag{1.2}$$

where $\omega=\exp(\pi i/p)$, then $f(z)=e^{P(z)}$, where $P(z)$ is a polynomial of degree p, or else

$$\lim_{r\to\infty}\frac{|\log|f(r)||}{r^p}=+\infty . \tag{1.3}$$

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We show that there exists an entire function satisfying the hypotheses of Theorem 1 and (1.3).

As an immediate consequence of Theorem 1,

COROLLARY 1. *Suppose that $f(z)$ is an entire function of order one, and that $f(z)$ has no zeros in a sector $\{z; |\arg z|<\pi/2+\eta\}$ $(\eta>0)$ and $\delta(0, f)=1$. If there exists a Jordan curve l joining $z=0$ to $z=\infty$ such that*

$$(1.4) \qquad f(z)\cdot f(-z)=O(1) \qquad (z\in l),$$

then $f(z)=Ae^{Bz}$, where A, B are constants, or else

$$(1.5) \qquad \lim_{r\to\infty}\frac{-\log|f(r)|}{r}=+\infty .$$

We show that there exists an entire function satisfying the hypotheses of Corollary 1 and (1.5). Observing the function $\cos z$, we note that we can remove neither the condition on the defect nor the condition on the location of zeros in Corollary 1.

To prove Theorem 2, we need the following Lemma.

LEMMA 1. *Suppose that $g(z)=e^{Q(z)}g_1(z)$ is an entire function of finite order having only negative zeros, where $Q(z)$ is a polynomial and $g_1(z)$ is a canonical product. Then the sign of $\log|g(r)|$ is definite for $r\geqq r_0$ where r_0 is a positive number, unless*

$$(1.6) \qquad \deg(\operatorname{Re}Q(r))=0 \quad and \quad g_1(z)\equiv 1 .$$

THEOREM 2. *Suppose that $f(z)$ is an entire function of order $q=2p+1$ having only negative zeros and $\delta(0, f)=1$. Further setting $\phi(z^2)=f(z)\cdot f(-z)$, $g(z)=\phi(-z)/\phi(0)$ we assume that there is an arbitrarily small $\beta>0$ such that*

$$(1.7) \qquad \left|\log|g(re^{i\beta})g(re^{-i\beta})|-2\left(\cos\frac{\beta q}{2}\right)\log|g(r)|\right| \\ \leqq \varepsilon(r)|\log|g(r)||$$

for all sufficiently large r where $0\leqq\varepsilon(r)=O(1/r^{\varepsilon_0})$, $\varepsilon_0>0$ unless $g(z)$ is in case (1.6). Then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree q, or else

$$(1.8) \qquad \lim_{r\to\infty}\frac{-\log|f(r)|}{r^q}=+\infty .$$

We show that there exists an entire function satisfying the hypotheses of Theorem 2 and (1.8).

We can remove the condition $\delta(0, f)=1$ in Theorem 2, by giving some conditions which are stronger than (1.7) and have a variant of Theorem 2.

THEOREM 3. *Suppose that $f(z)$ is an entire function of order $q=2p+1$ having only negative zeros. Setting $\phi(z^2)=f(z)\cdot f(-z)$, $g(z)=\phi(-z)/\phi(0)$, we assume that $g(z)$ is a canonical product. Further we assume that there is an arbitrarily small β such that if $|g(r)|\geqq 1$,*

$$\log|g(re^{i\beta})|\leqq(\cos\beta q/2)\log|g(r)| \tag{1.9}$$

for all sufficiently large r and if $|g(r)|\leqq 1$,

$$\log|g(re^{i\beta})|\geqq(\cos\beta q/2)\log|g(r)| \tag{1.10}$$

for all sufficiently large r. Then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree q, or else

$$\lim_{r\to\infty}\frac{\log M(r,f)}{r^q}=+\infty\,. \tag{1.11}$$

We are unable to decide whether there exist functions satisfying the hypotheses of Theorem 3 and (1.11).

Arguing as in the proof of Theorem 3, we have the following.

THEOREM 4. *If $f(z)$ is an entire function of order one having only negative zeros such that*

$$f(r)\cdot f(-r)=O(1)\qquad(r\to\infty)\,, \tag{1.12}$$

then $f(z)=Ae^{Bz}$, where A, B are constants, or else

$$\lim_{r\to\infty}\frac{\log M(r,f)}{r}=+\infty\,. \tag{1.13}$$

We show that there exists an entire function satisfying the hypotheses of Theorem 4 and (1.13).

Finally we can see the following result which is obtained under some conditions on the value distribution.

THEOREM 5. *Suppose that $f(z)$ is an entire function having only negative zeros and that $\phi(z)=f(z)\cdot f(-z)$ is real for real z. Further assume that $\phi(z)=w$ for any real number w has either only real roots or only non-real roots. Then $f(z)=(Az+B)e^{P(z)}$ or else $f(z)=Ae^{P(z)}f_1(z)$, where A, B are real or pure imaginary constants, $P(z)$ is an odd function and $f_1(z)$ is a canonical product of genus one such that $n(r,0,f_1)\sim Cr$, with a constant C.*

2. Proof of Theorem 1. We need two known results.

LEMMA A [5]. *Let $f(z)=\exp(\alpha_0 z^p+\alpha_1 z^{p-1}+\cdots+\alpha_p)\prod_{\nu=1}^{\infty}E\left(\frac{z}{a_\nu},p\right)$ be an entire function of order p and $\delta(0,f)=1$. Then for any $\varepsilon>0$ we have*

$$\log|f(z)|-\mathrm{Re}\, c_j z^p<4\varepsilon|c_j|r^p \qquad (j\geqq j_0(\varepsilon)), \tag{2.1}$$

for $z=re^{i\theta}\in\Gamma_j$ *and*

$$\log|f(z)|-\mathrm{Re}\, c_j z^p>-4\varepsilon|c_j|r^p \qquad (j\geqq j_0(\varepsilon)), \tag{2.2}$$

for $z=re^{i\theta}\in\Gamma_j-E_j$ *where* $c_j=\alpha_0+\sum_{|a_\nu|\leqq\alpha^j}a_\nu^{-p}$ $(\alpha=\exp(1/(p+1)))$, $\Gamma_j=\{z\,;\,\alpha^j<|z|<\alpha^{j+3/2}\}$ *and* E_j *is an exceptional set which is confined in a finite number of disks, the sum of whose radii is at most* $4e\delta\alpha^{j+2}$ *with an arbitrary small* $\delta>0$.

