T. HASEGAWA
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# THE INDEX THEOREM OF GEODESICS ON A RIEMANNIAN MANIFOLD WITH BOUNDARY

BY TAKUICHI HASEGAWA

Let $M$ be a Riemannian manifold with boundary (condimension one submanifold), $p$, $q$ two points in the interior, and $\Omega_1$ the set of piecewise $C^\infty$ curves from $p$ to $q$ which at some points of them lie on the boundary. Let $\gamma$ be an element of $\Omega_1$. By *a variation of* $\gamma$ *in* $\Omega_1$, we mean such a broken $C^\infty$ rectangle in $M$ that $\gamma$ is its base curve and its each longitudinal curve is an element of $\Omega_1$. (See [1] for the definition of "rectangle"). Then the tangent space $T_\gamma\Omega_1$ to $\Omega_1$ at $\gamma$ may be considered as the set of continuously piecewise $C^\infty$ vector fields along $\gamma$ which do not point to the outward at their boundary points and having at least one vector tangent to the boundary.

We consider the length functional $L$ over $\Omega_1$. From the first variation formula, the critical paths of $L$, when they are parametrized proportionally to arc length, are just geodesics of $M$ (there are no breaks) if they have more than one boundary point, or are geodesics which are reflected at a point of the boundary if they have only one boundary point.

We show here what is called the Index Theorem still holds in these circumstances.

Now let $\gamma$ be a critical point of $L$ in $\Omega_1$, which is parametrized by arc length and reflected strictly at $\gamma(a)\in\partial M$. $\gamma:[0, b]\rightarrow M$. Let $\tilde{\gamma}$ be any variation of $\gamma$ in $\Omega_1$, and $W$ its variation vector field. Then $W(a)$ is tangent to the boundary, the second variation $L''_W(0)$ of $L$ in the direction $W$ becomes

$$L''_W(0)=\langle S_{T(a-)-T(a+)}W(a), W(a)\rangle+\int_0^b\{\langle R(W, T)W, T\rangle+\langle W^{\perp\prime}, W^{\perp\prime}\rangle\}dt$$

here $T(t)=\dot{\gamma}(t)$, $W^\perp=W-\langle W, T\rangle T$. $S$ is the second fundamental form of $\partial M$ with respect to $T(a-)-T(a+)$, which is normal to the boundary from the reflection condition. Therefore the index form $I$ at $\gamma$ is

$$I(V, W)=\langle S_{T(a-)-T(a+)}V(a), W(a)\rangle+\int_0^b\{\langle R(T, V)T, W\rangle+\langle V^{\perp\prime}, W^{\perp\prime}\rangle\}dt$$

$$\text{for}\quad V, W\in T_\gamma\Omega_1$$

If $0<t_1<\cdots<t_k<a<t_{k+1}<\cdots<t_l<b$ is a partition of the interval $[0, b]$ so that $V$

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is $C^\infty$ on each subinterval, then

$$I(V, W)=\int_0^b \langle R(T, V^\perp)T-V^{\perp\prime\prime}, W\rangle dt+\sum_{i=1}^{l}\langle V'(t_i^-)-V'(t_i^+), W^\perp(t_i)\rangle$$
$$+\langle V^{\perp\prime}(a-)-V^{\perp\prime}(a+)+S_{T(a-)-T(a+)}V(a), W(a)\rangle$$
$$\text{for} \quad V, W\in T_\gamma\Omega_1$$

Thus the null space of $I$ consists of such vector fields $V$ that

$V$; a Jacobi field on $[0, a]$, $[a, b]$ and

$$V^{\perp\prime}(a-)-V^{\perp\prime}(a+)+S_{T(a-)-T(a+)}V(a)\text{; normal to } \partial M$$

We shall say in general a field $V\in T_\gamma\Omega_1$ along $\gamma$ an *admissible Jacobi field* if $V$ is a usual Jacobi field on each interval $[0, a]$, $[a, b]$, and at $\gamma(a)$, $V'(a-)-V'(a+)+S_{T(a-)-T(a+)}V(a)$ is normal to the boundary. Then as in the ordinary case a field is an admissible Jacobi field if and only if it is generated by one parameter family of geodesics which are reflected on the boundary. Note that in such a field $V$, $V'(a+)$ is determined by the values $V(a)$, $V'(a-)$ i.e. $V|_{[a,b]}$ is determined by $V|_{[0,a]}$. ("reflection of Jacobi fields").

