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GENERIC SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES WITH PARALLEL MEAN CURVATURE VECTOR

Dedicated to Professor Shigeru Ishihara on his sixtieth birthday

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A submanifold M of a Kaehlerian manifold $\tilde{M}$ is called a generic submanifold (an anti-holomorphic submanifold) if the normal space $N_P(M)$ of M at any point $P \in M$ is always mapped into the tangent space $T_P(M)$ under the action of the almost complex structure tensor F of the ambient manifold, that is, $FN_P(M) \subset T_P(M)$ for all $P \in M$ (see [4], [9], [10] and [12]). The typical examples of generic submanifolds are real hypersurfaces of a Kaehlerian manifold. So many authors, for example, Kon [12], Okumura [9], Pak [9] and Yano [12] etc., have studied generic submanifolds of a Kaehlerian manifold by using the method of Riemannian fibre bundles and developed this method of Lawson [2], Maeda [5] or Okumura [8] extensively for real hypersurfaces.

In particular, two of the present authors [4] have studied generic submanifolds with parallel mean curvature vector of an even-dimensional Euclidean space under the condition that the f-structure induced on M is normal (see section 2).

The purpose of the present paper is to characterize generic submanifolds of complex projective space CP^m.

In § 1, we investigate fundamental properties and structure equations for generic submanifolds immersed in a complex projective space CP^m. And we find the condition that the f-structure induced on M is normal.

In § 2, we recall the theory of fibrations and some relations between the second fundamental tensor of M in CP^m and that of $\bar{M} = \tilde{\pi}^{-1}(M)$ in S^{2m+1}, and then establish some equations for the connections in the normal bundles of M and of $\bar{M}$, where $\tilde{\pi}$ is the projection induced from the Hopf-fibrations $S^1 \to S^{2m+1} \to CP^m$.

In the last § 3, we characterize generic submanifolds of a complex projective space CP^m by the method of Riemannian fibration. In characterizing the submanifolds, we shall use the following theorem:

THEOREM A ([11]). *Let M be a complete n-dimensional submanifold of S^m with flat normal connection. If the second fundamental form of M is parallel, then M is a small sphere, a great sphere or a pythagorean product of a certain number of spheres. Moreover, if M is of essential codimension $m-n$, then M is a pythagorean product of the form*

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$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\ r_1^2+\cdots+r_N^2=1,\ N=m-n+1,$$

or a pythagorean product of the form

$$S^{p_1}(r_1)\times\cdots\times S^{p_{N'}}(r_{N'})\subset S^{m-1}(r)\subset S^m,\ r_1^2+\cdots+r_{N'}^2=r^2<1,\ N'=m-n.$$

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper are assumed to be differentiable and of C^∞, We use in the present paper the systems of indices as follows:

$$\kappa,\ \mu,\ \nu,\ \lambda=1,2,\cdots,2m+1;\quad h,\ i,\ j,\ k=1,2,\cdots,2m,$$

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon=1,2,\cdots,n+1;\quad a,\ b,\ c,\ d,\ e=1,2,\cdots,n,$$

$$u,\ v,\ w,\ x,\ y,\ z=1,2,\cdots,p,\ n+p=2m.$$

The summation convention will be used with respect to those systems of indices.

§1. Generic submanifolds of Kaehlerian manifolds.

Let $\tilde{M}$ be a $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U};y^h\}$ and denote by g_{ji} components of the Hermitian metric tensor and by $F_j{}^h$ those of the almost complex structure of $\tilde{M}$. Then we have

$$F_j{}^tF_t{}^h=-\delta_j^h, \tag{1.1}$$

$$F_j{}^tF_i{}^sg_{ts}=g_{ji}, \tag{1.2}$$

δ_j^h being the Kronecker delta.

And denoting by ∇_j the operator of covariant differentiation with respect to g_{ji}, we get

$$\nabla_jF_i{}^h=0. \tag{1.3}$$

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U;x^a\}$ and immersed isometrically in M by the immersion $i:M\to\tilde{M}$. We identify $i(M)$ with M itself and represent the immersion $i:M\to\tilde{M}$ by

$$y^h=y^h(x^a) \tag{1.4}$$

We put

$$B_b{}^h=\partial_by^h,\qquad \partial_b=\partial/\partial x^b \tag{1.5}$$

and denote by $C_x{}^h$ mutually orthogonal unit normals to M. Then denoting by g_{cb} the fundamental metric tensor of M, we have $g_{cb}=g_{ji}B_c{}^jB_b{}^i$ since the immersion is isometric. Therefore, denoting by ∇_c the operator of van der Waerden-

Bortolotti covariant differentiation with respect to g_{cb}, equations of Gauss and Weingarten for M are given by

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h\,, \tag{1.6}$$

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h \tag{1.7}$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cbx} h^{ba} = h_{cb}{}^y g^{ba} g_{yx}$, $g_{yx} = C_y{}^j C_x{}^i g_{ji}$ being the metric tensor of the normal bundle of M and $(g^{ba}) = (g_{ba})^{-1}$.

Equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}{}^a = K_{kji}{}^h B_{dcbh}^{kjia} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x\,, \tag{1.8}$$

$$K_{kji}{}^h B_{dcb}^{kji} C^x{}_h = \nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x\,, \tag{1.9}$$

$$K_{dcy}{}^x = K_{kji}{}^h B_{dc}^{kj} C_y{}^i C^x{}_h + h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y\,, \tag{1.10}$$

where $B_{dcbh}^{kjia} = B_d{}^k B_c{}^j B_b{}^i B^a{}_h$, $B_{dcb}^{kji} = B_d{}^k B_c{}^j B_b{}^i$, $B^a{}_h = B_b{}^j g^{ba} g_{jh}$, $C^x{}_h = C_y{}^j g^{yx} g_{jh}$, and $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are the curvature tensor of M and that of the connection induced in the normal bundle of M respectively.

From now on, we consider generic submanifolds of a Kaehlerian manifold $\tilde{M}$. Then we can put in each neighborhood

$$F_j{}^h B_c{}^j = f_c{}^a B_a{}^h - f_c{}^x C_x{}^h\,, \tag{1.11}$$

$$F_j{}^h C_x{}^j = f_x{}^a B_a{}^h\,, \tag{1.12}$$

where $f_c{}^a$ is a tensor field of type (1.1) defined on M, $f_c{}^x$ that of mixed type and $f_x{}^a = f_c{}^y g^{ca} g_{yx}$.