LEMMA B [8]. *If* $\phi(z)$ *is a non-constant entire function such that*

$$\log m(r,\,\phi)<\cos\pi\lambda\log M(r,\,\phi)+O(1) \tag{2.3}$$

as $r\to\infty$, *where* $0<\lambda<1$, *then*

$$\lim_{r\to\infty}\frac{\log M(r,\,\phi)}{r^\lambda}=\beta \tag{2.4}$$

where $0<\beta\leqq+\infty$.

Let $f(z)$ be an entire function satisfying the hypotheses in Theorem 1. We suppose that (1.3) is false, i.e.,

$$\liminf_{r\to\infty}\frac{|\log|f(r)||}{r^p}=K<+\infty\,. \tag{2.5}$$

At first, we show that the genus of the canonical product of $f(z)$ is not greater than $p-1$. Since the order of $f(z)$ is equal to p, we can write

$$f(z)=e^{\alpha_0 z^p+\cdots+\alpha_p}\prod_{\nu=1}^{\infty}E\Big(\frac{z}{a_\nu}\Big)\,. \tag{2.6}$$

Hence

$$c_j=\alpha_0+\sum_{|a_\nu|\leqq\alpha^j}a_\nu^{-p}=\alpha_0+a_1^{-p}+\cdots+a_{\nu_j}^{-p}\,.$$

Case (1). p is odd. Setting $a_\nu=|a_\nu|\exp(i(\pi+\theta_\nu))$ $(p|\theta_\nu|<\pi/2-\eta_0,\ \eta_0=p\eta)$, we have

$$\begin{aligned}\mathrm{Re}\,(c_jr^p)&=r^p(\mathrm{Re}\,\alpha_0-|a_1|^{-p}\cos p\theta_1-\cdots-|a_{\nu_j}|^{-p}\cos p\theta_{\nu_j})\\&\leqq r^p\,\mathrm{Re}\,\alpha_0-r^p(|a_1|^{-p}+\cdots+|a_{\nu_j}|^{-p})\cos(\pi/2-\eta_0)\,.\end{aligned}$$

Using (2.1), we see

$$\log|f(r)|\leqq r^p\,\mathrm{Re}\,\alpha_0-r^p(|a_1|^{-p}+\cdots+|a_{\nu_j}|^{-p})\cos(\pi/2-\eta_0)+4\varepsilon|c_j|r\,.$$

Therefore we have

$$\begin{aligned}&(|a_1|^{-p}+\cdots+|a_{\nu_j}|^{-p})\cos(\pi/2-\eta_0)-\mathrm{Re}\,\alpha_0\\&\qquad-4\varepsilon(|\alpha_0|+|a_1|^{-p}+\cdots+|a_{\nu_j}|^{-p})\leqq-(\log|f(r)|)/r^p\,.\end{aligned}$$

Hence (2.5) yields

$$\sum_{\nu=1}^{\infty} |a_\nu|^{-p} < +\infty .$$

Thus the genus of the canonical product of $f(z)$ is not greater than $p-1$ and we can rewrite (2.6) as follows

$$f(z)=\exp(\alpha_0 z^p + \cdots + \alpha_p) f_1(z) , \tag{2.7}$$

where the genus of $f_1(z)$ is at most $p-1$. By the well known estimation [6, p. 29],

$$\log M(r, f_1)=o(r^p) .$$

Now, it is easy to see that

$$\phi_1(z)=f_1(z)f_1(\omega z)\cdots f_1(\omega^{2p-1}z) , \qquad (\omega=\exp(\pi i/p)) , \tag{2.8}$$

is a function of z^{2p}. Hence setting

$$\phi(\zeta)=\phi(z^{2p})=f_1(z)f_1(\omega z)\cdots f_1(\omega^{2p-1}z) ,$$

we have $\log M(r^{2p}, \phi) \leqq 2p \log M(r, f_1)=o(r^p)$. Therefore it follows that

$$\lim_{\rho\to\infty} \frac{\log M(\rho, \phi)}{\rho^{1/2}}=0 , \qquad (\rho=|\zeta|=|z|^{2p}) . \tag{2.9}$$

On the other hand, by the assumption (1.2) we have

$$m(\rho, \phi) \leqq K < +\infty ,$$

and it follows that ϕ satisfies hypothesis (2.3) in Lemma B with $\lambda=1/2$ or else

$$\phi(z)=K'=\text{constant}. \tag{2.10}$$

Hence, if $\phi(z)$ is not constant, we have

$$\lim_{\rho\to\infty} \frac{\log M(\rho, \phi)}{\rho^{1/2}}=\beta , \qquad 0<\beta \leqq +\infty ,$$

which contradicts (2.9).

Now we deal with case (2.10). Suppose first that $K'=0$. Then (2.8) shows that $f_1(z)=0$ for every z, and so we have $f(z)=0$ for every z from (2.7). This contradicts the hypothesis of Theorem 1 that $f(z)$ is an entire function of positive integral order. Thus $K' \neq 0$. Then (2.8) shows that $f(z)$ does not take the value 0. From this and (2.10) we have $f(z)=e^{P(z)}$, where $P(z)$ is a polynomial of degree p.

Case (2). p is even. Similarly we can prove the conclusions of Theorem 1 using (2.2) instead of (2.1). In fact, for any $\delta_0>0$, setting $\delta=(4e\alpha^2)^{-1}\sin\delta_0$ we have $4e\delta\alpha^{j+2}=\alpha^j \sin\delta_0$ and so in view of $E_j \subset \Gamma_j=\{z\,;\,\alpha^j<|z|<\alpha^{j+3/2}\}$, E_j is confined in $\{z ; |\arg z| \geqq \pi/2\}$ $(j \geqq j_0(\varepsilon))$.

EXAMPLES. We show two examples satisfying the hypotheses of Corollary 1 and (1.5), and we show an example satisfying the hypotheses of Theorem 1 and (1.3).

