T. OTSUKI
KODAI MATH. J.
4 (1981), 28—70

A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (III)

BY TOMINOSUKE OTSUKI

§ 0. Introduction.

This is a continuation of Part (I) and Part (II) with the same title written by the present author. We shall use the same notation in them.

The period T of any non-constant solution $x(t)$ of the non-linear differential equation of order 2:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$\text{(0.1)} \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

We shall study the following conjecture in the present work, which is in place of Conjecture B in Part (II), implying the inequality:

$$\text{(U)} \qquad T<\sqrt{2}\,\pi.$$

CONJECTURE C. The period function T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to n ($\geqq 2$) for any fixed τ ($0<\tau<1$).

§ 1. Fundamental formulas.

Using a closed curve γ on the Riemann surface $\mathfrak{F}$: $z(n-z)^{n-1}-w^2=c$ as in [11], T can be written as

$$\text{(1.1)} \qquad T=T(c,\ n)=-\frac{\sqrt{nc}}{2}\int_\gamma\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}},$$

from which we obtain

$$\text{(1.2)} \qquad \frac{\partial T}{\partial c}=-\frac{1}{4}\sqrt{\frac{n}{c}}\int_\gamma\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}$$

and

Received November 23, 1979

(1.3) $$\frac{\partial T}{\partial n}=-\frac{1}{4}\sqrt{\frac{c}{n}}\int_{\Gamma}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}$$
$$+\frac{\sqrt{nc}}{4}\int_{\Gamma}\frac{1}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\left[\frac{1}{n-z}+\frac{z(n-z)^{n-1}\lambda(z)}{z(n-z)^{n-1}-c}\right]dz,$$

where

(1.4) $$\lambda(z)=\log(n-z)+\frac{n-1}{n-z}.$$

(See § 1 in Part (I)).

Now, we denote T by $\Omega(\tau, n)$, considering it as a function of τ and n. From $\tau=(n-x_1)/(n-1)$, we obtain

$$x_1=\tau+n-\tau n.$$

From $c=x_1(n-x_1)^{n-1}$, we obtain

$$\frac{\partial c}{\partial \tau}=\frac{\partial c}{\partial x_1}\frac{\partial x_1}{\partial \tau}=n(n-1)(x_1-1)(n-x_1)^{n-2}$$

and

$$\frac{\partial c}{\partial n}=\frac{\partial c}{\partial x_1}\frac{\partial x_1}{\partial n}+\frac{\partial c}{\partial n}$$
$$=n(1-x_1)(n-x_1)^{n-2}(1-\tau)+c\lambda(x_1)$$
$$=-\frac{n(x_1-1)^2(n-x_1)^{n-2}}{n-1}+c\lambda(x_1).$$

By (1.2) and (1.3), we obtain

$$\frac{\partial \Omega(\tau, n)}{\partial n}=\frac{\partial T(c, n)}{\partial c}\frac{\partial c}{\partial n}+\frac{\partial T(c, n)}{\partial n}$$
$$=-\frac{1}{4}\sqrt{\frac{n}{c}}\left\{-\frac{n(x_1-1)^2(n-x_1)^{n-2}}{n-1}+c\lambda(x_1)\right\}\int_{\Gamma}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{\phi(z)-c\}^3}}$$
$$-\frac{1}{4}\sqrt{\frac{c}{n}}\int_{\Gamma}\frac{dz}{z\sqrt{(n-z)\{\phi(z)-c\}}}$$
$$+\frac{\sqrt{nc}}{4}\int_{\Gamma}\frac{1}{z\sqrt{(n-z)\{\phi(z)-c\}}}\left[\frac{1}{n-z}+\frac{\phi(z)\lambda(z)}{\phi(z)-c}\right]dz,$$

where

(1.5) $$\phi(z)=z(n-z)^{n-1}.$$

Furthermore, this can be written as

(1.6) $$\frac{\partial \Omega(\tau, n)}{\partial n}=\frac{n\sqrt{n}}{4\sqrt{c}}\frac{(x_1-1)^2(n-x_1)^{n-2}}{n-1}\int_{\Gamma}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{\phi(z)-c\}^3}}$$

$$+\frac{\sqrt{nc}}{4}\int_\gamma \frac{(n-z)^{n-3/2}\{\lambda(z)-\lambda(x_1)\}\,dz}{\sqrt{\{\phi(z)-c\}^3}}+\frac{\sqrt{c}}{4\sqrt{n}}\int_\gamma \frac{dz}{\sqrt{(n-z)^3\{\phi(z)-c\}}}$$

The 1st term in the right hand side is negative by Proposition 1 in [10] or Theorem D in [12] and the 3rd term is also negative because

$$\int_\gamma \frac{dz}{\sqrt{(n-z)^3\{\phi(z)-c\}}}=-2\int_{x_0}^{x_1}\frac{dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}<0\,. \tag{1.7}$$

In the following, we shall change the right hand side of (1.6) into a real integral.

Setting

$$\begin{aligned} J_1(\gamma) &:=\int_\gamma \frac{(n-z)^{n-3/2}\{\lambda(z)-\lambda(x_1)\}\,dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\,, \\ J_2(\gamma) &:=\int_\gamma \frac{dz}{\sqrt{(n-z)^3\{z(n-z)^{n-1}-c\}}}\,, \\ J_3(\gamma) &:=\int_\gamma \frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}, \end{aligned} \tag{1.8}$$

(1.6) can be written as

$$\frac{\partial\Omega(\tau,\,n)}{\partial n}=\frac{\sqrt{c}}{4\sqrt{n}}\left[\{nJ_1(\gamma)+J_2(\gamma)\}+\frac{n^2(x_1-1)^2(n-x_1)^{n-2}}{(n-1)c}J_3(\gamma)\right]. \tag{1.9}$$

Since the Riemann surface $\mathfrak{F}$ is defined by the equation:

$$z(n-z)^{n-1}-w^2=c\,,$$

we have on $\mathfrak{F}$ the equality:

$$n(1-z)(n-z)^{n-2}dz=2w\,dw$$

and so

$$\frac{(n-z)^{n-3/2}\{\lambda(z)-\lambda(x_1)\}\,dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}=\frac{2}{n}\,\frac{\sqrt{n-z}\{\lambda(z)-\lambda(x_1)\}}{1-z}\cdot\frac{dw}{w^2}$$

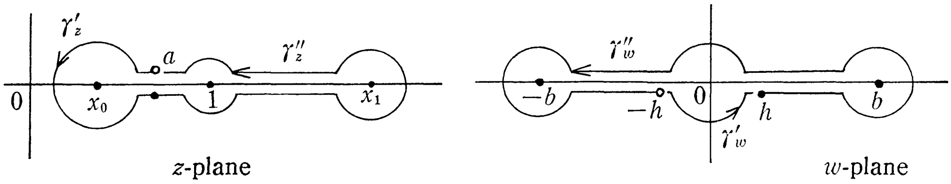
$$=-\frac{2}{n}d\left[\frac{\sqrt{n-z}\{\lambda(z)-\lambda(x_1)\}}{1-z}\cdot\frac{1}{w}\right]$$

$$+\frac{2}{n}\,\frac{1}{w}\left[-\frac{1}{\sqrt{(n-z)^3}}+\{\lambda(z)-\lambda(x_1)\}\left\{\frac{\sqrt{n-z}}{(1-z)^2}-\frac{1}{2(1-z)\sqrt{n-z}}\right\}\right]dz$$

$$=-\frac{2}{n}d\left[\frac{\sqrt{n-z}\{\lambda(z)-\lambda(x_1)\}}{1-z}\cdot\frac{1}{w}\right]-\frac{2}{n}\,\frac{dz}{\sqrt{(n-z)^3\{z(n-z)^{n-1}-c\}}}$$

$$+\frac{(2n-1-z)\{\lambda(z)-\lambda(x_1)\}\,dz}{n(1-z)^2\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\,.$$

Now, we divide γ into the subarcs γ' and γ'' by taking a real constant a $(x_0<a<1)$ as is shown in Fig. 1.



$(b=\sqrt{(n-1)^{n-1}-c},\ h=\sqrt{a(n-a)^{n-1}-c})$

Fig. 1.

Then, we have

$$J_1(\gamma)=J_1(\gamma')+J_1(\gamma''),$$

$$\begin{aligned}J_1(\gamma')=&-\frac{2}{n}\left[\frac{\sqrt{n-z}\{\lambda(z)-\lambda(x_1)\}}{1-z}\cdot\frac{1}{w}\right]_{\partial\gamma'}\\&-\frac{2}{n}\int_{\gamma'}\frac{dz}{\sqrt{(n-z)^3\{z(n-z)^{n-1}-c\}}}\\&+\frac{1}{n}\int_{\gamma'}\frac{(2n-1-z)\{\lambda(z)-\lambda(x_1)\}\,dz}{(1-z)^2\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\\=&-\frac{4}{n}\frac{\sqrt{n-a}\{\lambda(x_1)-\lambda(a)\}}{(1-a)\sqrt{a(n-a)^{n-1}-c}}+\frac{4}{n}\int_{x_0}^{a}\frac{dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}\\&+\frac{2}{n}\int_{x_0}^{a}\frac{(2n-1-x)\{\lambda(x_1)-\lambda(x)\}\,dx}{(1-x)^2\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}}\end{aligned}$$

and

$$J_1(\gamma'')=2\int_a^{x_1}\frac{(n-x)^n\{\lambda(x_1)-\lambda(x)\}\,dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}^3}}.$$

Hence, from these equalities and (1.7), we obtain

$$\begin{aligned}nJ_1(\gamma)+J_2(\gamma)=&-\frac{4\sqrt{n-a}\{\lambda(x_1)-\lambda(a)\}}{(1-a)\sqrt{a(n-a)^{n-1}-c}}\\&+2\int_{x_0}^{a}\frac{(1-x)^2+(n-x)(2n-1-x)\{\lambda(x_1)-\lambda(x)\}}{(1-x)^2\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}\,dx\\&+2\int_a^{x_1}\frac{-x(n-x)^{n-1}+c+n(n-x)^n\{\lambda(x_1)-\lambda(x)\}}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}^3}}.\end{aligned}\tag{1.10}$$

The 2nd term of the right hand side of (1.10) tends to 0 as $a\to x_0+0$, hence we obtain

$$\begin{aligned}nJ_1(\gamma)+J_2(\gamma)=2\lim_{a\to x_0+0}\Bigg[&-\frac{2\sqrt{n-a}\{\lambda(x_1)-\lambda(a)\}}{(1-a)\sqrt{a(n-a)^{n-1}-c}}\\&+\int_a^{x_1}\frac{-x(n-x)^{n-1}+c+n(n-x)^n\{\lambda(x_1)-\lambda(x)\}}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}^3}}\Bigg].\end{aligned}\tag{1.11}$$

§2. An expression of $nJ_1(\gamma)+J_2(\gamma)$.

For simplicity, setting

$$\tilde{F}(x)=\tilde{F}(x,\ x_1):=-x(n-x)^{n-1}+x_1(n-x_1)^{n-1} \\ +n(n-x)^n\{\lambda(x_1)-\lambda(x)\}, \tag{2.1}$$

we have easily

$$\tilde{F}(x_0)=n(n-x_0)^n\{\lambda(x_1)-\lambda(x_0)\}>0 \quad \text{and} \quad \tilde{F}(x_1)=0$$

by Lemma 2.2 in [11]. Since we have

$$\tilde{F}'(x)=-n(1-x)(n-x)^{n-2}-n^2(n-x)^{n-1}\{\lambda(x_1)-\lambda(x)\} \\ +n(n-x)^n\frac{1-x}{(n-x)^2}=-n^2(n-x)^{n-1}\{\lambda(x_1)-\lambda(x)\}, \tag{2.2}$$

we have

$$\tilde{F}'(x)<0 \quad \text{for } x_0\leqq x<x_1,\ \tilde{F}'(x)>0 \quad \text{for } x_1<x<n$$

and $\tilde{F}'(x_1)=0$. Hence $\tilde{F}(x)>0$ for $x_0\leqq x<n,\ x\neq x_1$.

Next we have

$$\frac{\sqrt{n-x}\{\lambda(x_1)-\lambda(x)\}}{(1-x)\sqrt{\phi(x)-c}}=\frac{\sqrt{n-x}}{(1-x)\sqrt{\phi(x)-c}}\cdot\frac{\tilde{F}(x)+\phi(x)-c}{n(n-x)^n}$$
$$=\frac{\tilde{F}(x)}{n(1-x)(n-x)^{n-1/2}\sqrt{\phi(x)-c}}+\frac{\sqrt{\phi(x)-c}}{n(1-x)(n-x)^{n-1/2}},$$

and

$$\lim_{a\to x_0}\frac{\sqrt{\phi(a)-c}}{n(1-a)(n-a)^{n-1/2}}=0, \quad \lim_{a\to x_0}\frac{B-\phi(a)}{B-c}=1,$$

where $B=(n-1)^{n-1}$. Hence, (1.11) can be written as

$$nJ_1(\gamma)+J_2(\gamma)=2\lim_{a\to x_0}\left[-\frac{2}{nb^2}\cdot\frac{\tilde{F}(a)\{B-\phi(a)\}}{(1-a)(n-a)^{n-1/2}\sqrt{\phi(a)-c}} \right. \\ \left. +\int_a^{x_1}\frac{\tilde{F}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\right]. \tag{2.3}$$

Using the function $\mu(x)$ defined by (4.8) in [11]:

$$\mu(x)=\begin{cases} \dfrac{B-\phi(x)}{(x-1)^2} & \text{for } 0<x<n,\ x\neq 1 \\ \dfrac{n(n-1)^{n-2}}{2} & \text{for } x=1, \end{cases}$$

we define an auxiliary function

$$\text{(2.4)} \qquad L(x) := \begin{cases} \dfrac{(1-x)\mu(x)\tilde{F}(x)}{(n-x)^{n-1/2}} & \text{for } 0<x<n,\ x\neq 1, \\ 0 & \text{for } x=1, \end{cases}$$

which is real analytic in (0, n), and we get

$$\tilde{L}(x)>0 \quad \text{for } x_0 \leqq x<1 \quad \text{and} \quad \tilde{L}(x)<0 \quad \text{for } 1<x<n,\ x\neq x_1,$$

and

$$\tilde{L}(1)=\tilde{L}(x_1)=\tilde{L}'(x_1)=0.$$

Using $\tilde{L}(x)$, (2.1) can be written as

$$\text{(2.3')} \qquad nJ_1(\gamma)+J_2(\gamma)=2\lim_{a\to x_0}\left[-\frac{2}{nb^2}\frac{\tilde{L}(a)}{\sqrt{\phi(a)-c}}+\int_a^{x_1}\frac{\tilde{F}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\right].$$

