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KODAI MATH. J.
1 (1978), 133—158

ENTIRE FUNCTIONS WITH THREE LINEARLY DISTRIBUTED VALUES

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1. Introduction. A complex number w will be called a linearly distributed value of the entire function $f(z)$ if there is a straight line L of the complex plane on which all the solutions of $f(z)=w$ lie. For the exponential function, every value is linearly distributed. Conversely, Baker proved that a transcendental entire function for which every value is linearly distributed must have the form $a+b\exp(cz)$, where a, b and c are constants.

In this connection we have shown the following result in our previous paper [4].

Let $f(z)$ be a transcendental entire function. Assume that there are three distinct finite complex numbers a_j and three distinct straight lines L_j of the complex plane on which all the solutions of $f(z)=a_j$ lie $(j=1, 2, 3)$. Assume further that $f(z)$ has a finite deficient value other than a_1, a_2 and a_3. Then $f(z)=P(\exp Az)$ with a quadratic polynomial $P(z)$ and a non-zero constant A.

The object of this paper is to give a further substantial improvement, which gives an essentially sharp form of the above our result.

THEOREM. *Let $f(z)$ be a transcendental entire function which has three distinct finite linearly distributed values c_1, c_2 and c_3. If these three values never lie on any straight line of the complex plane, then*

$$f(z)=P(\exp Az),$$

where $P(z)$ is a quadratic polynomial and A is a non-zero constant.

Considering the sine function or the cosine function, we easily assure that the assumptions of our theorem cannot be improved, in general.

Let $f(z)$ be an entire function having three distinct finite linearly distributed values c_1, c_2 and c_3. By L_j, we denote the straight line on which all the c_j-points of $f(z)$ lie $(j=1, 2, 3)$. With these conventions, we shall show the following four propositions.

PROPOSITION 1. *If at least two of the three lines L_j coincide with each other*

Received November 8, 1976.

and if the three values c_j do not lie on any straight line of the complex plane, then either

$$f(z)=A+B\exp(Cz)$$

with constants A, B and C, or $f(z)$ is a polynomial.

PROPOSITION 2. *Assume that the three lines L_j are distinct from one another and that no two of which run parallel with each other. Then $f(z)$ reduces to a polynomial.*

PROPOSITION 3. *If the three lines L_j are distinct from one another and if exactly two of which run parallel with each other, then*

$$f(z)=A+B\exp(Cz)$$

with constants A, B and C, unless $f(z)$ is a polynomial.

PROPOSITION 4. *Assume that the three lines L_j are distinct and parallel with one another. Then*

$$f(z)=P(\exp Az)$$

with a quadratic polynomial $P(z)$ and a non-zero constant A, provided that $f(z)$ does not reduce to a polynomial.

In conclusion, our Theorem is an immediate consequence of the above four propositions.

2. Preliminary results. Using the well known Bohr and Landau's theorem, we have proved the following fact [4, Theorem 6].

LEMMA 2.1. *Let $f(z)$ be an entire function with three distinct finite linearly distributed values. Then the order of $f(z)$ must be finite.*

The next lemma is due to Edrei and Fuchs [3].

LEMMA 2.2. *Let $f(z)$ be a transcendental entire function having only real zeros. Then $f(z)$ has at most one finite deficient value. Further if $f(z)$ has a finite deficient value other than zero, then the order of $f(z)$ is not greater than one.*

Let $f(z)$ be an entire function and let w be a complex number. We say that the value w is a radially distributed value of $f(z)$ if there exists a half straight line of the complex plane on which all the w-points of $f(z)$ lie. With this definition, Theorem B of [4] takes the following form.

LEMMA 2.3. *Let $f(z)$ be an entire function of finite genus q with $q \geqq 1$. If the value 0 is a radially distributed value of $f(z)$, then $f(z)$ has zero as a deficient* *value.*

Combining Lemmas 2.2 and 2.3, we can prove the following.