(1). $f(z)=1/(z\Gamma(z))$, where $\Gamma(z)$ is the Gamma function. Since it is well known [11, p. 151] that,

$$\log \Gamma(z)=(z-1/2)\log z-z+1/2\log 2\pi+O(1/z), \quad (-\pi+\delta \leqq \arg z \leqq \pi-\delta) \tag{2.11}$$

we have

$$\log f(z)=-(z+1/2)\log z+z-1/2\log 2\pi+O(1/z). \tag{2.12}$$

Now it is also well known [3, p. 21] that,

$$\exp\{\log \Gamma(z)\}=\exp\{\log \pi-\log(\sin \pi z)-\log \Gamma(1-z)\}$$

and so we have

$$\exp\{-\log \Gamma(-z)\}=\exp\{-\log \pi+\log(-\sin \pi z)+\log \Gamma(1-z)\}. \tag{2.13}$$

From (2.11) and (2.13), it follows that

$$\begin{aligned}\exp\{-\log \Gamma(-z)\}=\exp\{&-\log \pi+\log(-\sin \pi z)+(z+1/2)\log(z+1)\\ &-z-1+1/2\exp 2\pi+O(1/z)\}, \quad (-\pi+\delta \leqq \arg z \leqq \pi-\delta).\end{aligned}$$

Therefore we obtain in $\{z ; -\pi+\delta \leqq \arg z \leqq \pi-\delta\}$,

$$\begin{aligned}\exp\{\log f(-z)\}=\exp\{&\log(-\sin \pi z)+(z+1/2)\log(z+1)\\ &-z-\log z-\log \pi-1+1/2\log 2\pi+i\pi+O(1/z)\}.\end{aligned} \tag{2.14}$$

Combining (2.12) and (2.14) we have

$$f(r)\cdot f(-r) \longrightarrow 0 \qquad (r \to +\infty). \tag{2.15}$$

This is a stronger condition than (1.4). Condition (1.5) follows from (2.12).

(2). If we set $g(z)=f(z/\pi-1/2)$ where $f(z)=1/(z\Gamma(z))$, then $g(z)\ g(-z)=\pi^{-1}\cos z$. Hence $g(z)$ satisfies (1.4) and (1.5). However $g(z)$ does not satisfy (2.15).

(3). Let

$$f(z)=\prod_{\nu=1}^{\infty}\left(1-\frac{z}{a_\nu}\right)\exp\left\{\frac{z}{a_\nu}+\cdots+\frac{1}{p}\left(\frac{z}{a_\nu}\right)^p\right\}$$

where $a_\nu=-((2\nu-1)\pi/2)^{1/p}$. Then, setting $\rho_\nu=(2\nu-1)\pi/2$ we obtain

$$\phi_1(z)=f(z)f(\omega z)\cdots f(\omega^{2p-1}z)=\prod_{\nu=1}^{\infty}\left(1-\frac{z^{2p}}{a_\nu^{2p}}\right)$$

$$=\prod_{\nu=1}^{\infty}\left(1-\frac{\zeta^2}{\rho_\nu^2}\right)=\cos\zeta\,.$$

Thus $\phi_1(r)=O(1)$ $(r\to\infty)$. Hence $f(z)$ is an entire function satisfying the hypotheses of Theorem 1 and (1.3).

3. **Proof of Lemma 1.** Denoting the genus of $g_1(z)$ by k, we have

$$\log|g_1(re^{i\theta})|=(-1)^k r^{k+1}\int_0^\infty \frac{n(x)}{x^{k+1}}\frac{x\cos(k+1)\theta+r\cos k\theta}{x^2+r^2+2xr\cos\theta}dx \tag{3.1}$$

If k is odd, then (3.1) yields

$$-\log|g_1(r)|=r^{k+1}\int_0^\infty \frac{n(x)}{x^{k+1}}\frac{dx}{x+r}\geqq\frac{1}{2}r^k\int_0^r\frac{n(x)}{x^{k+1}}dx\,.$$

Thus we have

$$\frac{-\log|g_1(r)|}{r^k}\geqq\frac{1}{2}\int_0^r\frac{n(x)}{x^{k+1}}dx\longrightarrow+\infty\qquad(r\to+\infty)\,.$$

If k is even, then (3.1) yields similarly $(\log|g_1(r)|)/r^k\to+\infty$ as $r\to+\infty$. Hence, if $k\geqq l=\deg(\operatorname{Re}Q(r))$, then the sign of $\log|g(r)|$ coincides with the one of $\log|g_1(r)|$ for all sufficiently large r.

On the other hand, if $k<l$, then the sign of $\log|g(r)|$ coincides, with the one of $\operatorname{Re}Q(r)$ for all sufficiently large r. In fact, from (3.1) we have

$$||\log|g_1(r)||=r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{dx}{x+r}$$

$$\leqq r^k\int_0^r\frac{n(x)}{x^{k+1}}dx+r^{k+1}\int_r^\infty\frac{n(x)}{x^{k+2}}dx\,,$$

and so $||\log|g_1(r)||=o(\operatorname{Re}Q(r))$. Thus the sign of $\log|g(r)|$ is definite for $r\geqq r_0$, with the exception of case (1.6).

To prove Theorem 2, we shall make use of the following Baernstein's result [2].

LEMMA C. *Let $B(t)$ be a nondecreasing convex function of* $\log t$ *on* $(0,\infty)$ *with* $B(0)=B(0+)=0$. *Let $l(\theta)$ be a bounded and measurable function on* $(0,\pi)$. *Let $b(z)$ be the function which is bounded and harmonic in the half disk* $\{z\,;\,|z|<R,\ \operatorname{Im}z>0\}$, *and which has the following boundary values:*

$$b(Re^{i\theta})=l(\theta),\quad b(r)=0,\quad b(-r)=B(r)\qquad(0<r<R)\,.$$

Let $\sigma\in(0,1)$, $\alpha\in(0,1)$. *Suppose* $0<r<s=\alpha R$. *Then*

$$\int_r^s \frac{b_\theta(-t)-(\cos \pi\sigma)b_\theta(t)}{t^{1+\sigma}}\,dt \tag{3.2}$$

$$>K_1\frac{b_\theta(r)}{r^\sigma}-K(\alpha,\ \sigma)\frac{B(\alpha^{-1}R)+M_1}{s^\sigma},$$

where K_1 *is a positive constant depending only on* σ, $K(\alpha,\ \sigma)$ *is a positive constant depending only on* α *and* σ, *and* $M_1=\sup_{0<\theta<\pi}|l(\theta)|$.