We have

$$\lim_{x\to x_1}\frac{\tilde{L}(x)}{\sqrt{\phi(x)-c}}=0,$$

since $\tilde{L}(x)$ has a zero point of order at least 2 at $x=x_1$, and

$$\left(\frac{\tilde{L}(x)}{\sqrt{\phi(x)-c}}\right)'=\frac{2\{\phi(x)-c\}\tilde{L}'(x)-n(1-x)(n-x)^{n-2}\tilde{L}(x)}{2\sqrt{\{\phi(x)-c\}^3}}.$$

Hence we have

$$\begin{aligned}
&-\frac{2}{nb^2}\frac{\tilde{L}(a)}{\sqrt{\phi(a)-c}}+\int_a^{x_1}\frac{\tilde{F}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}} \\
&=\frac{2}{nb^2}\int_a^{x_1}\left(\frac{\tilde{L}(x)}{\sqrt{\phi(x)-c}}\right)'dx+\int_a^{x_1}\frac{\tilde{F}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}} \\
&=\frac{1}{b^2}\int_a^{x_1}\frac{1}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\Big[b^2\tilde{F}(x)+\frac{2}{n}(n-x)^{3/2}\{\phi(x)-c\}\tilde{L}'(x) \\
&\qquad\qquad -(1-x)(n-x)^{n-1/2}\tilde{L}(x)\Big]dx.
\end{aligned}$$

The expression in the brackets of the last equality can be written as

$$\begin{aligned}
&b^2\tilde{F}(x)+\frac{2}{n}(n-x)^{3/2}\{\phi(x)-c\}\tilde{L}'(x)-(1-x)(n-x)^{n-1/2}\tilde{L}(x) \\
&=\left\{\tilde{F}(x)+\frac{2}{n}(n-x)^{3/2}\tilde{L}'(x)\right\}\{\phi(x)-c\}.
\end{aligned}$$

By (2.1) and (2.2) we have

$$\tilde{F}(x)+\frac{2}{n}(n-x)^{3/2}\tilde{L}'(x)$$

$$=\tilde{F}(x)+\frac{2}{n}\frac{\{B-\phi(x)\}\tilde{F}(x)}{(1-x)(n-x)^{n-2}}\Big[-\frac{n^2(n-x)^{n-1}\{\lambda(x_1)-\lambda(x)\}}{\tilde{F}(x)}$$

$$-\frac{n(1-x)(n-x)^{n-2}}{B-\phi(x)}+\frac{4n-1-(2n+1)x}{2(1-x)(n-x)}\Big]$$

$$=-\tilde{F}(x)-\frac{2n(n-x)\{B-\phi(x)\}\{\lambda(x_1)-\lambda(x)\}}{1-x}$$

$$+\frac{\{4n-1-(2n+1)x\}\{B-\phi(x)\}\tilde{F}(x)}{n(1-x)^2(n-x)^{n-1}}.$$

Now, setting

$$\tilde{M}(x):=\tilde{F}(x)+\frac{2n(n-x)\{B-\phi(x)\}\{\lambda(x_1)-\lambda(x)\}}{1-x}\tag{2.5}$$

$$-\frac{\{4n-1-(2n+1)x\}\{B-\phi(x)\}\tilde{F}(x)}{n(1-x)^2(n-x)^{n-1}},$$

we obtain

$$-\frac{2}{nb^2}\frac{\tilde{L}(a)}{\sqrt{\phi(a)-c}}+\int_a^{x_1}\frac{\tilde{F}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}=-\frac{1}{b^2}\int_a^{x_1}\frac{\tilde{M}(x)dx}{\sqrt{(n-x)^3\{\phi(x)-c)}}.$$

By the properties of $\tilde{F}(x)$ and (2.5), we see easily that

$$\tilde{M}(x_1)=0\tag{2.6}$$

and

$$\tilde{M}(x_0)=n(n-x_0)^n\{\lambda(x_1)-\lambda(x_0)\}+\frac{2n(n-x_0)(B-c)\{\lambda(x_1)-\lambda(x_0)\}}{1-x_0}$$

$$-\frac{\{4n-1-(2n+1)x_0\}(n-x_0)(B-c)\{\lambda(x_1)-\lambda(x_0)\}}{(1-x_0)^2}$$

$$=\frac{(n-x_0)\{\lambda(x_1)-\lambda(x_0)\}}{(1-x_0)^2}[n(1-x_0)^2(n-x_0)^{n-1}$$

$$+2n(1-x_0)(B-c)-\{4n-1-(2n+1)x_0\}(B-c)],$$

i. e.

$$\tilde{M}(x_0)=\frac{(n-x_0)\{\lambda(x_1)-\lambda(x_0)\}}{(1-x_0)^2}[(n-x_0+(n-1)x_0^2)(n-x_0)^{n-1}\tag{2.7}$$

$$-(2n-1-x_0)B].$$

Since $\tilde{M}(x)$ is real analytic in $0<x<n$, we obtain finally the formula

$$nJ_1(\gamma)+J_2(\gamma)=-\frac{2}{b^2}\int_{x_0}^{x_1}\frac{\tilde{M}(x)dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}.\tag{2.8}$$

As in the argument in [11], we consider a complex valued function for $\tilde{M}(x)$ by

$$M(z,\ x_1)=\left[1-\frac{\{4n-1-(2n+1)z\}\{B-\psi(z)\}}{n(1-z)^2(n-z)^{n-1}}\right]\times[-\psi(z)+\psi(x_1)+n(n-z)^n\{\lambda(x_1)-\lambda(z)\}]+\frac{2n(n-z)\{B-\psi(z)\}\{\lambda(x_1)-\lambda(z)\}}{1-z}, \tag{2.9}$$

which coincides with $M(z,\ x_0)$ defined in [12], §1, replaced x_0 by x_1, and $\tilde{M}(x)=M(x,\ x_1)$. Hence, using the auxiliary functions

$$\begin{aligned} f_0(z)&:=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\}, \\ f_1(z)&:=\{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\}, \end{aligned} \tag{2.10}$$

we obtain the equality

$$M(z,\ x_1)=-\frac{n-z}{(1-z)^2}f_0(z)\{\lambda(x_1)-\lambda(z)\}+\frac{1}{n(1-z)^2(n-z)^{n-1}}f_1(z)\{\psi(z)-\psi(x_1)\}$$

(see §1 in [12]).

Now, using the function $X=X_n(x)$ $(0\leqq x\leqq 1)$ defined by

$$x(n-x)^{n-1}=X(n-X)^{n-1},\quad 1\leqq X\leqq n \tag{2.12}$$

with

$$\frac{dX}{dx}=\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}, \tag{2.13}$$

we have

$$\int_{x_0}^{x_1}\frac{M(x,\ x_1)dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}=\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}K(x,\ x_1)dx, \tag{2.14}$$

where

$$K(x,\ x_1):=\begin{cases}\dfrac{xM(x,\ x_1)}{(1-x)\sqrt{n-x}}-\dfrac{XM(X,\ x_1)}{(1-X)\sqrt{n-X}} & \text{for } 0<x<1\\ 0 \quad \text{for } x=1.\end{cases} \tag{2.15}$$

By means of (2.11), we obtain easily

$$\begin{aligned} K(x,\ x_1)=&\frac{x\sqrt{n-x}}{(x-1)^3}f_0(x)\{\lambda(x_1)-\lambda(x)\}+\frac{x^2f_1(x)}{n(1-x)^3\sqrt{n-x}}\cdot\frac{\psi(x)-\psi(x_1)}{\psi(x)} \\ &-\frac{X\sqrt{n-X}}{(X-1)^3}f_0(X)\{\lambda(x_1)-\lambda(X)\}-\frac{X^2f_1(X)}{n(1-X)^3\sqrt{n-X}}\cdot\frac{\psi(X)-\psi(x_1)}{\psi(X)} \end{aligned}$$

and hence

$$K(x, x_1)=\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}$$
$$-\left[\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\{\lambda(x_1)-\lambda(X)\}$$
$$-\left[\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\frac{\{\phi(x)-\phi(x_1)\}}{n\phi(x)}. \tag{2.16}$$

REMARK. The first term and the functions in the two pairs of brackets are all positive by Lemma 2.2 in [11] and Proposition 2 and Proposition 4 in [12].

§ 3. An expression of $\partial\Omega(\tau, n)/\partial n$ and some constants.

LEMMA 3.1. *We have the formula*:

$$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{(1-x)W(x, x_1)dx}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}$$

where $c=x_1(n-x_1)^{n-1}$, $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ *and*

$$W(x, x_1):=\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}$$
$$+\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\cdot\left\{\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)+\lambda(X)\right\}$$
$$-\left[\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\cdot\frac{\phi(x)-\phi(x_1)}{n\phi(x)}. \tag{3.1}$$

Proof. From (1.9), (2.8) and (2.14), we have

$$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}K(x, x_1)dx$$
$$+\frac{n\sqrt{n}(x_1-1)^2(n-x_1)^{n-2}}{4(n-1)\sqrt{c}}J_3(\gamma).$$

On the other hand, by (6) in Appendix in [12] we have

$$J_3(\gamma)=-\frac{2}{nb^2}\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]dx.$$

Hence, we obtain

$$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}\Big[K(x, x_1)$$
$$+\frac{n}{n-1}\frac{(x_1-1)^2(n-x_1)^{n-2}}{c}\left\{\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right\}\Big]dx.$$

By (2.16), we have

$$K(x, x_1)+\frac{n}{n-1}\frac{(x_1-1)^2(n-x_1)^{n-2}}{c}\left\{\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\right\}$$

$$=\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}$$

$$+\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\right]\cdot\left\{\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)+\lambda(X)\right\}$$

$$-\left[\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\cdot\frac{\phi(x)-\phi(x_1)}{n\phi(x)}=W(x, x_1)$$

Q. E. D.

LEMMA 3.2. *On $W(x, x_1)$, we have*

$$W(x, X(x))>0 \quad and \quad \lim_{x_1\to n}W(x, x_1)=+\infty \quad for\ 0<x<1.$$

Proof. From (3.1) we get easily

$$W(x, X(x))=\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}$$

$$+\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^2}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\right]\cdot\frac{n}{n-1}\frac{(X-1)^2}{X(n-X)}>0,$$

because $\frac{f_0(x)}{(x-1)^3}>0$ and $\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}>0$ by Lemma 2.1 and Proposition 2 in [12], and $\lambda(X)-\lambda(x)>0$ by Lemma 2.2 in [11].

Next, we shall show that the 2nd factor of the 2nd term of (3.1) diverges to $+\infty$ as $x_1\to n-0$. Since we have

$$\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)=\frac{n(x_1-1)^2-(n-1)x_1\{(n-x_1)\log(n-x)+n-1\}}{(n-1)x_1(n-x_1)},$$

we obtain

$$\lim_{x_1\to n}\left[\frac{n(x_1-1)^2}{(n-1)x_1(n-x_1)}-\lambda(x_1)\right]$$

$$=\frac{1}{n(n-1)}\lim_{x_1\to n}\frac{n(x_1-1)^2-(n-1)x_1\{(n-x_1)\log(n-x_1)+n-1\}}{n-x_1}$$

$$=\frac{1}{n(n-1)}\lim_{x_1\to n}\frac{2n(x_1-1)-(n-1)\{(n-2x_1)\log(n-x_1)-x_1+n-1\}}{-1}$$

$$=-\frac{1}{n(n-1)}\lim_{x_1\to n}[2n(n-1)+n-1+n(n-1)\log(n-x_1)]=+\infty,$$

from which we obtain immediately

$$\lim_{x_1\to n}W(x, x_1)=+\infty.$$

Q. E. D.

LEMMA 3.3. *We have*

$$\lim_{x\to 1-0} W(x, x_1)=0 \quad for\ 1<x_1<n\,.$$

Proof. Since we have $\lim_{x\to 1} X(x)=1$, we obtain this assertion by Lemma 2.2 and Lemma 4.2 in [12]. Q. E. D.

Now, we consider the increasing ratio of the function $W(x, x_1)$ defined above in the domain: $0<x<1$, $X_n(x)\leqq x_1<n$, with respect to x_1. Using the auxiliary functions $F_0(x)$ and $F_1(x)$ defined by (2.3) and (4.4) in [12] respectively as follow:

$$F_0(x)=\begin{cases} (n-x)^{-n+3/2}(x-1)^{-3}f_0(x) & \text{for } 0\leqq x<n,\ x\neq 1 \\ n(2n-1)/6\sqrt{n-1} & \text{for } x=1 \end{cases} \tag{3.2}$$

and

$$F_1(x)=\begin{cases} (n-x)^{-2n+3/2}(1-x)^{-3}f_1(x) & \text{for } 0\leqq x<n,\ x\neq 1 \\ n(4n+1)/6(n-1)^{n+1/2} & \text{for } x=1, \end{cases} \tag{3.3}$$

we obtain the following

LEMMA 3.4. *We have* $\dfrac{\partial W(x, x_1)}{\partial x_1}\geqq 0$, *if and only if*

$$\frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}\geqq\frac{(n-1)x_1^2(n-x_1)^{n-1}}{n+(n-1)x_1}, \tag{3.4}$$

where $X(x)<x_1<n$.

Proof. We obtain from (3.1), (3.2) and (3.3)

$$\begin{aligned}\frac{W(x, x_1)}{\phi(x)}=&F_0(x)\{\lambda(X)-\lambda(x)\}\\ &+[F_0(X)-F_0(x)]\left\{\frac{n}{n-1}-\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)+\lambda(X)\right\}\\ &-\frac{1}{n}[F_1(X)-F_1(x)]\{\phi(x)-\phi(x_1)\}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\phi(x)}\frac{\partial W(x, x_1)}{\partial x_1}=&[F_0(X)-F_0(x)]\left\{\frac{n}{n-1}\frac{(x_1-1)\{n+(n-2)x_1\}}{x_1^2(n-x_1)^2}-\frac{x_1-1}{(n-x_1)^2}\right\}\\ &-[F_1(X)-F_1(x)](x_1-1)(n-x_1)^{n-2}\\ =&[F_0(X)-F_0(x)]\frac{(x_1-1)\{n+(n-1)x_1\}}{(n-1)x_1^2(n-x_1)}\\ &-[F_1(X)-F_1(x)](x_1-1)(n-x_1)^{n-2}.\end{aligned}$$

Since $F_0(X)-F_0(x)$ and $F_1(X)-F_1(x)$ are positive by Proposition 2 and Proposition 4 in [12], we obtain immediately from the last equality the statement in this lemma. Q. E. D.

