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KODAI MATH. J.
3 (1980), 429—441

A GENERALIZATION OF MULTIVARIATE POISSON DISTRIBUTION

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Summary

Historically we have treated many multivariate discrete data which did not have an unimodal probability density. We consider that we need to develope a new method analyzing these data. It is not so easy to make convenient tables of these multivariate discrete distributions. The treat of data is different every underlying distributions. It is important that it is better to develop the structure of discrete data and to use the personal computer which have recently been near ourself than before to get the statistical utilizable levels and regions than to wait the finish of general theory and its statistical tables. And under some hypothesis of structure we can simulate the data by computer and may be able todecide the hypotheses is true or not. It is a dynamic system of statistical decision theory.

In this paper we attempt to generalize the multivariate Poisson distribution and to investigate the detail of structure. Our purpose is to keep some of the property of Poisson distribution and to enlarge the class of Poisson distribution which we can treat.

Notations and Definitions

n	positive integer, dimension.
N	sample size.
$X=(X_1, X_2, \cdots, X_n)$	n dimensional random vector.
$x=(x_1, x_2, \cdots, x_n)$	observation of X.
$i=(i_1, i_2, \cdots, i_n)$	n dimensional vector with components of non-negative integers. We also use j and k.
$p(x, \lambda)$	usual univariate Poisson density with parameter λ.
$s=(s_1, s_2, \cdots, s_n)$	n dimensional vector.
$B(1, p_i)$	multivariate Bernoulli distribution.
$B(N, p_i)$	multivariate binomial distribution.
$P(\lambda_i)$	multivariate Poisson distribution.

Received September 20, 1979

Main Results

In this main results we attempt systematically to develop and represent a generalized multivariate Poisson distribution and to discuss the structure of the distribution.

1. GENERALIZED MULTIVARIATE BERNOULLI DISTRIBUTION GB(1, p_i).
An usual multivariate Bernoulli distribution is defined by $P(X=j)=p_j$ where j is a n dimensional vector with components 0 or 1 and p_j satisfies $p_j \geqq 0$ and $\sum_j p_j=1$. To generalize this Bernoulli distribution we have to replace the vector j with components 0 or 1 by the vector i with components of $0, 1, 2, \cdots$. Generalized multivariate Bernoulli distribution will be defined by $P(X=i)=p_i$ where $p_i \geqq 0$ and $\sum_i p_i=1$. We shall denote this distribution as GB(1, p_i).

The moment generating function (m. g. f.) is given by

$$g(s)=\sum_i p_i s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} .$$

The mean vector $E(X)$ is given by

$$E(X_j)=\sum_i i_j p_i \qquad (j=1, 2, \cdots, n),$$

or

$$E(X)=(\sum_i i_1 p_i, \sum_i i_2 p_i, \cdots, \sum_i i_n p_i) .$$

We can denote this mean vector as $\sum_i i p_i$, then

$$E(X)=\sum_i i p_i .$$

The covariance matrix of GB(1, p_i) is given by

$$\operatorname{Cov}(X_j, X_k)=\sum_i i_j i_k p_i-(\sum_i i_j p_i)(\sum_i i_k p_i),$$

$$\operatorname{Var}(X_j)=\sum_i i_j^2 p_i-(\sum_i i_j p_i)^2 .$$

The marginal distribution of this generalized multivariate Bernoulli distribution is also a generalized degenerated multivariated Bernoulli distribution.

Note. $\sum_i$ means the sum of all terms of varying i.

2. GENERALIZED MULTIVARIATE BINOMIAL DISTRIBUTION GB(N, p_i).
Generalized multivariate binomial distribution will be defined by convolution of N independent observations of GB(1, p_i). The probability density is given by

$$P(X=k)=\sum_{a_i \left(\begin{array}{l}\sum_i a_i i_1=k_1 \\ \sum_i a_i i_2=k_2 \\ \cdots \\ \sum_i a_i i_n=k_n \\ \sum_i a_i=N \\ a_i \geqq 0 \text{ integer}\end{array}\right)} \frac{N!}{\prod_i a_i!} \prod_i p_i^{a_i} ,$$

where k is a n dimensional vector with nonnegative components of integers and the notation $\sum$ means to sum up all terms verying integer $a_i \geqq 0$ with the conditions denoted after a_i. The m. g. f. of this distribution is given by

$$g_N(s)=[g(s)]^N=[\sum_i p_i s^i]^N .$$

The marginal distribution of this distribution is also a degenerated generalized multivariate binomial distribution.

