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ANOTHER PROOF OF EXISTENCE OF MOMENTUM MAPPINGS

Dedicated to Professor Sihgeo Sasaki on his 70th birthday

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1. Momentum mappings for the symplectic action G on a symplectic manifold are the group theoretical analogue of the linear and angular momentum associated with the translational and rotational invariance. Although momentum mappings are important in mechanics, not every symplectic action has a momentum mapping. We know some theorems concerning with the existence of (coadjoint equivariant) momentum mappings. These are as follows:

THEOREM A ([4], [5]). *If $H^1(\mathfrak{g}, R)=0$, then G has a momentum mapping.*

THEOREM B (cf. [2], [4], [5]). *If $H^1(\mathfrak{g}, R)=H^2(\mathfrak{g}, R)=0$, then G has a coadjoint equivariant momentum mapping.*

THEOREM C ([1]). *If the symplectic form $\Omega=d\theta$ for some G-invariant 1-form θ, then G has a coadjoint equivariant momentum mapping.*

THEOREM D (cf. [3]). *If $G=G_1 \underset{\sigma}{\rtimes} G_2$ where G_1 and G_2 have coadjoint equivariant momentum mappings, G_1 is connected and $H^1(\mathfrak{g}_1, R)=0$, then G has a coadjoint equivariant momentum mapping.*

Here $\mathfrak{g}$ and $\mathfrak{g}_1$ are the Lie algebras of G and G_1 respectively.

According to [2], we consider a map $\Psi: M \to Z^2(\mathfrak{g})$ defined by $\Psi(m)=\phi_m^*\Omega$, where Ω is the symplectic form and ϕ_m is the orbit map through m, that is, $\phi_m(g)=gm$. In this paper, we will give another constructive and elementary proof to each of the above theorems by using the properties of Ψ.

Remark. In the original theorems of Theorem B and Theorem D, the connectedness of the Lie group G was needed because the coadjoint equivariancy of momentum mappings comes from the Lie algebra homomorphism. But our proofs are all right whether G is connected or not.

2. Let (M, Ω) be a connected symplectic manifold, that is, M is a connected smooth manifold with a non-degenerate closed 2-form Ω. Let G be a symplectic

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action on (M, Ω), that is, G is acting on M and $\phi_g: M \to M: m \mapsto gm$ is a symplectomorphism, i.e., $\phi_g^*\Omega=\Omega$ for each g in G. By ψ_m, we mean the orbit map $\psi_m: G \to M: g \mapsto gm$ through m. Let $\mathrm{aut}(M, \Omega)$ be the Lie algebra of all Hamiltonian vector fields on (M, Ω), that is, $X \in \mathrm{aut}(M, \Omega)$ if and only if $L_X\Omega=0$. The symplectic action G induces a Lie algebra homomorphism $\rho: \mathfrak{g} \to \mathrm{aut}(M, \Omega)$ defined by

$$\rho(\xi)_m=(\psi_m)_{*e}(-\xi)=\frac{d}{dt}(\exp -t\xi\, m)_{t=0},$$

where $\mathfrak{g}$ is the Lie algebra of G consisting of left-invariant vector fields on G and e is the unit of G.

A momentum mapping for the symplectic action G is a mapping $J: M \to \mathfrak{g}^*$ ($=$the dual space of $\mathfrak{g}$) such that

$$d\langle J, \xi\rangle=\rho(\xi) \lrcorner\, \Omega .$$

A momentum mapping $J: M \to \mathfrak{g}^*$ for the symplectic action G is coadjoint equivariant if and only if J satisfies

$$J(gm)=Ad(g^{-1})^*J(m)$$

for each $g \in G$ and $m \in M$.

Since each left-invariant differential form on G is completely determined by its value on the left-invariant vector fields, we can identify the space of left-invariant p-forms on G with the space of p-cochains of $\mathfrak{g}$. The exterior differentiation d induces the coboundary operator δ. According to [2], we consider a map $\Psi(m)=\psi_m^*\Omega$ on M. Then $\langle\Psi(m), \xi\wedge\eta\rangle=\Omega(\rho(\xi), \rho(\eta))$ holds. For each $g \in G$ and $m \in M$, we have

$$\begin{aligned}
l_g^*\Psi(m)&=l_g^*\psi_m^*\Omega\\
&=(\psi_m l_g)^*\Omega\\
&=(\phi_g\psi_m)^*\Omega\\
&=\psi_m^*\phi_g^*\Omega\\
&=\psi_m^*\Omega\\
&=\Psi(m),\\
\Psi(gm)&=\psi_{gm}^*\Omega\\
&=(\psi_m r_g)^*\Omega\\
&=r_g^*\psi_m^*\Omega\\
&=r_g^*\Psi(m)\\
&=Ad(g^{-1})^*\Psi(m),
\end{aligned}$$