LEMMA 3.5. *The function* $\dfrac{x^2(n-x)^{n-1}}{n+(n-1)x}$ *takes its maximum for* $0 \leqq x \leqq n$ *at* $\beta=\beta(n)$ *given by*

$$\beta := (\sqrt{2n^2-2n+1}-1)/(n-1)\,, \tag{3.5}$$

which is increasing with respect to n *and tends to* $\sqrt{2}$ *as* $n \to \infty$.

Proof. We have

$$\frac{d}{dx}\frac{x^2(n-x)^{n-1}}{n+(n-1)x} = -\frac{nx(n-x)^{n-2}\{(n-1)x^2+2x-2n\}}{\{n+(n-1)x\}^2}$$

from which we see that the given function is increasing in $[0, \beta]$ and decreasing in $[\beta, n]$, where β is the positive root of the quadratic equation: $(n-1)x^2+2x-2n=0$ given by (3.5). We get easily

$$1<\beta<2\,. \tag{3.6}$$

Next, we have

$$\frac{d\beta(n)}{dn} = \frac{\sqrt{2n^2-2n+1}-n}{(n-1)^2\sqrt{2n^2-2n+1}} > 0 \quad \text{for } n>1\,,$$

hence $\beta(n)$ is increasing with respect to n. It is evident that

$$\beta(n) \uparrow \sqrt{2} \quad \text{as } n \to \infty\,. \tag{3.7}$$

Q. E. D.

LEMMA 3.6. *When* $n>1$, *we have*

$$\sqrt{\frac{n}{n+1}}\sqrt{2} < \beta(n) < \sqrt{2} \tag{3.8}$$

Proof. Since we have

$$\begin{aligned}(n-1)x^2+2x-2n\Big|_{x=\sqrt{\frac{2n}{n+1}}} &= \frac{2n(n-1)}{n+1}+2\sqrt{\frac{2n}{n+1}}-2n \\ &= \frac{2\sqrt{2n}}{n+1}(\sqrt{n+1}-\sqrt{2n})<0\,.\end{aligned}$$

Q. E. D.

LEMMA 3.7. *The constant* $\nu=\nu(n)$ *defined by*

$$\nu := \frac{(n-1)\beta(n)}{n+(n-1)\beta(n)} \tag{3.9}$$

is increasing with respect to n *for* $n>1$ *and tends to* $2-\sqrt{2}$ *as* $n \to \infty$. *Using* ν *we have*

$$\left.\frac{(n-1)x^2(n-x)^{n-1}}{n+(n-1)x}\right|_{x=\beta}=\nu\phi(\beta)\,. \tag{3.10}$$

Proof. From (3.5) and (3.9) we obtain

$$\nu=\frac{\sqrt{2n^2-2n+1}-1}{\sqrt{2n^2-2n+1}+n-1}=\frac{2n-1-\sqrt{2n^2-2n+1}}{n} \tag{3.11}$$

and

$$\frac{d\nu(n)}{dn}=\frac{\sqrt{2n^2-2n+1}-(n-1)}{n^2\sqrt{2n^2-2n+1}}>0\,.$$

The rest of the statement of this lemma is evident. Q. E. D.

§ 4. Auxiliary function $F_0(x)-\nu\phi(\beta)F_1(x)$.

The author conjectured at first the function $W(x, x_1)$, $X(x)\leqq x_1<n$, would be positive and has made efforts to prove it as shown in §§ 4, 5, 6. But he found it was false. Being based on the facts described in these sections, especially Lemma 6.5, he will devise the expression of the main integral

$$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{c}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{(1-x)W(x, x_1)dx}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}$$

as the one to appear in § 7. Therefore, § 7 should be followed logically to the end of § 3. In order to show why he has to introduce the auxiliary function $V(x, x_1)$ by (7.10) in place of $W(x, x_1)$, he will describe them in brief, since these results in §§ 4, 5, 6 may be used in the arguments after § 7 in the future.

First of all, noticing Lemma 3.4, Lemma 3.5 and Lemma 3.7, we set an auxiliary function $G(x)$ for each n by

$$G(x):=F_0(x)-\nu\phi(\beta)F_1(x) \quad \text{for } 0\leqq x<n. \tag{4.1}$$

LEMMA 4.1. *We have $G'(x)\geqq 0$ if and only if*

$$(n-x)^nF_2(x)-\nu\phi(\beta)F_3(x)\geqq 0\,,$$

where

$$\begin{aligned} F_2(x)=&-\{(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\}B \\ &+(n-x)^{n-1}\{-(n-1)x^3+(2n^2-7n+8)x^2+(n-3)(4n-1)x+3n(2n-1)\} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} F_3(x)=&\{3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2\}B \\ &-(n-x)^{n-1}\{3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2+(n+1)(2n+1)x^3\}, \end{aligned} \tag{4.3}$$

which have been introduced in [12] by (2.6) and (4.8) respectively.

Proof. By means of (2.7) and (4.9) in [12], we obtain

$$G'(x)=F_0'(x)-\nu\psi(\beta)F_1'(x)$$
$$=\frac{1}{2}(x-1)^{-4}(n-x)^{-2n+1/2}\{(n-x)^n F_2(x)-\nu\psi(\beta)F_3(x)\}$$

for $0\leqq x<n$, $x\neq 1$, *from which we obtain immediately this lemma.* Q. E. D.

Now, by Lemma 4.1, $G'(x)\geqq 0$ at x $(0\leqq x<n,\ x\neq 1)$ if and only if

$$(n-x)^n F_2(x)-\nu\psi(\beta)F_3(x)\geqq 0\,.$$

Since $F_2(x)$ is positive for $0\leqq x\leqq n$, $x\neq 1$ by Proposition 1 in [12], we see that for $0\leqq x\leqq\beta$ this inequality is followed from the inequality:

$$(n-\beta)^n F_2(x)-\nu\psi(\beta)F_3(x)\geqq 0\,,$$

which is equivalent to

$$(n-\beta)F_2(x)-\nu\beta F_3(x)\geqq 0\,. \tag{4.4}$$

Now using the auxiliary polynomials in x:

$$P_2(x)=(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\,, \tag{4.5}$$

$$P_3(x)=-(n-1)x^3+(2n^2-7n+8)x^2+(4n^2-13n+3)x+3n(2n-1)\,, \tag{4.6}$$

$$\tilde{P}_2(x)=3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2, \tag{4.7}$$

$$\tilde{P}_3(x)=3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2+(n+1)(2n+1)x^3, \tag{4.8}$$

which have been introduced in [12], we can write $F_2(x)$ and $F_3(x)$ as

$$F_2(x)=-P_2(x)B+(n-x)^{n-1}P_3(x) \tag{4.9}$$

and

$$F_3(x)=\tilde{P}_2(x)B-(n-x)^{n-1}\tilde{P}_3(x)\,. \tag{4.10}$$

Setting

$$\begin{cases} S_2(x):=(n-\beta)P_2(x)+\nu\beta\tilde{P}_2(x)\,, \\ S_3(x):-(n-\beta)P_3(x)+\nu\beta\tilde{P}_3(x)\,, \end{cases} \tag{4.11}$$

we have

$$(n-\beta)F_2(x)-\nu\beta F_3(x)=(n-x)^{n-1}S_3(x)-S_2(x)B\,.$$

Since $\dfrac{8(n-1)}{2n+1}\geqq\dfrac{8}{5}$ for $n\geqq 2$, it must be $\beta<\gamma_0$ by Lemma 3.5 and Lemma 3.3 in [12], where γ_0 is the smaller root of the quadratic equation in x: $P_2(x)=0$. Hence we have $P_2(x)>0$ for $0\leqq x\leqq\beta$. By Lemma 5.1 in [12], we have $\tilde{P}_2(x)>0$ for $-\infty<x<+\infty$. Therefore, the condition (4.4) is equivalent to

$$(4.12) \qquad \frac{(n-x)^{n-1}S_3(x)}{S_2(x)} \geqq B \quad \text{for } 0 \leqq x \leqq \beta .$$

In the following we compute the derivative of the left hand side of (4.12). We have first

$$\frac{d}{dx}\frac{(n-x)^{n-1}S_3(x)}{S_2(x)} = \frac{(n-x)^{n-2}}{S_2^2}[-(n-x)S_2'S_3-S_2T_3],$$

where

$$T_3=-(n-x)S_3'+(n-1)S_3=(n-\beta)Q_3(x)+\nu\beta\tilde{Q}_3(x),$$

and

$$Q_3(x)=-(n-x)P_3'(x)+(n-1)P_3(x), \quad \tilde{Q}_3(x)=-(n-x)\tilde{P}_3'(x)+(n-1)\tilde{P}_3(x)$$

are given explicitly by (3.5) and (5.4) in [12] as follow

$$(4.13) \qquad Q_3(x)=2n^2(n+2)+n(n-13)x+2(n^3-n^2-n+4)x^2-(n-1)(n+2)x^3,$$

$$(4.14) \qquad \tilde{Q}_3(x)=6n^2(n+1)+n(22n^2-15n-13)x-2(n+1)^2(7n-4)x^2 +(n+1)(n+2)(2n+1)x^3.$$

Hence, we have

$$\begin{aligned}
&-(n-x)S_2'S_3-S_2T_3\\
&\quad=-(n-x)\{(n-\beta)P_2'+\nu\beta\tilde{P}_2'\}\{(n-\beta)P_3+\nu\beta\tilde{P}_3\}\\
&\qquad-\{(n-\beta)P_2+\nu\beta\tilde{P}_2\}\{(n-\beta)Q_3+\nu\beta\tilde{Q}_3\}\\
&\quad=-(n-\beta)^2\{(n-x)P_2'P_3+P_2Q_3\}\\
&\qquad-(n-\beta)\nu\beta\{(n-x)(P_2'\tilde{P}_3+\tilde{P}_2'P_3)+P_2\tilde{Q}_3+\tilde{P}_2Q_3\}\\
&\qquad-\nu^2\beta^2\{(n-x)\tilde{P}_2'\tilde{P}_3+\tilde{P}_2\tilde{Q}_3\}
\end{aligned}$$

and by means of the computations done for (3.7) and (5.6) in [12]

$$(4.15) \qquad (n-x)P_2'(x)P_3(x)+P_2(x)Q_3(x)=n(n-1)(1-x)^3Q_2(x),$$

$$(4.16) \qquad (n-x)\tilde{P}_2'(x)\tilde{P}_3(x)+\tilde{P}_2(x)\tilde{Q}_3(x)=-n(1-x)^3\tilde{Q}_2(x),$$

where $Q_2(x)$ and $\tilde{Q}_2(x)$ are given explicitly by (3.6) and (5.5) in [12] as follow

$$(4.17) \qquad Q_2(x)=4n(2n^2-2n+3)-(8n^2-2n+9)x+(2n+1)x^2,$$

$$(4.18) \qquad \tilde{Q}_2(x)=6n(28n^3-16n^2+2n+1)-3(2n+1)(16n^3+10n^2-9n+3)x +(n+1)(2n+1)^2(4n+1)x^2.$$

Next, by (4.5)$\sim$(4.8), (4.13) and (4.14) we have

$$(n-x)\{P_2'(x)\tilde{P}_3(x)+\tilde{P}_2'(x)P_3(x)\}+P_2(x)\tilde{Q}_3(x)+\tilde{P}_2(x)Q_3(x) \tag{4.19}$$
$$=2n(1-x)^3\{-3n(12n^3-8n^2-4n+3)-(4n^4-46n^3+23n^2+10n-9)x$$
$$+(2n+1)(n^2-3n-1)x^2\}.$$

Thus, we can put

$$-(n-x)S_2'(x)S_3(x)-S_2(x)T_3(x)=n(x-1)^3\{a_{n,0}-a_{n,1}x-a_{n,2}x^2\}, \tag{4.20}$$

where we set

$$a_{n,0}:=n\{4(n-\beta)^2(n-1)(2n^2-2n+3) \tag{4.21}$$
$$-6(n-\beta)\nu\beta(12n^3-8n^2-4n+3)-6\nu^2\beta^2(28n^3-16n^2+2n+1)\},$$

$$a_{n,1}:=(n-\beta)^2(n-1)(8n^2-2n+9) \tag{4.22}$$
$$+2(n-\beta)\nu\beta(4n^4-46n^3+23n^2+10n-9)$$
$$-3\nu^2\beta^2(2n+1)(16n^3+10n^2-9n+3),$$

$$a_{n,2}:=-(n-\beta)^2(n-1)(2n+1)-2(n-\beta)\nu\beta(2n+1)(n^2-3n-1) \tag{4.23}$$
$$+\nu^2\beta^2(n+1)(2n+1)^2(4n+1).$$

LEMMA 4.2. *$a_{n,2}>0$ for $n\geqq 4$.*

Proof. By Lemma 3.5 and Lemma 3.7, $\nu(n)\beta(n)$ is increasing with respect to n, and from (3.5) and (3.11) we have

$$\nu(n)\beta(n)=\frac{2\{\sqrt{2n^2-2n+1}-n\}}{n-1}, \tag{4.24}$$

from which we obtain $\frac{2}{3}\leqq\nu(n)\beta(n)<2(\sqrt{2}-1)$ for $4\leqq n<\infty$. Thus, for $n\geqq 4$ we obtain from (4.23)

$$a_{n,2}>-(n-1)^3(2n+1)-2(n-1)(2n+1)(n^2-3n-1)+\frac{4}{9}(n+1)(2n+1)^2(4n+1)$$
$$=\frac{1}{9}(10n^4+315n^3+85n^2-45n-5)>0,$$

since $n-\beta<n-1$, $2(\sqrt{2}-1)<1$ and $n^2-3n-1>0$. Q. E. D.

LEMMA 4.3. *The function $\sigma=\sigma(n)$ of n given by*

$$\sigma=\sigma(n):=(n-\beta(n))/n \tag{4.25}$$

is increasing for $n>1$.

