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# A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS $O_n^2$ (IV)

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## § 0. Introduction.

This is exactly a continuation of Part (III) ([13]) with the same title written by the present author. We shall use the same notation in it.

The period $T$ of any non-constant solution of $x(t)$ of the non-linear differential equation of order 2:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$\text{(0.1)} \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

We shall show in the present work that the following conjecture is true for $n\geqq 84$.

Conjecture C. The period function $T$ as a function of $\tau=(x_1-1)/(n-1)$ and $n$ is monotone decreasing with respect to $n(\geqq 2)$ for any fixed $\tau$ $(0<\tau<1)$.

## § 1. Fundamental formulas.

Denoting $T$ by $\Omega(\tau, n)$, we have by (7.4) and Proposition 3 in [13] the following formula:

$$\text{(1.1)} \qquad \frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c/n}}{2b^2n}\int_{x_0}^{1}\frac{(1-x)\sqrt{x(n-x)^{n-1}-c}}{x^2(n-x)^n}V(x, x_1)dx,$$

where $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ and $V(x, x_1)$ is defined as follows: By (2.10), (4.2), (1.4), (1.5), (7.1), (7.7) and (7.10) in [13], respectively,

$$\text{(1.2)} \qquad f_0(z):=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\},$$

$$\text{(1.3)} \qquad F_2(z):=-\{(2n+1)z^2-2(2n^2+5n-4)z+16n^2-16n+3\}B$$
$$+(n-z)^{n-1}\{-(n-1)z^3+(2n^2-7n+8)z^2+(n-3)(4n-1)z+3n(2n-1)\},$$

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(1.4) $$\lambda(z):=\log(n-z)+\frac{n-1}{n-z}, \qquad \psi(z):=z(n-z)^{n-1},$$

(1.5) $$\tilde{\lambda}(z):=\lambda(z)-\frac{n}{n-1}\cdot\frac{(z-1)^2}{z(n-z)}=\log(n-z)+\frac{nz-1}{(n-1)z},$$

(1.6) $$N(z, x_1):=(n-z)F_2(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}+3(z-1)^2 f_0(z)$$
$$-2n(z-1)^3\{B-z(n-z)^{n-1}\},$$

(1.7) $$V(x, x_1):=\frac{x^2N(x, x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X, x_1)}{(X-1)^5\sqrt{n-X}},$$

where $X=X_n(x)$, $0<x<1<X<n$, defined by

(1.8) $$\psi(x)=\psi(X).$$

We know the following facts about $V(x, x_1)$ by Lemma 8.1 in [13].

(i) $\lim_{x_1\to n-0} V(x, x_1)=+\infty$ for $0<x<1$;

(ii) $V(x, x_1)$ is increasing with respect to $x_1$ for each $x$ $(0<x<1)$;

(iii) When $n>2$, $\lim_{x\to+0} V(x, X(x))=0$ and furthermore when $n>(5+\sqrt{13})/4\doteqdot 2.15$, $V(x, X(x))>0$ near $x=0$;

(iv) When $n>2$, $\lim_{x\to 1-0} V(x, X(x))=0$ and furthermore when $n>(1+\sqrt{13})/2\doteqdot 2.30$, $V(x, X(x))>0$ near $x=1$.

From these facts, we obtain $V(x, x_1)>0$ for $0<x<1$, $X(x)<x_1<n$ if we can prove

$$V(x, X(x))\geqq 0 \qquad \text{for } 0<x<1.$$

By (8.2) and (8.3) in [13], setting for $0<x<n$, $x\neq 1$

(1.9) $$U_1(x):=\frac{3x^2 f_0(x)}{(x-1)^3\sqrt{n-x}},$$

(1.10) $$U_2(x):=\frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}},$$

(1.11) $$U_3(x):=\frac{nxF_2(x)}{(n-1)(x-1)^3\sqrt{n-x}}$$

and for $0<x<1$

(1.12) $$U_0(x):=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\},$$

(1.13) $$U_4(x):=U_3(X(x)), \quad U_5(x):=U_1(X(x)), \quad U_6(x):=U_2(X(x)),$$

we have from (8.1) in [13] the expression of $V(x, X(x))$ as follows:

$$V(x, X(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x), \tag{1.14}$$

in which $U_i(x)$, $i=0, 1, 2, 4, 5, 6$, are all positive for $0<x<1$.

## § 2. Certain auxiliary inequalities (I).

PROPOSITION 1. *When $n>2$, we have $n-(n-1)x^{1/(n-1)}<X(x)$ for $0<x<1$.*

*Proof*. Setting $Y=Y(x):=n-(n-1)x^{1/(n-1)}$, we have

$$Y(1)=1=X(1), \qquad Y(0)=n=X(0).$$

Setting $x=1-t$ near $x=1$, we have

$$\begin{aligned}Y(1-t)&=n-(n-1)(1-t)^{1/(n-1)}\\&=n-(n-1)\left\{1-\frac{t}{n-1}-\frac{(n-2)t^2}{2(n-1)^2}-\frac{(n-2)(2n-3)t^3}{6(n-1)^3}-\cdots\right\}\\&=1+t+\frac{(n-2)t^2}{2(n-1)}+\frac{(n-2)(2n-3)t^3}{6(n-1)^2}+\cdots\end{aligned}$$

and (8.12) in [13]

$$X(1-t)=1+t+\frac{2(n-2)t^2}{3(n-1)}+\cdots,$$

which implies $X(1-t)>Y(1-t)$ for sufficiently small $t$.

Now, we suppose that the above inequality is not true for the interval $0<x<1$ and let $\xi$ be the maximum value of $x$ such that $X(x)=Y(x)$, i.e.

$$X(\xi)=Y(\xi) \quad \text{and} \quad X(x)>Y(x) \qquad \text{for } \xi<x<1.$$

Then, we must have

$$0>\frac{dX}{dx}\geqq\frac{dY}{dx} \qquad \text{at } x=\xi.$$

Since we have

$$\frac{dX}{dx}=\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}, \qquad \frac{dY}{dx}=-x^{1/(n-1)-1},$$

we obtain at $x=\xi$

$$\begin{aligned}\frac{dX}{dx}&=\frac{1-\xi}{\xi(n-\xi)}\cdot\frac{Y(\xi)(n-Y(\xi))}{1-Y(\xi)}=-\frac{(1-\xi)Y(\xi)}{\xi(n-\xi)}\cdot\frac{(n-1)\xi^{1/(n-1)}}{(n-1)(1-\xi^{1/(n-1)})}\\&=\frac{(1-\xi)Y(\xi)}{(n-\xi)(1-\xi^{1/(n-1)})}\frac{dY}{dx}.\end{aligned}$$

Hence it must be

$$\frac{(1-\xi)\{n-(n-1)\xi^{1/(n-1)}\}}{(n-\xi)(1-\xi^{1/(n-1)})}\leqq 1\,. \tag{2.1}$$

On the other hand, we shall show that

$$\frac{(1-x)\{n-(n-1)x^{1/(n-1)}\}}{(n-x)(1-x^{1/(n-1)})}>1 \qquad \text{for } 0<x<1\,, \tag{2.2}$$

which contradicts to (2.1). (2.2) is equivalent to

$$(n-1)x^{(n-2)/(n-1)}<1+(n-2)x \qquad \text{for } 0<x<1\,.$$

Since we have

$$((n-1)x^{(n-2)/(n-1)})'=(n-2)x^{-1/(n-1)}>0,\ ((n-1)x^{(n-2)/(n-1)})''=-\frac{n-2}{n-1}x^{-n/(n-1)}<0\,,$$

the above inequality is clear. Q. E. D.

Proposition 1 implies immediately the following

LEMMA 2.1. *When $n>2$, we have*

$$x=X_n^{-1}(X)>\left(\frac{n-X}{n-1}\right)^{n-1} \qquad \text{for } 1<X<n\,. \tag{2.3}$$

LEMMA 2.2. *When $n>2$, we have*

$$\frac{x^2}{(1-x)^2\sqrt{n-x}}>\frac{\sqrt{B}(n-X)^{2(n-1)}}{\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}} \qquad \textit{for } 0<x<1.$$

*Proof.* Since we have for $0<x<n,\ x\neq 1$

$$\frac{d}{dx}\frac{x^2}{(x-1)^2\sqrt{n-x}}=\frac{x(x^2+3x-4n)}{2(x-1)^3(n-x)^{3/2}}$$

and $x^2+3x-4n<0$ for $0<x<1$, the function $x^2/(x-1)^2\sqrt{n-x}$ is increasing in the interval $0<x<1$. Therefore, by Lemma 2.1 we obtain

$$\begin{aligned}\frac{x^2}{(1-x)^2\sqrt{n-x}}&>\left[\frac{t^2}{(1-t)^2\sqrt{n-t}}\right]_{t=(n-X/n-1)^{n-1}}\\&=\frac{(n-X)^{2n-2}}{B^2}\Big/\left\{1-\frac{(n-X)^{n-1}}{B}\right\}^2\sqrt{n-\frac{(n-X)^{n-1}}{B}}\\&=\frac{\sqrt{B}(n-X)^{2(n-1)}}{\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}}\,.\end{aligned}$$

Q. E. D.

LEMMA 2.3. *When $n>2$, we have*

$$\frac{X^2}{(X-1)^2\sqrt{n-X}}-\frac{x^2}{(1-x)^2\sqrt{n-x}}<\frac{X^2\{1-\nu(X)\}}{(X-1)^2\sqrt{n-X}} \qquad \text{for } 0<x<1\,,$$

*where*

$$\nu(X):=\frac{\sqrt{B}(X-1)^2(n-X)^{2n-3/2}}{X^2\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}} \qquad \textit{for } 1<X<n. \tag{2.4}$$

*Proof.* From Lemma 2.2 we obtain immediately

$$\frac{X^2}{(X-1)^2\sqrt{n-X}}-\frac{x^2}{(1-x)^2\sqrt{n-x}}$$

$$<\frac{X^2}{(X-1)^2\sqrt{n-X}}-\frac{\sqrt{B}(n-X)^{2(n-1)}}{\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}}$$

$$=\frac{X^2}{(X-1)^2\sqrt{n-X}}\left\{1-\frac{\sqrt{B}(n-X)^{2n-3/2}}{X^2\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}}\right\}. \quad \text{Q.E.D.}$$

LEMMA 2.4. *When $n>2$, we have*

$$U_5(x)-U_1(x)>\frac{3nXf_0(X)}{(n-1)(X-1)^2\sqrt{n-X}} \qquad \textit{for } 0<x<1.$$

*Proof.* From (1.9) and (1.13) we obtain

$$U_5(x)-U_1(x)=\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}-\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}$$

$$=\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}-\frac{3x}{n-x}\cdot\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}$$

and by means of Proposition 2 in [12]

$$<\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}-\frac{3x}{n-x}\cdot\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}$$

$$=\frac{3X\sqrt{n-X}\,f_0(X)}{(X-1)^3}\cdot\left(\frac{X}{n-X}-\frac{x}{n-x}\right)=\frac{3nX\sqrt{n-X}\,f_0(X)}{(X-1)^3}\cdot\left(\frac{1}{n-X}-\frac{1}{n-x}\right)$$

$$>\frac{3nX\sqrt{n-X}\,f_0(X)}{(X-1)^3}\left(\frac{1}{n-X}-\frac{1}{n-1}\right)=\frac{3nXf_0(X)}{(n-1)(X-1)^2\sqrt{n-X}} \qquad \text{for } 0<x<1.$$

Q. E. D.

LEMMA 2.5. *When $n>2$, we have*

$$-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x) \tag{2.5}$$

$$>\frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\cdot\{G_2(X)+3(X-1)f_0(X)\} \qquad \textit{for } 0<x<1,$$

*where*

$$G_2(x):=F_2(x)-2(n-1)x(x-1)\{B-\phi(x)\}\{1-\nu(x)\}, \tag{2.6}$$

*and when* $n>(13+\sqrt{153})/2\doteqdot 12.685$, $G_2(x)+3(x-1)f_0(x)$ *is positive for* $x>1$ *sufficiently near* $x=1$.