4. Proof of Theorem 2. Let $f(z)$ be an entire function satisfying the hypotheses in Theorem 2. We suppose that (1.8) is false. Proceeding as in § 3 we obtain

$$f(z)=e^{P(z)}\cdot f_1(z)\,, \tag{4.1}$$

where $P(z)$ is a polynomial of degree at most q and the genus of $f_1(z)$ is not greater than $q-1=2p$. If we set

$$\phi(z^2)=f(z)\cdot f(-z)=e^{R(z)}f_1(z)\cdot f_1(-z)\,,$$

then the degree of $R(z)$ is not greater than $2p=q-1$. Hence we have

$$\log M(r^2,\ \phi)\leqq K\,r^{2p}+2\log M(r,\ f_1)=o(r^q)\,.$$

Since $g(z)=\phi(-z)/\phi(0)$, we obtain

$$\lim_{r\to\infty}\frac{\log M(r,\ g)}{r^{q/2}}=0\,. \tag{4.2}$$

Now we can write

$$g(z)=e^{Q(z)}\cdot g_1(z)\,, \tag{4.3}$$

where $Q(z)$ is a polynomial of degree at most p and the genus of the canonical product $g_1(z)$ is not greater than p.

We can easily deal with case (1.6). Since $g(z)$ has only negative zeros, it follows from (4.3) and (1.6) that

$$g(z)=\phi(-z)/\phi(0)=\exp\{i(\alpha_{k'}z^{k'}+\cdots+\alpha_1 z)\}\,,$$

where α_j $(j=0,\ \cdots,\ k')$ are all real. Hence by (4.1) we deduce $f(z)=\exp(P(z))$ where $P(z)$ is a polynomial of degree q, which is the desired result.

Now we consider the other cases than (1.6).

Case (1). $\log|g(r)|\geqq 0$ and $\log|g(re^{i\beta})g(re^{-i\beta})|-2(\cos\beta q/2)\log|g(r)|\leqq \varepsilon(r)\log|g(r)|$ for all sufficiently large r.

We set

$$Q(z)=a_{k'}z^{k'}+\cdots+a_1 z,\quad \deg(\operatorname{Re}Q(r))=l\quad(\leqq k')$$

and

$$\arg a_j=\theta_j \qquad (j=1, \cdots, k').$$

Let β be a sufficiently small positive number. We define in $D=\{z\,;0<|z|<R, 0<\arg z<\beta\}$,

$$H(re^{i\theta})=\int_{-\theta}^{\theta}\log|g(re^{i\phi})|\,d\phi$$

$$=\int_{-\theta}^{\theta}\mathrm{Re}\,(Q(re^{i\phi}))d\phi+2\int_0^{\theta}\log|g_1(re^{i\phi})|\,d\phi\,,$$

Then we have

$$H(re^{i\theta})=\frac{2}{l}|a_l|r^l\sin l\theta\cos\theta_l+\cdots \tag{4.4}$$

$$+2|a_1|r\sin\theta\cos\theta_1+2\int_0^{\theta}\log|g_1(re^{i\phi})|\,d\phi\,.$$

Since $g(z)$ has only negative zeros, we can show that $H(re^{i\theta})$ is harmonic in D by arguments similar to those in the proof of Theorem 1 in [1].

Furthermore we consider the subcases.

Case (1-1). $k\geqq l$. In this case the sign of $\log|g(r)|$ coincides with the one of $\log|g_1(r)|$ for all sufficiently large r.

Setting $I_1=[0, \pi/2)\cup(3\pi/2, 2\pi]$, $I_2=(\pi/2, 3\pi/2)$ we define

$$H_1(re^{i\theta})=\sum_{\theta_j\in I_1}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j+H_3(re^{i\theta})\,, \tag{4.5}$$

$$H_2(re^{i\theta})=\sum_{\theta_j\in I_2}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j$$

where

$$H_3(re^{i\theta})=2\int_0^{\theta}\log|g_1(re^{i\phi})|\,d\phi\,.$$

Then we have $H(re^{i\theta})=H_1(re^{i\theta})+H_2(re^{i\theta})$.

At first we show that $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0, \infty)$ with $H_1(0)=H_1(0+)=0$, for all sufficiently small positive numbers β. Since it is trivial from (4.5) that $H_1(re^{i\beta})-H_3(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0, \infty)$ with $H_1(0)-H_3(0)=0$, it is sufficient to show that $H_3(re^{i\beta})$ is so.

We can see that $\log|g_1(re^{i\theta})|$ is monotone decreasing for $0\leqq\theta\leqq 2\pi/(q+1)$. In fact, setting

$$G_k(t,\ \theta)=\frac{t\cos(k+1)\theta+\cos k\theta}{t^2+2t\cos\theta+1}$$

we have from (3.1),

(4.6) $$\frac{\partial}{\partial\theta}(\log|g_1(re^{i\theta})|)=\int_0^\infty \frac{n(rt)}{t^{k+1}}\frac{\partial G_k}{\partial\theta}dt,$$

where

$$\frac{\partial G_k}{\partial\theta}=-\frac{t\sin(k+1)\theta\ T_k(t,\theta)+\sin k\theta\ T_{k+1}(t,\theta)}{(t^2+2t\cos\theta+1)^2}$$

and

$$T_j(t,\theta)=(j+1)t^2+2jt\cos\theta+j-1.$$

Since $T_j(t,\theta)>0$ for $0\leqq\theta\leqq(1-1/2j)\pi$ [13], observing $2\pi/(q+1)\leqq\pi/(k+1)$, we easily deduce that

$$T_k(t,\theta)>0,\quad T_{k+1}(t,\theta)>0\qquad \text{for } 0\leqq\theta\leqq 2\pi/(q+1).$$

Thus $\log|g_1(re^{i\theta})|$ is monotone decreasing for $0\leqq\theta\leqq 2\pi/(q+1)$.