Proof. From (3.5) we have

$$\sigma=\frac{n^2-n+1-\sqrt{2n^2-2n+1}}{n(n-1)}=\frac{t+1-\sqrt{2t+1}}{t}, \text{ where } t=n(n-1).$$

Hence we obtain

$$\frac{d\sigma}{dn}=\frac{d\sigma}{dt}\frac{dt}{dn}=\frac{t+1-\sqrt{2t+1}}{t^2\sqrt{2t+1}}\cdot(2n-1)>0 \quad \text{for } n>1,$$

which implies this lemma. Q. E. D.

LEMMA 4.4. *$a_{n,0}>0$ for $n\geqq 11$.*

Proof. From (4.21) and (4.25) we obtain

$$\frac{1}{n}a_{n,0}=4\sigma^2(2n^5-4n^4+5n^3-3n^2)$$
$$-6\nu\beta\sigma(12n^4-8n^3-4n^2+3n)-6\nu^2\beta^2(28n^3-16n^2+2n+1)\,. \tag{4.26}$$

Using Lemma 3.5, Lemma 3.7 and Lemma 4.3 and dividing into the two cases: $13\leqq n$ and $11\leqq n<13$, we shall prove this lemma.

Case I: $13\leqq n$. Since we have

$$4\sigma^2\geqq 4(\sigma(13))^2=3.189804\cdots>3.189,$$

$$6\nu\beta\sigma<\lim_{n\to\infty}6\nu\beta\sigma=12\sqrt{2}-12=4.970562\cdots<4.971,$$

$$6\nu^2\beta^2<\lim_{n\to\infty}6\nu^2\beta^2=24(3-2\sqrt{2})=4.117749\cdots<4.118,$$

we obtain

$$\frac{1}{n}a_{n,0}>3.189(2n^5-4n^4+5n^3-3n^2)-4.971(12n^4-8n^3-4n^2+3n)$$
$$-4.118(28n^3-16n^2+2n+1)$$
$$=6.378n^5-72.408n^4-59.591n^3+76.205n^2-23.149n-4.118>0$$

for $n\geqq 13$.

Case II: $11\leqq n<13$. Since we have

$$4\sigma^2\geqq 4\sigma^2(11)=3.055118\cdots>3.055,$$

$$6\nu\beta\sigma<6(\nu\beta\sigma)(13)=4.189788\cdots<4.190,$$

$$6\nu^2\beta^2<6(\nu^2\beta^2)(13)=3.668840\cdots<3.669,$$

we obtain

$$\frac{1}{n}a_{n,0}>3.055(2n^5-4n^4+5n^3-3n^2)-4.190(12n^4-8n^3-4n^2+3n)$$

$$-3.669(28n^3-16n^2+2n+1)$$

$$=6.11n^5-62.5n^4-53.937n^3+66.299n^2-19.908n-3.669>0$$

for $11\leqq n<13$. Q. E. D.

LEMMA 4.5. *$a_{n,0}<0$ for $8\leqq n\leqq 10$.*

Proof. Since we have

$$4\sigma^2\leqq 4\sigma^2(10)=2.969600\cdots<2.970,$$

$$6\nu\beta\sigma\geqq 6(\nu\beta\sigma)(8)=3.733453\cdots>3.733,$$

$$6\nu^2\beta^2>6(\nu^2\beta^2)(8)=3.388245\cdots>3.388,$$

we obtain

$$\frac{1}{n}a_{n,0}<2.970(2n^5-4n^4+5n^3-3n^2)-3.733(12n^4-8n^3-4n^2+3n)$$

$$-3.388(28n^3-16n^2+2n+1)$$

$$=5.94n^5-56.676n^5-50.15n^3+60.23n^2-17.975n-3.388<0$$

for $8\leqq n\leqq 10$. Q. E. D.

Remark. By a computation we see that $a_{n,0}$ change its sign at a point in the interval $10.6<n<10.7$, i. e. $n=10.69931\cdots$.

LEMMA 4.6. *$a_{n,1}>0$ for $n\geqq 12$.*

Proof. From (4.22) and (4.25) we obtain

$$a_{n,1}=\sigma^2(8n^5-10n^4+11n^3-9n^2) \tag{4.27}$$

$$+2\nu\beta\sigma(4n^5-46n^4+23n^3+10n^2-9n)-3\nu^2\beta^2(32n^4+36n^3-8n^2-3n+3).$$

Since we have for $n\geqq 12$

$$\sigma^2\geqq\sigma^2(12)=0.781900\cdots>0.781,$$

$$2\nu\beta\sigma>2(\nu\beta\sigma)(12)=1.375838\cdots>1.375,$$

$$3\nu^2\beta^2<\lim_{n\to\infty}3\nu^2\beta^2=12(3-2\sqrt{2})=2.058874\cdots<2.059,$$

we obtain

$$a_{n,1}>0.781(8n^5-10n^4+11n^3-9n^2)+1.375(4n^5-46n^4+23n^3+10n^2-9n)$$

$$-2.059(32n^4+36n^3-8n^2-3n+3)$$

$$=11.748n^5-136.948n^4-33.908n^3+23.193n^2-6.198n-6.177>0$$

for $n \geqq 12$. Q. E. D.

LEMMA 4.7. *$a_{n,1}<0$ for $9 \leqq n \leqq 11$.*

Proof. Since we have

$$\sigma^2 \leqq \sigma^2(11)=0.763779\cdots<0.764,$$

$$2\nu\beta\sigma \leqq 2(\nu\beta\sigma)(11)=1.351492\cdots<1.352,$$

$$3\nu^2\beta^2 \geqq 3\nu^2\beta^2(9)=1.734618\cdots>1.734,$$

we obtain

$$\begin{aligned} a_{n,1}&<0.764(8n^5-10n^4+11n^3-9n^2)+1.352(4n^5-46n^4+23n^3+10n^2-9n)\\ &\quad -1.734(32n^4+36n^3-8n^2-3n+3)\\ &=11.52n^5-125.32n^4-22.924n^3+20.516n^2-6.966n-5.202<0 \end{aligned}$$

for $9 \leqq n \leqq 11$. Q. E. D.

Remark. By an analogous argument to the proof of Lemma 4.7, we can prove that $a_{n,1}<0$ for $2 \leqq n<9$ and see that $a_{n,1}$ changes its sign at a point in the interval $11.2<n<11.3$, i. e. $n=11.21186\cdots$.

§ 5. Quadratic polynomial $a_{n,0}-a_{n,1}x-a_{n,2}x^2$.

In this section, setting

$$Q^*_{n,2}(x):=a_{n,0}-a_{n,1}x-a_{n,2}x^2 \tag{5.1}$$

we shall investigate its sign in the interval $X^{-1}(\beta) \leqq x \leqq \beta$, when $n \geqq 11$.

First we obtain from (4.23) and (4.25)

$$\begin{aligned} a_{n,2}=&-\sigma^2(2n^4-n^3-n^2)-2\nu\beta\sigma(2n^4-5n^3-5n^2-n)\\ &+\nu^2\beta^2(16n^4+36n^3+28n^2+9n+1). \end{aligned} \tag{5.2}$$

LEMMA 5.1. $a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2>0$ for $n \geqq 11$.

Proof. We shall prove this, dividing into three cases according to the size of n.

Case I: $n \geqq 12$. Since we have

$$4\sigma^2 \geqq 4\sigma^2(12)=3.127601\cdots>3.127,$$

$$6\nu\beta\sigma<\lim_{n\to\infty} 6\nu\beta\sigma=12\sqrt{2}-12=4.970562\cdots<4.971,$$

$$6\nu^2\beta^2<\lim_{n\to\infty} 6\nu^2\beta^2=24(3-2\sqrt{2})=4.117749\cdots<4.118,$$

we obtain from (4.26)

$$a_{n,0}>3.127(2n^6-4n^5+5n^4-3n^3)-4.971(12n^5-8n^4-4n^3+3n^2)$$
$$-4.118(28n^4-16n^3+2n^2+n)$$
$$=6.254n^6-72.16n^5-59.901n^4+76.391n^3-23.149n^2-4.118n \tag{5.3}$$

for $n\geqq 12$. Since we have

$$\sigma^2<\lim_{n\to\infty}\sigma^2=1,$$
$$2\nu\beta\sigma<\lim_{n\to\infty}2\nu\beta\sigma=4\sqrt{2}-4=1.656854\cdots<1.657,$$
$$3\nu^2\beta^2>3(\nu^2\beta^2)(12)=1.815702\cdots>1.815,$$

we obtain from (4.27)

$$a_{n,1}<8n^5-10n^4+11n^3-9n^2+1.657(4n^5-46n^4+23n^3+10n^2-9n)$$
$$-1.815(32n^4+36n^3-8n^2-3n+3)$$
$$=14.628n^5-144.302n^4-16.229n^3+22.09n^2-9.468n-5.445 \tag{5.4}$$

for $n\geqq 12$. Since we have

$$\sigma^2\geqq\sigma^2(12)=0.781900\cdots>0.781,$$
$$2\nu\beta\sigma>2(\nu\beta\sigma)(12)=1.375838\cdots>1.375,$$
$$\nu^2\beta^2<\lim_{n\to\infty}\nu^2\beta^2=4(3-3\sqrt{2})=0.686291\cdots<0.687,$$

we obtain from (5.2)

$$a_{n,2}<-0.781(2n^4-n^3-n^2)-1.375(2n^4-5n^3-5n^2-n)$$
$$+0.687(16n^4+36n^3+28n^2+9n+1)$$
$$=6.68n^4+32.388n^3+26.892n^2+7.558n+0.687 \tag{5.5}$$

for $n\geqq 12$. From Lemma 3.5, we have $\beta<1.415$.

Hence, by (5.3), (5.4), (5.5) and Lemma 4.6 we obtain

$$a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2>6.254n^6-72.16n^5-59.901n^4+76.391n^3$$
$$-23.149n^2-4.118n-1.415(14.628n^5-144.302n^4-16.229n^3+22.09n^2$$
$$-9.468n-5.445)-2(6.68n^4+32.388n^3+26.892n^2+7.558n+0.687)$$
$$=6.254n^6-92.85862n^5+130.92633n^4+34.579035n^3-108.19035n^2$$

$$-5.83678n+6.330675$$

$$>6.254n^6-92.859n^5+130.926n^4+34.579n^3-108.191n^2-5.837n+6.33$$

$$>0 \quad \text{for } n\geqq 14.$$

Next, since we have for $12\leqq n<14$

$$6\nu\beta\sigma<6(\nu\beta\sigma)(14)=4.243517\cdots<4.244,$$

$$6\nu^2\beta^2<6(\nu^2\beta^2)(14)=3.700940\cdots<3.701,$$

$$\sigma^2<\sigma^2(14)=0.810940\cdots<0.811,$$

$$2\nu\beta\sigma<2(\nu\beta\sigma)(14)=1.414505\cdots<1.415,$$

$$\nu^2\beta^2<\nu^2\beta^2(14)=0.616823\cdots<0.617,$$

we obtain

(5.3′) $$a_{n,0}>3.127(2n^6-4n^5+5n^4-3n^3)-4.244(12n^5-8n^4-4n^3+3n^2)$$
$$-3.701(28n^4-16n^3+2n^2+n)$$
$$=6.254n^6-63.436n^5-54.041n^4+66.811n^3-20.134n^2-3.701n,$$

(5.4′) $$a_{n,1}<0.811(8n^5-10n^4+11n^3-9n^2)+1.415(4n^5-46n^4+23n^3+10n^2-9n)$$
$$-1.815(32n^4+36n^3-8n^2-3n+3)$$
$$=12.148n^5-131.28n^4-23.874n^3+21.371n^2-7.29n-5.445,$$

(5.5′) $$a_{n,2}<-0.781(2n^4-n^3-n^2)-1.375(2n^4-5n^3-5n^2-n)$$
$$+0.617(16n^4+36n^3+28n^2+9n+1)$$
$$=5.56n^4+29.868n^3+24.932n^2+6.928n+0.617.$$

Furthermore, from Lemma 3.5 we have

$$\beta<\beta(14)=1.392690\cdots<1.393, \quad \beta^2<\beta^2(14)=1.939568\cdots<1.940$$

for $n<14$. Hence, by (5.3′), (5.4′), (5.5′) and Lemma 4.6 we obtain

$$a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2$$

$$>6.254n^6-80.358164n^5+118.04564n^4+42.123562n^3-98.271883n^2$$

$$-6.98635n+6.387905$$

$$>6.254n^6-80.359n^5+118.045n^4+42.123n^3-98.272n^2-6.987n+6.387>0$$

for $12\leqq n<14$.

Case II: $11 \leqq n < 12$. Since we have

$$4\sigma^2 \geqq 4\sigma^2(11) = 3.055118\cdots > 3.055,$$

$$6\nu\beta\sigma < 6(\nu\beta\sigma)(12) = 4.127515\cdots < 4.128$$

$$6\nu^2\beta^2 < 6(\nu^2\beta^2)(12) = 3.631405\cdots < 3.632,$$

we obtain from (4.26)

$$(5.6) \qquad \begin{aligned} a_{n,0} &> 3.055(2n^6 - 4n^5 + 5n^4 - 3n^3) - 4.128(12n^5 - 8n^4 - 4n^3 + 3n^2) \\ &\quad - 3.632(28n^4 - 16n^3 + 2n^2 + n) \\ &= 6.11n^6 - 61.756n^5 - 53.397n^4 + 65.459n^3 - 19.648n^2 - 3.632n \end{aligned}$$

for $11 \leqq n < 12$. Since we have

$$\sigma^2 < \sigma^2(12) = 0.781900\cdots < 0.782,$$

$$2\nu\beta\sigma < 2(\nu\beta\sigma)(12) = 1.375838\cdots < 1.376,$$

$$3\nu^2\beta^2 \geqq 3(\nu^2\beta^2)(11) = 1.793579\cdots > 1.793,$$

we obtain from (4.27)

$$(5.7) \qquad \begin{aligned} a_{n,1} &< 0.782(8n^5 - 10n^4 + 11n^3 - 9n^2) + 1.376(4n^5 - 46n^4 + 23n^3 + 10n^2 - 9n) \\ &\quad - 1.793(32n^4 + 36n^3 - 8n^2 - 3n + 3) \\ &= 11.76n^5 - 128.492n^4 - 24.298n^3 + 21.066n^2 - 7.005n - 5.379 \end{aligned}$$

for $11 \leqq n < 12$. Since we have

$$\sigma^2 \geqq \sigma^2(11) = 0.763779\cdots > 0.763,$$

$$2\nu\beta\sigma \geqq 2(\nu\beta\sigma)(11) = 1.351492\cdots > 1.351,$$

$$\nu^2\beta^2 < \nu^2\beta^2(12) = 0.605234\cdots < 0.606$$

for $11 \leqq n < 12$, we obtain from (5.2)

$$(5.8) \qquad \begin{aligned} a_{n,2} &< -0.763(2n^4 - n^3 - n^2) - 1.351(2n^4 - 5n^3 - 5n^2 - n) \\ &\quad + 0.606(16n^4 + 36n^3 + 28n^2 + 9n + 1) \\ &= 5.468n^4 + 29.334n^3 + 24.486n^2 + 6.805n + 0.606 \end{aligned}$$

for $11 \leqq n < 12$.