Applying F to (1.11) and (1.12) respectively and using (1.1) and those equations, we can easily find

$$f_c{}^e f_e{}^a = -\delta_c^a + f_c{}^x f_x{}^a\,, \tag{1.13}$$

$$f_c{}^e f_e{}^x = 0\,, \qquad f_x{}^e f_e{}^a = 0\,, \tag{1.14}$$

$$f_e{}^x f_y{}^e = \delta_y^x\,. \tag{1.15}$$

Therefore, equations (1.13)~(1.15) show that M admits the so-called f-structure satisfying $f^3 + f = 0$ (cf. [6] and [7] etc.).

Using $F_{ji} = -F_{ij}$, $F_{ji} = F_j{}^h g_{ih}$, we have from (1.11) and (1.12),

$$f_{cb} = -f_{bc}\,, \qquad f_{cx} = f_{xc}\,, \tag{1.16}$$

where we have put $f_{cb} = f_c{}^a g_{ba}$, $f_{bx} = f_b{}^y g_{yx}$ and $f_{xb} = f_x{}^a g_{ba}$.

If we apply the operator ∇_c of the covariant differentiation to (1.11) and take account of (1.3), then we obtain

$$F_i^{\ h}\nabla_c B_b^{\ j}=(\nabla_c f_b^{\ a})B_a^{\ h}+f_b^{\ a}\nabla_c B_a^{\ h}-(\nabla_c f_b^{\ x})C_x^{\ h}-f_b^{\ x}\nabla_c C_x^{\ h},$$

or, substituting (1.6) and (1.7),

$$\nabla_c f_b^{\ a}=h_{cb}^{\ \ x}f_x^{\ a}-h_c^{\ a}{}_x f_b^{\ x}, \tag{1.17}$$

$$\nabla_c f_b^{\ x}=h_{ce}^{\ \ x}f_b^{\ e}. \tag{1.18}$$

By the same way we have from (1.12)

$$\nabla_c f_x^{\ a}=h_{cex}f^{ae}, \tag{1.19}$$

$$f_x^{\ e}h_{be}^{\ \ y}=h_b^{\ e}{}_x f_e^{\ y} \tag{1.20}$$

with the help of (1.6) and (1.7).

We now assume that the ambient manifold $\tilde{M}$ is of constant holomorphic sectional curvature c. Then it is well known that its curvature tensor $K_{kji}^{\ \ \ h}$ has the form

$$K_{kji}^{\ \ \ h}=\frac{c}{4}(\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k^{\ h}F_{ji}-F_j^{\ h}F_{ki}-2F_{kj}F_i^{\ h}). \tag{1.21}$$

Therefore, substituting (1.21) into (1.8), (1.9) and (1.10), we can see that the equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}^{\ \ \ a}=\frac{c}{4}(\delta_d^a g_{cb}-\delta_c^a g_{db}+f_d^{\ a}f_{cb}-f_c^{\ a}f_{db}-2f_{dc}f_b^{\ a})+h_d^{\ a}{}_x h_{cb}^{\ \ x}-h_c^{\ a}{}_x h_{db}^{\ \ x}, \tag{1.22}$$

$$\nabla_d h_{cb}^{\ \ x}-\nabla_c h_{db}^{\ \ x}=\frac{c}{4}(-f_d^{\ x}f_{cb}+f_c^{\ x}f_{db}+2f_{dc}f_b^{\ x}), \tag{1.23}$$

$$K_{dcy}^{\ \ \ x}=\frac{c}{4}(f_d^{\ x}f_{cy}-f_c^{\ x}f_{dy})+h_{de}^{\ \ x}h_c^{\ e}{}_y-h_{ce}^{\ \ x}h_d^{\ e}{}_y. \tag{1.24}$$

We now consider a tensor field S of type (1, 2) of the form

$$S_{cb}^{\ \ a}=[f,f]_{cb}^{\ \ a}+(\nabla_c f_b^{\ x}-\nabla_b f_c^{\ x})f_x^{\ a},$$

where

$$[f,f]_{cb}^{\ \ a}=f_c^{\ e}\nabla_e f_b^{\ a}-f_b^{\ e}\nabla_e f_c^{\ a}-(\nabla_c f_b^{\ e}-\nabla_b f_c^{\ e})f_e^{\ a}$$

is the Nijenhuis tensor formed with $f_c^{\ a}$.

Substituting (1.17) and (1.18) into this, we find

$$S_{cb}^{\ \ a}=(h_c^{\ e}{}_x f_e^{\ a}-f_c^{\ e}h_e^{\ a}{}_x)f_b^{\ x}-(h_b^{\ e}{}_x f_e^{\ a}-f_b^{\ e}h_e^{\ a}{}_x)f_c^{\ x}. \tag{1.25}$$

The induced f-structure on M is said to be *normal* if $S_{cb}^{\ \ a}$ vanishes identically (cf. [4]).

The left hand side of (1.25) does not depend on the choice of the unit normals $C_x^{\ h}$. Indeed, if we choose another set of mutually orthogonal unit normals $'C_x^{\ h}$, then we have

(1.26) $$'C_x{}^h = \sigma_x{}^y C_y{}^h ,$$

$(\sigma_x{}^y)$ being a special orthogonal matrix of degree $2m-n$.

Defining the second fundamental tensor $'h_{cb}{}^x$ with respect to $'C_x{}^h$ by $\nabla_c B_b{}^h = 'h_{cb}{}^x{}'C_x{}^h$, then we have from (1.6) and (1.26)

(1.27) $$'h_{cb}{}^x = \sigma_y{}^x h_{cb}{}^y .$$

Also, we have from (1.11) and (1.26)

$$'f_c{}^x = \sigma_y{}^x f_c{}^y$$

Consequently we have

$$(h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x) f_b{}^x - (h_b{}^e{}_x f_e{}^a - f_b{}^e h_e{}^a{}_x) f_c{}^x$$
$$= ('h_c{}^e{}_x f_e{}^a - f_c{}^{e\prime} h_e{}^a{}_x)' f_b{}^x - ('h_b{}^e{}_x f_e{}^a - f_b{}^{e\prime} h_e{}^a{}_x)' f_c{}^x$$

because of $\sigma_z{}^x \sigma_{yx} = g_{zy}$. This shows that the condition imposed on M is of intrinsic character.

Suppose that $S_{cb}{}^a$ vanishes identically on M, we have from (1.25)

$$(h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x) f_b{}^x - (h_b{}^e{}_x f_e{}^a - f_b{}^e h_e{}^a{}_x) f_c{}^x = 0 ,$$

from which, transvecting f^{cy},

$$h_{ae}{}^y f_b{}^e + h_{be}{}^y f_a{}^e = h_{cex} f_a{}^e f^{cy} f_b{}^x ,$$

from which, taking the skew-symmetric part and then transvecting $f_z{}^b$, we get $h_{cez} f_a{}^e f^{cy} = 0$.