*Proof.* From (1.9), (1.10), (1.11) and (1.13) and by Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
&-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)\\
&=U_4(x)-\{U_6(x)-U_2(x)\}+\{U_5(x)-U_1(x)\}\\
&=\frac{nXF_2(X)}{(n-1)(X-1)^3\sqrt{n-X}}-2n\{B-\phi(X)\}\left\{\frac{X^2}{(X-1)^2\sqrt{n-X}}-\frac{x^2}{(1-x)^2\sqrt{n-x}}\right\}\\
&\quad+\{U_5(x)-U_1(x)\}>\frac{nXF_2(X)}{(n-1)(X-1)^3\sqrt{n-X}}-2n\{B-\phi(X)\}\frac{X^2\{1-\nu(X)\}}{(X-1)^2\sqrt{n-X}}\\
&\quad+\frac{3nXf_0(X)}{(n-1)(X-1)^2\sqrt{n-X}}=\frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\\
&\quad\cdot\{F_2(X)-2(n-1)X(X-1)\cdot\{B-\phi(X)\}\{1-\nu(X)\}+3(X-1)f_0(X)\}\\
&=\frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\cdot\{G_2(X)+3(X-1)f_0(X)\}.
\end{aligned}$$

Now, setting $x=1+t$, we obtain from (8.14) in [13]

$$F_2(1+t)=t^4\left\{\frac{n(n^2-n+1)B}{6(n-1)}+O(t)\right\}. \tag{2.7}$$

Since we have

$$\begin{aligned}
&\phi(x)=x(n-x)^{n-1}, \qquad \phi'(x)=-n(x-1)(n-x)^{n-2},\\
&\phi''(x)=n(n-1)(x-2)(n-x)^{n-3},\\
&\phi'''(x)=-n(n-1)(n-2)(x-3)(n-x)^{n-4},
\end{aligned} \tag{2.8}$$

we obtain immediately

$$\phi(1+t)=B-\frac{nB}{2(n-1)}t^2+\frac{n(n-2)B}{3(n-1)^2}t^3+\cdots. \tag{2.9}$$

Next, since we have easily

$$(n-1-t)^{n-1}=B-Bt+\frac{(n-2)B}{2(n-1)}t^2+\cdots,$$

and

$$(n-1-t)^{2n-3/2}=(n-1)^{2n-3/2}-\left(2n-\frac{3}{2}\right)(n-1)^{2n-5/2}t+$$

$$+\frac{1}{2}\left(2n-\frac{3}{2}\right)\left(2n-\frac{5}{2}\right)(n-1)^{2n-7/2}t^2+\cdots$$

$$=(n-1)^{2n-3/2}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{(4n-3)(4n-5)}{8(n-1)^2}t^2+\cdots,\right.$$

we obtain from (2.4)

$$\nu(1+t)=\sqrt{B}\,t^2(1+t)^{-2}\cdot(n-1)^{2n-3/2}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{(4n-3)(4n-5)}{8(n-1)^2}t^2\cdots\right\}$$

$$\times B^{-2}t^{-2}\left\{1-\frac{n-2}{2(n-1)}t+\cdots\right\}^{-2}$$

$$\times B^{-1/2}\left\{n-1+t-\frac{n-2}{2(n-1)}t^2+\cdots\right\}^{-1/2}$$

$$=1-\frac{3n-1}{n-1}t+\cdots.$$

Thus, we obtain

$$G_2(1+t)=F_2(1+t)-2(n-1)(1+t)t\{B-\phi(1+t)\}\{1-\nu(1+t)\}$$

$$=Bt^4\left\{\frac{n(n^2-n+1)}{6(n-1)}+O(t)\right\}$$

$$-2(n-1)(1+t)t\cdot\frac{nBt^2}{2(n-1)}\left\{1-\frac{2(n-2)}{3(n-1)}t+\cdots\right\}\cdot\frac{(3n-1)t}{n-1}\{1+\cdots\}$$

$$=Bt^4\left\{\frac{n(n^2-19n+7)}{6(n-1)}+O(t)\right\}$$

and by (8.17) in [13]

$$G_2(1+t)+3tf_0(1+t)=Bt^4\left\{\frac{n(n^2-19n+7)}{6(n-1)}+O(t)\right\}$$

$$+Bt^4\left\{\frac{n(2n-1)}{2(n-1)}+O(t)\right\}=Bt^4\left\{\frac{n(n^2-13n+4)}{6(n-1)}+O(t)\right\}.$$

Since $n^2-13n+4>0$ for $n>(13+\sqrt{153})/2\doteqdot 12.685$, we obtain $G_2(x)+3(x-1)f_0(x)$ is positive for $x>1$ sufficiently near $x=1$. Q.E.D.

## § 3. Certain auxiliary inequalities (II).

In this section, we shall investigate the sign of the function $G_2(x)+3(x-1)f_0(x)$ for the interval $2\leqq x<n$.

LEMMA 3.1. *When* $n\geqq 6$, $G_2(x)+3(x-1)f_0(x)$ *is positive for* $2\leqq x<n$.

*Proof.* By (2.6) we see that $G_2(x)+3(x-1)f_0(x)$ is positive for $1<x<n$ if and only if

$$(3.1)\qquad 2(n-1)x(x-1)\{B-\psi(x)\}\nu(x)>2(n-1)x(x-1)\{B-\psi(x)\}$$
$$-F_2(x)-3(x-1)f_0(x)\qquad \text{for } 1<x<n\,.$$

Now, we rewrite the right hand side of (3.1) as follows.

$$\begin{aligned}
&2(n-1)x(x-1)\{B-\psi(x)\}-F_2(x)-3(x-1)f_0(x)\\
&=2(n-1)x(x-1)\{B-x(n-x)^{n-1}\}\\
&\quad+\{(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\}B\\
&\quad-(n-x)^{n-1}\{-(n-1)x^3+(2n^2-7n+8)x^2+(4n^2-13n+3)x+3n(2n-1)\}\\
&\quad-3(x-1)(2n-1-x)B+3(n-x)^{n-1}(x-1)\{n-x+(n-1)x^2\}\\
&=-2(n-x)\{(2n+1)x-(8n-5)\}B-2(n-x)^n\{(n-1)x^2+(2n-5)x+3n\}\,.
\end{aligned}$$

Hence, (3.1) is equivalent to

$$(3.2)\qquad (n-1)(x-1)^3(n-x)^{2n-5/2}\cdot\frac{\{B-x(n-x)^{n-1}\}\sqrt{B}}{x\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}}$$
$$>-\{(2n+1)x-(8n-5)\}B-(n-x)^{n-1}\{(n-1)x^2+(2n-5)x+3n\}$$
$$\text{for } 1<x<n\,.$$

We see easily that the left hand side is positive for $1<x<n$ and tends to zero as $x\to 1+0$ and the right hand side becomes zero at $x=1$.

On the right hand side we see that

$$(2n+1)x-(8n-5)\geqq 0\qquad \text{for } x\geqq\frac{8n-5}{2n+1}=4-\frac{9}{2n+1}$$

and

$$(n-1)x^2+(2n-5)x+3n>0\qquad \text{for } -\infty<x<+\infty\,,$$

when $n>2$. Hence, the right hand side is negative for $x\geqq(8n-5)/(2n+1)$. Therefore (3.2) holds for $(8n-5)/(2n+1)\leqq x<n$.

Next, we investigate (3.2) for $1<x<(8n-5)/(2n+1)$. The condition that the right hand side of (3.2) is non positive is equivalent to

$$(3.3)\quad \rho(x):=\frac{(n-x)^{n-1}\{(n-1)x^2+(2n-5)x+3n\}}{8n-5-(2n+1)x}\geqq B \qquad \text{for } 1<x<\frac{8n-5}{2n+1}.$$

We see easily that $\rho(1)=B$ and have

$$(3.4)\qquad \frac{d}{dx}\rho(x)=\frac{n(n-1)(x-1)^2(n-x)^{n-2}\{(2n+1)x-2(n+5)\}}{\{8n-5-(2n+1)x\}^2}.$$

Hence, we see that $\rho(x)$ is increasing in the interval

$$\frac{2(n+5)}{2n+1}<x<\frac{8n-5}{2n+1},$$

when $n>5/2$. Furthermore, we have $2(n+5)/(2n+1)\leqq 2$ if and only if $n\geqq 4$ and we have $(8n-5)/(2n+1)\geqq 2$ if and only if $n\geqq 7/4$.

In the following we suppose $n\geqq 4$, then $\rho(x)$ is increasing in the interval $2\leqq x<(8n-5)/(2n+1)$.

Finally we estimate the value $\rho(2)$. From the definition of $\rho(x)$ we have

$$\rho(2)=\frac{(n-2)^{n-1}(11n-14)}{4n-7}$$

and

$$(3.5)\qquad \frac{\rho(2)}{B}=\left(\frac{n-2}{n-1}\right)^{n-1}\cdot\frac{11n-14}{4n-7}>1 \qquad \text{for } n\geqq 6,$$

which will be proved in the next lemma. Then, we obtain

$$\rho(x)\geqq\rho(2)>B \qquad \text{for } 2\leqq x<\frac{8n-5}{2n+1},$$

and we see that the present lemma is true. Q.E.D.

LEMMA 3.2. *We have*

$$\left(\frac{n-2}{n-1}\right)^{n-1}\cdot\frac{11n-14}{4n-7}>1 \qquad \textit{for } n\geqq 6.$$

*Proof.* First of all, we show that the left hand side of the above inequality is increasing for $n>2$ as a function of $n$. We have

$$\frac{d}{dn}\log\left\{\left(\frac{n-2}{n-1}\right)^{n-1}\cdot\frac{11n-14}{4n-7}\right\}=\log\frac{n-2}{n-1}+\frac{1}{n-2}-\frac{21}{(11n-14)(4n-7)}.$$

Setting $t=1/(n-1)$, the last expression of $n$ can be written as

$$\log(1-t)+\frac{t}{1-t}-\frac{21}{44}\cdot t^2/\left(1-\frac{3t}{11}\right)\left(1-\frac{3t}{4}\right).$$

Since $0<t<1$, the above function of $t$ can be written in a power series of $t$ as

$$=-\left(t+\frac{1}{2}t^2+\frac{1}{3}t^3+\cdots+\frac{1}{m}t^m+\cdots\right)$$

$$+(t+t^2+t^3+\cdots+t^m+\cdots)$$

$$-\frac{21}{44}t^2\left\{1+\frac{3}{11}t+\left(\frac{3}{11}\right)^2t^2+\cdots+\left(\frac{3}{11}\right)^m t^m+\cdots\right\}$$

$$\times\left\{1+\frac{3}{4}t+\left(\frac{3}{4}\right)^2t^2+\cdots+\left(\frac{3}{4}\right)^m t^m+\cdots\right\}$$

$$=\left(\frac{1}{2}t^2+\frac{2}{3}t^3+\frac{3}{4}t^4+\cdots+\frac{m+1}{m+2}t^{m+2}+\cdots\right)-\frac{21}{44}t^2\sum_{m=0}^{\infty}a_m t^{m+2},$$

where

$$a_m=3^m\left(\frac{1}{4^m}+\frac{1}{4^{m-1}\cdot 11}+\cdots+\frac{1}{4\cdot 11^{m-1}}+\frac{1}{11^m}\right),\qquad m=0,1,2,\cdots.$$

It the following we shall prove $(m+1)/(m+2)>(21/44)\,a_m$, $m=0,1,2,\cdots$. In fact, we have

$$\frac{1}{2}-\frac{21}{44}a_0=\frac{1}{2}-\frac{21}{44}=\frac{1}{44}>0,$$

$$\frac{2}{3}-\frac{21}{44}a_1=\frac{2}{3}-\frac{21}{44}\cdot\frac{3\cdot 15}{44}=\frac{3872-2835}{3\cdot 44\cdot 44}=\frac{1037}{3\cdot 44\cdot 44}>0,$$

and for $m\geqq 2$

$$\frac{m+1}{m+2}-\frac{21}{44}a_m>\frac{m+1}{m+2}-\frac{1}{2}a_m>\frac{m+1}{m+2}-\frac{1}{2}\cdot\left(\frac{3}{4}\right)^m\left(1+\frac{1}{2}+\cdots+\frac{1}{2^m}\right)$$

$$>\frac{m+1}{m+2}-\left(\frac{3}{4}\right)^m=\frac{m+1}{m+2}\cdot\left(\frac{3}{4}\right)^m\left\{\left(\frac{4}{3}\right)^m-\frac{m+2}{m+1}\right\}$$

$$>\frac{m+1}{m+2}\cdot\left(\frac{3}{4}\right)^m\left\{1+\frac{m}{3}-\frac{m+2}{m+1}\right\}=\left(\frac{3}{4}\right)^m\cdot\frac{m^2+m-3}{3(m+2)}>0.$$

Thus we have proved

$$\log(1-t)+\frac{t}{1-t}-\frac{21}{44}\cdot t^2\Big/\left(1-\frac{3t}{11}\right)\left(1-\frac{3t}{4}\right)>0\qquad\text{for } 0<t<1$$

and so $((n-2)/(n-1))^{n-1}\cdot(11n-14)/(4n-7)$ is increasing with respect $n$ for $n>2$.

Finally we have

$$\left[\left(\frac{n-2}{n-1}\right)^{n-1}\cdot\frac{11n-14}{4n-7}\right]_{n=6}=\left(\frac{4}{5}\right)^5\cdot\frac{52}{17}=\frac{53248}{53125}>1$$

and

$$\left[\left(\frac{n-2}{n-1}\right)^{n-1}\cdot\frac{11n-14}{4n-7}\right]_{n=5}=\left(\frac{3}{4}\right)^{4}\cdot\frac{41}{13}=\frac{3321}{3328}<1\,.$$

From these we see that this lemma is true. Q. E. D.

## §4. A property of the auxiliary function $\sigma(x)$.

Using the arguments in the proof of Lemma 3.1, when $n>2$, the condition $G_2(x)+3(x-1)f_0(x)>0$ for $1<x\leqq 2$ is equivalent to

$$\sigma(x)>B-\rho(x) \qquad \text{for } 1<x\leqq 2\,, \tag{4.1}$$

where

$$\sigma(x):=\frac{(n-1)\sqrt{B}(x-1)^3(n-x)^{n-5/2}\{B-x(n-x)^{n-1}\}}{x\{8n-5-(2n+1)x\}\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}}\,. \tag{4.2}$$

In this section, we shall prove that $\sigma(x)$ is monotone increasing in the interval $1<x\leqq 2$, when $n\geqq 6$.