Now, noticing Lemma 4.6 and Lemma 4.7, we can prove that the last side of (5.7) is positive for $11.1 \leqq n < 12$, by considering the two cases: $11.1 \leqq n < 11.2$ and $11.2 \leqq n < 12$. Thus, since we have from Lemma 3.5

$$\beta<\beta(12)=1.388983\cdots<1.389, \quad \beta^2<\beta^2(12)=1.929275\cdots<1.930$$

for $n<12$, we obtain

$$a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2$$

$$>6.11n^6-78.09064n^5+114.525148n^4+42.594302n^3-96.166654n^2$$

$$-7.035705n+6.301851$$

$$>6.11n^6-78.091n^5+114.525n^4+42.594n^3-96.167n^2-7.036n+6.301>0$$

for $11.1\leqq n<12$.

Finally, we shall prove the above inequality for $11\leqq n<11.1$. First, we shall show that $a_{n,1}<0$ in this interval of n. Since we have

$$\sigma^2<\sigma^2(11.1)=0.765725\cdots<0.766,$$

$$2\nu\beta\sigma<2(\nu\beta\sigma)(11.1)=1.354115\cdots<1.355,$$

$$3\nu^2\beta^2\geqq 3(\nu^2\beta^2)(11)=1.793579\cdots>1.793$$

for $11\leqq n<11.1$, we obtain from (4.27)

$$a_{n,1}<0.766(8n^5-10n^4+11n^3-9n^2)+1.355(4n^5-46n^4+23n^3+10n^2-9n) \tag{5.9}$$

$$-1.793(32n^4+36n^3-8n^2-3n+3)$$

$$=11.548n^5-127.366n^4-24.957n^3+21n^2-6.816n-5.379<0$$

for $11\leqq n<11.1$.

Next, we shall make the evaluations of $a_{n,0}$ and $a_{n,2}$ a little more sharper than (5.6) and (5.8). Since we have

$$6\nu\beta\sigma<6(\nu\beta\sigma)(11.1)=4.062345\cdots<4.063,$$

$$6\nu^2\beta^2<6(\nu^2\beta^2)(11.1)=3.591943\cdots<3.592,$$

$$\nu^2\beta^2<0.598657\cdots<0.599$$

for $n<11.1$, we obtain from (4.26) and (5.2)

$$a_{n,0}>3.055(2n^6-4n^5+5n^4-3n^3)-4.063(12n^5-8n^4-4n^3+3n^2) \tag{5.10}$$

$$-3.592(28n^4-16n^3+2n^2+n)$$

$$=6.11n^6-60.976n^5-52.797n^4+64.559n^3-19.373n^2-3.592n$$

and

$$a_{n,2}<-0.763(2n^4-n^3-n^2)-1.351(2n^4-5n^3-5n^2-n) \tag{5.11}$$

$$+0.599(16n^4+36n^3+28n^2+9n+1)$$

$$=5.356n^4+29.082n^3+24.29n^2+6.742n+0.599$$

for $11 \leqq n < 11.1$. Furthermore, since we have

$$\beta \geqq \beta(11)=1.386606\cdots>1.386, \quad \beta^2<\beta^2(11.1)=1.923393\cdots<1.924,$$

we obtain from (5.10), (5.9) and (5.11)

$$\begin{aligned} & a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2 \\ & >6.11n^6-76.981528n^5+113.427332n^4+43.195634n^3-95.21296n^2 \\ & \quad -7.116632n+6.302818 \\ & >6.11n^6-76.982n^5+113.427n^4+43.195n^3-95.213n^2-7.117n+6.302>0 \end{aligned}$$

for $11 \leqq n < 11.1$.

Thus, we have finished the verification of Case II. Q. E. D.

Remark. By a numerical computation we see that $a_{n,0}-a_{n,1}\beta-a_{n,2}\beta^2<0$ at $n=10$.

LEMMA 5.2. *When* $n \geqq 11$, $a_{n,0}-a_{n,1}x-a_{n,2}x^2>0$ *for* $0 \leqq x \leqq \beta$.

Proof. By Lemma 4.2, Lemma 4.4 and Lemma 5.1, the statement of this lemma is evident. Q. E. D.

Regarding (4.12) we have the following

LEMMA 5.3.
$$\left.\frac{(n-x)^{n-1}S_3(x)}{S_2(x)}\right|_{x=1}=B\,.$$

Proof. By means of (4.11) and (4.5)$\sim$(4.8), we have

$$S_2(1)=S_3(1)=12(n-1)^2\{(n-\beta)+\nu\beta\}\,. \qquad \text{Q. E. D.}$$

§6. Properties of $W(x, x_1)$.

PROPOSITION 1. $W(x, x_1)>0$ *for* $X^{-1}(\beta) \leqq x<1$ *and* $X(x) \leqq x_1<n$, *when* $n \geqq 11$.

Proof. By means of Lemma 5.2, we see from (4.20) that the inequality (4.12) is true for $0 \leqq x \leqq \beta$, hence (4.4) is so. Thus, we obtain

$$(n-x)^n F_2(x)-\nu\phi(\beta)F_3(x)>0$$

and hence

$$G'(x)>0 \quad \text{for } 0 \leqq x \leqq \beta,\ x \neq 1$$

by the argument in §4. $G(x)$ must be strictly increasing in $0<x<\beta$. By Lemma

3.5 and Lemma 3.7 we see that the inequality (3.4) must be true for $X^{-1}(\beta)\leqq x<1$ and $X(x)<x_1<n$. Accordingly $W(x, x_1)$ is increasing with respect to x_1 for $X(x)<x_1<n$, when $X^{-1}(\beta)\leqq x<1$. From this fact and Lemma 3.2 we obtain

$$W(x, x_1)>0 \quad \text{for } X^{-1}(\beta)\leqq x<1 \quad \text{and } X(x)\leqq x_1<n. \qquad \text{Q.E.D.}$$

LEMMA 6.1. *The function $W(x, x_1)$ is convex downward with respect to x_1 for a fixed x $(0<x<1)$, when*

i) $\max(X(x), 2)\leqq x_1<n$;

and

ii) *for* $X(x)\leqq x_1=Y<2$, $X^{-1}(2)<x<1$,

where

$$\frac{\partial}{\partial x_1}W(x, x_1)|_{x_1=Y}=0,$$

provided $n\geqq 11$.

Proof. In the proof of Lemma 3.4, we get

$$\begin{aligned}(6.1)\qquad &\frac{1}{\phi(x)}\frac{\partial}{\partial x_1}W(x, x_1)\\ &=[F_0(X)-F_0(x)]\frac{(x_1-1)n+(n-1)x_1}{(n-1)x_1^2(n-x_1)}-[F_1(X)-F_1(x)](x_1-1)(n-x_1)^{n-2},\end{aligned}$$

from which we obtain

$$\begin{aligned}(6.2)\qquad &\frac{1}{\phi(x)}\frac{\partial^2}{\partial x_1^2}W(x, x_1)\\ &=[F_0(X)-F_0(x)]\frac{2(n-x_1)^2+(n-1)x_1^3}{(n-1)x_1^3(n-x_1)^2}\\ &\quad-[F_1(X)-F_1(x)](n-1)(2-x_1)(n-x_1)^{n-3}.\end{aligned}$$

Since $F_0(X)-F_0(x)$ and $F_1(X)-F_1(x)$ are positive for $0<x<1$, we obtain easily that

$$\frac{\partial^2 W(x, x_1)}{\partial x_1^2}>0 \quad \text{for } 2\leqq x_1<n.$$

Next, regarding the second statement, we see immediately from (6.1) that $\frac{\partial}{\partial x_1}W(x, x_1)$ vanishes if and only if

$$(6.3)\qquad \frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}=\frac{(n-1)x_1^2(n-x_1)^{n-1}}{n+(n-1)x_1}\ (:=\eta(x_1)).$$

From (6.2) we see that $\partial^2 W(x, x_1)/\partial x_1^2>0$ if and only if

$$(6.4)\qquad \frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}>\frac{(n-1)^2x_1^3(2-x_1)(n-x_1)^{n-1}}{2(n-x_1)^2+(n-1)x_1^3}.$$

Now, we factorize the right hand side of (6.4) as

$$\frac{(n-1)x_1(2-x_1)\{n+(n-1)x_1\}}{2(n-x_1)^2+(n-1)x_1^3}\cdot\frac{(n-1)x_1^2(n-x_1)^{n-1}}{n+(n-1)x_1},$$

the first factor of which is decreasing for $\frac{\sqrt{3}}{\sqrt{2}}<x_1$ and takes the value 1 at $x_1=\beta$. In fact, we can easily show that $x_1(2-x_1)\{n+(n-1)x_1\}$ is decreasing and $2(n-x_1)^2+(n-1)x_1^3$ is increasing for $\frac{\sqrt{3}}{\sqrt{2}}<x_1$. Next, using the equality

$$(6.5)\qquad (n-1)\beta^2=2(n-\beta)$$

by Lemma 3.5, we have

$$\frac{(n-1)\beta(2-\beta)\{n+(n-1)\beta\}}{2(n-\beta)^2+(n-1)\beta^3}=\frac{(n-1)(2-\beta)\{n\beta+2(n-\beta)\}}{2(n-\beta)^2+2\beta(n-\beta)}$$

$$=\frac{(n-1)\{4n-4\beta-(n-2)\beta^2\}}{2n(n-\beta)}=1.$$

Going back to the proof of ii), we assume that (6.3) holds at $x_1=Y$, $X(x)\leqq Y<n$, $0<x<1$. When $Y\geqq 2$, $W(x, x_1)$ is convex downward at $x_1=Y$ by i). When $Y<2$, it must be $X^{-1}(2)<x<1$.

On the other hand, from Lemma 3.6 we have

$$\beta(n)>\sqrt{\frac{2n}{n+1}}\quad\text{for } n>1$$

and

$$\sqrt{\frac{2n}{n+1}}>\frac{\sqrt{3}}{\sqrt{2}}\quad\text{for } n\geqq 3.$$

Now, assuming $n\geqq 11$, we see that it must be $X^{-1}(2)<x<X^{-1}(\beta)$ by Proposition 1. Hence we have

$$\frac{\sqrt{3}}{\sqrt{2}}<\beta\quad\text{and}\quad \beta<X(x)\leqq Y<2$$

and so

$$0<\frac{(n-1)Y(2-Y)\{n+(n-1)Y\}}{2(n-Y)^2+(n-1)Y^3}<1.$$

Therefore we obtain

$$\frac{F_0(X)-F_0(x)}{F_0(X)-F_0(x)}>\frac{(n-1)^2Y^3(2-Y)(n-Y)^{n-1}}{2(n-Y)^2+(n-1)Y^3},$$

which shows that $W(x, x_1)$ is convex downward with respect to x_1 at $x_1=Y$.

Q. E. D.

This lemma and Lemma 3.2 imply immediately the following

LEMMA 6.2. *When* $n \geqq 11$, *for a fixed* x $(0<x<1)$ *the function* $W(x, x_1)$ *takes its critical value with respect* x_1 *at most one point in the interval* $X(x) \leqq x_1 < n$, *which becomes the minimum value.*

LEMMA 6.3. *For a sufficiently small fixed* x $(0<x<1)$, $W(X, x)$ *as function of* x_1 *takes its critical value in the interval* $X(x)<x_1<n$.

Proof. By means of Lemma 3.4 or Lemma 6.1, we see that $W(x, x_1)$ is critical with respect to x_1 at $x_1=Y$ if and only if

$$F_0(X)-F_0(x)=\frac{(n-1)Y^2(n-Y)^{n-1}}{n+(n-1)Y}\{F_1(X)-F_1(x)\}, \tag{6.6}$$

$$X(x) \leqq Y < n\,.$$

For $0<x<X^{-1}(\beta)$, we have $\beta<X(x)$ and then from the fact described in the proof of Lemma 3.5, we see that there exists Y satisfying (6.6) if and only if

$$F_0(X)-F_0(x) \leqq \frac{(n-1)X^2(n-X)^{n-1}}{n+(n-1)X}\{F_1(X)-F_1(x)\},$$

which is equivalent to

$$\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3} \tag{6.7}$$

$$\leqq \frac{(n-1)X}{n+(n-1)X}\left\{\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right\}$$

by (3.2) and (3.3).

Now, we obtain easily from (2.10)

$$\lim_{x\to 0}\left\{\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right\}=0$$

and

$$\lim_{x\to 0}\frac{(n-1)X}{n+(n-1)X}\left\{\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right\}=\frac{n(2n-1)B}{n-1}\lim_{x\to 0}\frac{1}{\sqrt{n-X}}$$

$$=+\infty.$$

Hence, (6.7) must hold for sufficiently small $x>0$ without the equality sign.

Q. E. D.

Taking these lemmas into consideration, for each x $(0<x<1)$ let $Y=Y(x)$ be the value of x_1 where $W(x, x_1)$ takes its minimum in the interval $X(x) \leqq x_1 < n$.