Therefore, we obtain

(1.28) $$h_{be}{}^x f_a{}^e + h_{ae}{}^x f_b{}^e = 0 .$$

Hence we have

PROPOSITION 1.1. *Let M be a generic submanifold of a Kaehlerian manifold $\tilde{M}$. In order for the f-structure induced on M to be normal, it is necessary and sufficient that the second fundamental tensors $h_{cb}{}^x$ and $f_c{}^a$ commute.*

§ 2. Submersion $\tilde{\pi}: S^{2m+1} \to CP^m$ and immersion $\imath: M \to CP^m$.

Let $S^{2m+1}(1)$ be the hypersphere $\{(c^1, \cdots, c^{m+1}) \mid |c^1|^2 + \cdots + |c^{m+1}|^2 = 1\}$ of radius 1 in the $(m+1)$-dimensional complex space C^{m+1}, which will be identified naturally with $R^{2(m+1)}$. The sphere $S^{2m+1}(1)$ will be simply denoted by S^{2m+1}. Let $\tilde{\pi}: S^{2m+1} \to CP^m$ be the natural projection of S^{2m+1} onto a complex projective space CP^m which is defined by the Hopf-fibration.

We consider a Riemannian submersion $\pi: \bar{M} \to M$ compatible with the Hopf-fibration $\tilde{\pi}: S^{2m+1} \to CP^m$, where M is a submanifold of codimension p in CP^m and

$\bar{M}=\tilde{\pi}^{-1}(M)$ that of S^{2m+1}. More precisely speaking, $\pi:\bar{M}\rightarrow M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram is commutative:

$$\begin{tikzcd} \bar{M} \arrow[r, "\bar{\imath}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\curvearrowright"] & S^{2m+1} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "\imath"] & CP^m \end{tikzcd}$$

where $\bar{\imath}:\bar{M}\rightarrow S^{2m+1}$ and $\imath: M\rightarrow CP^m$ are certain isometric immersions.

Covering S^{2m+1} by a system of coordinate neighborhoods $\{\hat{U};\, y^\kappa\}$ such that $\tilde{\pi}(\hat{U})=U$ are coordinate neighborhoods of CP^m with local coordinate (y^h), we represent the projection $\tilde{\pi}: S^{2m+1}\rightarrow CP^m$ by

$$y^h=y^h(y^\kappa) \tag{2.1}$$

and put

$$E_\kappa{}^h=\partial_\kappa y^h\,, \qquad \partial_\kappa=\partial/\partial y^\kappa\,, \tag{2.2}$$

the rank of the matrix $(E_\kappa{}^h)$ being always $2m$.

Let's denote by $\tilde{\xi}^\kappa$ components of $\tilde{\xi}$ the unit Sasakian structure vector in S^{2m+1}. Since the unit vector field $\tilde{\xi}$ is always tangent to the fibre $\tilde{\pi}^{-1}(\tilde{P})$, $\tilde{P}\in CP^m$ everywhere, $E_\kappa{}^h$ and $\tilde{\xi}_\kappa$ form a local coframe in S^{2m+1}, where $\tilde{\xi}_\kappa=g_{\kappa\mu}\tilde{\xi}^\mu$ and $g_{\kappa\mu}$ denote the Riemannian metric tensor of S^{2m+1}. We denote by $\{E^\kappa{}_j,\, \tilde{\xi}^\kappa\}$ the frame corresponding to the coframe $\{E_\kappa{}^h,\, \tilde{\xi}_\kappa\}$. We then have

$$E_\kappa{}^j E^\kappa{}_i=\delta_i^j\,, \qquad E_\kappa{}^j\tilde{\xi}^\kappa=0\,, \qquad \tilde{\xi}_\kappa E^\kappa{}_i=0\,. \tag{2.3}$$

We now take coordinate neighborhoods $\{\bar{U};\, x^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinate (x^a). Let the isometric immersions $\bar{\imath}$ and $\imath$ be locally expressed by $y^\kappa=y^\kappa(x^\alpha)$ and $y^h=y^h(x^a)$ in terms of local coordinates (x^α) in $\bar{U}\ (\subset\bar{M})$ and (x^a) in $U\ (\subset M)$ respectively. Then the commutativity $\tilde{\pi}\circ\bar{\imath}=\imath\circ\pi$ of the diagram implies

$$y^h(x^a(x^\alpha))=y^h(y^\kappa(x^\alpha))\,,$$

where we expressed the submersion π by $x^a=x^a(x^\alpha)$ locally, and hence

$$B_a{}^j E_\alpha{}^a=E_\kappa{}^j B_\alpha{}^\kappa\,, \tag{2.4}$$

$B_a{}^j=\partial_a y^j$, $B_\alpha{}^\kappa=\partial_\alpha y^\kappa$ and $E_\alpha{}^a=\partial_\alpha x^a$.

For an arbitrary point $P\in M$ we choose unit normal vector fields $C_x{}^j$ to M defined in a neighborhood U of P in such a way that $\{B_a{}^j,\, C_x{}^j\}$ spans the tangent space of CP^m at $i(P)$. Let $\bar{P}$ be an arbitrary point of the fibre $\pi^{-1}(P)$ over P

then the lifts $C_x{}^\kappa = C_x{}^j E^\kappa{}_j$ of $C_x{}^j$ are unit normal vector fields to $\bar{M}$ defined in the tubular neighborhood over U because of (2.4). Since $\tilde{\xi}^\kappa E_\kappa{}^j = 0$, we can represent by

$$\tilde{\xi}^\kappa = \xi^\alpha B_\alpha{}^\kappa, \tag{2.5}$$

where ξ^α is a local vector field in $\bar{M}$. Using (2.4) and (2.5), we find

$$\xi_\alpha \xi^\alpha = 1, \qquad \xi^\alpha E_\alpha{}^a = 0, \tag{2.6}$$

where $\xi_\alpha = \xi^\beta g_{\beta\alpha}$ and $g_{\beta\alpha}$ is the Riemannian metric tensor of $\bar{M}$ induced from that of S^{2m+1}. Therefore, $\{E_\alpha{}^a, \xi_\alpha\}$ is a local coframe in $\bar{M}$ induced from that of S^{2m+1}. Denote by $\{E^\alpha{}_a, \xi^\alpha\}$ the frame corresponding to this coframe $\{E_\alpha{}^a, \xi_\alpha\}$, we have

$$E_\alpha{}^b E^\alpha{}_a = \delta_a^b, \qquad \xi_\alpha E^\alpha{}_b = 0, \tag{2.7}$$

and consequently

$$E^\kappa{}_j B_b{}^j = B_\alpha{}^\kappa E^\alpha{}_b \tag{2.8}$$

with the help of (2.4) and (2.6).