Hence we have

$$\frac{\partial^2 H_3}{\partial\theta^2}\leqq 0,$$

and therefore we have

$$\frac{\partial^2 H_3}{\partial(\log r)^2}=r^2\Big(\frac{\partial^2 H_3}{\partial r^2}+\frac{1}{r}\frac{\partial H_3}{\partial r}\Big)\geqq 0$$

from the harmonicity of $H_3(re^{i\theta})$. Hence $H_3(re^{i\beta})$ is a convex function of $\log r$ on $(0,\infty)$. On the other hand, $H_3(re^{i\beta})$ is a nondecreasing function of $\log r$ on $(0,\infty)$ from the fact that $\log|g_1(re^{i\beta})|$ is positive and (3.1). Thus $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0,\infty)$ with $H_1(0)=H_1(0+)=0$.

Next we show that $H(re^{i\beta})$ is an increasing convex function of $\log r$ for all sufficiently large r for all sufficiently small positive numbers β. Since $\log|g_1(re^{i\theta})|$ is a decreasing function of θ $(0\leqq\theta\leqq\beta)$, we have from (3.1)

$$\begin{aligned}H_3(re^{i\beta})&=2\int_0^\beta \log|g_1(re^{i\phi})|\,d\phi\geqq 2\beta\log|g_1(re^{i\beta})|\\&=2\beta r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{x\cos(k+1)\beta+r\cos k\beta}{x^2+r^2+2rx\cos\beta}dx\\&\geqq\beta r^k\cos(k+1)\beta\int_0^r\frac{n(x)}{x^{k+1}}dx>0.\end{aligned}$$

Hence $H_3(re^{i\beta})/r^k\to+\infty$, as $r\to+\infty$ and $H(re^{i\beta})$ is unbounded. From (4.6) we have

$$\frac{\partial}{\partial\theta}(\log|g_1(re^{i\theta})|)=-r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{x\sin(k+1)\theta\ T_k^*+r\sin k\theta\ T_{k+1}^*}{(x^2+2rx\cos\theta+r^2)^2}dx$$

where $T_k^*=(k+1)x^2+2krx\cos\theta+(k-1)r^2$. Hence we have

$$\frac{\partial}{\partial\theta}(\log|g_1(re^{i\theta})|)=-k\sin k\theta\, r^{k+1}\int_0^r \frac{n(x)}{x^{k+1}}\frac{r^3}{(x+r)^4}dx$$

$$\leqq -\frac{k}{16}\sin k\theta\, r^k\int_0^r \frac{n(x)}{x^{k+1}}dx$$

for all sufficiently small positive numbers θ. Hence $(\partial^2 H/\partial\theta^2)_{\theta=\beta}$ is negative and $(\partial^2 H/(\partial\log r)^2)_{\theta=\beta}$ is positive for all sufficiently large r. Thus $H(re^{i\beta})$ is an increasing convex function of $\log r$ for all sufficiently large r.

Let $\gamma=\beta/\pi$. Now we can define a function $B^*(t)$ satisfying the hypotheses on $B(t)$ of Lemma C such that $B^*(t)\geqq H(t^\gamma e^{i\beta})$ on $(0,\infty)$. In fact, choosing a sufficiently large r_1, we define

$$\begin{aligned} B^*(t^{1/\gamma}) &= H_1(te^{i\beta}) : 0\leqq t\leqq r_1 \\ &= H_1(r_1e^{i\beta})+r_1H_1'(r_1e^{i\beta})\log\frac{t}{r_1} : r_1\leqq t\leqq r_2 \\ &= H(te^{i\beta}) : t\geqq r_2, \end{aligned}$$

where $H(r_2e^{i\beta})=H_1(r_1e^{i\beta})+r_1H_1'(r_1e^{i\beta})\log r_2/r_1$.

Fix $R>r_2$. Let $H^*(z)$ be the bounded harmonic function in $D=\{z\,;0<|z|<R, 0<\arg z<\beta\}$, which has the following boundary values:

$$H^*(r)=0,\qquad H^*(re^{i\beta})=B^*(r^{1/\gamma})\qquad (0\leqq r<R),$$
$$H^*(Re^{i\theta})=H(Re^{i\theta}).$$

We define $b(z)$ in $D'=\{z\,;0<|z|<R^{1/\gamma}, 0<\arg z<\pi\}$ by $b(z)=H^*(z^\gamma)$. Then $b(z)$ is the function considered in Lemma C, with $B(t)=B^*(t)$, the R there replaced by $R^{1/\gamma}$ and

$$l(\theta)=b(R^{1/\gamma}e^{i\theta})=H^*(Re^{i\gamma\theta})=H(Re^{i\gamma\theta})=\int_{-\gamma\theta}^{\gamma\theta}\log|g(Re^{i\phi})|\,d\phi\,.$$

Let $s=2^{-1/2}R$ and $r_2\leqq r<s$. Using (3.2) with $\sigma=\gamma q/2$ $(=\beta q/2\pi<\beta(q+1)/2\pi<1)$ and $\alpha=2^{-1/2\gamma}$ (<1), we obtain

$$\text{(4.7)}\qquad \int_{r^{1/\gamma}}^{s^{1/\gamma}}\frac{b_\theta(-t)-(\cos\pi\sigma)b_\theta(t)}{t^{1+\sigma}}dt>K_1\frac{b_\theta(r^{1/\gamma})}{r^{q/2}}$$
$$-K_2\frac{B^*(2^{1/2\gamma}R^{1/\gamma})+2\beta\log M(R,g)}{s^{q/2}},$$

where K_1, K_2 depend only on β and q. Now $b_\theta(t)=\gamma H_\theta^*(t^\gamma)$, $b_\theta(-t)=\gamma H_\theta^*(t^\gamma e^{i\beta})$. Changing variables in (4.7) and using $B^*(2^{1/2\gamma}R^{1/\gamma})=H^*(2^{1/2}Re^{i\beta})=H(2se^{i\beta})=\int_{-\beta}^{\beta}\log|g(2se^{i\phi})|\,d\phi\leqq 2\beta\log M(2s,g)$, we obtain

$$\int_r^s\frac{H_\theta^*(te^{i\beta})-(\cos\beta q/2)H_\theta^*(t)}{t^{1+q/2}}dt>K_1\frac{\gamma H_\theta^*(r)}{r^{q/2}}-K_2\frac{4\beta\log M(2s,g)}{s^{q/2}}\,.$$