Next we say $\gamma(t)$ *is conjugate to* $\gamma(0)$ along $\gamma|_{[0,t]}$ if there exists a nonzero admissible Jacobi field along $\gamma|_{[0,t]}$ which vanishes both at $\gamma(0)$ and $\gamma(t)$. Its multiplicity is defined by the dimension of the space of such Jacobi fields. If $\gamma(t)$ is not conjugate to $\gamma(0)$, $\gamma|_{[0,t]}$ is said to be *nondegenerate.*

Now let $\tilde{T}$ be $\{V^\perp | V\in T_\gamma\Omega_1\}$, and $\tilde{I}$ be a symmetric bilinear form on $\tilde{T}$ defined by $\tilde{I}(V^\perp, W^\perp)=I(V, W)$ for $V, W\in T_\gamma\Omega_1$. Then the index of $I$ is equal to the index of $\tilde{I}$. We shall say elements of the null space of $\tilde{I}$ *admissible normal Jacobi fields* along $\gamma$. Owing to the fact that $\gamma$ is reflected strictly at its boundary point, with each admissible normal Jacobi field is associated such a unique admissible Jacobi field that its normal part is equal to the given one.

LEMMA *So long as $\varepsilon>0$, $\delta>0$ are sufficiently small, there exist neighborhoods $U_1$ and $U_2$ respectively of $\gamma(a-\varepsilon)$ and $\gamma(a+\delta)$ such that*

(1) *$\gamma|[a-\varepsilon, a+\delta]$ is a nondegenerate, minimizing critical path in $\Omega_1(\gamma(a-\varepsilon), \gamma(a+\delta))$*

(2) *for each $u_i\in U_i (i=1, 2)$ there exists a critical and minimizing path $c(u_1, u_2)$ in $\Omega_1(u_1, u_2)$, which is unique in the neighborhood of $\gamma|[a-\varepsilon, a+\delta]$ and depends smoothly on $u_1$ and $u_2$.*

From this follows immediately

COROLLARY *For any two vectors $v\in \perp T(a-\varepsilon)$, $w\in \perp T(a+\delta)$, there exists a unique aamissible normal Jacobi field $Y$ along $\gamma|[a-\varepsilon, a+\delta]$ having the given values $v$, $w$ at $\gamma(a-\varepsilon)$ and $\gamma(a+\delta)$. $Y$ depends smoothly upon $\varepsilon$, $\delta$, $v$ and $w$. Moreover for any normal field $V$ along $\gamma|[a-\varepsilon, a+\delta]$ having the same values $V(a-\varepsilon)=v$, $V(a+\delta)=w$,*

$$\tilde{I}_{a-\varepsilon}^{a+\delta}(V) \geqq \tilde{I}_{a-\varepsilon}^{a+\delta}(Y)$$

*where equality occurs if and only if* $V=Y$, *and* $\tilde{I}_{a-\varepsilon}^{a+\delta}(V)$ *is defined by*

$$\tilde{I}_{a-\varepsilon}^{a+\delta}(V) := \langle S_{T(a-)-T(a+)} \tilde{V}(a),\ \tilde{V}(a) \rangle + \int_{a-\varepsilon}^{a+\delta} (\langle R(T, V)T, V \rangle + \langle V', V' \rangle) dt$$

$\tilde{V}(a)$ *is such a unique vector tangent to the boundary at* $\gamma(a)$ *that its orthogonal projection to* $\perp T(a-)$ *is equal to* $V(a-)$. ($\perp T(a-)$ *is the orthogonal conplement of* $T(a-)$)