Denoting by $\left\{ {\lambda \atop \mu\ \nu} \right\}$, $\left\{ {i \atop j\ h} \right\}$, $\left\{ {\alpha \atop \beta\ \gamma} \right\}$ and $\left\{ {a \atop c\ b} \right\}$ the Christoffel symbols formed with the Riemannian metric $g_{\mu\lambda}$, g_{ji}, $g_{\beta\alpha}$ and g_{ba} respectively, we put

$$D_\mu E_\lambda{}^i = \partial_\mu E_\lambda{}^i - \left\{ {\kappa \atop \mu\ \lambda} \right\} E_\kappa{}^i + \left\{ {h \atop j\ i} \right\} E_\mu{}^j E_\lambda{}^h,$$

$$D_\mu E^\lambda{}_i = \partial_\mu E^\lambda{}_i + \left\{ {\lambda \atop \mu\ \kappa} \right\} E^\kappa{}_i - \left\{ {h \atop j\ i} \right\} E_\mu{}^j E^\lambda{}_h,$$

and

$$\bar{\nabla}_\beta E_\alpha{}^a = \partial_\beta E_\alpha{}^a - \left\{ {\gamma \atop \beta\ \alpha} \right\} E_\gamma{}^a + \left\{ {a \atop b\ c} \right\} E_\beta{}^b E_\alpha{}^c,$$

$$\bar{\nabla}_\beta E^\alpha{}_a = \partial_\beta E^\alpha{}_a + \left\{ {\alpha \atop \beta\ \gamma} \right\} E^\gamma{}_a - \left\{ {c \atop b\ a} \right\} E_\beta{}^b E^\alpha{}_c.$$

Since the metrics $g_{\lambda\mu}$ and $g_{\alpha\beta}$ are both invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of $E_\lambda{}^i$, $E^\lambda{}_i$ and $E_\alpha{}^a$, $E^\alpha{}_a$ are given by

$$\begin{cases} D_\mu E_\lambda{}^i = h_j{}^i (E_\mu{}^j \tilde{\xi}_\lambda + \tilde{\xi}_\mu E_\lambda{}^j), \\ D_\mu E^\lambda{}_i = h_{ji} E_\mu{}^j \tilde{\xi}^\lambda - h_i{}^j \tilde{\xi}_\mu E^\lambda{}_j, \end{cases} \tag{2.9}$$

$$\begin{cases} \bar{\nabla}_\beta E_\alpha{}^a = h_b{}^a (E_\beta{}^b \xi_\alpha + \xi_\beta E_\alpha{}^b), \\ \bar{\nabla}_\beta E^\alpha{}_a = h_{ba} E_\beta{}^b \xi^\alpha - h_a{}^b \xi_\beta E^\alpha{}_b \end{cases} \tag{2.10}$$

respectively, where $h_j{}^i = h_{jh} g^{ih}$, $h_b{}^a = h_{bc} g^{ca}$, h_{ji} and h_{ba} being the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (see Ishihara and Konishi [3]).

On the other hand the equations of Gauss and Weingarten for the immersion $\bar{\imath}: \bar{M} \to S^{2m+1}$ are given by

$$\bar{\nabla}_\beta B_\alpha{}^\kappa = h_{\beta\alpha}{}^x C_x{}^\kappa, \qquad \bar{\nabla}_\beta C_x{}^\kappa = -h_\beta{}^\alpha{}_x B_\alpha{}^\kappa, \tag{2.11}$$

and those for the immersion $\imath: M \to CP^m$ by

$$\nabla_b B_a{}^h = h_{ba}{}^x C_x{}^h, \qquad \nabla_b C_x{}^h = -h_b{}^a{}_x B_a{}^h, \tag{2.12}$$

where $h_\beta{}^\alpha{}_x = h_{\beta\gamma}{}^y g^{\gamma\alpha} g_{yx}$, $h_{\beta\alpha}{}^x$ and $h_{ba}{}^x$ are the second fundamental tensors of $\bar{M}$ and M with respect to the unit normals $C_x{}^\kappa$ and $C_x{}^h$ respectively. Moreover, in such a case, (2.4) and (2.8) imply

$$\nabla_b = E^\alpha{}_b \bar{\nabla}.$$

We now put $F_\mu{}^\lambda = D_\mu \tilde{\xi}^\lambda$. Then we have by definition of the Sasakian structure

$$F_\mu{}^\lambda F_\kappa{}^\mu = -\delta_\kappa^\lambda + \tilde{\xi}_\kappa \tilde{\xi}^\lambda, \quad F_\mu{}^\lambda \tilde{\xi}^\mu = 0, \quad \tilde{\xi}_\lambda F_\mu{}^\lambda = 0, \quad F_{\mu\lambda} + F_{\lambda\mu} = 0 \tag{2.13}$$

and

$$D_\mu F_\lambda{}^\kappa = \tilde{\xi}_\lambda \delta_\mu^\kappa - \tilde{\xi}^\kappa g_{\mu\lambda}, \qquad D_\mu \tilde{\xi}^\lambda = F_\mu{}^\lambda, \tag{2.14}$$

where $F_{\mu\lambda} = g_{\kappa\lambda} F_\mu{}^\lambda$. Denoting by $\mathcal{L}$ the Lie differentiation with respect to the vector field $\tilde{\xi}$, we find

$$\mathcal{L} F_\mu{}^\lambda = 0. \tag{2.15}$$

Putting in each neighborhood U

$$F_j{}^i = F_\mu{}^\lambda E^\mu{}_j E_\lambda{}^i, \tag{2.16}$$

we can see that $F_j{}^i$ defines a global tensor field of the same type of $F_\mu{}^\lambda$, which will be denoted by the same letter, with the help of (2.15), $\mathcal{L} E^\lambda{}_j = 0$ and $\mathcal{L} E_\lambda{}^i = 0$. Moreover, using (2.9), (2.14) and (2.16), we easily see

$$F_j{}^i = -h_j{}^i, \tag{2.17}$$

which satisfies

$$F_j{}^h F_h{}^i = -\delta_j^i. \tag{2.18}$$

Differentiating (2.6) covariantly along CP^m and using (2.9) and (2.14), we have

$$\nabla_i F_j{}^h = 0 \tag{2.19}$$

where ∇ denotes the projection of D. Hence the base space CP^m admits a Kaehlerian structure $\{F_j{}^i, g_{ji}\}$ which is represented by the structure tensor $h_j{}^i$

of the submersion $\tilde{\pi}: S^{2m+1} \to CP^m$ defined by the Hopf-fibration.