LEMMA 6.4. *We have*

$$\lim_{x\to+0} W(x,\ X(x))=\infty \quad (n>2).$$

Proof. From $x(n-x)^{n-1}=X(n-X)^{n-1}$, we have

$$x\fallingdotseq\frac{1}{n^{n-2}}(n-X)^{n-1} \quad \text{near } x=0, \tag{6.8}$$

from which we obtain

$$\lim_{x\to+0}\frac{x}{n-X}=0\,.$$

We have also from (1.4)

$$\lim_{x\to+0}(n-X)\lambda(X)=\lim_{x\to n}\{(n-X)\log(n-X)+n-1\}=n-1\,. \tag{6.9}$$

Now, since we have

$$\begin{aligned}W(x,\ X(x))&=\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}\\&\quad+\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\frac{n(X-1)^2}{(n-1)X(n-X)}\\&=\frac{\sqrt{n-x}\,f_0(x)}{(x-1)^3}\left\{(n-X)\lambda(X)\frac{x}{n-X}-x\lambda(x)\right\}\\&\quad+\frac{nf_0(X)}{(n-1)(X-1)}\cdot\frac{1}{\sqrt{n-X}}-\frac{n\sqrt{n-x}\,f_0(x)(X-1)^2}{(n-1)(x-1)^3X}\cdot\frac{x}{n-X},\end{aligned}$$

we obtain

$$\begin{aligned}\lim_{x\to+0}W(x,\ X(x))&=-\sqrt{n}\,f_0(0)\left[(n-1)\lim_{x\to+0}\frac{x}{n-X}-0\right]\\&\quad+\frac{nf_0(n)}{(n-1)^2}\lim_{x\to+0}\frac{1}{\sqrt{n-X}}+(n-1)\sqrt{n}\,f_0(0)\lim_{x\to+0}\frac{x}{n-X}\\&=\frac{nf_0(n)}{(n-1)^2}\lim_{x\to+0}\frac{1}{\sqrt{n-X}}=+\infty\end{aligned}$$

Q. E. D.

LEMMA 6.5. *We have*

$$\lim_{x\to+0}W(x,\ Y(x))=-\infty\,.$$

Proof. By Lemma 6.3 we have (6.6) at $Y=Y(x)$ for sufficiently small $x>0$. Hence, we have

$$\begin{aligned}\frac{\phi(Y)}{\phi(X)}&=\frac{n+(n-1)Y}{(n-1)Y}\cdot\frac{F_0(X)-F_0(x)}{\{F_1(X)-F_1(x)\}\phi(x)}\\&=\frac{n+(n-1)Y}{(n-1)Y}\cdot\frac{\dfrac{f_0(X)}{(n-X)^{n-3/2}(X-1)^3}-\dfrac{f_0(x)}{(n-x)^{n-3/2}(x-1)^3}}{\left\{\dfrac{f_1(X)}{(n-X)^{2n-3/2}(1-X)^3}-\dfrac{f_1(x)}{(n-x)^{2n-3/2}(1-x)^3}\right\}\phi(x)}\end{aligned}$$

$$=\frac{n+(n-1)Y}{(n-1)Y}\cdot\frac{\dfrac{f_0(X)}{(X-1)^3}-\dfrac{f_0(x)}{(x-1)^3}\left(\dfrac{n-X}{n-x}\right)^{n-3/2}}{\dfrac{Xf_1(X)}{(1-X)^3}-\dfrac{xf_1(X)}{(1-x)^3}\left(\dfrac{n-X}{n-x}\right)^{n-1/2}}\cdot(n-X),$$

from which we obtain near $x=0$

$$\frac{\psi(Y)}{\psi(X)}=\frac{n^2}{(n-1)n}\cdot\frac{f_0(n)}{-nf_1(n)}\cdot(n-X)=\frac{-f_0(n)}{(n-1)f_1(n)}(n-X).$$

Since we have from (2.10)

$$f_0(n)=(n-1)B,\quad f_1(n)=-(n-1)(2n-1)B,$$

we obtain

$$\frac{\psi(Y)}{\psi(X)}=\frac{Y(n-Y)^{n-1}}{X(n-X)^{n-1}}=\frac{1}{(n-1)(2n-1)}(n-X) \tag{6.10}$$

near $x=0$, which implies

$$\lim_{x\to+0}\frac{\psi(Y)}{\psi(X)}=+0. \tag{6.11}$$

Now, we have from (3.1)

$$\begin{aligned}W(x,\ Y(x))=&\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}\\&+\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\left\{\frac{n}{n-1}\frac{(Y-1)^2}{Y(n-Y)}-\lambda(Y)+\lambda(X)\right\}\\&-\frac{1}{n}\left[\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\left\{1-\frac{\psi(Y)}{\psi(X)}\right\}\\=&\frac{\sqrt{n-x}\,f_0(X)}{(x-1)^3}(n-X)\lambda(X)\frac{x}{n-X}-\frac{x\sqrt{n-x}\,f_0(x)\lambda(x)}{(x-1)^3}\\&+\left[(n-X)\left\{\frac{n}{n-1}\frac{(Y-1)^2}{Y(n-Y)}-\lambda(Y)\right\}+(n-X)\lambda(X)\right]\\&\cdot\left[\frac{Xf_0(X)}{(X-1)^3}\cdot\frac{1}{\sqrt{n-X}}-\frac{\sqrt{n-x}\,f_0(x)}{(x-1)^3}\cdot\frac{x}{n-X}\right]\\&-\frac{1}{n}\left\{1-\frac{\psi(Y)}{\psi(X)}\right\}\frac{X^2f_1(X)}{(1-X)^3}\cdot\frac{1}{\sqrt{n-X}}+\frac{1}{n}\left\{1-\frac{\psi(Y)}{\psi(X)}\right\}\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}.\end{aligned}$$

On the other hand, near $x=0$ we obtain from (6.10)

$$n-Y=\left\{\frac{1}{(n-1)(2n-1)}\right\}^{1/(n-1)}(n-X)^{n/(n-1)} \tag{6.12}$$

or

(6.12′) $$n-X \fallingdotseq \{(n-1)(2n-1)\}^{1/n}(n-Y)^{(n-1)/n}$$

and

$$\frac{n}{n-1}\frac{(Y-1)^2}{Y}-(n-Y)\lambda(Y)$$

$$=\frac{1}{n-1}\frac{(n-1)^2-2(n-1)(n-Y)+(n-Y)^2}{1-\dfrac{n-Y}{n}}-(n-Y)\log(n-Y)-(n-1)$$

$$=(n-1)\left\{1-\frac{2(n-Y)}{n-1}+\frac{(n-Y)^2}{(n-1)^2}\right\}\left\{1+\frac{n-Y}{n}+\left(\frac{n-Y}{n}\right)^2+\cdots\right\}$$

$$-(n-Y)\log(n-Y)-(n-1)$$

$$=(n-Y)\left[-\log(n-Y)-\frac{n+1}{n}+\frac{1}{n(n-1)}\left\{\frac{n-Y}{n}+\left(\frac{n-Y}{n}\right)^2+\cdots\right\}\right].$$

Hence we have

$$\lim_{x\to+0}(n-X)\left\{\frac{n}{n-1}\cdot\frac{(Y-1)^2}{Y(n-Y)}-\lambda(Y)\right\}$$

$$=\lim_{x\to+0}\frac{n-X}{n-Y}\left\{\frac{n}{n-1}\frac{(Y-1)^2}{Y}-(n-Y)\lambda(Y)\right\}$$

$$=\lim_{x\to+0}(n-X)\left[-\log(n-Y)-\frac{n+1}{n}+\frac{1}{n(n-1)}\left\{\frac{n-Y}{n}+\left(\frac{n-Y}{n}\right)^2+\cdots\right\}\right]$$

$$=\lim_{x\to+0}-(n-X)\log(n-Y)=-\{(n-1)(2n-1)\}^{1/n}\lim_{x\to+0}(n-Y)\log(n-Y)$$

$$=0,$$

i. e.

(6.13) $$\lim_{x\to+0}(n-X)\left\{\frac{n}{n-1}\frac{(Y-1)^2}{Y(n-Y)}-\lambda(Y)\right\}=0.$$

Using these facts, we obtain from the expression above of $W(x, Y(x))$

$$\lim_{x\to+0}W(x, Y(x))=\frac{\sqrt{n}f_0(0)}{-1}(n-1)\lim_{x\to+0}\frac{x}{n-X}-0$$

$$+\lim_{x\to+0}\left\{\left[(n-X)\left\{\frac{n}{n-1}\frac{(Y-1)^2}{Y(n-Y)}-\lambda(Y)\right\}+(n-X)\lambda(X)\right]\frac{Xf_0(X)}{(X-1)^3\sqrt{n-X}}\right.$$

$$\left.-\frac{1}{n}\left\{1-\frac{\phi(Y)}{\phi(X)}\right\}\frac{X^2f_1(X)}{(1-X)^3}\right\}\cdot\frac{1}{\sqrt{n-X}}+0$$

$$=\lim_{x\to+0}\left\{(n-1)\frac{nf_0(n)}{(n-1)^3}-\frac{1}{n}\frac{n^2f_1(n)}{(1-n)^3}\right\}\frac{1}{\sqrt{n-X}}$$

$$=\frac{n}{(n-1)^3}\{(n-1)f_0(n)+f_1(n)\}\lim_{x\to+0}\frac{1}{\sqrt{n-X}}.$$

Since we have

$$(n-1)f_0(n)+f_1(n)=\{(n-1)^2-(n-1)(2n-1)\}B=-n(n-1)B<0,$$

we obtain finally

$$\lim_{x\to+0}W(x,\ Y(x))=-\infty \qquad \text{Q.E.D.}$$

§ 7. A device in the main integral.

By virtue of the properties of $W(x,\ x_1)$, especially described in Lemma 6.4 and Lemma 6.5, which are not desirable for our way to prove Conjecture C by making use of Lemma 3.1, we shall try to find a more convenient form of the main integral.

First of all, we show that $W(x,\ x_1)$ given by (3.1) can be written as

$$\begin{aligned}W(x,\ x_1)=&\psi(x)F_0(x)\{\lambda(X)-\lambda(x)\}\\&+\psi(x)[F_0(X)-F_0(x)]\cdot\left\{\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)+\lambda(X)\right\}\\&-\frac{\psi(x)}{n}[F_1(X)-F_1(x)]\cdot\{\psi(x)-\psi(x_1)\}\\&=\psi(X)\left[F_0(X)\{\lambda(X)-\tilde{\lambda}(x_1)\}-\frac{1}{n}F_1(X)\{\psi(X)-\psi(x_1)\}\right]\\&-\psi(x)\left[F_0(x)\{\lambda(x)-\tilde{\lambda}(x_1)\}-\frac{1}{n}F_1(x)\{\psi(x)-\psi(x_1)\}\right],\end{aligned}$$

where we set

(7.1) $$\tilde{\lambda}(x):=\lambda(x)-\frac{n}{n-1}\cdot\frac{(x-1)^2}{x(n-x)}=\log(n-x)+\frac{nx-1}{(n-1)x}.$$

Here we introduce a complex valued function by

(7.2) $$E(z,\ x_1):=F_0(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}-\frac{1}{n}F_1(z)\{\psi(z)-\psi(x_1)\},$$

which is analytic with respect to z with singularity only at $z=n$. Then, we have

(7.3) $$W(x,\ x_1)=\psi(x)\{E(X,\ x_1)-E(x,\ x_1)\}$$

and going back to § 3 we obtain easily the formula:

$$\frac{\partial\Omega(\tau,\ n)}{\partial n}=\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{x_1}\frac{(1-x)(n-x)^{n-2}E(x,\ x_1)dx}{\sqrt{x(n-x)^{n-1}-c}}\tag{7.4}$$
$$=-\frac{\sqrt{c}}{4b^2\sqrt{n}}\int_{\gamma}\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{z(n-z)^{n-1}-c}}.$$

PROPOSITION 2. *We have*

$$\int_{\gamma}\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{z(n-z)^{n-1}-c}}=\frac{2}{n}\int_{x_0}^{x_1}\frac{\sqrt{x(n-x)^{n-1}-c}\ N(x,\ x_1)dx}{(x-1)^4(n-x)^{n+1/2}}$$

where

$$N(x,\ x_1)=(n-x)F_2(x)\{\lambda(x)-\tilde{\lambda}(x_1)\}+3(x-1)^2f_0(x)-2n(x-1)^3\{B-\phi(x)\}.$$

Proof. Noticing $c=\phi(x_1)$ on the Riemann surface $\mathfrak{F}$ and setting $c^*=\tilde{\lambda}(x_1)$, we obtain

$$\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{\phi(z)-c}}$$
$$=\frac{(1-z)(n-z)^{n-2}}{w}\Big[F_0(z)\{\lambda(z)-c^*\}-\frac{1}{n}F_1(z)\{\phi(z)-c\}\Big]dz$$
$$=\frac{2}{n}F_0(z)\{\lambda(z)-c^*\}\,dw-\frac{1}{n}(1-z)(n-z)^{n-2}F_1(z)w\,dz$$
$$=\frac{2}{n}d[F_0(z)\{\lambda(z)-c^*\}\,w]$$
$$-\frac{w}{n}[2d(F_0(z)\{\lambda(z)-c^*\})+(1-z)(n-z)^{n-2}F_1(z)dz].$$

Since we have from (3.2), (3.3) and (1.5), (2.7) in [12]

$$\frac{d}{dz}(F_0(z)\{\lambda(z)-c^*\})=\frac{F_2(z)\{\lambda(z)-c^*\}}{2(z-1)^4(n-z)^{n-1/2}}+F_0(z)\frac{z-1}{(n-z)^2}$$

and

$$2d(F_0(z)\{\lambda(z)-c^*\})+(1-z)(n-z)^{n-2}F_1(z)dz$$
$$=\Big[\frac{F_2(z)\{\lambda(z)-c^*\}}{(z-1)^4(n-z)^{n-1/2}}+\frac{2f_0(z)}{(z-1)^2(n-z)^{n+1/2}}+\frac{f_1(z)}{(1-z)^2(n-z)^{n+1/2}}\Big]dz$$
$$=\frac{1}{(z-1)^4(n-z)^{n+1/2}}[(n-z)F_2(z)\{\lambda(z)-c^*\}+(z-1)^2\{2f_0(z)+f_1(z)\}]dz,$$

we obtain the formula

$$\int_{\gamma}\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{z(n-z)^{n-1}-c}}\tag{7.5}$$

$$=-\frac{1}{n}\int_{\Gamma}\frac{\sqrt{z(n-z)^{n-1}-c}}{(z-1)^4(n-z)^{n+1/2}}[(n-z)F_2(z)\{\lambda(z)-c^*\}+(z-1)^2\{2f_0(z)+f_1(z)\}]dz\,.$$

On the other hand, we have from (2.10)

$$f_1(z)=f_0(z)-2n(z-1)\{B-\psi(z)\}, \tag{7.6}$$

and so

$$2f_0(z)+f_1(z)=3f_0(z)-2n(z-1)\{B-\psi(z)\}.$$

Hence setting

$$\begin{aligned} N(z,\ x_1):=&(n-z)F_2(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}+3(z-1)^2f_0(z)\\ &-2n(z-1)^3\{B-\psi(z)\}, \end{aligned} \tag{7.7}$$

(7.5) becomes

$$\begin{aligned} &\int_{\Gamma}\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{z(n-z)^{n-1}-c}}\\ &=-\frac{1}{n}\int_{\Gamma}\frac{\sqrt{z(n-z)^{n-1}-c}\,N(z,\ x_1)dz}{(z-1)^4(n-z)^{n+1/2}}=\frac{2}{n}\int_{x_0}^{x_1}\frac{\sqrt{x(n-x)^{n-1}-c}\,N(x,\ x_1)dx}{(x-1)^4(n-x)^{n+1/2}} \end{aligned} \tag{7.8}$$

Q. E. D.

Remark. By means of the remark at the end of §2 in [12], $F_2(x)$ has a zero point of order at least 4 at $x=1$. And $f_0(x)$ has a zero point of order 3 by Lemma 2.2 and $B-\psi(x)$ has a zero point of order 2 at $x=1$.