LEMMA 4.1. *The function $(x-1)(n-x)^{n-2}/x\{8n-5-(2n+1)x\}$ is monotone increasing in the interval $1\leqq x\leqq 2$, when $n\geqq 3$.*

*Proof.* We have

$$\frac{d}{dx}\frac{(x-1)(n-x)^{n-2}}{x\{8n-5-(2n+1)x\}}=\frac{(n-x)^{n-3}}{x^2\{8n-5-(2n+1)x\}^2}$$

$$\times\{n(8n-5)+(4n^2-31n+15)x-(8n^2-29n+6)x^2+(n-3)(2n+1)x^3\}\,,$$

the last factor of which is positive for $1<x<2$. In fact, its derivative with respect to $x$ is the quadratic polynomial of $x$:

$$4n^2-31n+15-2(8n^2-29n+6)x+3(n-3)(2n+1)x^2\,,$$

whose values at $x=1$ and $x=2$ are $-6(n-1)^2<0$ and $-4n^2+25n-45<0$, respectively, and so whose value is negative for $1\leqq x\leqq 2$. Therefore, the above cubic function of $x$ is decreasing in the interval $1\leqq x\leqq 2$. Since, its value at $x=2$ is $9(n-2)>0$, it must be positive for $1\leqq x\leqq 2$. Thus, we have proved that the given function in this lemma is increasing in the interval $1\leqq x\leqq 2$, when $n\geqq 3$. Q. E. D.

LEMMA 4.2. *The function $(n-x)^{n-1/2}\{B-x(n-x)^{n-1}\}$ is monotone increasing in the interval $1\leqq x\leqq 2$, when $n\geqq 6$.*

*Proof.* For $0<x<n$, the derivative of this function is

$$\frac{1}{2}(n-x)^{n-3/2}[\{(4n-1)x-2n\}(n-x)^{n-1}-(2n-1)B],$$

the expression in the brackets of which is positive for $1<x\leqq 2$ as is shown in the following. Since we have

$$\frac{d}{dx}\{(4n-1)x-2n\}(n-x)^{n-1}=n(n-x)^{n-2}\{3(2n-1)-(4n-1)x\}$$

and

$$1<\frac{3(2n-1)}{4n-1}<2,$$

$\{(4n-1)x-2n\}(n-x)^{n-1}$ takes its minimum in the interval $1\leqq x\leqq 2$ at $x=1$ or $x=2$. And its value at $x=1$ and $x=2$ are $(2n-1)B$ and $2(3n-1)(n-2)^{n-1}$, respectively. Furthermore, we have

$$2(3n-1)(n-2)^{n-1}-(2n-1)B=(2n-1)(n-2)^{n-1}\left\{\frac{2(3n-1)}{2n-1}-\left(\frac{n-1}{n-2}\right)^{n-1}\right\},$$

and we can prove the function $((n-1)/(n-2))^{n-1}$ of $n$ is decreasing for $n>2$.

In fact, we have

$$\frac{d}{dn}\log\left(\frac{n-1}{n-2}\right)^{n-1}=\log\frac{n-1}{n-2}+(n-1)\left(\frac{1}{n-1}-\frac{1}{n-2}\right)$$

$$=-\log\left(1-\frac{1}{n-1}\right)+1-1/\left(1-\frac{1}{n-1}\right)$$

$$=\left\{\frac{1}{n-1}+\frac{1}{2(n-1)^2}+\frac{1}{3(n-1)^3}+\cdots\right\}-\left\{\frac{1}{n-1}+\frac{1}{(n-1)^2}+\frac{1}{(n-1)^3}+\cdots\right\}<0.$$

Using this fact, we have for $n\geqq 7$

$$\frac{2(3n-1)}{2n-1}-\left(\frac{n-1}{n-2}\right)^{n-1}>3-\left(\frac{6}{5}\right)^6=3-\frac{46656}{15625}=\frac{219}{15625}>0$$

and for $6\leqq n<7$

$$\frac{2(3n-1)}{2n-1}-\left(\frac{n-1}{n-2}\right)^{n-1}>\frac{40}{13}-\left(\frac{5}{4}\right)^5=\frac{40}{13}-\frac{3125}{1024}=\frac{335}{13312}>0,$$

and

$$\left[\frac{2(3n-1)}{2n-1}-\left(\frac{n-1}{n-2}\right)^{n-1}\right]_{n=5}=\frac{28}{9}-\left(\frac{4}{3}\right)^4=-\frac{4}{81}<0.$$

Thus, we have proved that

$$\{(4n-1)x-2n\}(n-x)^{n-1}>(2n-1)B \qquad \text{for } 1<x<2,$$

when $n\geqq 6$, which implies this lemma. Q. E. D.

LEMMA 4.3. *The function* $(B-(n-x)^{n-1})/(x-1)$ *is monotone decreasing in the interval* $1<x<n$, *when* $n>2$.

*Proof.* For $1<x<n$, the derivative of this function is

$$\frac{(n-x)^{n-2}\{(n-2)x+1\}-B}{(x-1)^2}.$$

Since we have

$$\frac{d}{dx}(n-x)^{n-2}\{(n-2)x+1\}=-(n-1)(n-2)(n-x)^{n-3}<0$$

for $1<x<n$, we obtain

$$(n-x)^{n-2}\{(n-2)x+1\}-B<(n-1)^{n-1}-B=0 \qquad \text{for } 1<x<n,$$

which implies this lemma. Q. E. D.

LEMMA 4.4. *The function* $\{(B-(n-x)^{n-1})/(x-1)\}\cdot\{nB-(n-x)^{n-1}\}$ *is monotone decreasing in the interval* $1<x<n$, *when* $n\geqq 4$.

*Proof.* For $1<x<n$, the derivative of this function is $p(x)/(x-1)^2$, where

$$p(x):=\{(n-x)^{n-2}((n-2)x+1)-B\}\{nB-(n-x)^{n-1}\}$$
$$+(n-1)(x-1)(n-x)^{n-2}\{B-(n-x)^{n-1}\}.$$

Since we have

$$p'(x)=(n-1)(x-1)(n-x)^{n-3}\{2(2n-3)(n-x)^{n-1}-(n-2)(n+1)B\}$$

and for $1<x<n$

$$2(2n-3)(n-x)^{n-1}-(n-2)(n+1)B<[2(2n-3)(n-x)^{n-1}-(n-2)(n+1)B]_{x=1}$$
$$=-(n^2-5n+4)B=-(n-1)(n-4)B\leqq 0,$$

when $n\geqq 4$, we obtain $p'(x)<0$ for $1<x<n$, which implies this lemma. Q.E.D.

PROPOSITION 2. *The function* $\sigma(x)$ *defined by* (4.2) *is monotone increasing in the interval* $1<x<2$, *when* $n\geqq 6$.

*Proof.* First we factorize $\sigma(x)$ as

$$\sigma(x)=(n-1)\sqrt{B}\times\frac{(x-1)(n-x)^{n-2}}{x\{8n-5-(2n+1)\}}\times(n-x)^{n-1/2}\{B-x(n-x)^{n-1}\}$$
$$\times\left\{\frac{B-(n-x)^{n-1}}{x-1}\right\}^{-3/2}\times\left\{\frac{B-(n-x)^{n-1}}{x-1}\cdot\{nB-(n-x)^{n-1}\}\right\}^{-1/2}.$$

By means of Lemma 4.1-Lemma 4.4, we see that $\sigma(x)$ is increasing in the interval $1<x<2$, when $n\geqq 6$. Q. E. D.

## § 5. The inequality $\sigma(x)>B-\rho(x)$.

In this section we shall prove that $G_2(x)+3(x-1)f_0(x)$ is positive for $(2n+10)/(2n+1)\leqq x<2$.

LEMMA 5.1. *We have*

$$\rho(x)-B>-\frac{n(n+2)B(x-1)^3\{2n+13-(2n+1)x\}}{36(n-1)(2n+1)(2n-5)} \quad \textit{for } 1<x\leqq\frac{2n+10}{2n+1}, \tag{5.1}$$

*when* $n\geqq 3$.

*Proof*. When $n\geqq 3$, from (3.3) and (3.4) we have

$$\begin{aligned}\rho(x)-B&=\rho(x)-\rho(1)=\int_1^x\rho'(t)dt\\ &=-n(n-1)\int_1^x\frac{(t-1)^2(n-t)^{n-2}\{2n+10-(2n+1)t\}\,dt}{\{8n-5-(2n+1)t\}^2}\\ &>-n(n-1)^{n-2}\int_1^x\frac{n-t}{\{8n-5-(2n+1)t\}^2}\cdot(t-1)^2\{2n+10-(2n+1)t\}\,dt\end{aligned}$$

for $1<x<(2n+10)/(2n+1)$. In this interval the function $(n-x)/\{8n-5-(2n+1)x\}^2$ is increasing, because

$$\frac{d}{dx}\frac{n-x}{\{8n-5-(2n+1)x\}^2}=\frac{4n^2-6n+5-(2n+1)x}{\{8n-5-(2n+1)x\}^3}$$

and

$$\begin{aligned}4n^2-6n+5-(2n+1)x&\geqq 4n^2-6n+5-(2n+1)\cdot\frac{2n+10}{2n+1}\\ &=(2n+1)(2n-5)>0\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{n-t}{8n-5-(2n+1)t^2}&<\frac{n-\dfrac{2n+10}{2n+1}}{\{8n-5-(2n+10)\}^2}=\frac{2n^2-n-10}{9(2n+1)(2n-5)^2}\\ &=\frac{n+2}{9(2n+1)(2n-5)} \qquad \text{for } 1<t<x\leqq\frac{2n+10}{2n+1}\,.\end{aligned}$$

Using this inequality, we obtain

$$\rho(x)-B>-\frac{n(n+2)(n-1)^{n-2}}{9(2n+1)(2n-5)}\int_1^x(t-1)^2\{2n+10-(2n+1)t\}\,dt$$

$$=-\frac{n(n+2)B}{9(n-1)(2n+1)(2n-5)}\int_0^{x-1} t^2\{9-(2n+1)t\}\,dt$$

$$=-\frac{n(n+2)B(x-1)^3\{2n+13-(2n+1)x\}}{36(n-1)(2n+1)(2n-5)} \quad \text{for } 1<x\leqq\frac{2n+10}{2n+1}.$$

Q. E. D.

LEMMA 5.2. *We have*

$$\sigma(x)>\frac{n\sqrt{n-1}}{36B^2}\cdot\frac{n+2}{n+5}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{2n-13/2}(x-1)^3(3n-1-2x) \tag{5.2}$$

*for* $1<x\leqq(2n+10)/(2n+1)$, *when* $n\geqq 25/4$.

*Proof*. As in the proof of Proposition 2, we represent $\sigma(x)$ as a product:

$$\frac{\sigma(x)}{\sqrt{n-1}\,B}=\frac{(x-1)(n-x)^{n-2}}{x\{8n-5-(2n+1)x\}}\times(n-x)^{n-1/2}\{B-x(n-x)^{n-1}\}$$

$$\times\left\{\frac{B-(n-x)^{n-1}}{x-1}\right\}^{-3/2}\times\left\{\frac{B-(n-x)^{n-1}}{x-1}\cdot\{nB-(n-x)^{n-1}\}\right\}^{-1/2}.$$

First of all, we estimate the first factor. We see easily

$$\frac{2n+10}{2n+1}\leqq\frac{8n-5}{2(2n+1)} \qquad \text{when } n\geqq\frac{25}{4}.$$

Therefore, assuming $n\geqq 25/4$, we have

$$6(n-1)<x\{8n-5-(2n+1)x\}<\frac{2n+10}{2n+1}\cdot(6n-15)=\frac{6(n+5)(2n-5)}{2n+1}$$

for $1<x<(2n+10)/(2n+1)$ and hence

$$\frac{(x-1)(n-x)^{n-2}}{x\{8n-5-(2n+1)x\}}>\left(n-\frac{2n+10}{2n+1}\right)^{n-2}(x-1)\Big/\frac{6(n+5)(2n-5)}{2n+1}$$

i. e.

$$\frac{(x-1)(n-x)^{n-2}}{x\{8n-5-(2n+1)x\}}>\frac{(n+2)^{n-2}(2n-5)^{n-3}}{6(n+5)(2n+1)^{n-3}}\cdot(x-1) \quad \text{for } 1<x<\frac{2n+10}{2n+1}, \tag{5.3}$$

when $n\geqq 25/4$.

Second, we estimate the second factor as follows. Assuming $n\geqq 3$, we have for $1<x<(2n+10)/(2n+1)$

$$B-x(n-x)^{n-1}=\phi(1)-\phi(x)=\int_x^1\phi'(t)dt$$

$$=n\int_1^x(t-1)(n-t)^{n-2}dt=n\int_1^x(n-t)^{n-3}(t-1)(n-t)dt$$

$$>n\left(n-\frac{2n+10}{2n+1}\right)^{n-3}\int_1^x (t-1)(n-t)dt$$

$$=\frac{n}{6}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{n-3}(x-1)^2(3n-1-2x).$$

Hence, we obtain

$$(n-x)^{n-1/2}\{B-x(n-x)^{n-1}\}>\frac{n}{6}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{2n-7/2}(x-1)^2(3n-1-2x) \tag{5.4}$$

for $1<x<(2n+10)/(2n+1)$, when $n\geqq 3$.

Third, we estimate the third and fourth factors by Lemma 4.3 and Lemma 4.4 as follows:

$$\left\{\frac{B-(n-x)^{n-1}}{x-1}\right\}^{-3/2}>\{(n-1)^{n-1}\}^{-3/2}=B^{-3/2}$$

and

$$\left\{\frac{B-(n-x)^{n-1}}{x-1}\cdot\{nB-(n-x)^{n-1}\}\right\}^{-1/2}>\{B\cdot(n-1)B\}^{-1/2}=(n-1)^{-1/2}B^{-1}$$

for $1<x<n$, when $n\geqq 4$.