Let's denote by $K_{\kappa\mu\nu}{}^{\lambda}$ and $K_{kji}{}^{h}$ components of the curvature tensors of $(S^{2m+1}, g_{\lambda\mu})$ and (CP^m, g_{ji}) respectively. Since the unit sphere is a space of constant curvature 1, using the equations of co-Gauss, we have

$$K_{kji}{}^{h} = K_{\kappa\mu\lambda}{}^{\lambda} E^{\kappa}{}_{k} E^{\mu}{}_{j} E^{\nu}{}_{i} E_{\lambda}{}^{h} + h_{k}{}^{h} h_{ji} - h_{j}{}^{h} h_{ki} - 2h_{kj} h_{i}{}^{h}$$

and together with (2.17)

$$K_{kji}{}^{h} = \delta_k^h g_{ji} - \delta_j^h g_{ki} + F_{k}{}^{h} F_{ji} - F_{j}{}^{h} F_{ki} - 2F_{kj} F_{i}{}^{h}.$$

Hence CP^m is a Kaehlerian manifold with constant holomorphic sectional curvature 4 (cf. Ishihara and Konishi [3]).

Putting

$$F_{i}{}^{h} B_{b}{}^{i} = f_{b}{}^{a} B_{a}{}^{h} - f_{b}{}^{x} C_{x}{}^{h}, \qquad F_{i}{}^{h} C_{x}{}^{i} = f_{x}{}^{a} B_{a}{}^{h}, \tag{2.20}$$

as already shown in § 1, we can easily find the algebraic relations (1.13)～(1.16) are the structure equations (1.17)～(1.24) with $c=4$ which will be very useful. Now we put in each neighborhood $\bar{U}$ of $\bar{M}$

$$f_{\beta}{}^{\alpha} = f_{b}{}^{a} E_{\beta}{}^{b} E^{\alpha}{}_{a}. \qquad f_{x}{}^{\alpha} = f_{x}{}^{a} E^{\alpha}{}_{a}, \qquad f_{\alpha}{}^{x} = f_{a}{}^{x} E_{\alpha}{}^{a}, \tag{2.21}$$

where here and in the sequel we denote the lifts of functions by the same letters as those the given functions. Then, using (2.4), (2.8), (2.20) and (2.21) and taking account of $C_{x}{}^{\kappa} = C_{x}{}^{j} E^{\kappa}{}_{j}$, we obtain

$$F_{\mu}{}^{\kappa} B_{\alpha}{}^{\mu} = f_{\alpha}{}^{\beta} B_{\beta}{}^{\kappa} - f_{\alpha}{}^{x} C_{x}{}^{\kappa}, \tag{2.22}$$

$$F_{\mu}{}^{\kappa} C_{x}{}^{\mu} = f_{x}{}^{\alpha} B_{\alpha}{}^{\kappa}. \tag{2.23}$$

Transvecting $F_{\kappa}{}^{\mu}$ to (2.22) and (2.23) respectively and using (2.13), (2.22) and (2.23) in the usual way, we can easily obtain that

$$f_{\alpha}{}^{\gamma} f_{\gamma}{}^{\beta} = -\delta_\beta^\alpha + f_{x}{}^{\beta} f_{\alpha}{}^{x} + \xi_{\alpha} \xi^{\beta}, \tag{2.24}$$

$$f_{\alpha}{}^{\gamma} f_{\gamma}{}^{x} = 0, \qquad f_{x}{}^{\gamma} f_{\gamma}{}^{\alpha} = 0, \tag{2.25}$$

$$f_{x}{}^{\gamma} f_{\gamma}{}^{y} = \delta_x^y, \tag{2.26}$$

$$f_{\alpha}{}^{\gamma} \xi_{\gamma} = 0, \qquad \xi^{\gamma} f_{\gamma}{}^{\alpha} = 0, \tag{2.27}$$

$$f_{\gamma}{}^{x} \xi^{\gamma} = 0, \qquad \xi_{\gamma} f_{x}{}^{\gamma} = 0, \tag{2.28}$$

$$f_{\beta\alpha} = -f_{\alpha\beta}, \qquad f_{\alpha x} = f_{x\alpha}, \tag{2.29}$$

where we have put $f_{\beta\alpha} = f_{\beta}{}^{\gamma} g_{\gamma\alpha}$, $f_{\alpha x} = f_{\alpha}{}^{y} g_{yx}$, $f_{x\alpha} = f_{x}{}^{\beta} g_{\beta\alpha}$. Applying the operator $\bar{\nabla}_{\beta} = B_{\beta}{}^{\kappa} D_{\kappa}$ to (2.22) and (2.23) respectively and making use of (2.11), (2.14), (2.22) and (2.23), we also find

$$\bar{\nabla}_{\gamma} f_{\beta}{}^{\alpha} = -g_{\gamma\beta} \xi^{\alpha} + \delta_\gamma^\alpha \xi_{\beta} + h_{\gamma\beta}{}^{x} f_{x}{}^{\alpha} - h_{\gamma}{}^{\alpha}{}_{x} f_{\beta}{}^{x}. \tag{2.30}$$

$$\bar{\nabla}_\beta f_\alpha{}^x = h_{\beta\gamma}{}^x f_\alpha{}^\gamma, \qquad \bar{\nabla}_\beta f_x{}^\alpha = -h_\beta{}^\gamma{}_x f_\gamma{}^\alpha, \tag{2.31}$$

$$h_{\beta\alpha}{}^y f_x{}^\alpha = h_\beta{}^\alpha{}_x f_\alpha{}^y. \tag{2.32}$$

Also, applying the operator $\bar{\nabla}_\beta$ to (2.5) and taking account of (2.11) and (2.14), we have

$$\bar{\nabla}_\beta \xi^\alpha = f_\beta{}^\alpha, \qquad \xi^\alpha h_{\beta\alpha}{}^x = -f_\beta{}^x, \qquad h_\beta{}^\alpha{}_x \xi^\beta = -f_x{}^\alpha, \tag{2.33}$$

which and (2.9) and (2.21) imply

$$f_b{}^a = -h_b{}^a. \tag{2.34}$$

Moreover, in such a submanifold $\bar{M}$, equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{\delta\gamma\beta}{}^\alpha = \delta_\delta^\alpha g_{\gamma\beta} - \delta_\gamma^\alpha g_{\delta\beta} + h_\delta{}^\alpha{}_x h_{\gamma\beta}{}^x - h_\gamma{}^\alpha{}_x h_{\delta\beta}{}^x, \tag{2.35}$$

$$\bar{\nabla}_\gamma h_{\beta\alpha}{}^x - \bar{\nabla}_\beta h_{\gamma\alpha}{}^x = 0, \tag{2.36}$$

$$K_{\beta\alpha y}{}^x = h_{\beta\gamma}{}^x h_\alpha{}^\gamma{}_y - h_{\alpha\gamma}{}^x h_\beta{}^\gamma{}_y, \tag{2.37}$$

where $K_{\delta\gamma\beta}{}^\alpha$ and $K_{\beta\alpha y}{}^\alpha$ are components of the curvature tensor of $\bar{M}$ and those of the normal bundle of $\bar{M}$ respectively because the ambient manifold S^{2m+1} is a space of constant curvature 1.