The mean values and the covariance of our GB(N, p_i) will be given by

$$E(X_j)=N\sum_i i_j p_i ,$$

$$\mathrm{Cov}\,(X_j,\ X_k)=N[(\sum_i i_j i_k p_i)-(\sum_i i_j p_i)(\sum_i i_k p_i)]$$

and

$$\mathrm{Var}\,(X_j)=N[(\sum_i i_j^2 p_i)-(\sum_i i_j p_i)^2] .$$

3. GENERALIZED MULTIVARIATE POISSON DISTRIBUTION GP(λ_i).

In this section, a generalized multivariate Poisson distribution will be introduced as a limiting distribution of our GB(N, p_i). To get a limiting distribution we have to assume that only a finite number of p_i including p_0 are positive such that $Np_i=\lambda_i>0$ ($i\neq 0$) and another p_i equal to zero. In this assumption λ_i ($i\neq 0$) are nonnegative fixed parameters. Exactly we have to denote $p_i(N)$ instead of p_i in our assumptions. So that our assumption about p_i becomes

$$p_0(N)>0 \quad \text{and} \quad Np_i(N)=\lambda_i \geqq 0 ,$$

where λ_i are nonnegative fixed parameters and the number $\sharp$ of positive λ_i will be assumed as finite.

If a random variable X_N has this generalized multivariate binomial distribution GB(N, $p_i(N)$) and we assume

$$p_0(N)>0 \quad \text{and} \quad Np_i(N)=\lambda_i \geqq 0$$

$$\sharp\ \{i : Np_i(N)=\lambda_i>0\} < \infty$$

then we can derive that

$$\lim_{N\to\infty} P(X_N=k)=\sum_{a_i \left(\begin{array}{l}\Sigma_i a_i i_1=k_1\\ \Sigma_i a_i i_2=k_2\\ \cdots\\ \Sigma_i a_i i_n=k_n\\ \Sigma_i a_i=N\\ a_i\geqq 0 \ \text{integer}\end{array}\right)} \prod_{i\neq 0} p(a_i,\ \lambda_i)$$

where $p(a_i,\ \lambda_i)$ is an usual univariate Poisson probability density. The notation $\sum$ means the sum of the products with a_i varying nonnegative integer and satisfying the denoted $n+1$ equalities. For the simplicity of notation we write the restriction including $n+1$ equalities as $*$.

THEOREM 1. *If a sequence of random variables X_N has a sequence of distri-*

butions GB(N, $p_i(N)$) ($N=1, 2, \cdots$) *respectively and we assume that* $N\to\infty$ *and* $p_0(N)>0$, $Np_i(N)=\lambda_i\geqq 0$ *and* $\#\{i:\lambda_i>0\}<\infty$, *then weh ave a limiting distribution*

$$\lim_{N\to\infty} P(X_N=k)=\textstyle\sum_{a_i*}\prod_{i\neq 0} p(a_i, \lambda_i)\,.$$

Proof. From our assumption that X_N has a distribution GB(N, $p_i(N)$), we can express

$$P(X_N=k)=\sum_{a_i*}\frac{N!}{\prod_i a_i!}\prod_i p_i(N)^{a_i}\,.$$

From each term of the sum we can pull the next limiting value

$$\lim_{\substack{N\to\infty\\ Np_i(N)=\lambda_i\\ \Sigma_i a_i=N}} \frac{N!}{\prod_i a_i!}\prod_i p_i(N)^{a_i}$$

$$=\lim_{N\to\infty}\frac{N!}{a_0!\prod_{i\neq 0}a_i!}(1-\sum_{i\neq 0}p_i(N))^{a_0}\prod_{i\neq 0}p_i(N)^{a_i}$$

$$=\lim_{N\to\infty}\frac{N!}{a_0!\prod_{i\neq 0}a_i!}\left(1-\frac{\sum_{i\neq 0}\lambda_i}{N}\right)^{N-\sum_{i\neq 0}a_i}\prod_{i\neq 0}\frac{\lambda_i^{a_i}}{N^{a_i}}$$

$$=\prod_{i\neq 0}p(a_i, \lambda_i)\,.$$

Therefore, under the assumptions of the theorem, we have

$$\lim_{\substack{N\to\infty\\ Np_i(N)=\lambda_i}} P(X_N=k)=\textstyle\sum_{a_i*}\prod_{i\neq 0}p(a_i, \lambda_i)\,.$$

This is our conclusion of this theorem and we shall call this limiting distribution as generalized multivariate Poisson and we shall denote it as GP(λ_i).