Thus, by means of (5.3), (5.4) and the above two inequalities we obtain

$$\sigma(x)>\sqrt{n-1}\sqrt{B}\times\frac{(n+2)^{n-2}(2n-5)^{n-3}(x-1)}{6(n+5)(2n+1)^{n-3}}$$

$$\times\frac{n}{6}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{2n-7/2}(x-1)^2(3n-1-2x)\times B^{-3/2}\times(n-1)^{-1/2}B^{-1}$$

$$=\frac{n\sqrt{n-1}}{36B^2}\cdot\frac{n+2}{n+5}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{3n-13/2}(x-1)^3(3n-1-2x)$$

for $1<x<(2n+10)/(2n+1)$, when $n\geqq 25/4$. Q. E. D.

LEMMA 5.3. *When $n\geqq 13$, we have $\sigma(x)>B-\rho(x)$ for $(2n+10)/(2n+1)\leqq x<2$.*

*Proof.* $\sigma(x)$ is positive for $1<x<2$ by Proposition 2, when $n\geqq 6$. And $\rho(x)-B$ is negative for $1<x<(2n+10)/(2n+1)$ by (3.4), when $n\geqq 5/2$.

Assuming $n\geqq 25/4$ in the following, by means of Lemma 5.1 and Lemma 5.2 we have

$$\frac{B-\rho(x)}{\sigma(x)}<\left[\frac{n(n+2)B(x-1)^3\{2n+13-(2n+1)x\}}{36(n-1)(2n+1)(2n-5)}\right]$$

$$\Big/\left[\frac{n\sqrt{n-1}}{36B^2}\cdot\frac{n+2}{n+5}\left\{\frac{(n+2)(2n-5)}{2n+1}\right\}^{3n-13/2}(x-1)^3(3n-1-2x)\right]$$

$$=\frac{B^3(n+5)}{(n-1)^{3/2}(2n+1)(2n-5)}\cdot\left\{\frac{2n+1}{(n+2)(2n-5)}\right\}^{3n-13/2}\cdot\frac{2n+13-(2n+1)x}{3n-1-2x},$$

i. e.

$$B-\rho(x) \over \sigma(x)} < \frac{(n+5)(n-1)^2}{(2n+1)(2n-5)} \cdot \left\{\frac{(n-1)(2n+1)}{(n+2)(2n-5)}\right\}^{3n-13/2} \cdot \frac{2n+13-(2n+1)x}{3n-1-2x} \tag{5.5}$$

for $1<x\leqq(2n+10)/(2n+1)$. Since we have $(2n+13-(2n+1)x)/(3n-1-2x)=(2n+1)/(2n^2-n-7)$ at $x=(2n+10)/(2n+1)$, we obtain from (5.5)

$$B-\rho\left(\frac{2n+10}{2n+1}\right)\Big/\sigma\left(\frac{2n+10}{2n+1}\right)<b_n c_n \tag{5.6}$$

where

$$b_n:=\frac{4(n+5)(n-1)}{(2n+1)(2n-5)}\cdot\left\{\frac{(n-1)(2n+1)}{(n+2)(2n-5)}\right\}^{3n-13/2}, \tag{5.7}$$

$$c_n:=\frac{2n^2-n-1}{4(2n^2-n-7)}=\frac{1}{4}\left(1+\frac{6}{2n^2-n-7}\right). \tag{5.8}$$

We see easily that

$$c_n>\frac{1}{4} \qquad \text{for } n>\frac{1+\sqrt{57}}{4}\doteqdot\frac{1+7.549}{4}=2.13725$$

and $c_n\downarrow 1/4$ as $n\to\infty$. We shall prove in the next lemma that for $n\geqq 8$ $b_n\downarrow 1$ as $n\to\infty$. Here, supposing this fact, we have

$$\begin{aligned} b_{12}c_{12}&=\frac{4\cdot 17\cdot 11}{25\cdot 19}\cdot\left(\frac{11\cdot 25}{14\cdot 19}\right)^{29.5}\times\frac{1}{4}\cdot\frac{275}{269}\\ &=\frac{17\cdot 11\cdot 11}{19\cdot 269}\cdot\left(\frac{275}{269}\right)^{29.5}\doteqdot\frac{2057}{5111}\times(1.0338345)^{29.5}\\ &=0.4024652\times 2.66874=1.074075>1 \end{aligned}$$

and

$$\begin{aligned} b_{13}c_{13}&=\frac{4\cdot 18\cdot 12}{27\cdot 21}\cdot\left(\frac{12\cdot 27}{15\cdot 21}\right)^{32.5}\times\frac{1}{4}\cdot\frac{324}{318}\\ &=\frac{8\cdot 18}{7\cdot 53}\times\left(\frac{36}{35}\right)^{32.5}\doteqdot\frac{144}{371}\times(1.0285714)^{32.5}\\ &=0.3881401\times 2.49816=0.969636<1. \end{aligned}$$

Therefore, it must be

$$b_n c_n<1 \qquad \text{for } n\geqq 13. \tag{5.9}$$

Hence, we obtain from (5.6) and (5.9)

$$\sigma\left(\frac{2n+10}{2n+1}\right)>B-\rho\left(\frac{2n+10}{2n+1}\right), \qquad \text{when } n\geqq 13. \tag{5.10}$$

On the other hand, we see that $B-\rho(x)$ is decreasing in the interval

$(2n+10)/(2n+1)\leqq x\leqq 2$, when $n\geqq 4$, by (3.4) and $\sigma(x)$ is increasing in the interval $1<x<2$, when $n\geqq 6$, by Proposition 2. Hence, we obtain from these facts and (5.10)

$$\sigma(x)>B-\rho(x) \qquad \text{for } \frac{2n+10}{2n+1}\leqq x<2, \text{ when } n\geqq 13. \qquad \text{Q.E.D.}$$

LEMMA 5.4. *$b_n$ defined by* (5.7) *is monotone decreasing with respect to $n$ for $n\geqq 8$ and tends to* 1 *as $n\to\infty$.*

*Proof*. We have from (5.7)

$$b_n=4\cdot\left[\frac{(n+5)^2(n+2)^7(2n-5)^5}{(n-1)^5(2n+1)^9}\right]^{1/2}\cdot\left[\left\{\frac{(n-1)(2n+1)}{(n+2)(2n-5)}\right\}^{n-1}\right]^3.$$

In the following, we shall show that the both functions of $n$ in the above pairs of brackets are decreasing for $n\geqq 8$.

First, we have

$$\frac{d}{dn}\frac{(n+5)^2(n+2)^7(2n-5)^5}{(n-1)^5(2n+1)^9}=-\frac{(n+5)(n+2)^6(2n-5)^4}{(n-1)^6(2n+1)^{10}}$$
$$\times(48n^3-216n^2-1197n+555)$$

and

$$48n^3-216n^2-1197n+555>0 \qquad \text{for } n\geqq 8.$$

Therefore, we see that $\dfrac{(n+5)^2(n+2)^7(2n-5)^5}{(n-1)^5(2n+1)^9}$ is decreasing for $n\geqq 8$ and

$$\lim_{n\to\infty}\frac{(n+5)^2(n+2)^7(2n-5)^5}{(n-1)^5(2n+1)^9}=\frac{1}{16}. \tag{5.11}$$

Second, setting $1/(n-1)=t$, we have

$$\left\{\frac{(n-1)(2n+1)}{(n+2)(2n-5)}\right\}^{n-1}=\left\{\frac{2+3t}{(1+3t)(2-3t)}\right\}^{1/t}.$$

We show that the function of $t$ of the right hand side is increasing with respect to $t$ for $0<t<2/3$.

$$\frac{d}{dt}\left\{\frac{1}{t}\log\frac{2+3t}{(1+3t)(2-3t)}\right\}=-\frac{1}{t^2}\log\frac{2+3t}{(1+3t)(2-3t)}$$
$$+\frac{3}{t}\left\{\frac{1}{2+3t}-\frac{1}{1+3t}+\frac{1}{2-3t}\right\}$$
$$=\frac{1}{t^2}\log\left(1-\frac{9t^2}{2+3t}\right)+\frac{9(4+3t)}{(1+3t)(2+3t)(2-3t)}.$$

For $t>0$, the condition $9t^2/(2+3t)<1$ is equivalent to $9t^2-3t-2=(3t-2)(3t+1)<0$,

i. e. $t<2/3$, which is equivalent to $n>5/2$. Then, we have

$$\frac{1}{t^2}\log\left(1-\frac{9t^2}{2+3t}\right)=-\frac{1}{t^2}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{9t^2}{2+3t}\right)^m$$

$$>-\frac{9}{2+3t}-\frac{81t^2}{2(2+3t)^2}\sum_{m=0}^{\infty}\left(\frac{9t^2}{2+3t}\right)^m=-\frac{9}{2+3t}-\frac{81t^2}{2(2+3t)^2}$$

$$\times\left\{1/\left(1-\frac{9t^2}{2+3t}\right)\right\}=-\frac{9}{2+3t}-\frac{81t^2}{2(2+3t)(2+3t-9t^2)}$$

$$=-\frac{9(4+6t-9t^2)}{2(1+3t)(2+3t)(2-3t)}$$

and hence

$$\frac{1}{t^2}\log\left(1-\frac{9t^2}{2+3t}\right)+\frac{9(4+3t)}{(1+3t)(2+3t)(2-3t)}$$

$$>\frac{9(4+9t^2)}{2(1+3t)(2+3t)(2-3t)}>0\,.$$

Therefore, $\left\{\frac{2+3t}{(1+3t)(2-3t)}\right\}^{1/t}$ is increasing with respect to $t$ for $0<t<2/3$ and

$$\lim_{t\to 0}\left\{\frac{2+3t}{(1+3t)(2-3t)}\right\}^{1/t}=\lim_{n\to\infty}\left\{\frac{n-1}{n+2}\cdot\frac{2n+1}{2n-5}\right\}^{n-1}$$

$$=\lim_{n\to\infty}\left(\frac{1}{1+3/(n-1)}\right)^{n-1}\cdot\lim\left[\left(1+\frac{6}{2n-5}\right)^{(2n-5)/6}\right]^{6(n-1)/(2n-5)}=\frac{1}{e^3}\cdot e^3=1\,.$$

Thus, we have proved that $\left\{\frac{(n-1)(2n+1)}{(n+2)(2n-5)}\right\}^{n-1}$ is decreasing for $n>5/2$ and tends to 1 as $n\to\infty$. Q. E. D.

PROPOSITION 3. *When $n\geqq 13$, we have*

$$G_2(x)+3(x-1)f_0(x)>0 \qquad \text{for } \frac{2n+10}{2n+1}\leqq x<n\,.$$

*Proof.* By Lemma 3.1, this inequality is true for $2\leqq x<n$ and it is also true for $(2n+10)/(2n+1)\leqq x<2$ by (4.1) and Lemma 5.3. Q. E. D.

## § 6. Evaluations of some constants.

Setting

$$\alpha_0=\alpha_0(n):=X_n^{-1}(2)\,,\qquad \alpha_1=\alpha_1(n):=X_n^{-1}\left(\frac{2n+10}{2n+1}\right), \tag{6.1}$$

we shall evaluate $\alpha_0$ and $\alpha_1$ in this section.

LEMMA 6.1. *When* $n\geqq 6$, *we have*

$$\frac{1}{3}<\alpha_0(n)<\frac{5}{12}=0.41\dot{6}\,. \tag{6.2}$$

*Proof.* In order to prove the inequality $a<\alpha_0(n)$ for $0<a<1$, it is sufficient to prove $a(n-a)^{n-1}<2(n-2)^{n-1}$, i.e.

$$\left(\frac{n-a}{n-2}\right)^{n-1}<\frac{2}{a}\,. \tag{6.3}$$

For the constant $a$, the function of $n$ of the left hand side of (6.3) is decreasing, because

$$\frac{d}{dn}\left\{\log\left(\frac{n-a}{n-2}\right)^{n-1}\right\}=\log\frac{n-a}{n-2}-\left(\frac{1-a}{n-a}+\frac{1}{n-2}\right)$$

$$=-u\left[\frac{1-a}{1-au}+\frac{1}{1-2u}-\frac{1}{u}\{\log(1-au)-\log(1-2u)\}\right],$$

where $u=1/n$, and

$$\frac{1-a}{1-au}+\frac{1}{1-2u}-\frac{1}{u}\log(1-au)+\frac{1}{u}\log(1-2u)$$

$$=\sum_{m=1}^{\infty}\left\{\left(1-\frac{m}{m+1}a\right)a^m+\frac{m-1}{m+1}2^m\right\}u^m>0\,.$$

Hence, if $((6-a)/4)^5<2/a$, then (6.3) holds for $n\geqq 6$. Now, $((6-a)/4)^5<2/a$ is equivalent to $a(6-a)^5<2048$ and for $a=1/3$ we have

$$a(6-a)^5=\frac{17^5}{3^6}=\frac{1419857}{729}\doteqdot 1947.678\,.$$

Hence it must be $\alpha_0(n)>1/3$.

Next, in order to prove the inequality $\alpha_0(n)<b$ for $0<b<1$, it is sufficient to prove

$$\left(\frac{n-b}{n-2}\right)^{n-1}>\frac{2}{b}\,. \tag{6.4}$$

Since we have

$$\lim_{n\to\infty}\left(\frac{n-b}{n-2}\right)^{n-1}=\lim_{n\to\infty}\left(1+\frac{2-b}{n-2}\right)^{n-1}=e^{2-b}$$

(6.4) holds for $n>2$, provided $e^{2-b}\geqq 2/b$, i.e.

$$e^b\leqq\frac{e^2}{2}b\,. \tag{6.5}$$

Now, for $b=5/12$ we have

$$e^b \doteqdot 1.517\,, \qquad \frac{e^2}{2}b=\frac{5}{24}e^2 \doteqdot 1.539\,,$$

Hence (6.4) holds for $n>2$ and $b=5/12$, and it must be $\alpha_0(n)<5/12$. Q. E. D.