Now we apply the operator $\nabla_b = B_b{}^j \nabla_j = E^\gamma{}_b \bar{\nabla}_\gamma$ to (2.4). Then, using (2.11) and (2.12), we have

$$h_{ba}{}^x C_x{}^j E_\alpha{}^a + B_a{}^j E^\gamma{}_b \bar{\nabla}_\gamma E_\alpha{}^a = B_b{}^j E^\mu{}_i (D_\mu E_\kappa{}^j) B_\alpha{}^\kappa + E_\kappa{}^j E^\gamma{}_b h_{\gamma\alpha}{}^x C_x{}^\kappa,$$

from which, taking account of (2.9), (2.10) and (2.34),

$$h_{ba}{}^x C_x{}^j E_\alpha{}^a - f_b{}^a B_a{}^j \xi_\alpha = -F_i{}^j B_b{}^j \xi_\alpha + (h_{\beta\alpha}{}^x E^\beta{}_b) C_x{}^j,$$

or, using (2.20),

$$h_{\beta\alpha}{}^x E^\beta{}_b = h_{ba}{}^x E_\alpha{}^a - f_b{}^x \xi_\alpha. \tag{2.38}$$

Transvecting (2.38) with $E_\gamma{}^b$ and changing the index γ with β, we get

$$h_{\beta\alpha}{}^x = h_{ba}{}^x E_\beta{}^b E_\alpha{}^a - f_\beta{}^x \xi_\alpha - \xi_\beta f_\alpha{}^x \tag{2.39}$$

with the help of (2.21) and (2.33).

Thus we have

LEMMA 2.1. *The mean curvature of $\bar{M}$ is the same as that of M.*

Therefore, from now on, we write $h_e{}^{ex}$ and $h_\alpha{}^{\alpha x}$ as the same letter h^x Moreover, the mean curvature vector $\bar{M}$ is given by

$$H^{\kappa}=\frac{1}{n+1}h^{x}C_{x}{}^{\kappa}.$$

The mean curvature vector H is said to be parallel in the normal bundle of $\bar{M}$ if $\bar{\nabla}_{\beta}h^{x}=0$. Hence, as a direct consequence of Lemma 2.1, we have

LEMMA 2.2. *The mean curvature vector of $\bar{M}$ is parallel in the normal bundle of $\bar{M}$ if and only if the mean curvature vector of M is parallel in the normal bundle of M.*

Transvecting $h_{\gamma}{}^{\alpha}{}_{y}$ to (2.39) and using (2.21), (2.26), (2.28) and (2.38) imply

$$h_{\beta\gamma}{}^{x}h_{\alpha}{}^{\gamma}{}_{y}=(h_{ba}{}^{x}h_{c}{}^{a}{}_{y}+f_{b}{}^{x}f_{cy})E_{\beta}{}^{b}E_{\alpha}{}^{c}-h_{ba}{}^{x}f_{y}{}^{a}E_{\beta}{}^{b}\xi_{\alpha}$$
$$-h_{b}{}^{a}{}_{y}f_{a}{}^{x}\xi_{\beta}E_{\alpha}{}^{b}+(f_{\gamma}{}^{x}f_{y}{}^{\gamma})\xi_{\beta}\xi_{\alpha}, \tag{2.40}$$

which and (1.20) gives

$$h_{\beta\gamma}{}^{x}h_{\alpha}{}^{\gamma}{}_{y}-h_{\alpha\gamma}{}^{x}h_{\beta}{}^{\gamma}{}_{y}=(f_{d}{}^{x}f_{cy}-f_{c}{}^{x}f_{dy}+h_{de}{}^{x}h_{c}{}^{e}{}_{y}-h_{ce}{}^{x}h_{d}{}^{e}{}_{y})E_{\beta}{}^{d}E_{\alpha}{}^{c},$$

that is,

$$K_{\beta\alpha y}{}^{x}=K_{dcy}{}^{x}E_{\beta}{}^{d}E_{\alpha}{}^{c}.$$

Thus we obtain

LEMMA 2.3. *In order that the connection in the normal bundle of $\bar{M}$ in S^{2m+1} is flat, it is necessary and sufficient that the connection in the normal bundle of M in CP^{m} is flat.*

§ 3. Generic submanifolds of a complex projective space admitting the normal f-structure.

In this section we assume that the f-structure induced on M in CP^{m} is normal and the normal connection of M is flat, that is,

$$h_{be}{}^{x}f_{a}{}^{e}+h_{ae}{}^{x}f_{b}{}^{e}=0 \tag{3.1}$$

and

$$f_{d}{}^{x}f_{cy}-f_{c}{}^{x}f_{dy}+h_{de}{}^{x}h_{c}{}^{e}{}_{y}-h_{ce}{}^{x}h_{d}{}^{e}{}_{y}=0 \tag{3.2}$$

with the help of (1.24) with $c=4$.

Transvecting (3.1) with $f_{c}{}^{a}$ and making use of (1.13), we obtain

$$h_{cb}{}^{x}-(h_{be}{}^{x}f_{y}{}^{e})f_{c}{}^{y}+h_{de}{}^{x}f_{c}{}^{d}f_{b}{}^{e}=0,$$

from which, taking the skew-symmetric part with respect to c and b,

$$(h_{ce}{}^{x}f_{y}{}^{e})f_{b}{}^{y}-(h_{be}{}^{x}f_{y}{}^{e})f_{c}{}^{y}=0.$$

Transvection $f_{z}{}^{c}$ gives

(3.3) $$h_{be}{}^{x}f_{y}{}^{e}=P_{yz}{}^{x}f_{b}{}^{z},$$

where we have put

(3.4) $$P_{yz}{}^{x}=h_{cb}{}^{x}f_{y}{}^{c}f_{z}{}^{b}.$$

Putting $P_{yzx}=P_{yz}{}^{w}g_{wx}$, we see that P_{yzx} is symmetric for all indices because of (1.20) and (3.4). Also, transvecting (3.2) with $f_{z}{}^{c}$ and using (3.3), we find