LEMMA 7.1. *$\tilde{\lambda}(x)$ $(0<x<n)$ has the following properties:*

(i) $\lim_{x\to+0}\tilde{\lambda}(x)=\lim_{x\to n-0}\tilde{\lambda}(x)=-\infty$,

(ii) $\dfrac{d\tilde{\lambda}(x)}{dx}=-\dfrac{(x-1)\{n+(n-1)x\}}{(n-1)x^2(n-x)}$,

(iii) $\dfrac{d^2\tilde{\lambda}(x)}{dx^2}=-\dfrac{2}{(n-1)x^3}-\dfrac{1}{(n-x)^2}<0$.

Proof. We obtain this lemma immediately from (7.1).

LEMMA 7.2. *$N(x,\ x_1)$ $(0<x<n,\ 1\leq x_1<n)$ has the following properties:*

(i) $\lim_{x\to n-0}N(x,\ x_1)=+\infty$ *for* $x\neq 1$,

(ii) $N(1,\ x_1)=0$ *for* $1\leq x_1<n$,

(iii) $\lim_{x\to+0}N(x,\ X(x))=+\infty$,

(iv) $\lim_{x\to n-0} N(x, x)=0$,

(v) $N(x, x)<0$ *for* $x>1$ *sufficiently near* $x=1$,

(vi) $\dfrac{\partial N(x, x_1)}{\partial x_1}>0$ *for* $x\neq 1$, $1<x_1<n$.

Proof. (i) is derived easily from Lemma 7.1 and the fact $F_2(x)>0$ for $0\leqq x<n$, $x\neq 1$ (Proposition 1 in [12]). (ii) is evident.

Next, we shall prove (iii) and (iv). We have easily

$$\lim_{x\to+0} N(x, X(x))=nF_2(0)\lim_{x\to 0}(-\lambda(X(x)))+nF_2(0)\lambda(0)+3f_0(0)+2nB=+\infty,$$

and by (7.1) and $F_2(n)=(n-1)^2(2n-3)B$ we have

$$\lim_{x\to n-0} N(x, x)=\lim_{x\to n-0} F_2(x)\frac{n(x-1)^2}{(n-1)x}+3(n-1)^2f_0(n)-2n(n-1)^3B$$
$$=(n-1)F_2(n)+3(n-1)^3B-2n(n-1)^3B=0.$$

From Lemma 2.3 in [12] and Lemma 4.1 in [11] we obtain

$$\lim_{x\to 1}\frac{f_0(x)}{(x-1)^2}=\frac{1}{6}n(2n-1)(n-1)^{n-2},\quad \lim_{x\to 1}\frac{B-\phi(x)}{(x-1)^2}=\frac{n(n-1)^{n-2}}{2}.$$

Since we have

$$N(x, x)=\frac{n}{n-1}\,\frac{(x-1)^2}{x}F_2(x)+3(x-1)^2f_0(x)-2n(x-1)^3\{B-\phi(x)\}, \tag{7.9}$$

we obtain

$$\lim_{x\to 1+0}\frac{N(x, x)}{(x-1)^5}=\frac{n}{n-1}\lim_{x\to 1+0}\frac{F_2(x)}{(x-1)^3}+\frac{1}{2}n(2n-1)(n-1)^{n-2}$$
$$-n^2(n-1)^{n-2}=0-\frac{1}{2}n(n-1)^{n-2}<0,$$

because $F_2(x)$ has a zero point of order at least 4. This fact implies immediately (v).

Finally, from Lemma 7.1 we obtain

$$\frac{\partial N(x, x_1)}{\partial x_1}=(n-x)F_2(x)\frac{(x_1-1)\{n+(n-1)x_1\}}{(n-1)x_1^2(n-x_1)}>0$$

for $0<x<n$, $x\neq 1$ and $1<x_1<n$. Q. E. D.

Taking note of the facts in Lemma 7.2, we shall provide for the following

PROPOSITION 3. *We have*

$$\int_{r}\frac{(1-z)(n-z)^{n-2}E(z,\ x_1)dz}{\sqrt{z(n-z)^{n-1}-c}}=\frac{2}{n}\int_{x_0}^{1}\frac{(1-x)\sqrt{x(n-x)^{n-1}-c}}{x^2(n-x)^n}V(x,\ x_1)dx\,,$$

where

$$V(x,\ x_1):=\frac{x^2N(x,\ x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X(x),\ x_1)}{(X-1)^5\sqrt{n-X}}\quad \textit{for } 0<x<1. \tag{7.10}$$

Proof. In fact we have

$$\int_{x_0}^{x_1}\frac{\sqrt{\phi(x)-c}\,N(x,\ x_1)dx}{(x-1)^4(n-x)^{n+1/2}}=\int_{x_0}^{1}\frac{x\sqrt{\phi(x)-c}\,N(x,\ x_1)}{(1-x)^5(n-x)^{n-1/2}}\cdot\frac{1-x}{x(n-x)}dx$$
$$+\int_{1}^{x_1}\frac{X\sqrt{\phi(X)-c}\,N(X,\ x_1)}{(X-1)^5(n-X)^{n-1/2}}\cdot\frac{X-1}{X(n-X)}dX.$$

By means of (2.12) and (2.13), the second integral of the right hand side of the above equality becomes

$$\int_{x_0}^{1}\frac{X\sqrt{\phi(x)-c}\,N(X,\ x_1)}{(X-1)^5(n-X)^{n-1/2}}\cdot\frac{1-x}{x(n-x)}dx$$
$$=\int_{x_0}^{1}\frac{\sqrt{\phi(x)-c}\,X^2N(X,\ x_1)}{\phi(x)(X-1)^5\sqrt{n-X}}\cdot\frac{1-x}{x(n-x)}dx\,.$$

Hence, we obtain

$$\int_{x_0}^{x_1}\frac{\sqrt{\phi(x)-c}\,N(x,\ x_1)dx}{(x-1)^4(n-x)^{n+1/2}}=\int_{x_0}^{1}\frac{(1-x)\sqrt{\phi(x)-c}}{x^2(n-x)^n}$$
$$\cdot\left[\frac{x^2N(x,\ x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X,\ x_1)}{(X-1)^5\sqrt{n-X}}\right]dx=\int_{x_0}^{1}\frac{(1-x)\sqrt{\phi(x)-c}\ V(x,\ x_1)dx}{x^2(n-x)^n},$$

which implies this proposition. Q. E. D.

Formula (7.4), Proposition 2, Proposition 3 and Lemma 7.2 imply $\frac{\partial\Omega(\tau,\ n)}{\partial n}<0$, if we can prove that $V(x,\ X(x))\geqq 0$ for $0<x<1$.

§ 8. Properties of $V(x,\ x_1)$.

LEMMA 8.1. *$V(x,\ x_1)$ $(0<x<1,\ 1<x_1<n)$ has the following properties*:

(i) $\lim_{x_1\to n-0}V(x,\ x_1)=+\infty$ *for* $0<x<1$;

(ii) *$V(x,\ x_1)$ is increasing with respect to x_1 for each x* $(0<x<1)$;

(iii) *When* $n>2$, $\lim_{x\to+0}V(x,\ X(x))=0$ *and furthermore when* $n>\frac{5+\sqrt{13}}{4}=2.15$, $V(x,\ X(x))>0$ *near* $x=0$;

(iv) *When* $n>2$, $\lim_{x\to 1-0}V(x,\ X(x))=0$ *and furthermore when* $n>\frac{1+\sqrt{13}}{2}=2.30$, $V(x,\ X(x))>0$ *near* $x=1$.

Proof. From (7.10) and Lemma 7.2 we obtain easily (i) and (ii).

Now, we shall estimate $V(x, X(x))$ near $x=0$. From (7.7), (7.9) and (7.10) we have

$$(8.1)\qquad V(x, X(x))=\frac{x^2N(x, X)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X, X)}{(X-1)^5\sqrt{n-X}}$$
$$=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X)\}-\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}+\frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}}$$
$$+\frac{n}{n-1}\frac{XF_2(X)}{(X-1)^3\sqrt{n-X}}+\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}-\frac{2nX^2\{B-\psi(X)\}}{(X-1)^2\sqrt{n-X}}.$$

In the following, we set for simplicity

$$(8.2)\qquad \begin{cases} U_0(x):=\dfrac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\}, \\ U_1(x):=\dfrac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}, \quad U_2(x):=\dfrac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}}, \\ U_3(x):=\dfrac{nxF_2(x)}{(n-1)(x-1)^3\sqrt{n-x}}, \end{cases}$$

and

$$(8.3)\qquad U_4(x):=U_3(X(x)),\quad U_5(x):=U_1(X(x)),\quad U_6(x):=U_2(X(x)),$$

then we have

$$V(x, X(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x).$$

By means of Proposition 3, Lemma 2.1 in [12] and Lemma 7.1 we see easily that $U_i(x)$, $i=0, 1, 2, 4, 5, 6$, are all positive.

Now, taking an auxiliary parameter $t=n-X(x)$, we can put

$$x=t^{n-1}(b_1+b_2t+\cdots)\quad \text{near } x=0$$

by the equality: $x(n-x)^{n-1}=X(n-X)^{n-1}=nt^{n-1}-t^n$. Therefore from the equality

$$(b_1+b_2t+\cdots)(n-b_1t^{n-1}-b_2t^n+\cdots)^{n-1}=n-t,$$

we obtain easily

$$b_1=\frac{1}{n^{n-2}},\quad b_2=-\frac{1}{n^{n-1}},$$

hence

$$(8.4)\qquad x=t^{n-1}\Big(\frac{1}{n^{n-2}}-\frac{t}{n^{n-1}}+\cdots\Big),$$

supposing $n>2$. Next, we shall compute $\lambda(x)-\tilde{\lambda}(X)$. From (1.4) and (7.1), we have

$$\lambda(x)-\tilde{\lambda}(X)=\log(n-x)+\frac{n-1}{n-x}-\log(n-X)-\frac{nX-1}{(n-1)X}$$

$$=\log n+\frac{n-1}{n}+O(t^{n-1})-\log t-\frac{n+1}{n}+O(t)$$

$$=-\log t+\log n-\frac{2}{n}+O(t).$$

Hence we obtain

$$U_0(x)=\{\sqrt{n}\,F_2(0)+O(t^{n-1})\}\,x^2\left\{-\log t+\log n-\frac{2}{n}+O(t)\right\} \tag{8.5}$$

$$=O(t^{2n-2-m}),\quad 0<m<1,$$

since we have

$$\lim_{t\to+0} t^m \log t=0$$

for any small positive constant m. Analogously we have

$$U_1(x)=O(t^{2n-2}),\quad U_2(x)=O(t^{2n-2}). \tag{8.6}$$

Next, we shall compute $U_4(x)$, $U_5(x)$, $U_6(x)$. We have

$$U_4(x)=\frac{n}{n-1}\,\frac{XF_2(X)}{(X-1)^3\sqrt{n-X}}=\frac{n}{(n-1)\sqrt{t}}\left\{\frac{nF_2(n)}{(n-1)^3}-\frac{d}{dX}\,\frac{XF_2(X)}{(X-1)^3}\bigg|_{x=n}t+\cdots\right\}$$

and

$$\frac{d}{dX}\,\frac{XF_2(X)}{(X-1)^3}\bigg|_{x=n}=-\frac{(2n+1)F_2(n)}{(n-1)^4}+\frac{nF_2'(n)}{(n-1)^3}.$$

From (4.2) we have

$$F_2(n)=-P_2(n)B=(n-1)^2(2n-3)B,$$

$$F_2'(n)=-P_2'(n)B=8(n-1)B$$

and hence

$$\frac{d}{dX}\,\frac{XF_2(X)}{(X-1)^3}\bigg|_{x=n}=-\frac{(4n^2-12n-3)B}{(n-1)^2}.$$

Thus we obtain

$$U_4(x)=\frac{n^2(2n-3)B}{(n-1)^2\sqrt{t}}+\frac{n(4n^2-12n-3)B}{(n-1)^3}\sqrt{t}\ \{1+O(t)\} \tag{8.7}$$

Next, we have

$$U_5(x)=\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}=\frac{3}{\sqrt{t}}\left\{\frac{n^2f_0(n)}{(n-1)^3}-\frac{d}{dX}\,\frac{X^2f_0(X)}{(X-1)^3}\bigg|_{x=n}t+\cdots\right\}$$

and

$$\frac{d}{dX}\left.\frac{X^2 f_0(X)}{(X-1)^3}\right|_{x=n}=-\frac{n(n+2)f_0(n)}{(n-1)^4}+\frac{n^2 f_0'(n)}{(n-1)^3}.$$

From (2.10) we have $f_0(n)=(n-1)B$, $f_0'(n)=-B$ and hence

$$\frac{d}{dX}\left.\frac{X^2 f_0(X)}{(X-1)^3}\right|_{x=n}=-\frac{2n(n+1)B}{(n-1)^3}.$$

Thus we obtain

$$U_5(x)=\frac{3n^2B}{(n-1)^2\sqrt{t}}+\frac{6n(n+1)B}{(n-1)^3}\sqrt{t}\,\{1+O(t)\}. \tag{8.8}$$

Next, we have

$$U_6(x)=\frac{2nX^2\{B-\phi(X)\}}{(X-1)^2\sqrt{n-X}}=\frac{2n}{\sqrt{t}}\left\{\frac{n^2B}{(n-1)^2}-\frac{d}{dX}\left.\frac{X^2\{B-\phi(X)\}}{(X-1)^2}\right|_{x=n}t+\cdots\right\}$$

and

$$\frac{d}{dX}\left.\frac{X^2\{B-\phi(X)\}}{(X-1)^2}\right|_{x=n}=-\frac{2nB}{(n-1)^3}.$$

Thus we obtain

$$U_6(x)=\frac{2n^3B}{(n-1)^2\sqrt{t}}+\frac{4n^2B}{(n-1)^3}\sqrt{t}\,\{1+O(t)\}. \tag{8.9}$$

By means of (8.5)$\sim$(8.9), we have

$$V(x,\,X(x))=O(t^{2n-2-m})+O(t^{2n-2})+\frac{B}{\sqrt{t}}\left\{\frac{n^2(2n-3)}{(n-1)^2}+\frac{3n^2}{(n-1)^2}-\frac{2n^3}{(n-1)^2}\right\}$$
$$+B\sqrt{t}\left\{\frac{n(4n^2-12n-3)}{(n-1)^3}+\frac{6n(n+1)}{(n-1)^3}-\frac{4n^2}{(n-1)^3}+O(t)\right\},$$

i. e.

$$V(x,\,X(x))=\frac{n(4n^2-10n+3)B}{(n-1)^3}\sqrt{t}\,(1+O(t)), \tag{8.10}$$

which implies immediately

$$\lim_{x\to+0}V(x,\,X(x))=0$$

and $V(x,\,X(x))>0$ sufficiently near $x=0$, when $n>\dfrac{5+\sqrt{13}}{4}=2.15138\cdots$.