*Remark.* (6.5) does not hold for $b=2/5=0.4$. In fact, $2^b=e^{2/5}\doteqdot 1.492$, $(e^2/2)b=(2/10)e^2\doteqdot 1.479$.

LEMMA 6.2. *When $n>2$, we have*

$$x>2-X_n(x) \qquad \text{for } 0<x<1\,. \tag{6.6}$$

*Proof.* By virtue of Lemma 2.1, it is sufficient to prove

$$\left(\frac{n-X}{n-1}\right)^{n-1}>2-X \qquad \text{for } 1<x<n\,. \tag{6.7}$$

Since we have

$$\frac{d}{dX}\left[\left(\frac{n-X}{n-1}\right)^{n-1}+X\right]=1-\left(\frac{n-X}{n-1}\right)^{n-2}>0\,,$$

we obtain easily (6.7). Q. E. D.

LEMMA 6.3. *When $n\geqq 10$, we have*

$$\frac{2n-8}{2n+1}<\alpha_1(n)=X_n^{-1}\left(\frac{2n+10}{2n+1}\right)<\frac{2n-6}{2n+1}\,. \tag{6.8}$$

*Proof.* The left side inequality is evident by Lemma 6.2.
The right side inequality is equivalent to

$$\frac{2n-6}{2n+1}\cdot\left(n-\frac{2n-6}{2n+1}\right)^{n-1}>\frac{2n+10}{2n+1}\cdot\left(n-\frac{2n+10}{2n+1}\right)^{n-1},$$

that is

$$\left(\frac{2n^2-n+6}{2n^2-n-10}\right)^{n-1}>\frac{n+5}{n-3}\,. \tag{6.9}$$

Now, since we have

$$\left(\frac{2n^2-n+6}{2n^2-n-10}\right)^{n-1}=\left(1+\frac{16}{2n^2-n-10}\right)^{n-1}$$

$$>1+\frac{16(n-1)}{2n^2-n-10}+\frac{8\cdot 16\cdot(n-1)(n-2)}{(2n^2-n-10)^2}+\frac{8\cdot 16^2\cdot(n-1)(n-2)(n-3)}{3(2n^2-n-10)^3}\,,$$

(6.9) is implied from the following inequality:

$$\frac{2(n-1)}{2n^2-n-10}+\frac{16(n-1)(n-2)}{(2n^2-n-10)^2}+\frac{16^2(n-1)(n-2)(n-3)}{3(2n^2-n-10)^3}\geqq\frac{1}{n-3},$$

which is equivalent to

$$12n^5-92n^4-813n^3+6908n^2-16116n+12288\geqq 0\,. \tag{6.10}$$

First, we show that (6.10) holds for $n\geqq 13$. In fact, we have for $n\geqq 13$

$$12n^2-92n-813\geqq 12\cdot 13^2-92\cdot 13-813=19$$

and

$$19n^3+6908n^2-16116n+12288>0\,.$$

Second, for $12\leqq n<13$, we have

$$12n^2-92n-813\geqq 12\cdot 12^2-92\cdot 12-813=-189$$

and

$$-189n^3+6908n^2-16116n+12288$$

$$\geqq n(-189\cdot 12^2+6908\cdot 12-16116)+12288=39564n+12288>0\,,$$

which implies (6.10) for $12\leqq n<13$.

Third, for $11\leqq n<12$, we have

$$12n^2-92n-813\geqq 12\cdot 11^2-92\cdot 11-813=-373$$

and

$$-373n^3+6908n^2-16116n+12288$$

$$>n(-373\cdot 12^2+6908\cdot 12-16116)+12288=13068n+12288>0\,,$$

since $6908/2\times 373\doteqdot 9.260<11$. Therefore (6.10) holds also for $11\leqq n<12$.

Fourth, for $10.2\leqq n<11$, we have

$$12n^2-92n-813\geqq 12\cdot 10.2^2-92\cdot 10.2-813=-502.92$$

and

$$-502.92n^3+6908n^2-16116n+12288$$

$$>n(-502.92\cdot 11^2+6908\cdot 11-16116)+12288=-981.32n+12288>0$$

since $6908/2\times 502.92\doteqdot 6.868<10.2$. Therefore (6.10) holds also for $10.2\leqq n<11$.

Last, for $10\leqq n<10.2$, we have

$$12n^2-92n-813\geqq -533$$

and

$$-533n^3+6908n^2-16116+12288$$

$$>n(-533\cdot 10.2^2+6908\cdot 10.2-16116)+12288$$

$$=-1107.72n+12288>0$$

since $6908/2\times 533 \doteqdot 6.480<10$. Therefore (6.10) holds also for $10\leqq n<10.2$.

Thus, we have proved that (6.10) is true for $n\geqq 10$. Hence (6.9) holds for $n\geqq 10$. Q. E. D.

*Remark.* We check (6.9) for $n=9$.

$$\left(\frac{2n^2-n+6}{2n^2-n-10}\right)^{n-1}=\left(\frac{159}{143}\right)^8 \doteqdot 2.336\,, \qquad \frac{n+5}{n-3}=\frac{14}{6}\doteqdot 2.333\,,$$

which shows that (6.9) may be true for $9\leqq n<10$. But the method taken in Lemma 6.3 does not go well. Next, for $n=8$ we have

$$\left(\frac{2n^2-n+6}{2n^2-n-10}\right)^{n-1}=\left(\frac{126}{110}\right)^7 \doteqdot 2.587, \qquad \frac{n+5}{n-3}=\frac{13}{5}=2.600\,,$$

which shows (6.9) is not true for $n=8$.

## § 7. Evaluations of $U_0(x)$ and $U_4(x)$ for $\alpha_1(n)\leqq x<1$.

In the following two sections, we shall evaluate the functions $U_i(x)$, $i=0$, 1, 2, 4, 5, 6, defined in § 1 and appeared in $V(x, X_n(x))$, for the interval $\alpha_1(n)\leqq x<1$, considering an effect of Proposition 3.

In the following, we set for simplicity

$$\begin{aligned} &t=1-x\,, \qquad s=X_n(x)-1 \qquad \text{for } 0<x<1 \\ &t_0=1-\alpha_1(n)\,, \qquad s_0=\frac{2n+10}{2n+1}-1=\frac{9}{2n+1} \end{aligned} \tag{7.1}$$

and we obtain by Lemma 6.2 and Lemma 6.3 the inequalities

$$0<t<s\,, \tag{7.2}$$

$$\frac{7}{2n+1}<t_0<\frac{9}{2n+1}\,. \tag{7.3}$$

LEMMA 7.1. *When $n\geqq 10$, we have*

$$\begin{aligned} U_0(x)>\frac{16B}{3}\cdot\frac{n(n-1)^3(n-4)^2(n^2-n+1)(2n+1)^{3/2}(2n^2-n+8)^{1/2}}{(8n^3+18n+1)^2} \\ \times\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\cdot\left(\frac{s}{t}\right)^2\right]t \end{aligned} \tag{7.4}$$

*for $\alpha_1(n)\leqq x<1$.*

*Proof.* First, we find a lower bound of $\lambda(x)-\tilde{\lambda}(X(x))$. By means of (1.4), (1.5), (1.5) in [12] and Lemma 7.1 in [13] we have

$$\lambda(x)-\tilde{\lambda}(X(x))=\lambda(x)-\lambda(1)-\{\tilde{\lambda}(X)-\tilde{\lambda}(1)\}$$

$$=\int_x^1 \frac{(1-u)du}{(n-u)^2}+\int_1^X \frac{(u-1)\{n+(n-1)u\}\,du}{(n-1)u^2(n-u)}$$

$$=\int_0^t \frac{udu}{(n-1+u)^2}+\frac{1}{n-1}\int_0^s \frac{u\{2n-1+(n-1)u\}\,du}{(1+u)^2(n-1-u)}\,.$$

On the other hand, we see that the function $\dfrac{2n-1+(n-1)x}{(1+x)^2(n-1-x)}$ is decreasing for $0\leqq x\leqq\dfrac{9}{2n+1}$, because

$$\frac{d}{dx}\frac{2n-1+(n-1)x}{(1+x)^2(n-1-x)}=-\frac{3n^2-6n+2+(n^2-8n+4)x-2(n-1)x^2}{(1+x)^3(n-1-x)^2}<0\,.$$

Hence, we obtain for $0\leqq x\leqq 9/(2n+1)$

$$\frac{2n-1+(n-1)x}{(1+x)^2(n-1-x)}\geqq\left[\frac{2n-1+(n-1)x}{(1+x)^2(n-1-x)}\right]_{x=9/(2n+1)}=\frac{(2n+1)^2(4n^2+9n-10)}{4(n+5)^2(2n^2-n-10)}\,.$$

Thus, from the above expression of $\lambda(x)-\tilde{\lambda}(x)$ in integral we obtain

$$\lambda(x)-\tilde{\lambda}(X(x))>\frac{t^2}{2(n-1+t_0)^2}+\frac{(2n+1)^2(4n^2+9n-10)s^2}{8(n-1)(n+5)^2(2n^2-n-10)}$$

and by (7.3)

$$\lambda(x)-\tilde{\lambda}(X(x))>\frac{(2n+1)^2t^2}{2(2n^2-n+8)^2}+\frac{(2n+1)^2(4n^2+9n-10)s^2}{8(n-1)(n+5)^2(2n^2-n-10)} \tag{7.5}$$

$$\text{for } \alpha_1(n)\leqq x<1\,.$$

Second, we find a lower bound of $F_2(x)$. By means of the facts in § 3 in [12], we have for $0<x<1$

$$F_2(x)=-P_2(x)B+(n-x)^{n-1}P_3(x)=P_2(x)\left\{\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}-B\right\}$$

$$>P_2(1)\int_1^x\frac{d}{du}\left\{\frac{(n-u)^{n-1}P_3(u)}{P_2(u)}\right\}du$$

$$=12(n-1)^2\int_1^x\frac{n(n-1)(n-u)^{n-2}(u-1)^3Q_2(u)du}{(P_2(u))^2}$$

$$=12n(n-1)^3\int_x^1\frac{(n-u)^{n-2}(1-u)^3Q_2(u)du}{(P_2(u))^2}\,.$$

Since we obtain from (3.2) and (3.6) in [12]

$$Q_2(u)=(2n+1)u^2-(8n^2-2n+9)u+4n(2n^2-2n+3)$$

$$>Q_2(1)=8(n-1)(n^2-n+1)\,,$$

$$P_2(u)=(2n+1)u^2-2(2n^2+5n-4)u+16n^2-16n+3$$

$$<P_2(x)<P_2(\alpha_1)<P_2\left(\frac{2n-8}{2n+1}\right)=\frac{3(8n^3+18n+1)}{2n+1}$$

$$\text{for } x<u<1, \qquad \alpha_1(n)\leqq x<1.$$

Thus, we obtain

$$F_2(x)>12n(n-1)^3\cdot\frac{(n-1)^{n-2}\cdot 8(n-1)(n^2-n+1)\cdot(2n+1)^2}{9(8n^3+18n+1)^2}\cdot\int_x^1(1-u)^3du$$

i. e.

$$F_2(x)>\frac{8Bn(n-1)^3(n^2-n+1)(2n+1)^2t^4}{3(8n^3+18n+1)^2} \qquad \text{for } \alpha_1(n)\leqq x<1. \tag{7.6}$$

Third, since $x^2\sqrt{n-x}$ is increasing for $0<x<(4/5)n$, we obtain

$$x^2\sqrt{n-x}\geqq\alpha_1^2\sqrt{n-\alpha_1}<\left(\frac{2n-8}{2n+1}\right)^2\sqrt{n-\frac{2n-8}{2n+1}}$$

$$=\frac{4(n-4)^2\sqrt{2n^2-n+8}}{(2n+1)^{5/2}} \qquad \text{for } \alpha_1(n)\leqq x<1.$$

Finally, combining these evaluations, we obtain

$$U_0(x)=\frac{x^2\sqrt{n-x}}{(1-x)^5}\cdot F_2(x)\cdot\{\lambda(x)-\tilde{\lambda}(X(x))\}$$

$$>\frac{1}{t^5}\frac{4(n-4)^2\sqrt{2n^2-n+8}}{(2n+1)^{5/2}}\cdot\frac{8Bn(n-1)^3(n^2-n+1)(2n+1)^2t^4}{3(8n^3+18n+1)^2}$$

$$\times\frac{(2n+1)^2}{2}\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\left(\frac{s}{t}\right)^2\right]t^2$$

$$=\frac{16B}{3}\cdot\frac{n(n-1)^3(n-4)^2(n^2-n+1)(2n+1)^{3/2}(2n^2-n+8)^{1/2}}{(8n^3+18n+1)^2}$$

$$\times\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\left(\frac{s}{t}\right)^2\right]t$$

for $\alpha_1(n)\leqq x<1$. Q. E. D.