THEOREM 2. *The moment generating function of the generalized multivariate Poisson distribution is given by*

$$h(s)=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i s^i\}$$

$$=\prod_{i\neq 0}\exp\{-\lambda_i+\lambda_i s^i\}\,.$$

Proof. We shall derive the m. g. f. from $g(s)^N$.

$$h(s)=\lim_{\substack{N\to\infty\\ Np_i(N)=\lambda_i}} g(s)^N=\lim_{\substack{N\to\infty\\ Np_i(N)=\lambda_i}}[\textstyle\sum_i p_i s^i]^N$$

$$=\lim_{N\to\infty}(1-\sum_{i\neq 0}p_i(N)+\sum_{i\neq 0}p_i(N)s^i)^N$$

$$=\lim_{N\to\infty}\left(1-\sum_{i\neq 0}\frac{\lambda_i}{N}+\sum_{i\neq 0}\frac{\lambda_i}{N}s^i\right)^N$$

$$=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i s^i\}$$

where we have denoted $s^i=s_1^{i_1}s_2^{i_2}\cdots s_n^{i_n}$.

THEOREM 3. *If a random vector X has the generalized multivariate Poisson distribution then we have an unique decomposition of the random vector X as $X_j=\sum_i i_j Y_i$ $(j=1, 2, \cdots, n)$ where Y_i $(i\neq 0)$ are mutually independent univariate Poisson variables with parameter λ_i.*

Proof. If Y_i $(i\neq 0)$ are mutually independent univariate Poisson variables with parameter λ_i then the random vector X with components $X_j=\sum_i i_j Y_i$ has a generalized multivariate Poisson probability density

$$P(X=k)=\sum_{a_i*}\prod_{i\neq 0} p(a_i, \lambda_i).$$

And if we assume X has the generalized multivariate Poisson density $GP(\lambda_i)$ then X has a m. g. f. $h(s)$ as described in the preceding theorem.

$$h(s)=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i s^i\}$$
$$=\prod_{i\neq 0}\exp\{-\lambda_i+\lambda_i s^i\}.$$

For simplicity of our proof we assume $n=2$ and only two of λ_i $\{i=(1, 2), (2, 1)\}$ are positive then $h(s)$ becomes

$$h(s)=\exp\{-\lambda_{12}-\lambda_{21}+\lambda_{12}s_1{}^1s_2{}^2+\lambda_{21}s_1{}^2s_2{}^1\}$$
$$=\exp\{-\lambda_{12}+\lambda_{12}s_1{}^1s_2{}^2\}\exp\{-\lambda_{21}+\lambda_{21}s_1{}^2s_2{}^1\}.$$

This means there exist two independent univariate Poisson random variables X_{12}, X_{21} with parameter λ_{12}, λ_{21} respectively and X has a decomposition

$$X=(1, 2)X_{12}+(2, 1)X_{21}.$$

In another way of proof, if we put $s_2=1$ then

$$h(s)=\exp\{-\lambda_{12}-\lambda_{21}+\lambda_{12}s_1+\lambda_{21}s_1{}^2\}$$
$$=\exp\{-\lambda_{12}+\lambda_{12}s_1\}\exp\{-\lambda_{21}+\lambda_{21}s_1{}^2\}.$$

The marginal distribution of X_1 is given by $X_{12}+2X_{21}$ and in the same way, our X_2 is given by $2X_{12}+X_{21}$. This means

$$X=(1, 2)X_{12}+(2, 1)X_{21} \quad \text{or} \quad [X_1=X_{12}+2X_{21} \quad \text{and} \quad X_2=2X_{12}+X_{21}].$$

And in general case, we can prove our result of this theorem by the same way.

Note. In this proof we have denoted $X_{(1,2)}$, $X_{(2,1)}$ as X_{12}, X_{21} and $\lambda_{(1,2)}$, $\lambda_{(2,1)}$ as λ_{12}, λ_{21} for our simplicity of notation. And we shall use this notations in the following lines.