*Proof of Lemma* We take a small neighborhood centered at $\gamma(a)$. In it exist such a neighborhood $V_1$ of $\gamma(a-\varepsilon)$ in Int$(M)$, a neighborhood $V_2$ of $\gamma(a+\delta)$ in Int$(M)$, and a neighborhood $W$ of $\gamma(a)$ in $\partial M$ that three have mutually no intersection, and any two points of $V_1$ and $W$ are joined by minimizing geodesics which depend smoothly on the end points. The same property is assumed for $V_2$ and $W$. Let $K$ be a function on $W \times V_1 \times V_2$ defined by $K(w, v_1, v_2) = d(v_1, w) + d(w, v_2)$, $w \in W$, $v_i \in V_i (i=1, 2)$, here $d$ is the distance defined in the usual way by the Riemannian metric of $M$. $K$ is a smooth function. Let $D_1 K$ be the gradient of $K$ with respect to the first variable. Then, from the assumption, $D_1 K(0, \gamma(a-\varepsilon), \gamma(a+\delta))=0$, hence $D_1(K^2)(0, \gamma(a-\varepsilon), \gamma(a+\delta))=0$. As the hessian of $K^2$ at $(0; 0, 0)$ with respect to the first variable, $D_1^2(K^2)(0; 0, 0)$, is positive definite, so is $D_1^2(K^2)(0; \gamma(a-\varepsilon), \gamma(a+\delta))$ for small $\varepsilon \geqq 0$, $\delta \geqq 0$. Accordingly, $D_1^2(K)(0; \gamma(a-\varepsilon), \gamma(a+\delta))$, the hessian of $K$ at $(0; \gamma(a-\varepsilon), \gamma(a+\delta))$, is also positive definite. Then, from the inverse function theorem, there exist neighborhoods $U_1[U_2]$ of $\gamma(a-\varepsilon)[\gamma(a+\delta)]$, $U_i \subset V_i$, and the unique smooth function $F$ satisfying

$$F;\ U_1 \times U_2 \longrightarrow W, \qquad F(\gamma(a-\varepsilon), \gamma(a+\delta))=0$$

$$D_1 K(F(u_1, u_2); u_1, u_2)=0, \qquad u_i \in U_i (i=1, 2)$$

$$D_1^2 K(F(u_1, u_2); u_1, u_2): \text{positive definite matrix } (u_i \in U_i)$$

Now it suffices to define $c(u_1, u_2)(u_i \in U_i)$ as the minimizing geodesic from $u_1$ to $F(u_1, u_2)$ plus the minimizing geodesic from $F(u_1, u_2)$ to $u_2$. Q. E. D.

THEOREM *Let* $\gamma$ *be a critical path of* $L$ *in* $\Omega_1(p, q)$, *which is parametrized by arc length and reflected strictly at* $\gamma(a) \in \partial M$. $\gamma: [0, b] \rightarrow M$, $0 < a < b$. *Let* $I$ *be the index form on* $T_\gamma \Omega_1$ *derived from the second variation of* $L$. *Then the index of* $I$ *is equal to the number of conjugate points* $\gamma(t)$ *to* $\gamma(0)$ *along* $\gamma(0 < t < b)$, *counted with their multiplicities.*

The index, geometrically, is the number of independent directions towards which the geodesic $\gamma$ becomes shorter curves through the path space $\Omega_1$.

*Proof of Theorem* Let $\gamma_\tau$ be the restriction of $\gamma$ to $[0, \tau]$, $0 \leqq \tau \leqq b$, and $i(\tau)$ the index of $\gamma_\tau$. Let $0 = t_0 < t_1 < \cdots < t_k = \tau$ be such a partition of $[0, \tau]$ that each segment $\gamma|_{[t_i, t_{i+1}]}$ is contained in a convex normal neighborhood. In case of $a < \tau$, we always take some $t_j$ and $t_{j+1}$ so that $t_j<a<t_{j+1}$ and $\gamma|_{[t_j,t_{j+1}]}$ is a minimizing path in $\Omega_1(\gamma(t_j), \gamma(t_{j+1}))$ as in Lemma.