LEMMA 7.2. *When $n\geqq 10$, we have*

$$U_4(x)>\frac{1}{96}\cdot\frac{n^2(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^4(2n+1)^n}\cdot(8n^3-24n^2-18n+61) \tag{7.7}$$

$$\times(8n^4-12n^3-28n^2+27n+5)\cdot\frac{(1+s)s}{\sqrt{n-1-s}} \qquad \textit{for } \alpha_1(n)\leqq x<1.$$

*Proof.* As in Lemma 7.1, we have

$$F_2(X)>P_2\Big(\frac{2n+10}{2n+1}\Big)\int_1^x \frac{n(n-1)(n-u)^{n-2}(u-1)^3Q_2(u)du}{\{P_2(u)\}^2}$$

$$>\frac{1}{\{P_1(1)\}^2}\cdot P_2\Big(\frac{2n+10}{2n+1}\Big)\cdot Q_2\Big(\frac{2n+10}{2n+1}\Big)\cdot n(n-1)\Big(n-\frac{2n+10}{2n+1}\Big)^{n-2}\cdot\frac{s^4}{4}$$

for $\alpha_1(n)\leqq x<1$. Since we have

$$P_2(1)=12(n-1)^2,$$

$$P_2\Big(\frac{2n+10}{2n+1}\Big)=\frac{3(8n^3-24n^2-18n+61)}{2n+1},$$

$$Q_2\Big(\frac{2n+10}{2n+1}\Big)=\frac{2(8n^4-12n^3-28n^2+27n+5)}{2n+1},$$

we obtain

$$F_2(X)>\frac{1}{12^2(n-1)^4}\cdot\frac{3(8n^3-24n^2-18n+61)}{2n+1}$$

$$\times\frac{2(8n^4-12n^3-28n^2+27n+5)}{2n+1}\cdot\frac{n(n-1)(2n^2-n-10)^{n-2}}{(2n+1)^{n-2}}\cdot\frac{s^4}{4},$$

i. e.

$$F_2(X)>\frac{1}{96}\cdot\frac{n(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^3(2n+1)^n}\cdot(8n^3-24n^2-18n+61) \tag{7.8}$$

$$\times(8n^4-12n^3-28n^2+27n+5)s^4 \qquad \text{for } 1<X\leqq\frac{2n+10}{2n+1}.$$

Thus, we obtain easily from the expressions of $U_3(x)$ and $U_4(x)$ the inequality (7.7). Q. E. D.

## §8. Evaluations of $U_1(x)$, $U_2(x)$, $U_5(x)$ and $U_6(x)$ for $\alpha_1(n)\leqq x<1$.

LEMMA 8.1. *When $n\geqq 4$, we have*

$$U_1(x)<\frac{n(n-1)(1-t)^2(n-1+t)^{n-5/2}}{2(n-1)+t}\cdot\Big(2n-1+\frac{3}{4}t\Big) \quad \textit{for } 0<x<1. \tag{8.1}$$

*Proof.* By means of the facts regarding $f_0(x)$ in § 2 of [12] and (1.9) we have

$$U_1(x)=\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}$$

$$=\frac{3x^2(2n-1-x)}{(x-1)^3\sqrt{n-x}}\Big[B-\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}\Big]$$

$$=\frac{3x^2(2n-1-x)}{(x-1)^3\sqrt{n-x}}\int_x^1\frac{n(n-1)(n-u)^{n-2}(u-1)^2(u-2n)du}{(2n-1-u)^2}$$

$$=\frac{3n(n-1)(1-t)^2(2n-2+t)}{t^3\sqrt{n-1+t}}\int_0^t\frac{(n-1+u)^{n-2}u^2(2n-1+u)du}{(2n-2+u)^2}.$$

Since the function $\dfrac{(n-1+x)^{n-2}}{(2n-2+x)^2}$ is increasing, because

$$\frac{d}{dx}\frac{(n-1+x)^{n-2}}{(2n-2+x)^2}=\frac{(n-1+x)^{n-3}\{2(n-1)(n-3)+(n-4)x\}}{(2n-2+x)^3}>0$$

for $x>0$, when $n\geqq 4$, we have

$$\frac{(n-1+u)^{n-2}}{(2n-2+u)^2}<\frac{(n-1+t)^{n-2}}{(2n-2+t)^2}\qquad \text{for } 0<u<t.$$

Hence, we obtain

$$U_1(x)<\frac{3n(n-1)(1-t)^2(n-1+t)^{n-5/2}}{t^3\{2(n-1)+t\}}\int_0^t u^2(2n-1+u)du$$

$$=\frac{n(n-1)(1-t)^2(n-1+t)^{n-5/2}}{2(n-1)+t}\cdot\left(2n-1+\frac{3}{4}t\right).$$

Q. E. D.

LEMMA 8.2. *When $n>3$, we have*

$$U_2(x)>\frac{B}{3}\cdot\frac{n^2}{(n-1)^2}\cdot\frac{(1-t)^2}{\sqrt{n-1+t}}\cdot\{3(n-1)+2t\}\qquad \textit{for } 0<x<1. \tag{8.2}$$

*Proof.* By means of (2.8) we have

$$B-\phi(x)=\phi(1)-\phi(x)=n\int_x^1(1-u)(n-u)^{n-2}du$$

$$=n\int_0^t u(n-1+u)^{n-2}du>n(n-1)^{n-3}\int_0^t u(n-1+u)du$$

$$=n(n-1)^{n-3}t^2\left(\frac{n-1}{2}+\frac{t}{3}\right),$$

from which we obtain immediately (8.2) by (1.10). Q. E. D.

LEMMA 8.3. *When $n\geqq 4$, we have*

$$U_5(x)>\frac{n(n-1)(1+s)^2(n-1-s)^{n-5/2}}{2(n-1)-s}\cdot\left(2n-1-\frac{3}{4}s\right) \tag{8.3}$$

*for $\alpha_1(n)\leqq x<1$.*

*Proof.* By means of an analogous way as to $U_1(x)$, we obtain from (1.9) and (1.13)

$$U_5(x)=\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}$$
$$=\frac{3n(n-1)(1+s)^2(2n-2-s)}{s^3\sqrt{n-1-s}}\int_0^s\frac{(n-1-u)^{n-2}u^2(2n-1-u)du}{(2n-2-u)^2}.$$

Since the function $\dfrac{(n-1-x)^{n-2}}{(2n-2-x)^2}$ is decreasing for $0<x<1$, when $n\geqq 4$ and $0<s\leqq s_0=9/(2n+1)$, we obtain

$$U_5(x)>\frac{3n(n-1)(1+s)^2(2n-2-s)}{s^3\sqrt{n-1-s}}\cdot\frac{(n-1-s)^{n-2}}{(2n-2-s)^2}\int_0^s u^2(2n-1-u)du$$
$$=\frac{n(n-1)(1+s)^2(n-1-s)^{n-5/2}}{2(n-1)-s}\cdot\left(2n-1-\frac{3}{4}s\right)\quad\text{for } \alpha_1(n)\leqq x<1.$$

Q. E. D.

LEMMA 8.4. *When $n\geqq 4$, we have*

$$U_6(x)<\frac{B}{3}\cdot\frac{n^2}{(n-1)^2}\cdot\frac{(1+s)^2}{\sqrt{n-1-s}}\{3(n-1)-2s\}\qquad \textit{for } \alpha_1(n)\leqq x<1. \tag{8.4}$$

*Proof.* By means of an analogous way as to $U_2(x)$, we obtain from (1.10) and (1.13)

$$U_6(x)=\frac{2nX^2\{B-\psi(X)\}}{(X-1)^2\sqrt{n-X}}=\frac{2n^2X^2}{(X-1)^2\sqrt{n-X}}\int_0^s u(n-1-u)^{n-2}du$$
$$<\frac{2n^2(n-1)^{n-3}X^2}{(X-1)^2\sqrt{n-X}}\int_0^s u(n-1-u)du$$
$$=\frac{B}{3}\cdot\frac{n^2}{(n-1)^2}\cdot\frac{(1+s)^2}{\sqrt{n-1-s}}\{3(n-1)-2s\},$$

if $1<X\leqq n-1$ and $n>3$. Then condition: $(2n+10)/(2n+1)\leqq n-1$ is equivalent to $2n^2-3n-11\geqq 0$, which is satisfied for $n\geqq 4$. Hence, we obtain (8.4), when $n\geqq 4$.

Q. E. D.

## §9. Evaluation of $V(x, X(x))$ for $\alpha_1(n)\leqq x<1$.

We shall evaluate $V(x, X(x))$ for $\alpha_1(n)\leqq x<1$ and show that it is positive, when $n\geqq 84$, which implies the result described in Introduction.

PROPOSITION 4. *The function $(X_n(x)-1)/(1-x)$ is monotone decreasing with respect to $x$ for $0<x<1$, when $n>2$.*

*Proof.* Since we have

$$\frac{d}{dx}\left(\frac{X_n(x)-1}{1-x}\right)=\frac{1}{(1-x)^2}\{(1-x)X_n'(x)+X_n(x)-1\}$$

$$=-\frac{X-1}{x(n-x)}\left\{\frac{X(n-X)}{(X-1)^2}-\frac{x(n-x)}{(1-x)^2}\right\},\quad X=X_n(x),$$

by $dX/dx=\{(1-x)/x(n-x)\}\cdot\{X(n-X)/(1-X)\}$, it is sufficient to prove

$$\frac{X(n-X)}{(X-1)^2}>\frac{x(n-x)}{(1-x)^2}\qquad \text{for } 0<x<1, \tag{9.1}$$

which is equivalent to

$$(X-1)^2(n-X)^{n-2}<(1-x)^2(n-x)^{n-2}\qquad \text{for } 0<x<1. \tag{9.2}$$

On the other hand, we have

$$\frac{d}{dx}(1-x)^2(n-x)^{n-2}=-(1-x)(n-x)^{n-3}(3n-2-nx).$$

Since we have

$$(1-x)^2(n-x)^{n-2}=4\left(\frac{n-1}{n}\right)^n(n-2)^{n-2}\qquad \text{at } x=\frac{3n-2}{n},$$

$$=n^{n-2}\qquad \text{at } x=0$$

and

$$n^{n-2}>4\left(\frac{n-1}{n}\right)^n(n-2)^{n-2}\qquad \text{for } n>2$$

because $((n-1)/n)^n((n-2)/n)^{n-2}$ is decreasing with respect to $n$ for $n>2$ as is easily proved and $\lim_{n\to 2}((n-1)/n)^n((n-2)/n)^{n-2}=1/4$, there exists a real number $\xi$ $(0<\xi<1)$ uniquely determined by

$$(1-\xi)^2(n-\xi)^{n-2}=4\left(\frac{n-1}{n}\right)^n(n-2)^{n-2}.$$

Then, (9.2) holds clearly for $0<x<\xi$ and $x<1$ sufficiently near 1 by means of (8.12) in [13]. Now, we suppose that (9.2) does not hold at some point of $x$, then there exist a value $y$ $(\xi\leqq y<1)$ such that

$$(Y-1)^2(n-Y)^{n-2}=(1-y)^2(n-y)^{n-2},\qquad Y=X_n(y) \tag{9.3}$$

and

$$\left[\frac{d}{dx}\{(X-1)^2(n-X)^{n-2}-(1-x)^2(n-x)^{n-2}\}\right]_{x=y}\leqq 0,$$

i.e.

$$\left[-(1-X)(n-X)^{n-3}(3n-2-nX)\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}\right.$$
$$\left.+(1-x)(n-x)^{n-3}(3n-2-nx)\right]_{x=y}$$

$$=\frac{1-y}{y(n-y)}[-Y(n-Y)^{n-2}(3n-2-nY)+y(n-y)^{n-2}(3n-2-ny)]\leqq 0\,,$$

i.e.

$$Y(3n-2-nY)(n-Y)^{n-2}\geqq y(3n-2-ny)(n-y)^{n-2}\,. \tag{9.4}$$

Adding (9.3) multiplied with $n-1$, (9.4) and $-Y(n-Y)^{n-1}=-y(n-y)^{n-1}$ in the both sides respectively, we obtain

$$(n-1)(n-Y)^{n-2}\geqq(n-1)(n-y)^{n-2}\,.$$

This is a contradiction, because $0<y<1<Y<n$, $n>2$. Thus we obtain this lemma. Q. E. D.