THEOREM 4. *The mean vector and the covariance matrix of the generalized multivariate Poisson distribution* GP(λ_i) *is given by*

$$E(X_j)=\sum_i i_j\lambda_i\,, \qquad \operatorname{Cov}(X_j,\ X_k)=\sum_i i_j i_k\lambda_i \qquad (j\neq k)$$

$$\textit{and} \qquad \operatorname{Var}(X_j)=\sum_i i_j{}^2\lambda_i\,.$$

Proof. We assume that X has our distribution GP(λ_i). First we shall calculate the mean value of X_j. We shall use the m. g. f. $h(s)$ of X. To differentiate the $h(s)$ by s_j we get

$$\frac{dh(s)}{ds_j}=h(s)\{\sum_{i\neq 0} i_j\lambda_i s_1{}^{i_1}\cdots s_{j-1}{}^{i_{j-1}}s_j{}^{i_j-1}s_{j+1}{}^{i_{j+1}}\cdots s_n{}^{i_n}\}$$

and if we put $s_1=s_2=\cdots=s_n=1$ then we have

$$E(X_j)=\left[\frac{dh(s)}{ds_j}\right]_{s_1=s_2=\cdots=s_n=1}=\sum_{i\neq 0} i_j\lambda_i\,.$$

In the same way we shall use the equality

$$E(X_jX_k)=\left[\frac{d^2h(s)}{ds_k ds_j}\right]_{s_1=s_2=\cdots=s_n=1},$$

where the differential is given by

$$\frac{d}{ds_k}\left\{\frac{dh(s)}{ds_j}\right\}=\frac{dh(s)}{ds_k}\{\sum_i i_j\lambda_i s_1{}^{i_1}\cdots s_{j-1}{}^{i_{j-1}}s_j{}^{i_j-1}s_{j+1}{}^{i_{j+1}}\cdots s_n{}^{i_n}\}$$

$$+\,h(s)\frac{d}{ds_k}\{\sum_i i_j\lambda_i s_1{}^{i_1}\cdots s_{j-1}{}^{i_{j-1}}s_j{}^{i_j-1}s_{j+1}{}^{i_{j+1}}\cdots s_n{}^{i_n}\}$$

$$=h(s)\{\sum_i i_j\lambda_i s_1{}^{i_1}\cdots\}\{\sum_i i_k\lambda_i s_1{}^{i_1}\cdots\}+h(s)\{\sum_i i_j i_k\lambda_i s_1{}^{i_1}\cdots\}.$$

To put $s_1=s_2=\cdots=s_n=1$ in this equality we can derive

$$E(X_jX_k)=(\sum_i i_j\lambda_i)(\sum_i i_k\lambda_i)+(\sum_i i_j i_k\lambda_i)\,.$$

And we can get our conclusion

$$\operatorname{Cov}(X_j,\ X_k)=E(X_jX_k)-E(X_j)E(X_k)=\sum_i i_j i_k\lambda_i\,.$$

To derive $\operatorname{Var}(X_j)$ we shall use the result of preceding Theorem 3. From Theorem 3 if X has the distribution GP(λ_i) then we have a Poisson decomposition of X.

$$X_j=\sum_i i_j Y_i\,, \qquad (j=1,\ 2,\ \cdots,\ n)\,.$$

Therefore we can conclude

$$\mathrm{Var}\,(X_j)=\mathrm{Var}\,(\textstyle\sum_i i_j Y_i)=\sum_i i_j{}^2\,\mathrm{Var}\,(Y_i)=\sum_i i_j{}^2\lambda_i\,.$$

In the following lines we shall consider the marginal distribution of our generalized multivariate Poisson distribution $\mathrm{GP}(\lambda_i)$.

THEOREM 5. *If a random vector X has a generalized multivariate Poisson distribution $GP(\lambda_i)$ then the marginal distribution is also a degenerated generalized multivariate Poisson distribution.*

Proof. Since X has the m. g. f. $h(s)$, it follows that a degenerated random vector of X denoted as

$$X^{(j)}=(X_1,\ \cdots,\ X_{j-1},\ X_{j+1},\ \cdots,\ X_n) \qquad (j=1,\ 2,\ \cdots,\ n)$$

has a m. g. f. $h(s)|_{s_j=1}$.

$$h(s)|_{s_j=1}=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i s^i\}\,|_{s_j=1}$$

$$=\exp\{-\textstyle\sum_{i^{(j)}\neq 0}(\sum_{i_j}\lambda_i)+\sum_{i^{(j)}\neq 0}(\sum_{i_j}\lambda_i)s^i\}\,.$$