Since $H_\theta^*(te^{i\beta})\leqq H_\theta(te^{i\beta})=\log|g(te^{i\beta})g(te^{-i\beta})|$ and $H_\theta^*(t)\geqq H_\theta(t)=2\log|g(t)|$ $(t\geqq r_2)$, $H_\theta^*(te^{i\theta})-(\cos\beta q/2)H_\theta^*(t)\leqq\varepsilon(t)\log|g(t)|$ from (1.7). Hence we have

$$\int_r^s\frac{\varepsilon(t)\log|g(t)|}{t^{1+q/2}}dt\geqq C_1\frac{\log|g(r)|}{r^{q/2}}-C_2\frac{\log M(2s,g)}{(2s)^{q/2}},\tag{4.8}$$

where C_1, C_2 depend only on β and q. From (4.2) we find a sequence of $r=\{r_n\}$ tending to infinity with n such that

$$C_1\frac{\log|g(r)|}{r^{q/2}}-C_2\frac{\log M(2s,g)}{(2s)^{q/2}}\leqq C'\frac{\log|g(r)|}{r^{q/2}}\int_r^s\frac{dt}{t^{1+\varepsilon_0}}$$

$$\leqq C\frac{\log|g(r)|}{r^{q/2+\varepsilon_0}},$$

where C', C are positive constants which do not depend on r. For each fixed r, if s tends to ∞, then we arrive at an impossible inequality from $\varepsilon_0>0$.

Case (1-2). $l>k$. In this case, since $\mathrm{Re}(Q(r))$ is positive for all sufficiently large r, θ_l lies in $I_1=[0,\pi/2]\cup(3\pi/2,2\pi]$.

Firstly we assume that k is even. In this case, we use the functions H, H_1, H_2 and H_3 defined by (4.5). $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0,\infty)$ with $H_1(0)=H_1(0+)=0$. Since the degree of $H_1(re^{i\beta})-H_3(re^{i\beta})$ is higher than one of $H_2(re^{i\beta})$, $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r. Hence arguments similar to those in case (1-1) lead to a contradiction.

Secondly we assume that k is odd. In this case we define

$$H_1(re^{i\theta})=\sum_{\theta_j\in I_1}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j,$$

$$H_2(re^{i\theta})=\sum_{\theta_j\in I_2}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j+H_3(re^{i\theta}),$$

where

$$H_3(re^{i\theta})=2\int_0^\theta\log|g_1(re^{i\phi})|d\phi.$$

Then we have $H(re^{i\theta})=H_1(re^{i\theta})+H_2(re^{i\theta})$.

It is trivial that $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0,\infty)$ with $H_1(0)=H_1(0+)=0$.

Now we show that $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r for all sufficiently small positive numbers β. Since $\log|g_1(re^{i\theta})|$ is an increasing function of θ $(0\leqq\theta\leqq\beta)$, we have from (3.1)

$$0\leqq -H_3(re^{i\beta})=-2\int_0^{\beta}\log|g_1(re^{i\phi})|\,d\phi\leqq -2\beta\log|g_1(r)|$$

$$=2\beta r^{k+1}\int_0^{\infty}\frac{n(x)}{x^{k+1}}\frac{dx}{x+r}\leqq 2\beta r^k\int_0^r\frac{n(x)}{x^{k+1}}dx+2\beta r^{k+1}\int_r^{\infty}\frac{n(x)}{x^{k+2}}dx=o(r^l)$$

$$(r\to\infty).$$

Hence $|H_3(re^{i\beta})|/r^l\to 0$ as $r\to+\infty$ and $H(re^{i\beta})$ is unbounded.

Proceeding as in case (1-1), we have

$$0\leqq\frac{(\partial/\partial\theta)(\log|g_1(re^{i\theta})|)}{r^{k+1}}\leqq\int_0^{\infty}\frac{n(x)}{x^{k+1}}\frac{x\sin(k+1)\theta\;T_k^*+r\sin k\theta\;T_{k+1}^*}{(x^2+2rx\cos\theta+r^2)^2}dx$$

$$\leqq\int_0^{\infty}\frac{n(x)}{x^{k+1}}\frac{x(k+1)(x+r)^2+(k+2)r(x+r)^2}{(\cos\theta)^2(x+r)^4}dx$$

$$\leqq\frac{k+2}{\cos^2\theta}\frac{1}{r}\int_0^r\frac{n(x)}{x^{k+1}}dx+\frac{k+2}{\cos^2\theta}\int_r^{\infty}\frac{n(x)}{x^{k+2}}dx\longrightarrow 0\qquad(r\to\infty)$$

for all sufficiently small positive θ.

Hence $(\partial^2H/\partial\theta^2)_{\theta=\beta}$ is negative and $(\partial^2H/(\partial\log r)^2)_{\theta=\beta}$ is positive for all sufficiently large r. Thus $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r. Thus arguments similar to those in case (1-1) lead to a contradiction.

Case (2). $\log|g(r)|<0$ and $\log|g(re^{i\beta})g(re^{-i\beta})|-2(\cos\beta q/2)\log|g(r)|\geqq\varepsilon(r)\log|g(r)|$ for all sufficiently large r.

Set $Q^*(z)=-Q(z)$, $g_1^*(z)=g_1(z)^{-1}$ and $g^*(z)=e^{Q^*(z)}g_1^*(z)$. Then (1.7) is equivalent to

$$|\log|g^*(re^{i\beta})g^*(re^{-i\beta})|-2(\cos\beta q/2)\log|g^*(r)||\leqq\varepsilon(r)|\log|g^*(r)||.$$

Thus our case is handled in a fashion almost similar to case (1).