LEMMA 2.4. *Let $f(z)$ be an entire function of finite order satisfying*

$$\liminf_{r\to\infty} \frac{T(r, f)}{r} \neq 0 .$$

Then $f(z)$ has at most one finite radially distributed value.

Proof. Assume that $f(z)$ has two finite radially distributed values, say 0 and 1. Then it is possible to choose two complex numbers a $(\neq 0)$ and b so that the function defined by

$$f^*(z)=f(az+b)$$

has only negative zeros. Of course, the value 1 is a radially distributed value of $f^*(z)$ and the characteristic function of $f^*(z)$ satisfies

$$\liminf_{r\to\infty} \frac{T(r, f^*)}{r} \neq 0 .$$

Hence the genera of $f^*(z)$ and $f^*(z)-1$ are finite and at least one. Therefore Lemma 2.3 implies $\delta(0, f^*)>0$ and $\delta(1, f^*)>0$. This contradicts Lemma 2.2.

Further by means of Lemma 2.2 and Theorem C of [4], we easily obtain the next fact.

LEMMA 2.5. *Let $f(z)$ be an entire function with three distinct finite linearly distributed values c_1, c_2 and c_3. Then the genera of $f(z)-c_j$ $(j=1, 2, 3)$ are at most one.*

Now let $g(z)$ be an entire function whose zeros $\{a_n\}$ lie on a straight line L. Express this line L as

$$L=\{z:\ \mathrm{Re}(uz)=r\},$$

where r is a suitable real number and u is a suitable complex number with $|u|=1$. Assume further that the genus of $g(z)$ is at most one. Then Lemma 5 of [4] gives

$$\overline{g(\bar{u}\bar{z}+\bar{u}r)}=g(-\bar{u}z+\bar{u}r)\exp(2Cz+iC')$$

and

$$\mathrm{Re}\frac{\bar{u}g'(z)}{g(z)}=C+\sum_n \frac{\mathrm{Re}(uz)-r}{|z-a_n|^2} \tag{*}$$

with suitable real constants C and C'. The proofs of our propositions are based on these relations.

If the real constant C is equal to zero and if $g(z)$ has at least one zero-point, then the relation (*) yields

$$\mathrm{Re}\frac{\bar{u}g'(z)}{g(z)}\neq 0$$

for $\mathrm{Re}(uz)>r$ and for $\mathrm{Re}(uz)<r$. In particular, $g'(z)$ fails to take the value 0 there. Therefore all the zeros of $g'(z)$, if exist, must lie on the line L.

Next let us assume that $C>0$. Then it follows from the relation (*) that

$$\mathrm{Re}\frac{\bar{u}g'(z)}{g(z)}\geqq C>0$$

for $\mathrm{Re}(uz)\geqq r$. Thus we find

$$\begin{aligned}\log|g(\bar{u}b)|-\log|g(\bar{u}a)|&=\int_a^b \mathrm{Re}\frac{\bar{u}g'(\bar{u}t)}{g(\bar{u}t)}dt\\ &\geqq C(b-a)\end{aligned}$$

for $r<a\leqq b$. From this inequality, $g(\bar{u}t)$ tends to infinity as t does to infinity along the positive real axis.

Similarly, if $C<0$, then $g(\bar{u}t)$ tends to infinity as t does to infinity along the negative real axis.

3. Principal lemmas. Let $G(z)$ be a transcendental entire function of finite lower order such that all the zero-points $\{a_n\}$ of $G(z)$ lie on the line $\mathrm{Re}\,z=0$, and all the one-points $\{b_n\}$ of $G(z)$ lie on the line $\mathrm{Re}\,z=1$. Then by Theorem 4 of [4], $G(z)$ has at most order one, mean type. So by means of Lemma 5 of [4], we find

$$\mathrm{Re}\frac{G'(z)}{G(z)}=A+\sum_n\frac{\mathrm{Re}\,z}{|z-a_n|^2},\tag{3.1}$$

$$\mathrm{Re}\frac{G'(z)}{G(z)-1}=B+\sum_n\frac{\mathrm{Re}\,z-1}{|z-b_n|^2}\tag{3.2}$$

and

$$\overline{G(\bar{z})}=G(-z)\exp(2Az+iA'),\tag{3.3}$$

$$\overline{G(\bar{z}+1)}-1=(G(-z+1)-1)\exp(2Bz+iB')\tag{3.4}$$

with suitable real constants A, B, A' and B'. The quantities A and B play an important role in what follows.