Where we have used a new notation $i^{(j)}$ which has been denoted likely as $X^{(j)}$. This equality means that if X has the generalized multivariate Poisson distribution, it follows that $X^{(j)}$ has also a degenerated generalized multivariate Poisson distribution $\mathrm{GP}(\sum_{i_j}\lambda_i)$. And if we put similarly

$$X^{(j_1,j_2,\cdots,j_k)}=(X_1,\ \cdots,\ X_{j_1-1},\ X_{j_1+1},\ \cdots,\ X_{j_2-1},\ X_{j_2+1},\ \cdots,\ X_{j_k-1},\ X_{j_k+1},\ \cdots,\ X_n)$$

where $j_1,\ j_2,\ \cdots,\ j_k$ are integers and $j_1\leqq j_2\leqq\cdots\leqq j_k$. This degenerated random vector of X has a m. g. f.

$$\begin{aligned}
&h(s)|_{s_{j_1}=s_{j_2}=\cdots=s_{j_k}=1}\\
&=\exp\{-\textstyle\sum_{i^{(j_1,j_2,\dots,j_k)}\neq 0}(\sum_{i_{j_1},i_{j_2},\cdots,i_{j_k}}\lambda_i)\\
&\qquad+\textstyle\sum_{i^{(j_1,j_2,\cdots,j_k)}\neq 0}(\sum_{i_{j_1},i_{j_2},\cdots,i_{j_k}}\lambda_i)s_1{}^{i_1}\cdots s_{j_1-1}{}^{i_{j_1-1}}\\
&\qquad\cdot s_{j_1+1}{}^{i_{j_1+1}}\cdots s_{j_2-1}{}^{i_{j_2-1}}s_{j_2+1}{}^{i_{j_2+1}}\cdots s_{j_k-1}{}^{i_{j_k-1}}s_{j_k+1}{}^{i_{j_k+1}}\cdots s_n{}^{i_n}\}\,,
\end{aligned}$$

where we used a new notation $i^{(j_1,j_2,\cdots,j_k)}$ as we had denoted $X^{(j_1,j_2,\cdots,j_k)}$. Therefore, the random vector $X^{(j_1,j_2,\cdots,j_k)}$ has a degenerated generalized multivariate Poisson distribution

$$\mathrm{GP}(\textstyle\sum_{i_{j_1},i_{j_2},\cdots,i_{j_k}}\lambda_i)$$

as to be proved.

COROLLARY 1. *The marginal distribution X_j of X is a univariate generalized*

Poisson with parameter $\sum_{i(j)}\lambda_i$.

COROLLARY 2. *If* $\mathrm{Cov}(X_j, X_k)=0$ $(j\neq k)$ *then* X_j *and* X_k *are mutually independent random variables.*

Proof. From Theorem 3 we have decompositions of X_j and X_k

$$X_j=\sum_i i_j Y_i\,, \qquad X_k=\sum_i i_k Y_i\,,$$

and from Theorem 4, we have

$$\mathrm{Cov}(X_j, X_k)=\sum_i i_j i_k \lambda_i=0 \qquad (j\neq k)$$

this means, for any fixed i if i_j and i_k are simultaneously positive integers then λ_i must be zero, that is, our $Y_i\equiv 0$. From this property we can conclude that X_j, X_k are mutually independent random variables.

THEOREM 6. *If* $X_1, X_2, \cdots, X_N$ *are mutually independent random vectors of the generalized multivariate Poisson distributions* $\mathrm{GP}(\lambda_{i_1}), \mathrm{GP}(\lambda_{i_2}), \cdots, \mathrm{GP}(\lambda_{i_N})$ *respectively then the sum vector* $\sum_{j=1}^N X_j$ *has a generalized multivariate Poisson distribution* $\mathrm{GP}(\sum_{i_1=i_2=\cdots=i_N=i}\lambda_{i_j})$.