Finally we shall prove the statement (iv). Taking an auxiliary parameter

$$t=1-x, \tag{8.11}$$

we can put near $x=1$ as

$$X=1+b_1t+b_2t^2+\cdots.$$

Substituting these into the equality

$$(1-x)(n-x)^{n-2}dx=(1-X)(n-X)^{n-2}dX,$$

we have

$$(n-1+t)^{n-2}=(b_1+b_2t+\cdots)(b_1+2b_2t+\cdots)(n-1-b_1t-b_2t^2+\cdots)^{n-2},$$

from which we obtain easily

$$b_1=1, \quad b_2=\frac{2(n-2)}{3(n-1)}$$

and hence

$$X=1+t+\frac{2(n-2)}{3(n-1)}t^2+\cdots. \tag{8.12}$$

We have

$$\frac{x^2\sqrt{n-x}}{(1-x)^5}=\frac{\sqrt{n-1}}{t^5}(1-2t+t^2)\left(1+\frac{t}{2(n-1)}-\frac{t^2}{8(n-1)^2}+\cdots\right) \tag{8.13}$$

$$=\frac{\sqrt{n-1}}{t^5}\left\{1-\frac{4n-5}{2(n-1)}t+\frac{8n^2-24n+15}{8(n-1)^2}t^2+\cdots\right\},$$

and from (4.5) and (4.6)

$$\begin{aligned}F_2(x)&=-P_2(x)B+(n-x)^{n-1}P_3(x)\\&=-P_2(1-t)B+B\left(1+\frac{t}{n-1}\right)^{n-1}P_3(1-t)\\&=-B\{12(n-1)^2+2(2n^2+3n-5)t+(2n+1)t^2\}\\&\quad+B\left\{1+t+\frac{n-2}{2(n-1)}t^2+\frac{(n-2)(n-3)}{6(n-1)^2}t^3+\frac{(n-2)(n-3)(n-4)}{24(n-1)^3}t^4+\cdots\right\}\\&\quad\times\{12(n-1)^2-2(n-1)(4n-11)t+(2n^2-10n+11)t^2+(n-1)t^3\},\end{aligned}$$

i. e.

$$F_2(1-t)=t^4\left\{\frac{n(n^2-n+1)B}{6(n-1)}+O(t)\right\}. \tag{8.14}$$

Then, setting

$$X=1+s, \quad s=t+\frac{2(n-2)}{3(n-1)}t^2+\cdots,$$

we obtain from (1.4) and (7.1)

$$\lambda(x)-\tilde{\lambda}(X)=\log(n-x)+\frac{n-1}{n-x}-\log(n-X)+\frac{1}{(n-1)X}-\frac{n}{n-1}$$

$$=\log\left(1+\frac{t}{n-1}\right)+\frac{1}{1+\frac{t}{n-1}}-\log\left(1-\frac{s}{n-1}\right)+\frac{1}{(n-1)(1+s)}-\frac{n}{n-1}$$

$$=\left(\frac{t}{n-1}-\frac{t^2}{2(n-1)^2}+\cdots\right)+\left(1-\frac{t}{n-1}+\frac{t^2}{(n-1)^2}-\cdots\right)-\frac{n}{n-1}$$

$$+\left(\frac{s}{n-1}+\frac{s^2}{2(n-1)^2}+\cdots\right)+\left(\frac{1}{n-1}-\frac{s}{n-1}+\frac{s^2}{n-1}+\cdots\right)$$

$$=\left(\frac{t^2}{2(n-1)^2}+\cdots\right)+\left(\frac{(2n-1)s^2}{2(n-1)^2}+\cdots\right),$$

hence

$$\lambda(x)-\tilde{\lambda}(X)=t^2\left\{\frac{n}{(n-1)^2}+O(t)\right\}. \tag{8.15}$$

Then, using (8.13), (8.14) and (8.15), we obtain

$$\begin{aligned} U_0(x)&=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\} \\ &=t\left\{\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}+O(t)\right\}. \end{aligned} \tag{8.16}$$

Next, we have from (2.10)

$$\begin{aligned} f_0(x)&=(2n-1-x)B-(n-x)^{n-1}\{n-x+(n-1)x^2\} \\ &=\{2(n-1)+t\}B-B\left(1+\frac{t}{n-1}\right)^{n-1}\{2(n-1)-(2n-3)t+(n-1)t^2\} \\ &=B\{2(n-1)+t\}-B\{2(n-1)-(2n-3)t+(n-1)t^2\} \\ &\quad\times\left\{1+t+\frac{n-2}{2(n-1)}t^2+\frac{(n-2)(n-3)}{6(n-1)^2}t^3+\frac{(n-2)(n-3)(n-4)}{24(n-1)^3}t^4+\cdots\right\}, \end{aligned}$$

i. e.

$$f_0(1-t)=-\frac{n(2n-1)B}{6(n-1)}t^3-\frac{n(n-2)(3n-1)B}{12(n-1)^2}t^4+\cdots. \tag{8.17}$$

Hence, we obtain

$$\begin{aligned} U_1(x)&=\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}=-\frac{3(1-2t+t^2)}{t^3\sqrt{n-1+t}}\cdot f_0(1-t) \\ &=\frac{1}{\sqrt{n-1}}(1-2t+t^2)\left(1-\frac{t}{2(n-1)}+\cdots\right) \\ &\quad\times\left(\frac{n(2n-1)B}{2(n-1)}+\frac{n(n-2)(3n-1)B}{4(n-1)^2}t+\cdots\right) \end{aligned}$$

i. e.

$$U_1(x)=\frac{1}{\sqrt{n-1}}\left\{\frac{n(2n-1)B}{2(n-1)}-\frac{n(5n^2-3n+1)B}{4(n-1)^2}t+\cdots\right\}. \tag{8.18}$$

Analogously, we obtain

$$U_5(x)=U_1(X(x))=U_1(1+s)$$
$$=\frac{1}{\sqrt{n-1}}\left\{\frac{n(2n-1)B}{2(n-1)}+\frac{n(5n^2-3n+1)B}{4(n-1)^2}s+\cdots\right\},$$

therefore

$$U_5(x)=\frac{1}{\sqrt{n-1}}\left\{\frac{n(2n-1)B}{2(n-1)}+\frac{n(5n^2-3n+1)B}{4(n-1)^2}t+\cdots\right\}. \tag{8.19}$$

Next, we have

$$\frac{x^2}{(1-x)^2\sqrt{n-x}}-\frac{X^2}{(X-1)^2\sqrt{n-X}}=\frac{1-2t+t^2}{t^2\sqrt{n-1+t}}-\frac{1+2s+s^2}{s^2\sqrt{n-1-s}}$$
$$=\frac{1}{t^2\sqrt{n-1}}(1-2t+t^2)\left\{1-\frac{t}{2(n-1)}+\frac{3t^2}{8(n-1)^2}+\cdots\right\}$$
$$-\frac{1}{s^2\sqrt{n-1}}(1+2s+s^2)\left\{1+\frac{s}{2(n-1)}+\frac{3s^2}{8(n-1)^2}+\cdots\right\}$$
$$=\frac{1}{t^2\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{8n^2-8n+3}{8(n-1)^2}t^2+\cdots\right\}$$
$$-\frac{1}{s^2\sqrt{n-1}}\left\{1+\frac{4n-3}{2(n-1)}s+\frac{8n^2-8n+3}{8(n-1)^2}s^2+\cdots\right\},$$

into which substituting (8.12) we obtain

$$\frac{x^2}{(1-x)^2\sqrt{n-x}}-\frac{X^2}{(X-1)^2\sqrt{n-X}}$$
$$=\frac{1}{t^2\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{8n^2-8n+3}{8(n-1)^2}t^2+\cdots\right\}$$
$$-\frac{1}{t^2\sqrt{n-1}}\left\{1-\frac{4(n-2)}{3(n-1)}t+\cdots\right\}\left\{1+\frac{4n-3}{2(n-1)}t+\cdots\right\}$$
$$=\frac{1}{t}\left\{-\frac{8n-1}{3(n-1)^{3/2}}+O(t)\right\}.$$

Furthermore, by Lemma 4.1 in [11] we have

$$B-\psi(x)=B-\psi(X)=t^2\left\{\frac{n(n-1)^{n-2}}{2}+O(t)\right\} \quad \text{near } x=1.$$

Thus, we obtain

$$U_2(x)-U_6(x)=t\left\{-\frac{n^2(8n-1)(n-1)^{n-7/2}}{3}+O(t)\right\}. \tag{8.20}$$

Next, using (8.14) we have

$$U_4(x)=\frac{nXF_2(X)}{(n-1)(X-1)^3\sqrt{n-X}}=\frac{n(1+s)}{(n-1)s^3}\cdot\frac{1}{\sqrt{n-1}}\left\{1+\frac{s}{2(n-1)}+\frac{3s^2}{8(n-1)^2}+\cdots\right\}$$
$$\times s^4\left\{\frac{n(n^2-n+1)B}{6(n-1)}+O(s)\right\}=s\left\{\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}+O(s)\right\}$$

and hence

$$U_4(x)=t\left\{\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}+O(t)\right\}. \tag{8.21}$$

By means of (8.16), (8.18), (8.19), (8.20) and (8.21), we have near $x=1$

$$\begin{aligned}V(x,\ X(x))=&\,t\left\{\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}+O(t)\right\}\\ &-\frac{n(2n-1)(n-1)^{n-5/2}}{2}+t\left\{\frac{n(5n^2-3n+1)(n-1)^{n-7/2}}{4}+O(t)\right\}\\ &+\frac{n(2n-1)(n-1)^{n-5/2}}{2}+t\left\{\frac{n(5n^2-3n+1)(n-1)^{n-7/2}}{4}+O(t)\right\}\\ &+t\left\{-\frac{n^2(8n-1)(n-1)^{n-7/2}}{3}+O(t)\right\}\\ &+t\left\{\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}+O(t)\right\}\\ =&\,t\left\{\frac{n(2n^3-3n^2-5n+3)(n-1)^{n-7/2}}{6}+O(t)\right\}\end{aligned}$$

i. e.

$$V(x,\ X(x))=\frac{n(2n-1)(n^2-n-3)(n-1)^{n-7/2}}{6}\,t\,\{1+O(t)\}, \tag{8.22}$$

where $x=1-t$, which implies immediately

$$\lim_{x\to1-0}V(x,\ X(x))=0$$

and

$$V(x,\ X(x))>0\quad\text{sufficiently near }x=1,$$

when $n>\dfrac{1+\sqrt{13}}{2}=2.30277\cdots$. Q. E. D.

References

[1] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields, Springer-Verlag, 1970, 60-75.

[2] S. Furuya, On periods of periodic solutions of a certain nonlinear differential equation, Japan-United States Seminar on Ordinary Differential and Functional

Equations, Lecture Notes in Mathematics, Springer-Verlag, **243** (1971), 320-323.
[3] W. Y. HSIANG AND H. B. LAWSON, Jr., Minimal submanifolds of low cohomogeneity, J. Diff. Geometry, **5** (1970), 1-38.
[4] T. OTSUKI, Minimal hypersurfaces in a Riemannian manifold of constant curvature, Amer. J. Math., **92** (1970), 145-173.
[5] T. OTSUKI, On integral inequalities related with a certain nonlinear differential equation, Proc. Japan Acad., **48** (1972), 9-12.
[6] T. OTSUKI, On a 2-dimensional Riemannian manifold, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 401-414.
[7] T. OTSUKI, On a family of Riemannian manifolds defined on an m-disk, Math. J. Okayama Univ., **16** (1973), 85-97.
[8] T. OTSUKI, On a bound for periods of solutions of a certain nonlinear differential equation (I), J. Math. Soc. Japan, **26** (1974), 206-233.
[9] T. OTSUKI, On a bound for periods of solutions of a certain nonlinear differential equation (II), Funkcialaj Ekvacioj, **17** (1974), 193-205.
[10] T. OTSUKI, Geodesics of O_n^2 and an analysis on a related Riemann surface, Tôhoku Math. J., **28** (1976), 411-427.
[11] T. OTSUKI, A certain property of geodesics of the family of Riemannian manifolds O_n^2 (I), Proc. of Japan-United States Seminar on Minimal Submanifolds, including Geodesics, Kaigai, Tokyo, 1978, 173-192.
[12] T. OTSUKI, A certain property of geodesics of the family of Riemannian manifolds O_n^2 (II), Kodai Math. J., **2** (1979), 211-242.
[13] M. MAEDA AND T. OTSUKI, Models of the Riemannian manifolds O_n^2 in the Lorentzian 4-space, J. Diff. Geometry, **9** (1974), 97-108.
[14] M. URABE, Computations of periods of a certain nonlinear autonomous oscilations, Study of algorithms of numerical computations, Sûrikaiseki Kenkyûsho Kôkyûroku, **149** (1972), 111-129 (Japanese).

DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO
WAKAMIYA-CHO 26, SHINJUKU-KU
TOKYO, JAPAN 162