We set $\tilde{T}_\tau=\{V\in\tilde{T}_\gamma | V=0 \text{ on } [\tau, b]\}$, $\tilde{I}_\tau$ be the restriction of $\tilde{I}$ to $\tilde{T}_\tau$. Then the index of $\tilde{I}_\tau$ on $\tilde{T}_\tau$ is just $i(\tau)$, its nullity $\nu(\tau)$ is the congugacy multiplicity of $\gamma(\tau)$ to $\gamma(0)$ along $\gamma$. We define

$$\tilde{T}_\tau(t_0, \cdots, t_k):=\{V\in\tilde{T}_\tau | V \text{ is an admissible normal Jacobi field which breaks only at } t_i's\}$$

$$T'_\tau:=\{V\in\tilde{T}_\tau | V(t_i)=0 \ (i=1, \cdots, k)\}$$

Then the following facts can be easily verified with the aid of Lemma and its Corollary (see § 15 of [2]). Therefore the proof of Theorem is completed

(1) $\tilde{T}_\tau\cong\tilde{T}_\tau(t_0, \cdots, t_k)\oplus T'_\tau$, orthogonal direct sum with respect to $\tilde{I}_\tau$. $\tilde{I}_\tau$ is positive definite on $T'_\tau$. Consequently $i(\tau)$ and $\nu(\tau)$ are respectively equal to the index and nullity of $\tilde{I}_\tau|_{\tilde{T}_\tau(t_0,\cdots,t_k)}$.

(2) $i$ is a monotone increasing function of $\tau\in[0, b]$, and $i(0)=0$, $i(\tau-\varepsilon)=i(\tau)$ for each $\tau$ and small $\varepsilon>0$.

(3) $i(\tau+\varepsilon)=i(\tau)+\nu(\tau)$ for each $\tau$ and small $\varepsilon>0$. Q. E. D.

In the same way we can prove the following general theorem.

Let $\gamma: [0, b]\to M$ be a geodesic from $p$ to $q$, which is parametrized by arc length and reflected strictly at $\gamma(t_i)$'s$\in\partial M (i=1, \cdots, k)$. Let $\Omega_k$ be the set of piecewise $C^\infty$ curves from $p$ to $q$ which have at least $k$ boundary points. Then the tangent space $T_\gamma\Omega_k$ to $\Omega_k$ at $\gamma$ may be considered as the set of continuously piecewise $C^\infty$ vector fields along $\gamma$ which are tangent to the boundary at $\gamma(t_i)$'s. $\gamma$ is a critical point of $L$ in $\Omega_k$ and the index form $I$ at $\gamma$ over $T_\gamma\Omega_k$ becomes

$$I(V, W)=\sum_{i=1}^{k}\langle S_{T(t_i^-)-T(t_i^+)}V(t_i), W(t_i)\rangle+\int_0^b\{\langle R(T, V)T, W\rangle+\langle V^{\perp\prime}, W^{\perp\prime}\rangle\}\,dt$$

for $V, W\in T_\gamma\Omega_k$. Admissible Jacobi fields, conjugate points and so on are defined in the analogous way. Then

THEOREM *In the situation described above, the index of $I$ over $T_\gamma\Omega_k$ is equal to the number of conjugate points $\gamma(t)$ to $\gamma(0)$ along $\gamma(0<t<b)$, counted with their multiplicities.*

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## REFERENCES

[1] R. BISHOP AND R. CRITTENDEN, Geometry of Manifolds, Academic Press, 1964.
[2] J. MILOR, Morse Theory, Princeton University Press, 1963.

TOKYO INSTITUTE OF TECHNOLOGY