Proof. If we assume all the parameters equals to a same λ_i

$$\lambda_{i_1}=\lambda_{i_2}=\cdots=\lambda_{i_N}=\lambda_i$$

then $\sum_{j=1}^N X_j$ has a generalized multivariate Poisson distribution with parameter $N\lambda_i$, because the m. g. f. of $\sum_{j=1}^N X_j$ becomes

$$\begin{aligned} h(s)^N &= N\exp\{-\sum_i\lambda_i+\sum_i\lambda_i s^i\} \\ &=\exp\{-\sum_i N\lambda_i+\sum_i N\lambda_i s^i\}\,. \end{aligned}$$

And, generally $\sum_{j=1}^N X_j$ has a m. g. f.

$$\begin{aligned} h(s) &= \prod_{j=1}^N \exp\{-\sum_{i_j}\lambda_{i_j}+\sum_{i_j}\lambda_{i_j}s^{i_j}\} \\ &=\exp\{-\sum_{j=1}^N \sum_{i_j}\lambda_{i_j}+\sum_{j=1}^N \sum_{i_j}\lambda_{i_j}s^{i_j}\} \\ &=\exp\{-\sum_i\sum_{i_1=i_2=\cdots=i_N=i}\lambda_{i_j}+\sum_i\sum_{i_1=i_2=\cdots=i_N=i}\lambda_{i_j}s^i\}\,. \end{aligned}$$

Therefore $h(s)$ is a m. g. f. of generalized multivariate Poisson distribution with parameter

$$\sum_{i_1=i_2=\cdots=i_N=i}\lambda_{i_j}\,.$$

4. SOME RESTRICTIONS ON THE PARAMETERS.

In preceding section 1, we have defined a generalized multivariate Bernoulli distribution GB$(1, p_i)$. If we assume

$$p_i \geqq 0 \text{ for } i \in \{0, 1\}^n \text{ and } p_i = 0 \text{ for } i \notin \{0, 1\}^n$$

then GB$(1, p_i)$ means $B(1, p_i)$ which is called a multivariate Bernoulli distribution. Under the same assumption, our generalized multivariate binomial distribution GB(N, p_i) defined in section 2 means $B(N, p_i)$, which is called a multivariate binomial distribution, see Kawamura [4].

In section 3, we have defined GP(λ_i). It is a generalized multivariate Poisson distribution because if we assume that λ_i is defined as nonnegative parameter for $i \neq 0$ and

$$\lambda_i \geqq 0 \quad \text{for} \quad i \in \{0, 1\}^n \quad \text{and} \quad \lambda_i = 0 \quad \text{for} \quad i \notin \{0, 1\}^n .$$

then our GP(λ_i) means $p(\lambda_i)$ which is called a multivariate Poisson distribution, see Kawamura [4].

THEOREM 7. *Given a generalized multivariate Bernoulli distribution* GB$(1, p_i)$, *if we restrict the parameter* p_i *as*

$$p_i \geqq 0 \quad \text{on } i \in \{0, 1\}^n \qquad \text{and} \qquad p_i = 0 \quad \text{on } i \notin \{0, 1\}^n .$$

Then GB$(1, p_i)$ *means* B$(1, p_i)$ *which is a multivariate Bernoulli distribution. And given a generalized multivariate binomial distribution* GB(N, p_i), *if we restrict* p_i *as above then* GB(N, p_i) *means* B(N, p_i) *which is a multivariate binomial distribution.*

THEOREM 8. *Given a generalized multivariate Poisson distribution* GP(λ_i), *if we restrict the parameter* λ_i $(i \neq 0)$ *as* $\lambda_i \geqq 0$ *on* $i \in \{0, 1\}^n$ *and* $\lambda_i = 0$ *on* $i \notin \{0, 1\}^n$ *then* GP(λ_i) *means* P(λ_i) *which is a multivariate Poisson distribution.*

5. EXAMPLES.

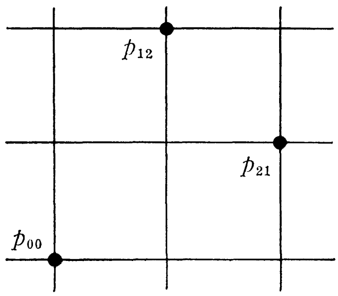
We shall discuss some examples in this section. For our simplicity of discussion, we treat only the bivariate case $(n=2)$.