LEMMA 9.1. *When $n\geqq 10$, we have*

$$\begin{aligned}
\frac{V(x,\,X(x))}{sB\sqrt{n-1}} &> \frac{112}{27}\cdot\frac{n(n-1)^{5/2}(n-4)^2(n^2-n+1)(2n+1)^{3/2}}{(8n^3+18n+1)^2}\\
&\quad \frac{(2n^2-n+8)^{1/2}}{}\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\right]\\
&\quad +\frac{1}{96}\cdot\frac{n^2(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{n+4}(2n+1)^n}\cdot(8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5)\\
&\quad -\frac{s+t}{s}\cdot\frac{n}{12(n-1)^{n-3/2}}\cdot\frac{1}{(2n+1)^{n-3/2}(n+2)^{3/2}(2n-5)^{3/2}(4n^2-2n+7)^2}\\
&\quad \times[4n(n-1)^{n-4}(n+5)(2n+1)^{n-3}(4n^2-2n+7)^2(48n^4-52n^3+342n^2-198n+616)\\
&\quad -3(n-4)(n+2)^{n-2}(2n-5)^{n-2}(256n^6-992n^5+80n^4+4120n^3\\
&\quad -3592n^2-5474n+3010)] \qquad \text{for } \alpha_1(n)\leqq x<1\,.
\end{aligned} \tag{9.5}$$

*Proof*. By virtue of Lemmas 7.1, 7.2, 8.1, 8.2, 8.3 and 8.4, we obtain an evaluation of $V(x,\,X(x))$ as follows: When $n\geqq 10$, for $\alpha_1(n)\leqq x<1$, we have

$$\begin{aligned}
V(x,\,X(x)) &= U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)\\
&> \frac{16B}{3}\cdot\frac{n(n-1)^3(n-4)^2(n^2-n+1)(2n+1)^{3/2}(2n^2-n+8)^{1/2}}{(8n^3+18n+1)^2}\\
&\quad \times\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\left(\frac{s}{t}\right)^2\right]t
\end{aligned}$$

$$+\frac{1}{96}\cdot\frac{n^2(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^4(2n+1)^n}\cdot(8n^3-24n^2-18n+61)(8n^4-12n^3 \tag{9.6}$$

$$-28n^2+27n+5)\cdot\frac{(1+s)s}{\sqrt{n-1-s}}$$

$$+n(n-1)\left[\frac{(1+s)^2(n-1-s)^{n-5/2}}{2(n-1)-s}\cdot\left(2n-1-\frac{3}{4}\right)\right.$$

$$\left.-\frac{(1-t)^2(n-1+t)^{n-5/2}}{2(n-1)+t}\left(2n-1+\frac{3}{4}t\right)\right]$$

$$-\frac{B}{3}\cdot\frac{n^2}{(n-1)^2}\left[\frac{(1+s)^2}{\sqrt{n-1-s}}\cdot\{3(n-1)-2s\}-\frac{(1-t)^2}{\sqrt{n-1+t}}\cdot\{3(n-1)+2t\}\right].$$

When $n\geqq 10$, we obtain from Lemma 6.2, Lemma 6.3 and Proposition 4

$$1<\frac{s}{t}<\frac{s_0}{t_0}<\frac{9}{7} \qquad \text{for } 0<t<t_0. \tag{9.7}$$

Regarding the right hand side of (9.6), we have the following facts. First, we have

$$\frac{1+s}{\sqrt{n-1-s}}>\frac{1}{\sqrt{n-1}} \qquad \text{for } 0<s<s_0. \tag{9.8}$$

Second, we have

$$\frac{d}{dx}\frac{(1+x)^2(n-1-x)^{n-5/2}\left(2n-1-\frac{3}{4}x\right)}{2(n-1)-x}=\frac{(1+x)(n-1-x)^{n-7/2}R_3(x)}{8\{2(n-1)-x\}^2}, \tag{9.9}$$

where

$$\begin{aligned} R_3(x):&=4(n-1)(8n^2+n-1)-(32n^3-40n^2+38n-10)x \\ &\quad+(28n^2-32n+9)x^2-3(2n-1)x^3. \end{aligned} \tag{9.10}$$

Furthermore, we have for $x\leqq 1$

$$\begin{aligned} R_3'(x)&=-(32n^3-40n^2+38n-10)+2(28n^2-32n+9)x-9(2n-1)x^2 \\ &<-32n^3+40n^2-38n+10+2(28n^2-32n+9) \\ &=-2(16n^3-48n^2+51n-14)<0, \end{aligned}$$

when $n>2$, and hence

$$R_3(x)\geqq R_3(1)=2(20n^2-42n+13)>0.$$

Thus, we see that $\dfrac{(1+x)^2(n-1-x)^{n-5/2}\left(2n-1-\frac{3}{4}x\right)}{2(n-1)-x}$ is increasing for $-1<x<1$. Using the mean value theorem and (9.9), we obtain

$$\frac{(1+s)^2(n-1-s)^{n-5/2}}{2(n-1)-s}\cdot\left(2n-1-\frac{3}{4}s\right)-\frac{(1-t)^2(n-1+t)^{n-5/2}}{2(n-1)+t}\cdot\left(2n-1+\frac{3}{4}t\right)$$

$$>(s+t)\cdot\frac{(1-t)(n-1-s)^{n-7/2}}{8\{2(n-1)+t\}^2}\cdot R_3(s)$$

$$>(s+t)\cdot\frac{(1-t_0)(n-1-s_0)^{n-7/2}}{8\{2(n-1)+t_0\}^2}\cdot R_3(s_0)$$

$$>(s+t)\cdot\frac{\left(1-\frac{9}{2n+1}\right)\left(n-1-\frac{9}{2n+1}\right)^{n-7/2}}{8\left\{2(n-1)+\frac{9}{2n+1}\right\}^2}\cdot R_3\left(\frac{9}{2n+1}\right)$$

$$=(s+t)\cdot\frac{(n-4)(2n^2-n-10)^{n-7/2}}{4(4n^2-2n+7)^2(2n+1)^{n-3/2}}\cdot(256n^6-992n^5+80n^4 \tag{9.11}$$

$$+4120n^3-3592n-5474n+3010)\,.$$

Third, we have

$$\frac{d}{dx}\frac{(1+x)^2\{3(n-1)-2x\}}{\sqrt{n-1-x}}=\frac{(1+x)\mathrm{R}_2(x)}{2(n-1-x)^{3/2}}, \tag{9.12}$$

where

$$R_2(x):=(n-1)(12n-13)-(21n-23)x+10x^2\,. \tag{9.13}$$

Since $(21n-23)^2-40(n-1)(12n-13)=-(39n^2-34n-9)<0$ for $n>2$, we have $R_2(x)>0$. Thus, we see that $\dfrac{(1+x)^2\{3(n-1)-2x\}}{\sqrt{n-1-x}}$ is increasing for $-1<x<1$. Using the mean value theorem and (9.12), we obtain

$$\frac{(1+s)^2\{3(n-1)-2s\}}{\sqrt{n-1-s}}-\frac{(1-t)^2\{3(n-1)+2t\}}{\sqrt{n-1+t}}$$

$$<(s+t)\cdot\frac{1+s}{2(n-1-s)^{3/2}}\cdot R_2(-t)<(s+t)\cdot\frac{(1+s)R_2(-s)}{2(n-1-s)^{3/2}}$$

$$<(s+t)\cdot\frac{\left(1+\frac{9}{2n+1}\right)R_2\left(-\frac{9}{2n+1}\right)}{2\left(n-1-\frac{9}{2n+1}\right)^{3/2}}$$

$$=(s+t)\cdot\frac{(n+5)(48n^4-52n^3+342n^2-198n+616)}{(2n^2-n-10)^{3/2}(2n+1)^{3/2}}\,. \tag{9.14}$$

Now, from (9.11) and (9.14) we obtain

$$n(n-1)\Big[\frac{(1+s)^2(n-1-s)^{n-5/2}}{2(n-1)-s}\cdot\Big(2n-1-\frac{3}{4}s\Big)$$

$$-\frac{(1-t)^2(n-1+t)^{n-5/2}}{2(n-1)+t}\cdot\Big(2n-1+\frac{3}{4}t\Big)\Big]$$

$$-\frac{B}{3}\cdot\frac{n^2}{(n-1)^2}\Big[\frac{(1+s)^2\{3(n-1)-2s\}}{\sqrt{n-1-s}}-\frac{(1-t)^2\{3(n-1)+2t\}}{\sqrt{n-1+t}}\Big]$$

$$>(s+t)\Big[\frac{n(n-1)(n-4)(2n^2-n-10)^{n-7/2}}{4(4n^2-2n+7)^2(2n+1)^{n-3/2}}\cdot(256n^6-992n^5+80n^4$$

$$+4120n^3-3592n^2-5474n+3010)$$

$$-\frac{n^2(n-1)^{n-3}(n+5)(48n^4-52n^3+342n^2-198n+616)}{3(2n^2-n-10)^{3/2}(2n+1)^{3/2}}\Big]$$

$$=-(s+t)B\sqrt{n-1}\cdot\frac{n}{12(n-1)^{n-3/2}}\cdot\frac{1}{(2n+1)^{n-3/2}(2n^2-n-10)^{3/2}} \tag{9.15}$$

$$\frac{}{(4n^2-2n+7)^2}\cdot[4n(n-1)^{n-4}(n+5)(2n+1)^{n-3}(4n^2-2n+7)^2$$

$$\times(48n^4-52n^3+342n^2-198n+616)-3(n-4)(2n^2-n-10)^{n-2}(256n^6-992n^5$$

$$+80n^4+4120n^3-3592n^2-5474n+3010)].$$

Thus, combining (9.7), (9.8) and (9.15) with (9.6), we obtain the inequality (9.5). Q. E. D.

## § 10. A proof of positiveness of $V(x, X(x))$ for $\alpha_1(n)\leqq x<1$.

First of all, we notice that the first two terms and the third term of the right hand side of (9.5) are of order 1 and 0 with respect to $n$, respectively.

LEMMA 10.1. *When* $n\geqq 10$, *we have*

$$\frac{112}{27}\cdot\frac{(n-1)^{5/2}(n-4)^2(n^2-n+1)(2n+1)^{3/2}(2n^2-n+8)^{1/2}}{(8n^3+18n+1)^2}$$

$$\times\Big[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\Big]>\frac{2}{3}\cdot\Big(\frac{4}{9}\Big)^4\cdot\frac{7\cdot 91\cdot\sqrt{110}}{101\cdot 101}. \tag{10.1}$$

*Proof*. Supposing $n\geqq 10$ and setting $1/n=u$, we have

(the left hand side of (10.1))

$$=\frac{4^2\cdot 7}{3^3}\cdot\frac{(1-u)^{5/2}(1-4u)^2(1-u+u^2)(2+u)^{3/2}(2-u+8u^2)^{1/2}}{(8+18u^2+u^3)^2}$$

$$\times\left[\frac{u}{(2-u+8u^2)^2}+\frac{4+9u-10u^2}{4(1-u)(1+5u)^2(2-u-10u^2)}\right]$$

$$>\frac{4^2\cdot 7}{3^3}\cdot\frac{\left(\frac{9}{10}\right)^{5/2}\cdot\left(\frac{3}{5}\right)^2\cdot\frac{91}{100}\cdot 2^{3/2}\cdot\frac{1}{10}(198)^{1/2}}{\frac{(8181)^2}{10^6}}\cdot\left[0+\frac{4}{4\cdot\frac{9^3}{2\cdot 10^2}}\right]$$

$$=\frac{2}{3}\cdot\left(\frac{4}{9}\right)^4\cdot\frac{7\cdot 91\cdot\sqrt{110}}{101\cdot 101}.$$

In this computation, we have used the fact the function $(1-u)(1+5u)^2(2-u-10u^2)$ is increasing for $0<u\leqq 1/10$. In fact, we have

$$\frac{d}{du}(1-u)(1+5u)^2(2-u-10u^2)$$

$$=(1+5u)\{3(3-5u)(2-u-10u^2)-(1-u)(1+5u)(1+20u)\}$$

$$>(1+5u)\left\{3\cdot\frac{5}{2}\cdot\left(2-\frac{1}{10}-\frac{1}{10}\right)-1\cdot\left(1+\frac{1}{2}\right)(1+2)\right\}=9(1+5u)>0$$

for $0<u\leqq 1/10$ and so

$$(1-u)(1+5u)^2(2-u-10u^2)\leqq\frac{9}{10}\cdot\frac{9}{4}\cdot\frac{9}{5}=\frac{9^3}{2\cdot 10^2}\qquad\text{for } 0<u\leqq\frac{1}{10}.$$

Q. E. D.

LEMMA 10.2. *When $n\geqq 10$, we have*

$$\frac{1}{96}\cdot\frac{n(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{n+4}(2n+1)^n}\cdot(8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5)$$

$$>\frac{3^4\cdot 7\cdot 203\cdot 291\cdot e_9\cdot(e_5)^2}{4^4\cdot\sqrt{2}\cdot 10^6\cdot e^3}. \tag{10.2}$$

*Proof.* Supposing $n\geqq 10$ and setting $1/n=u$, the left hand side of (10.2) is equal to

$$\frac{1}{2\cdot 3\cdot 4^2}\cdot\frac{(1+2u)^{n-2}(2-5u)^{n-2}}{(1-u)^{n+4}(2+u)^n}\cdot(8-24u-18u^2+61u^3)(8-12u-28u^2+27u^3+5u^4).$$

Using the notation

$$e(n)=e_n:=\left(1+\frac{1}{n}\right)^n\qquad\text{for } n>0,$$

we have

$$(1-u)^{n+4}=(1-u)^5/e(n-1),\quad (2+u)^n=2^n\cdot\sqrt{e(2n)},$$

$$(1+2u)^{n-2}=\frac{1}{(1+2u)^2}\left(e\left(\frac{n}{2}\right)\right)^2,\quad (2-5u)^{n-2}=2^{n-5/2}\sqrt{2-5u}\Big/\left(e\left(\frac{2n-5}{5}\right)\right)^{5/2}.$$

Hence, putting these expressions into the above one, we obtain

$$\frac{1}{2\cdot 3\cdot 4^3\sqrt{2}}\cdot\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}\cdot\frac{e(n-1)\cdot\left(e\left(\frac{n}{2}\right)\right)^2}{\sqrt{e(2n)}\cdot\left(e\left(\frac{2n-5}{5}\right)\right)^{5/2}} \tag{10.3}$$
$$\times(8-24u-18u^2+61u^3)(8-12u-28u^2+27u^3+5u^4).$$

Then, since we have

$$\frac{d}{du}\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}=-\frac{1-41u+130u^2}{2(1-u)^6(1+2u)^3\sqrt{2-5u}},$$

the function $\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}$ takes its minimum in the interval $0<u\leqq\frac{1}{10}$ at $u=\frac{41-\sqrt{1161}}{2\cdot 130}=0.0266405$, because $\frac{41+\sqrt{1161}}{2\cdot 130}>\frac{1}{10}$, and its minimum is computed as follows: For $u=0.0266405$, $\sqrt{2-5u}\doteqdot 1.3663079$, $(1-u)^5\doteqdot 0.8737080$, $(1+2u)^2\doteqdot 1.1094008$, and

$$\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}\doteqdot\frac{1.3663079}{0.9692923}\doteqdot 1.40959.$$

Therefore, we obtain

$$\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}>\frac{7}{5}\qquad\text{for } 0<u\leqq\frac{1}{10}. \tag{10.4}$$

Next, the functions $8-24u-18u^2+61u^3$ and $8-12u-28u^2+27u^3+5u^4$ are decreasing for $0<u\leqq 1/10$ are easily seen, and so we obtain

$$8-24u-18u^2+61u^3\geqq 8-\frac{24}{10}-\frac{18}{10^2}+\frac{16}{10^3}=\frac{3^3\cdot 203}{10^3},$$
$$8-12u-28u^2+27u^3+5u^4\geqq 8-\frac{12}{10}-\frac{28}{10^2}+\frac{27}{10^3}+\frac{5}{10^4}=\frac{3^2\cdot 5^2\cdot 291}{10^4} \tag{10.5}$$
$$\text{for } 0<u\leqq\frac{1}{10}.$$

Thus, combining (10.4) and (10.5) with (10.3) and using the inequality

$$\frac{e(n-1)\left(e\left(\frac{n}{2}\right)\right)^2}{\sqrt{e(2n)}\left(e\left(\frac{2n-5}{5}\right)\right)^{5/2}}>\frac{e_9\cdot(e_5)^2}{e^3}\qquad\text{for } n\geqq 10,$$

we obtain the inequality (10.2). Q. E. D.