We only show how to handle the inequality corresponding to (4.8). Proceeding as in case (1-1), we have

$$\int_r^s\frac{\varepsilon(t)\log|g^*(t)|}{t^{1+q/2}}dt\geqq C_1\frac{\log|g^*(r)|}{r^{q/2}}-C_2\frac{\log M_\beta(2s,g^*)}{(2s)^{q/2}},\tag{4.9}$$

where $M_\beta(2s,g^*)=\sup_{0<|\theta|<\beta}|g^*(2se^{i\theta})|$. In this inequality we must show that

$$\lim_{r\to\infty}\frac{\log M_\beta(r,g^*)}{r^{q/2}}=0.$$

Since $\log M_\beta(r,g^*)\leqq\sup_{0<|\theta|<\beta}\mathrm{Re}(Q^*(re^{i\theta}))+\log M_\beta(r,g_1^*)$ and $\lim_{r\to\infty}\{\sup_{0<\theta<\beta}\mathrm{Re}(Q^*(re^{i\theta}))\}/r^{q/2}=0$, it is sufficient to show that

$$\lim_{r\to\infty}\frac{\log M_\beta(r,\ g_1^*)}{r^{q/2}}=0 \tag{4.10}$$

in the case that the genus of $g_1^*(z)$ is not smaller than degree of Re $(Q(r))$. In this case we have from (3.1),

$$\log|g_1^*(re^{i\theta})|=\int_0^\infty\frac{n(rt)}{t^{k+1}}\frac{t\cos(k+1)\theta+\cos k\theta}{t^2+2t\cos\theta+1}dt.$$

Hence there exists a θ_0 for some $\delta_0>0$ such that

$$m_{\theta_0}(r,\ g_1^*)=\frac{1}{2\pi}\int_0^{\theta_0}\log^+|g_1^*(re^{i\theta})|\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{\theta_0}\log|g_1^*(re^{i\theta})|\,d\theta\geqq\frac{\theta_0\delta_0}{2\pi}\int_0^\infty\frac{n(rt)}{t^{k+1}}\frac{dt}{t+1}.$$

Since (4.2) implies $\lim_{r\to\infty}\{\log M(r,\ g_1)\}/r^{q/2}=0$, we have $m_{\theta_0}(r,\ g^*)/r^{q/2}\to 0$ as $r\to\infty$ and so we have

$$\frac{\log|g_1^*(r)|}{r^{q/2}}=\frac{1}{r^{q/2}}\int_0^\infty\frac{n(rt)}{t^{k+1}}\frac{dt}{t+1}\longrightarrow 0\quad(r\to+\infty).$$

Since $\log|g_1^*(re^{i\theta})|$ is monotone decreasing for $0\leqq\theta\leqq 2\pi/(q+1)$, we have $\log M_\beta(r,g_1^*)=\log|g_1^*(r)|$ for $0<\beta\leqq 2\pi/(q+1)$. Therefore we proved (4.10).

Proceeding as in case (1), we have a contradiction from (4.9) and (4.10).

An example. Let

$$f(z)=\prod_{n=1}^\infty\left(1-\frac{z}{a_n}\right)\exp\left\{\left(\frac{z}{a_n}+\cdots+\frac{1}{q}\left(\frac{z}{a_n}\right)^q\right\}\qquad(q=2p+1),$$

where $a_n=-n^{1/q}$. Since

$$\phi(z^2)=f(z)f(-z)=\prod_{n=1}^\infty\left(1-\frac{z^2}{a_n^2}\right)\exp\left\{\frac{z^2}{a_n^2}+\cdots+\frac{1}{p}\left(\frac{z^2}{a_n^2}\right)^p\right\},$$

we have

$$g(z)=\phi(-z)=\prod_{n=1}^\infty\left(1-\frac{z}{b_n}\right)\exp\left\{\frac{z}{b_n}+\cdots+\frac{1}{p}\left(\frac{z}{b_n}\right)^p\right\},$$

where $b_n=-n^{2/q}$. Now $n(r,\ 0:g)\sim r^{q/2}$ and hence we have in $\{z\,;\,|\arg z|<\pi-\delta\}$ $(0<\delta<\pi)$ the asympotic expansion [9, p. 232],

$$\log|g(re^{i\theta})|=(-1)^p\pi r^{p+1/2}\cos\{\theta(p+1/2)\}+O(r^p+\log r).$$

Therefore

$$|\log|g(re^{i\beta})|-(\cos\beta(p+1/2))\log|g(r)||\leqq\varepsilon(r)\log|g(r)|,$$

where $\varepsilon(r)=O(1/r^{\varepsilon_0})$ $(\varepsilon_0>0)$. Thus $f(z)$ satisfies the hypotheses of Theorem 2 and (1.8).

5. Proof of Theorem 3. Let $f(z)$ be an entire function satisfying the hypotheses in Theorem 3. We suppose that (1.11) is false, i.e.,

$$\liminf_{r\to\infty}\frac{\log M(r, f)}{r^q}<\infty .$$

Since $\phi(z^2)=f(z)\cdot f(-z)$, $g(z)=\phi(-z)/\phi(0)$ and $\log M(r^2, \phi)\leqq 2\log M(r, f)$, there exists a sequence $\{r_n\}=r$ which tends to infinity, such that

$$\frac{\log M(r, g)}{r^{q/2}}=O(1)\,. \tag{5.1}$$

Arguing as in §3, we see that the sign of $\log|g(r)|$ is definite for all sufficiently large r, with the exception of case (1.6) in which case we have the required function $f(z)=e^{P(z)}$, $\deg P(z)=q$.

If the sign of $\log|g(r)|$ is positive, (5.1) yields

$$\liminf_{r\to\infty}\frac{\log|g(r)|}{r^{q/2}}<+\infty\,. \tag{5.2}$$

If the sign of $\log|g(r)|$ is negative, then arguments similar to those in case (2) of §4 yields $\liminf\limits_{r\to\infty}(-\log|g(r)|)/r^{q/2}<+\infty$. Thus in the sequel we may assume that the sign of $\log|g(r)|$ is positive for all sufficiently large r, because the remaining case is similarly dealt with.