5-1. We assume X has a distribution GB$(1, p_i)$ and we restrict the space of X to three points $(0,0)$, $(1,2)$ and $(2,1)$, or in another words we restrict only three p_i on $i=(0,0)$, $(1,2)$ and $(2,1)$ are positive and otherwise $p_i=0$. Then our GB$(1, p_i)$ becomes

$$\text{(A)} \quad P(X=(0, 0)) = p_{(0,0)}, \; P(X=(1, 2)) = p_{(1,2)} \; \text{ and } \; P(X=(2, 1)) = p_{(2,1)} .$$

And we shall denote $p_{(0,0)} = p_{00}$, $P_{(1,2)} = p_{12}$ and $p_{(2,1)} = p_{21}$.

Of cause we can select these three points without selecting (0, 0) but to consider the limiting distribution to generalized Poisson we must remain (0, 0) in the space of X with large probability or more exactly near one. But in this GB$(1, p_i)$ case if there does not include (0, 0) in the space of X or $P(X=(0,0))=0$, there is no trouble theoretically.



The space of GP$(1, p_i)$

The mean value of X with this GB$(1, p_i)$ is given by

$$E(X)=(0p_{00}+1p_{12}+2p_{21},\ 0p_{00}+2p_{12}+1p_{21}).$$

And the covariance value is given by

$$\begin{aligned}\mathrm{Cov}\,(X_1,\ X_2)&=0\cdot 0p_{00}+1\cdot 2p_{12}+2\cdot 1p_{21}\\ &\quad-(0p_{00}+1p_{12}+2p_{21})(0p_{00}+2p_{12}+1p_{21})\\ &=2p_{12}+2p_{21}-(p_{12}+2p_{21})(2p_{12}+p_{21}).\end{aligned}$$

We consider the n convolution of GB$(1, p_i)$ defined in (A) in the followings. We shall rewrite again as X the convolution of n independent variables X_1, X_2, $\cdots$, X_N. Then the sum vector X has a distribution GB(N, p_i) by the discussion of section 2.

$$P(X=k)=\sum_{a_i^*}\frac{N!}{\prod_i a_i!}\prod_i p_i^{a_i}$$

(B)

$$=\sum_{a_i\left[\begin{smallmatrix}0a_{00}+1a_{12}+2a_{21}=k_1\\ 0a_{00}+2a_{12}+1a_{21}=k_2\\ a_{00}+a_{12}+a_{21}=N\\ a_{00},\, a_{12} \text{ and } a_{21}\geqq 0 \text{ integer}\end{smallmatrix}\right]}\frac{N!}{a_{00}!a_{12}!a_{21}!}p_{00}{}^{a_{00}}p_{12}{}^{a_{12}}p_{21}{}^{a_{21}}$$

We shall restrict in (B) only on $i=(1,2)$ and $(2,1)$, $Np_i(N)=\lambda_i>0$ and $N\rightarrow\infty$ where $p_{00}(N)+P_{12}(N)+p_{21}(N)=1$ and another $p_i\equiv 0$ then we can derive a generalized multivariate Poisson distribution GP(λ_i).

$$P(X=k)=\sum_{a_{12},\,a_{21}\left[\begin{smallmatrix}a_{12}+2a_{21}=k_1\\ 2a_{12}+a_{21}=k_2\\ a_{12},\, a_{21}\geqq 0 \text{ integer}\end{smallmatrix}\right]}p(a_{12},\ \lambda_{12})p(a_{21},\ \lambda_{21})$$

where X is rewrited again and $p(a_i,\ \lambda)$ is an univariate Poisson probability density. From our decomposition theory our X will be expressed as

$$X=(1,\ 2)Y_{12}+(2,\ 1)Y_{21}$$

where Y_{12} and Y_{21} are mutually independent univariate Poisson random variables with parameter λ_{12} and λ_{21} respectively. The m. g. f. of this X is given by

$$h(s)=\exp\{-\lambda_{12}-\lambda_{21}+\lambda_{12}s_1{}^1s_2{}^2+\lambda_{21}s_1{}^2s_2{}^1\}.$$

The mean value of X and the covariance matrix is given by

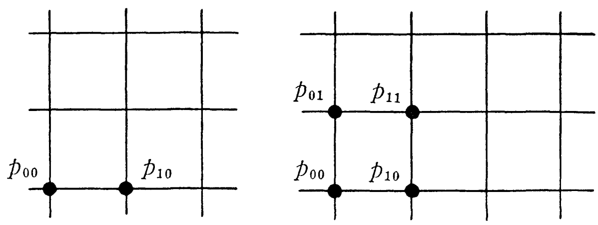
$$E(X)=(\lambda_{12}+2\lambda_{21},\ 2\lambda_{12}+\lambda_{21}),$$

$$\mathrm{Cov}\,(X_1,\ X_2)=2(\lambda_{12}+\lambda_{21}),$$

$$\mathrm{Var}\,(X_1)=\lambda_{12}+4\lambda_{21} \quad \text{and} \quad \mathrm{Var}\,(X_2)=4\lambda_{12}+\lambda_{21}.$$