where l_g and r_g are the left and right multiplications by g, and

$$\delta\Psi(m)=d\phi_m^*\Omega=\phi_m^* d\Omega=0\,.$$

Differentiating the equation $\langle\Psi(m),\ \xi\wedge\eta\rangle=\Omega(\rho(\xi),\ \rho(\eta))_m$, we have

$$v \lrcorner\, d\langle\Psi,\ \xi\wedge\eta\rangle=v \lrcorner -\rho[\xi,\ \eta] \lrcorner\, \Omega$$

for any tangent vector v on M. Thus we have

PROPOSITION 1 ([2]). *There is a coadjoint equivariant mapping* $\Psi: M\rightarrow Z^2(\mathfrak{g})$ *satisfying*

(i) $\langle\Psi(m),\ \xi\wedge\eta\rangle=\Omega(\rho(\xi),\ \rho(\eta))_m$

and

(ii) $d\langle\Psi,\ \xi\wedge\eta\rangle=-\rho[\xi,\ \eta] \lrcorner\, \Omega$

for each ξ, η *in* $\mathfrak{g}$, *where* $Z^2(\mathfrak{g})$ *is the subspace of* 2-*cocycles of* $\mathfrak{g}$.

Concerning to the 1-coboundary operator $\delta: C^1(\mathfrak{g})\rightarrow C^2(\mathfrak{g})$, we have

PROPOSITION 2. (i) δ *is coadjoint equivariant.*
(ii) δ *is injective if and only if* $H^1(\mathfrak{g},\ R)=0$.
(iii) $\delta(\mathfrak{g}^*)=Z^2(\mathfrak{g})$ *if and only if* $H^2(\mathfrak{g},\ R)=0$.

Proof. Choose any $\lambda\in C^1(\mathfrak{g})=\mathfrak{g}^*$. For each $g\in G$ and $\xi,\ \eta\in\mathfrak{g}$, we have

$$\begin{aligned}(Ad(g)^*\delta\lambda)\xi\wedge\eta&=\delta\lambda(Ad(g)\xi\wedge Ad(g)\eta)=-\lambda[Ad(g)\xi,\ Ad(g)\eta]\\&=-\lambda(Ad(g)[\xi,\ \eta])=-(Ad(g)^*\lambda)[\xi,\ \eta]\\&=(\delta Ad(g)^*\lambda)\xi\wedge\eta\,,\end{aligned}$$

thus (i) is proved. $\delta\lambda=0$ is equivalent to $\lambda[\mathfrak{g},\ \mathfrak{g}]=0$. If $\delta\lambda=0$ and $H^1(\mathfrak{g},\ R)=0$, that is, $\mathfrak{g}=[\mathfrak{g},\ \mathfrak{g}]$, then $\lambda=0$. This shows that δ is injective. Conversely, assume that δ is injective. If $\mathfrak{g}\neq[\mathfrak{g},\ \mathfrak{g}]$, then we can define 1-cochain λ such that $\delta\lambda=0$ and $\lambda\neq0$. This contradicts to the injectivity of δ. Thus (ii) is proved. (iii) is obvious because $H^2(\mathfrak{g},\ R)=Z^2(\mathfrak{g},\ R)/\delta(\mathfrak{g}^*)$.

3. First we prove the following lemma, which suggests how to construct a momentum mapping from Ψ.

LEMMA. *If* J *is a momentum mapping for a symplectic action* G, *then* $\Psi-\delta J$ *is constant on* M.

Proof. From the definition of momentum mapping J and (ii) of Proposition 1, we have

$$\begin{aligned} d\langle \delta J, \xi\wedge\eta\rangle &= d\langle J, -[\xi, \eta]\rangle \\ &= -\rho[\xi, \eta] \lrcorner\, \Omega \\ &= d\langle \Psi, \xi\wedge\eta\rangle \end{aligned}$$

for each ξ, η in $\mathfrak{g}$. Since M is connected, $\delta J-\Psi$ is constant on M.