LEMMA 10.3. *When $n\geqq 10$, we have*

$$\begin{aligned}(10.6)\quad &\frac{n}{12(n-1)^{n-3/2}}\cdot\frac{1}{(2n+1)^{n-3/2}(n+2)^{3/2}(2n-5)^{3/2}(4n^2-2n+7)^2}\\ &\times[4n(n-1)^{n-4}(n+5)(2n+1)^{n-3}(4n^2-2n+7)^2(48n^4-52n^3+342n^2-198n+616)\\ &-3(n-4)(n+2)^{n-2}(2n-5)^{n-2}(256n^6-992n^5+80n^4+4120n^3-3592n^2\\ &-5474n+3010)]<\frac{10\cdot 4267\cdot\sqrt{3\cdot 5\cdot 7}}{3^7\cdot 7^2}-\frac{3\sqrt{3}\cdot 199147}{7\cdot 8^3\cdot 10^3}\,.\end{aligned}$$

*Proof.* Supposing $n\geqq 10$ and setting $1/n=u$, the left hand side of (10.6) is equal to

$$\begin{aligned}&\frac{1}{12(1+2u)^{3/2}(2-5u)^{3/2}(4-2u+7u^2)^2}\cdot\Big[\frac{4(1+5u)(4-2u+7u^2)^2}{(1-u)^{5/2}(2+u)^{3/2}}\\ &\cdot(48-52u+342u^2-198u^3+616u^4)-\frac{3(1-4u)}{\sqrt{1-u}\sqrt{2+u}}\cdot\Big(\frac{n+2}{n-1}\Big)^{n-2}\cdot\Big(\frac{2n-5}{2n+1}\Big)^{n-2}\\ &\cdot(256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6)\Big]\,.\end{aligned}$$

Since we have

$$\Big(\frac{n+2}{n-1}\Big)^{n-2}=\frac{1-u}{1+2u}\Big(e\Big(\frac{n-1}{3}\Big)\Big)^3,\quad \Big(\frac{2n-5}{2n+1}\Big)^{n-2}=\sqrt{\frac{2-5u}{2+u}}\Big(e\Big(\frac{2n-5}{6}\Big)\Big)^{-3},$$

substituting these into the above expression, we obtain

$$\begin{aligned}(10.7)\quad &\frac{(1+5u)(48-52u+342u^2-198u^3+616u^4)}{3(1+2u)^{3/2}(2-5u)^{3/2}(1-u)^{5/2}(2+u)^{3/2}}\\ &-\frac{(1-4u)\sqrt{1-u}}{4(1+2u)^{5/2}(2-5u)(2+u)}\cdot\frac{1}{(4-2u+7u^2)^2}\cdot\Big(e\Big(\frac{n-1}{3}\Big)\Big/e\Big(\frac{2n-5}{6}\Big)\Big)^3\\ &\times(256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6)\,.\end{aligned}$$

Now, we estimate the terms in the above expression. The function $(1+5u)(48-52u+342u^2-198u^3+616u^4)$ is increasing for $0<u\leqq 1/10$, because

$$\begin{aligned}&\frac{d}{du}(1+5u)(48-52u+342u^2-198u^3+616u^4)\\ &=(1+5u)(-52+684u-594u^2+2464u^3)+5(48-52u+342u^2-198u^3+616u^4)\\ &>(1+5u)(-52)+5\Big(48-\frac{52\cdot 13}{171}+\frac{342\cdot 13\cdot 13}{171\cdot 171}-\frac{198}{1000}\Big)\\ &>-\frac{3}{2}\cdot 52+5(48-4+1-1)=142>0\,,\end{aligned}$$

and so we obtain

$$
\begin{aligned}
(10.8)\qquad & (1+5u)(48-52u+342u^2-198u^3+616u^4) \\
& \leqq \frac{3}{2}\left(48-\frac{52}{10}+\frac{342}{10^2}-\frac{198}{10^3}+\frac{616}{10^4}\right)=\frac{2\cdot 3^4\cdot 4267}{10^4} \quad \text{for } 0<u\leqq\frac{1}{10}.
\end{aligned}
$$

The function $(1+2u)^3(2-5u)^3(1-u)^5(2+u)^3$ is decreasing for $0<u\leqq 1/10$, because

$$
\begin{aligned}
& \frac{d}{du}(1+2u)^3(2-5u)^3(1-u)^5(2+u)^3 \\
& = -(1+2u)^2(2-5u)^2(1-u)^4(2+u)^2\{(7+8u)(1+2u)(2-5u) \\
& \quad +3(1+20u)(1-u)(2+u)\}<0
\end{aligned}
$$

and so we obtain

$$
\begin{aligned}
(1+2u)^3(2-5u)^3(1-u)^5(2+u)^3 & > \left(1+\frac{1}{5}\right)^3\left(2-\frac{1}{2}\right)^3\left(\frac{9}{10}\right)^5\left(\frac{21}{10}\right)^3 \\
& = \frac{2^3\cdot 3^{19}\cdot 7^3}{10^{11}},
\end{aligned}
$$

hence

$$
(10.9)\qquad (1+2u)^{3/2}(2-5u)^{3/2}(1-u)^{5/2}(2+u)^{3/2}\geqq\frac{2^2\cdot 3^9\cdot 7}{10^6}\cdot\sqrt{3\cdot 5\cdot 7} \quad \text{for } 0<u\leqq\frac{1}{10}.
$$

The function $256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6$ is decreasing for $0<u\leqq 1/10$, because its derivative is

$$
\begin{aligned}
& -992+160u+12360u^2-14368u^3-27370u^4+18060u^5 \\
& < -992+16+123.6+0.1806<0,
\end{aligned}
$$

and so we obtain

$$
\begin{aligned}
(10.10)\qquad & 256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6 \\
& > 256-\frac{992}{10}+\frac{8}{10}+\frac{412}{10^2}-\frac{3592}{10^4}-\frac{5474}{10^5}+\frac{301}{10^5}=\frac{3^4\cdot 199147}{10^5} \\
& \qquad \text{for } 0<u\leqq\frac{1}{10}.
\end{aligned}
$$

We obtain also easily

$$
(10.11)\qquad (1-4u)\sqrt{1-u}\geqq\left(1-\frac{2}{5}\right)\sqrt{1-\frac{1}{10}}=\frac{3^2}{5\cdot\sqrt{10}} \quad \text{for } 0<u\leqq\frac{1}{10},
$$

$$
(10.12)\qquad \left(e\left(\frac{n-1}{3}\right)\Big/e\left(\frac{2n-5}{6}\right)\right)^3>1.
$$

The function $(1+2u)^{5/2}(2-5u)(2+u)$ is increasing for $0<u\leqq 1/10$, because

$$\frac{d}{du}\{(1+2u)^{5/2}(2-5u)(2+u)\}$$
$$=(1+2u)^{3/2}\{(1+2u)(2-5u)+5(1-7u)(2+u)\}>0,$$

and so we obtain

$$(1+2u)^{5/2}(2-5u)(2+u)\leqq\left(\frac{6}{5}\right)^{5/2}\cdot\frac{3}{2}\cdot\frac{21}{10}=\frac{2^4\cdot 3^4\cdot 7\cdot\sqrt{3}}{10^3\sqrt{10}} \quad \text{for } 0<u\leqq\frac{1}{10}. \tag{10.13}$$

We obtain also easily

$$4-2u+7u^2<4 \qquad \text{for } 0<u\leqq\frac{1}{10}. \tag{10.14}$$

Finally, combining these inequalities (10.8)~(10.14) with (10.7), we obtain

$$\{\text{the expression (10.7) of } u\}$$
$$<\frac{\dfrac{2\cdot 3^4\cdot 4267}{10^4}}{3\times\dfrac{2^2\cdot 3^9\cdot 7}{10^6}\cdot\sqrt{3\cdot 5\cdot 7}}-\frac{\dfrac{3^2}{5\cdot\sqrt{10}}\times 1\times\dfrac{3^4\cdot 199147}{10^5}}{4\times\dfrac{2^4\cdot 3^4\cdot 7\cdot\sqrt{3}}{10^3\sqrt{10}}\times 4^2}$$
$$=\frac{10\cdot 4267\cdot\sqrt{3\cdot 5\cdot 7}}{3^7\cdot 7^2}-\frac{3\sqrt{3}\cdot 199147}{2^3\cdot 4^3\cdot 7\cdot 10^3}\quad(>0)\qquad \text{for } 0<u\leqq\frac{1}{10}.$$

Thus, we obtain the inequality (10.6). Q. E. D.

Now, we have reached to a step to evaluate the sign of $V(x, X(x))$ for $\alpha_1(n)\leqq x<1$ by Lemma 9.1, Lemma 10.1~Lemma 10.3 as follows.

When $n\geqq 10$, for $\alpha_1(n)\leqq x<1$ we obtain

$$\frac{V(x, X(x))}{sB\sqrt{n-1}}>n\left[\frac{2}{3}\cdot\left(\frac{4}{9}\right)^4\cdot\frac{7\cdot 91\cdot\sqrt{110}}{101\cdot 101}+\frac{3^4\cdot 7\cdot 203\cdot 291\cdot e_9\cdot(e_5)^2}{4^4\cdot\sqrt{2}\cdot 10^6\cdot e^3}\right] \tag{10.15}$$
$$-2\left[\frac{10\cdot 4267\sqrt{3\cdot 5\cdot 7}}{3^7\cdot 7^2}-\frac{3\sqrt{3}\cdot 199147}{2^3\cdot 4^3\cdot 7\cdot 10^3}\right].$$

In the following, we evaluate the right hand side of (10.15).

$$\frac{2}{3}\cdot\left(\frac{4}{9}\right)^4\cdot\frac{7\cdot 91\cdot\sqrt{110}}{101\cdot 101}=\frac{326144\sqrt{110}}{19683\cdot 10201}\doteqdot 0.017036.$$

Since we have

$$e_9\cdot(e_5)^2=\left(\frac{10}{9}\right)^9\left(\frac{6}{5}\right)^{10}=\frac{2^{20}}{3^8\cdot 10},$$

hence

$$\frac{3^4\cdot 7\cdot 203\cdot 291\cdot e_9\cdot (e_5)^2}{4^4\cdot\sqrt{2}\cdot 10^6\cdot e^3}=\frac{4^6\cdot 7\cdot 203\cdot 291}{3^4\cdot\sqrt{2}\cdot 10^7\cdot e^3}\doteqdot 0.073615\,.$$

Therefore, we obtain

$$\frac{2}{3}\cdot\left(\frac{4}{9}\right)^4\cdot\frac{7\cdot 91\cdot\sqrt{110}}{101\cdot 101}+\frac{3^4\cdot 7\cdot 203\cdot 291\cdot e_9\cdot (e_5)^2}{4^4\cdot\sqrt{2}\cdot 10^6\cdot e^3}\doteqdot 0.090651\,. \tag{10.16}$$

Next, we have

$$\frac{10\cdot 4267\cdot\sqrt{3\cdot 5\cdot 7}}{3^7\cdot 7^2}-\frac{3\sqrt{3}\cdot 199147}{2^3\cdot 4^3\cdot 7\cdot 10^3}=\frac{42670\cdot\sqrt{105}}{107163}-\frac{597441\cdot\sqrt{3}}{3584000} \tag{10.17}$$

$$\doteqdot 4.080115-0.288727=3.791388\,.$$

By means of (10.16) and (10.17), we obtain the following formula: When $n\geqq 10$,

$$V(x,\,X(x))>\frac{sB\sqrt{n-1}}{100}(9.0651n-379.1388\times 2) \tag{10.18}$$

$$\text{for } \alpha_1(n)\leqq x<1\,.$$

Therefore, if $n>\dfrac{379.138\times 2}{9.0651}\doteqdot 83.6480$, the right hand side of (10.18) is positive. Thus, we obtain

PROPOSITION 4. *When $n\geqq 84$, we have*

$$V(x,\,X(x))>0 \qquad \text{for } \alpha_1(n)\leqq x<1\,.$$

From Proposition 3 and Proposition 4, we obtain the main theorem.

THEOREM C. *The period function $T$ as a function of $\tau$ and $n$ is monotone decreasing with respect to $n\geqq 84$ for any fixed $\tau$ $(0<\tau<1)$.*

In the next paper [14], we shall prove that Conjecture C is also true for $16\leqq n\leqq 84$.

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