So that our covariance matrix is represented as

$$\begin{bmatrix} \lambda_{12}+4\lambda_{21} & 2(\lambda_{12}+\lambda_{21}) \\ 2(\lambda_{12}+\lambda_{21}) & 4\lambda_{12}+\lambda_{21} \end{bmatrix}.$$



5-2(1) 5-2(2)

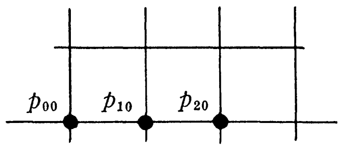
The space of GB$(1, p_i)$

5-2. (1) If we assume X has a distribution GB$(1, p_i)$ and we restrict the space of X to two points $(0, 0)$ and $(1, 0)$ only, then GB$(1, p_i)$ becomes to an univariate Bernoulli distribution and our GB(N, p_i) becomes to an usual univariate binomial distribution. Under our restriction of limitation $Np_{10}(N)=\lambda_{10}>0$ and $N\to\infty$ we can derive that GP(λ_i) becomes to an usual Poisson distribution with parameter p_{10}.

(2) If we restrict the space of X with a distribution GB$(1, p_i)$ to four points $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ only, then GB$(1, p_i)$ becomes to an usual bivariate binomial distribution $B(1,\ p_i)$. From N convolution of this GB$(1, p_i)$ we can derive that GB(N, p_i) becomes to an usual bivariate binomial distribution $B(N,\ p_i)$. To pull our limiting distribution of GB(N, p_i) we have to restrict $Np_i=\lambda_i$ $(i\neq 0)$ and $N\to\infty$. Our limiting distribution is an usual bivariate Poisson distribution $P(\lambda_i)$.

$$P((X_1,\ X_2)=(k,\ l))=\sum_{\substack{b+d=k \\ c+d=l \\ b, c \text{ and } d\geq 0 \text{ integer}}} \frac{\lambda_{10}{}^b\lambda_{01}{}^c\lambda_{11}{}^d}{b!c!d!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}$$

5-3. If we assume X has a distribution GB$(1, p_i)$ and we restrict the space of X to three points $(0, 0)$, $(1, 0)$ and $(2, 0)$ only, then our GB(N, p_i) becomes to a generalized (univariate) binomial distribution which is a degenerated case as we treated in 5-2 (1). We rewrite again X which we assume to have GB(N, p_i) distribution, then we can derive



The space of GB$(1, p_i)$

$$P(X=k)=\sum_{\left[\substack{a_{10}+2a_{20}=k_1,\ k_2=0 \\ a_{00}+a_{10}+a_{20}=N \\ a_{00},\, a_{10} \text{ and } a_{20}\geqq 0 \quad \text{integer}}\right]} \frac{N!}{\Pi_i a_i!} \Pi_i p_i^{a_i}$$

$$=\sum{}^{*} \frac{N!}{a_{00}!\,a_{10}!\,a_{20}!} p_{00}^{a_{00}} p_{10}^{a_{10}} p_{20}^{a_{20}}.$$

Under our restriction of $Np_{10}(N)=\lambda_{10}$, $Np_{20}(N)=\lambda_{20}$ and $N\to\infty$ we can derive a limiting degenerated generalized distribution GP(λ_i). We shall rewrite X again the random variable of GP(λ_i), then our decomposition theory states that

$$X=(1,\ 0)Y_{10}+(2,\ 0)Y_{20}$$

where Y_{10} and Y_{20} are mutually independent univariate Poisson random variables, and this X is a degenerated generalized bivariate Poisson random variable and this X rolls as an univariate generalized Poisson distribution and as an univariate compound Poisson distribution.

Acknowledgement

The author expresses harty thanks to Professor Bradley Efron who invited the author as a visiting scholor of Department of Statistics, Stanford University for one year from Oct. 78' to Sept. 79' and he provided for the author a good laboratory and a comfortable apartment; Escondido Village for the author's family.

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