(3.1) *Proof of Theorem B.* The assumption $H^1(\mathfrak{g}, R)=H^2(\mathfrak{g}, R)=0$, implies that $\delta: \mathfrak{g}^*\to Z^2(\mathfrak{g})$ is bijective from (ii) and (iii) of Proposition 2. So we can define a mapping $J: M\to\mathfrak{g}^*$ by

$$J(m)=\delta^{-1}\Psi(m).$$

Since

$$\begin{aligned} d\langle J, [\xi, \eta]\rangle &= -d\langle \delta J, \xi\wedge\eta\rangle \\ &= -d\langle \Psi, \xi\wedge\eta\rangle \\ &= \rho[\xi, \eta] \lrcorner\, \Omega \end{aligned}$$

and since $[\xi, \eta]$'s generate $\mathfrak{g}$, J is a momentum mapping. J is also coadjoint equivariant because

$$\begin{aligned} J(gm) &= \delta^{-1}\Psi(gm)=\delta^{-1}Ad(g^{-1})^*\Psi(m) \\ &= Ad(g^{-1})^*\delta^{-1}\Psi(m)=Ad(g^{-1})^*J(m). \end{aligned}$$

(3.2) *Proof of Theorem C.* Since $\Omega=d\theta$, we have $\Psi(m)=\phi_m^*\Omega=\delta(\phi_m^*\theta)$, where $\phi_m^*\theta\in\mathfrak{g}^*$. So we define $J(m)$ by $\phi_m^*\theta$. Coadjoint equivariancy of this mapping J comes just the same way as the proof of that of Ψ in Proposition 1. For any vector field X on M, we have

$$\begin{aligned} X \lrcorner\, d\langle J, \xi\rangle &= L_X(\rho(-\xi) \lrcorner\, \theta) \\ &= -[X, \rho(\xi)] \lrcorner\, \theta - \rho(\xi) \lrcorner\, X \lrcorner\, d\theta - \rho(\xi) \lrcorner\, d(X \lrcorner\, \theta) \\ &= X \lrcorner\, \rho(\xi) \lrcorner\, \Omega. \end{aligned}$$

Thus $J(m)=\phi_m^*\theta$ is a coadjoint equivariant momentum mapping.

(3.3) *Proof of Theorem A.* (ii) of Proposition 1 implies that

$$\langle \Psi, \xi\wedge\eta\rangle-\langle \Psi, \xi'\wedge\eta'\rangle=\text{constant on } M$$

if $[\xi, \eta]=[\xi', \eta']$. Since $\mathfrak{g}=[\mathfrak{g}, \mathfrak{g}]$ by the assumption, we can define a map $J: M\to\mathfrak{g}^*$ by

$$\langle J(m), [\xi, \eta]\rangle=-\langle \Psi(m)-\Psi(m_0), \xi\wedge\eta\rangle$$

where m_0 is any given point of M. Differentiating the above, we have

$$\begin{aligned} d\langle J, [\xi, \eta]\rangle &= -d\langle \Psi, \xi\wedge\eta\rangle \\ &= \rho[\xi, \eta] \lrcorner\, \Omega. \end{aligned}$$

Therefore J is a momentum mapping.

(3.4) *Proof of Theorem D.* G_1 and G_2 have coadjoint equivariant momentum mappings, say J_1 and J_2. It is easily seen that $J=J_1+J_2$ is a momentum mapping for $G=G_1 \rtimes_{\sigma} G_2$. To prove the coadjoint equivariancy of J, it suffices only to show

$$\langle J(gm)-Ad(g^{-1})^*J(m),\ \eta\rangle=0$$

for two cases (i) $g\in G_2$ and $\eta\in\mathfrak{g}_1$ or (ii) $g\in G_1$ and $\eta\in\mathfrak{g}_2$. For the case (i), since $H^1(\mathfrak{g}_1, R)=0$, we may assume $\eta=[\eta_1, \eta_2]$ where η_1 and η_2 are in $\mathfrak{g}_1$. Then we have

$$\begin{aligned}&\langle J(gm)-Ad(g^{-1})^*J(m),\ [\eta_1,\ \eta_2]\rangle\\ &=-\langle\Psi(gm)-Ad(g^{-1})^*\Psi(m),\ \eta_1\wedge\eta_2\rangle\\ &=0\end{aligned}$$

because of the coadjoint equivariancy of Ψ.

For the case (ii), consider the mapping

$$G_1\ni g\mapsto\langle J(gm)-Ad(g^{-1})^*J(m),\ \eta\rangle\,,$$

Differentiating this mapping to the direction $\xi\in\mathfrak{g}_1$, we have

$$\begin{aligned}&\rho(\xi)\lrcorner\, d\langle J,\ \eta\rangle-\langle J,\ ad(\xi)\eta\rangle\\ &=\rho(\xi)\lrcorner\,\rho(\eta)\lrcorner\,\Omega-\langle J,\ [\xi,\ \eta]\rangle\\ &=\Omega(\rho(\eta),\ \rho(\xi))+\langle\Psi,\ \xi\wedge\eta\rangle\\ &=0\,.\end{aligned}$$

Thus $\langle J(gm)-Ad(g^{-1})^*J(m),\ \eta\rangle$ is constant on G_1 if G_1 is connected and is 0 identically. This completes another proof of Theorem D.

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