$$P_{zy}{}^{u}P_{uw}{}^{x}f_{d}{}^{w}-P_{zu}{}^{x}P_{yw}{}^{u}f_{d}{}^{w}=\delta_{z}^{x}f_{dy}-f_{d}{}^{x}g_{yz},$$

or, using (1.15),

(3.5) $$P_{zy}{}^{u}P_{uw}{}^{x}-P_{zu}{}^{x}P_{yw}{}^{u}=\delta_{z}^{x}g_{yw}-\delta_{w}^{x}g_{yz},$$

from which,

(3.6) $$P_{zu}{}^{x}P_{yx}{}^{u}=P_{x}P_{yz}{}^{x}+(p-1)g_{yz},$$

where we have put

(3.7) $$P^{x}=g^{yz}P_{yz}{}^{x}.$$

Now we prove

LEMMA 3.1. *Let M be an n-dimensional generic submanifold of CP^{m} with flat normal connection. If the f-structure induced on M is normal, then we have*

(3.8) $$h_{\beta\alpha}{}^{x}h_{\gamma}{}^{\alpha}{}_{y}=P_{yz}{}^{x}h_{\beta\gamma}{}^{z}+g_{\beta\gamma}\delta_{y}^{x}.$$

Proof. Differentiating (3.3) covariantly along M and then taking the skew-symmetric part of what obtained thus, we have

$$(\nabla_{c}h_{be}{}^{x}-\nabla_{b}h_{ce}{}^{x})f_{y}{}^{e}+h_{ce}{}^{x}h_{b}{}^{a}{}_{y}f_{a}{}^{e}-h_{be}{}^{x}h_{c}{}^{a}{}_{y}f_{a}{}^{e}$$
$$=(\nabla_{c}P_{yz}{}^{x})f_{b}{}^{z}-(\nabla_{b}P_{yz}{}^{x})f_{c}{}^{z}+P_{yz}{}^{x}h_{ce}{}^{z}f_{b}{}^{e}-P_{yz}{}^{x}h_{be}{}^{z}f_{c}{}^{e}$$

with the help of (1.18). Substituting (1.23) with $c=4$ and making use of (3.2), then it must be that

(3.9) $$2f_{cb}\delta_{y}^{x}+2h_{ce}{}^{x}h_{a}{}^{e}{}_{y}f_{b}{}^{a}=(\nabla_{c}P_{yz}{}^{x})f_{b}{}^{z}-(\nabla_{b}P_{yz}{}^{x})f_{c}{}^{z}+2P_{yz}{}^{x}h_{ce}{}^{z}f_{b}{}^{e}$$

with the help of (3.1). Transvecting (3.9) with $f_{w}{}^{b}$ and using (1.14) and (1.15),

$$\nabla_{c}P_{yw}{}^{x}=f_{w}{}^{b}(\nabla_{b}P_{yz}{}^{x})f_{c}{}^{z}.$$

Therefore (3.9) reduces to

(3.10) $$f_{cb}\delta_{y}^{x}+h_{ce}{}^{x}h_{a}{}^{e}{}_{y}f_{b}{}^{a}=P_{yz}{}^{x}h_{ce}{}^{z}f_{b}{}^{e}$$

with the help of $P_{yz}{}^{x}=P_{zy}{}^{x}$. Transvecting (3.10) with $f_{d}{}^{b}$, we have

(3.11) $$(g_{cd}-f_{c}{}^{z}f_{dz})\delta_{y}^{x}-h_{ce}{}^{x}h_{d}{}^{e}{}_{y}+h_{ce}{}^{x}h_{a}{}^{e}{}_{y}f_{d}{}^{z}f_{z}{}^{a}=-P_{yz}{}^{x}h_{cd}{}^{z}+P_{yz}{}^{x}h_{ce}{}^{z}f_{d}{}^{w}f_{w}{}^{e}.$$

Taking account of (3.3) and (3.5), then (3.11) gives

$$(g_{cd}-f_c{}^z f_{dz})\delta_y^x-h_{ce}{}^x h_d{}^e{}_y+P_{wu}{}^x P_{zy}{}^w f_d{}^z f_c{}^u$$
$$=-P_{yz}{}^x h_{cd}{}^z+f_c{}^x f_{dy}-\delta_y^x f_c{}^z f_{zd}+P_{wu}{}^x P_{zy}{}^w f_d{}^z f_c{}^u .$$

Consequently, we obtain

$$h_{ce}{}^x h_b{}^e{}_y=P_{yz}{}^x h_{cb}{}^z+g_{cb}\delta_y^x-f_c{}^x f_{by} . \tag{3.12}$$

Substitution (3.12) into (2.41) yields

$$h_{\beta\alpha}{}^x h_\gamma{}^\alpha{}_y=P_{yz}{}^x h_{\beta\gamma}{}^z+g_{\beta\gamma}\delta_y^x$$

with the help of (2.21), (2.39) and (3.3). This completes the proof of our lemma.

On the other hand, by the straightforward computation we get

$$h_{\beta\alpha}{}^x f_\gamma{}^\alpha+h_{\gamma\alpha}{}^x f_\beta{}^\alpha=0 \tag{3.13}$$

with the help of (2.24), (2.25), (2.26), (2.27), (2.39) and (3.1). Transvection (2.39) with $f_y{}^\alpha$ gives

$$h_{\beta\alpha}{}^x f_y{}^\alpha=P_{yz}{}^x f_\beta{}^z-\delta_y^x\xi_\beta \tag{3.14}$$

with the help of (2.21) and (3.3), from which, transvecting $f_w{}^\beta$, we find

$$P_{yz}{}^x=h_{\beta\alpha}{}^x f_y{}^\beta f_z{}^\alpha , \tag{3.15}$$

or, transvecting g^{yz},

$$P^x=h_{\beta\alpha}{}^x f_y{}^\beta f^{y\alpha} . \tag{3.16}$$

Now, differentiating (3.13) covariantly and using (2.31), we find

$$(\overline{\nabla}_\delta h_{\beta\alpha}{}^x)f_\gamma{}^\alpha+h_{\beta\alpha}{}^x(-g_{\delta\gamma}\xi^\alpha+\delta_\delta^\alpha\xi_\gamma+h_{\delta\gamma}{}^y f_y{}^\alpha-h_\delta{}^\alpha{}_y f_\gamma{}^y)+(\overline{\nabla}_\delta h_{\gamma\alpha}{}^x)f_\beta{}^\alpha$$
$$+h_{\gamma\alpha}{}^x(-g_{\delta\beta}\xi^\alpha+\delta_\delta^\alpha\xi_\beta+h_{\delta\beta}{}^y f_y{}^\alpha-h_\delta{}^\alpha{}_y f_\beta{}^y)=0 ,$$