Fix $R>0$. We define in $D=\{z\,;\,0<|z|<R,\ 0<\arg z<\beta\}$ a harmonic function $H(z)$ as follows

$$H(re^{i\theta})=\int_0^\theta \log|g(re^{i\phi})|\,d\phi\,.$$

Let $\gamma=\beta/\pi$ and define $b(z)$ by $b(z)=H(z^\gamma)$ in $\{z\,;\,0<|z|<R^{1/\gamma},\ 0<\arg z<\pi\}$. Then $b(z)$ is the function considered in Lemma 4, with $B(t)=H(t^\gamma e^{i\beta})$, the R there replaced by $R^{1/\gamma}$ and

$$l(\theta)=b(R^{1/\gamma}e^{i\theta})=\int_0^{\gamma\theta}\log|g(Re^{i\phi})|\,d\phi\,.$$

It is easily verified that $B(t)$ satisfies the hypotheses of Lemma C. Now $b_\theta(t)=\gamma\log|g(t^\gamma)|$, $b_\theta(-t)=\gamma\log|g(t^\gamma e^{i\beta})|$. Hence arguing as in §4, we have

$$\int_r^s\frac{\log|g(te^{i\beta})|-(\cos\beta q/2)\log|g(t)|}{t^{1+q/2}}\,dt \tag{5.3}$$

$$>C_1\frac{\log|g(r)|}{r^{q/2}}-C_2\frac{\log|g(2s)|}{(2s)^{q/2}}\quad(0<r<s<\infty)\,.$$

Case (1).

$$B=\limsup_{r\to\infty}\frac{\log|g(r)|}{r^{q/2}}=+\infty\,.$$

From (5.2) we can find arbitrarily large values of r and s, with $r<s$, such that the righthand side of (5.3) is positive. Thus it follows that the inequality

$$\log|g(te^{i\beta})|-(\cos\beta q/2)\log|g(t)|>0$$

holds for some $t>r$ and this constradicts with our assumption (1.9).

Case (2). $B=0$. In any case, we have $(\log|g(r)|)/r^{q/2}>0$ for $r>0$. For each fixed r the right-hand side of (5.3) is positive for sufficiently large s, and again we have a contradiction.

Case (3). $0<B<+\infty$. Using the identity [2]:

$$b_\theta(r)=\int_0^\infty(b_\theta(t)+b_\theta(-t))Q(r,t)dt\,,$$

where $Q(r,t)=2r\pi^{-2}(r^2-t^2)^{-1}\log rt^{-1}$, we have

$$\log|g(r^\gamma)|\leqq\int_0^\infty(\log|g(t^\gamma)|+\log|g(t^\gamma e^{i\beta})|)Q(r,t)dt\,. \tag{5.4}$$

Dividing g by a large positive constant, if necessary, we can assume that (1.9) holds for all $t>0$. Putting (1.9) in (5.4), we obtain

$$\log|g(r^\gamma)|\leqq\int_0^\infty(1+\cos\beta q/2)\log|g(t^\gamma)|\cdot Q(r,t)dt\,.$$

Proceeding as in § 4 of [8], with $\gamma q/2$ in place of λ, we arrive at

$$\lim_{r\to\infty}\frac{\log|g(r^\gamma)|}{r^{\gamma q/2}}=B>0\,.$$

Hence, by Valiron's Tauberian Theorem [12], we have

$$n(r,0,g)\sim\frac{B}{\pi}r^{q/2}\,,$$

and

$$n(r,0,f)\sim\frac{B}{\pi}r^q\,.$$

Therefore we have $\delta(0,f)=1$. Proceeding as in the proof of Theorem 2, we have $B=0$, which is impossible.

6. Proof of Theorem 4. Put $\phi(z^2)=f(z)f(-z)$ and $g(z)=\phi(-z)/\phi(0)$, then $g(-r)$ is bounded i.e., $|g(-r)|\leqq C$. If $C>1$, then $h(z)=g(z)/C$ satisfies the assumption of Theorem 3, that is,

$$\log|h(re^{i\beta})|\leqq(\cos\beta q/2)|\log h(r)|$$

with $\beta=\pi$ and $q=1$.

Now we have the following fundamental inequality which corresponds to (5.3),

$$\int_r^s \frac{\log|h(-t)|}{t^{3/2}}dt > C_1\frac{\log|h(r)|+\log|C|}{r^{1/2}}$$

$$-C_2\frac{\log M(2s)+\log|C|}{(2s)^{1/2}}-2\log|C|(r^{-1/2}-s^{-1/2}).$$

We note that if $C\leqq 1$, then we use (5.3) with $q=1$, again. Proceeding as in the proof of Theorem 3, we have the desired result.

7. Proof of Theorem 5. Let $\mathcal{A}$ be the set of real numbers w for which $\phi(z)=w$ has only real roots.

We consider three cases.

Case (1). $\mathcal{A}$ consists of one element. In this case we have $\phi(z)=K$ (=constant). Suppose first that $K=0$, then $\phi(z)=f(z)\cdot f(-z)$ shows that $f(z)$ or $f(-z)$ is zero for every z, so that $f(z)=0$. Suppose next that $K\neq 0$, then we have $f(z)\neq 0$ and $f(z)=A\exp(P(z))$ where $P(z)$ is an odd function.

Case (2). $\mathcal{A}$ is unbounded. We need the following result [4].

LEMMA D. *Let $\phi(z)$ be an entire function. Assume that there exists an unbounded sequence $\{w_n\}$ such that all the roots of the equations $\phi(z)=w_n$ $(n=1,\cdots)$ are real. Then $\phi(z)$ is a polynomial of degree not greater than two.*

Since $\phi(z)=f(z)f(-z)$ is a polynomial of degree not greater than two by Lemma D and $f(z)$ has only negative zeros, it follows that $\phi(z)=K(z-\alpha)(z+\alpha)$, where K and α are real numbers. Hence $f(z)=A(z-\alpha)\exp(P(z))$ where $P(z)$ is an odd function.

Case 3. $\mathcal{A}$ consists of at least two elements and is bounded. In this case, we shall make use of the following result [10].

LEMMA E. *Let $\phi(z)$ be a transcendental entire function, real for real z. Assume that $\phi(z)=w$ has either only non-real roots or only real roots for all real numbers w. Then $\phi(z)=A\cos(Bz+C)+D$ with real constants A, B, C, D, $AB\neq 0$.*

Since $\phi(z)=f(z)f(-z)$ and since $f(z)$ has only negative zeros, Lemma E yields $n(r)\sim|B|r/\pi$, which is the desired result.

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