or, using (2.33), (3.8) and (3.14),

$$(\overline{\nabla}_\delta h_{\beta\alpha}{}^x)f_\gamma{}^\alpha+(\overline{\nabla}_\delta h_{\alpha\gamma}{}^x)f_\beta{}^\alpha=0 ,$$

from which, taking the skew-symmetric part with respect to the indices δ and β,

$$(\overline{\nabla}_\delta h_{\gamma\alpha}{}^x)f_\beta{}^\alpha-(\overline{\nabla}_\beta h_{\gamma\alpha}{}^x)f_\delta{}^\alpha=0$$

since the ambient manifold S^{2m+1} is a space of constant curvature 1. Hence the last two equations imply $(\overline{\nabla}_\gamma h_{\beta\alpha}{}^x)f_\delta{}^\alpha=0$, from which, transvecting $f_\varepsilon{}^\delta$, we find

$$\overline{\nabla}_\gamma h_{\beta\varepsilon}{}^x=(\overline{\nabla}_\gamma h_{\beta\alpha}{}^x)f_\varepsilon{}^y f_y{}^\alpha+(\overline{\nabla}_\gamma h_{\beta\alpha}{}^x)\xi_\varepsilon\xi^\alpha$$

by virtue of (2.24). Transvection this equation with $g^{\beta\varepsilon}$ gives

$$\overline{\nabla}_\gamma h^x = (\overline{\nabla}_\gamma h_{\beta\alpha}{}^x) f^{\beta y} f_y{}^\alpha + (\overline{\nabla}_\gamma h_{\beta\alpha}{}^x)\xi^\beta \xi^\alpha .$$

By the straightforward computation, we find

$$(\overline{\nabla}_\gamma h_{\beta\alpha}{}^x)\xi^\beta \xi^\alpha = 0$$

because of (2.28) and (2.33). Consequently, we have

$$\overline{\nabla}_\gamma h^x = (\overline{\nabla}_\gamma h_{\beta\alpha}{}^x) f^{\beta y} f_y{}^\alpha . \tag{3.17}$$

If we differentiate (3.16) covariantly and use (3.17), then we obtain

$$\overline{\nabla}_\gamma P^x = \overline{\nabla}_\gamma h^x + h_{\beta\alpha}{}^x (\overline{\nabla}_\gamma f^{\beta y}) f_y{}^\alpha + h_{\beta\alpha}{}^x f^{\beta y} \overline{\nabla}_\gamma f_y{}^\alpha ,$$

or, substitute (2.25) and make use of (3.14), we get

$$\overline{\nabla}_\gamma P^x = \overline{\nabla}_\gamma h^x .$$

Thus we have

LEMMA 3.2. *Under the same assumptions as those stated in Lemma* 3.1, *we have*

$$\overline{\nabla}_\gamma P^x = \overline{\nabla}_\gamma h^x . \tag{3.18}$$

Next, we prove

LEMMA 3.3. *Under the same assumptions as those stated in Lemma* 3.1, *we have*

$$\frac{1}{2}\Delta(h_{\beta\alpha}{}^x h^{\beta\alpha}{}_x) = (\overline{\nabla}_\beta \overline{\nabla}_\alpha h^x) h^{\beta\alpha}{}_x + \|\overline{\nabla}_\gamma h_{\beta\alpha}{}^x\|^2 , \tag{3.19}$$

where Δ *is the Laplacian given by* $\Delta = g^{\gamma\beta}\overline{\nabla}_\gamma \overline{\nabla}_\beta$.

Proof. From the Ricci identity, we have

$$\overline{\nabla}^\gamma \overline{\nabla}_\gamma h_{\beta\alpha}{}^x - \overline{\nabla}_\beta \overline{\nabla}_\alpha h^x = K_{\beta\gamma} h_\alpha{}^{\gamma x} - K_{\delta\beta\alpha\gamma} h^{\gamma\delta x} \tag{3.20}$$

with the help of (2.36), where $K_{\beta\gamma}$ is the Ricci tensor given by

$$K_{\beta\gamma} = n g_{\beta\gamma} + h^x h_{\beta\gamma x} - h_{\beta\alpha}{}^x h_\gamma{}^\alpha{}_x \tag{3.21}$$

by virtue of (2.35). If we transvect (3.20) with $h^{\beta\alpha}{}_x$ and take account of (2.37), (3.8) and (3.21), then we find

$$(\overline{\nabla}^\gamma \overline{\nabla}_\gamma h_{\beta\alpha}{}^x) h^{\beta\alpha}{}_x - (\overline{\nabla}_\beta \overline{\nabla}_\alpha h^x) h^{\beta\alpha}{}_x = 0$$

with the help of (3.6). Therefore, we have the Laplacian of the length of the second fundamental tensors $h_{\beta\alpha}{}^x$ as follows:

$$\frac{1}{2}\Delta(h_{\beta\alpha}{}^x h^{\beta\alpha}{}_x) = (\overline{\nabla}_\beta \overline{\nabla}_\alpha h^x) h^{\beta\alpha}{}_x + \|\overline{\nabla}_\gamma h_{\beta\alpha}{}^x\|^2 .$$

Thus we complete the proof of this lemma.

If the mean curvature vector of M is parallel in the normal bundle, then it follows that the mean curvature vector of $\bar{M}$ is also parallel in the normal bundle by means of Lemma 2.2. Therefore, $h_{\beta\alpha}{}^{x}h^{\beta\alpha}{}_{x}=h_{x}P^{x}+(n+1)p$, which is induced from (3.8), is a constant along $\bar{M}$ because of (3.18). Hence (3.19) reduces to $\bar{\nabla}_{\gamma}h_{\beta\alpha}{}^{x}=0$. Since $\bar{M}$ is of essential codimension $2m-n$ and does not admit umbilical sections because of (3.14), combining with Theorem A is § 0, we have

THEOREM 3.4. *Let M be an n-dimensional complete generic submanifold of a complex projective space CP^{m} with flat normal connection. If the f-structure induced on M is normal and if the mean curvature vector of M is parallel in the normal bundle, then M is of the form*

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),\ p_1,\cdots,\ p_N \text{ are odd numbers } \geqq 1,$$

$$p_1+p_2+\cdots+p_N=n+1,\ r_1^2+r_2^2+\cdots+r_N^2=1,\ N=2m-n+1,$$

where $S^{p_i}(r_i)$ is a p_i-dimensional sphere with radius r_i.

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