Indeed if $A=B\neq 0$, then

$$G(z)=P(\exp Az)$$

with a quadratic polynomial $P(z)$. This fact was proved in Lemma 11 of [4]. Further if $AB\leqq 0$, then $G(z)$ has no finite deficient values. This was also proved in Lemma 8 of [4].

3.1. The purpose of this section is to prove the following lemmas on which our proofs are based.

LEMMA A. *Assume that* $A=0$. *Then the quantity* B *must be positive. Assume further that* $G(1+iy)$ *is real and non-negative for some real number* y. *Then* $G(1+iy)=1$.

Proof. The fact that B must be positive was proved in Lemma 7 of [4]. So we shall prove the latter statement only. It follows from the relation (3.3) that

$$\overline{G(\bar{z})}=G(-z)\exp(iA')\,. \tag{3.5}$$

Here we can claim that

$$\exp(iA')\neq 1\,. \tag{3.6}$$

For otherwise,

$$\overline{G(\bar{z})}=G(-z)\,.$$

So, if z_* is a one-point of $G(z)$, then $-z_*$ is also a one-point. This contradicts the assumptions. Thus $G(z)$ has no one-points. Hence we can express $G(z)$ as

$$G(z)=1+K\exp(Dz)\,,$$

where D and K are non-zero constants. Since all the zero-points of $G(z)$ are distributed on the imaginary axis, the constant D must be real. Therefore

$$\mathrm{Re}\frac{G'(z)}{G(z)}=\frac{D}{2}$$

for values of z with $\mathrm{Re}\, z=0$ and $G(z)\neq 0$. Using $A=0$ and the relation (3.1), we thus conclude that $D=0$. This is absurd again. Accordingly, (3.6) is true.

Assume now that $0<G(1+iy_*)<1$ for some real number y_*. Then the relations (3.1) and (3.2) yield

$$\mathrm{Re}\frac{G'(1+iy_*)}{G(1+iy_*)}\geqq 0$$

and

$$\mathrm{Re}\frac{G'(1+iy_*)}{G(1+iy_*)-1}=B>0\,,$$

which are clearly absurd. Consequently, if $0\leqq G(1+iy_*)\leqq 1$ for some real number y_*, then $G(1+iy_*)=1$.

Next assume that $G(1+iy^*)>1$ for some real number y^*. Let us set

$G(1+\imath y^*)=x^*>1$ and denote the inverse function of $G(z)$ by $G^{-1}(w)$. Of course, $G^{-1}(w)$ is an infinitely many valued analytic function with algebraic character. By $E(w, x^*)$, we also denote the element of $G^{-1}(w)$ with center x^* and satisfying $E(x^*, x^*)=1+\imath y^*$. Now let us continue analytically $E(w, x^*)$ along the segment $I=\{1\leqq t\leqq x^*\}$ toward the point $t=1$. Then we have an analytic continuation $G^{-1}(I_r)$ with algebraic character along the segment I up to some point $t=r$ $(1\leqq r<x^*)$, with the possible exception of this end point. Thus using this continuation $G^{-1}(I_r)$, we can define the simple path $C^*=\{z(t): 0\leqq t<x^*-r\}$ such that $z(0)=1+\imath y^*$ and

$$G(z(t))=x^*-t \tag{3.7}$$

for $0\leqq t<x^*-r$. Since $G(z)$ assumes only positive real values on this path C^*, by means of (3.5) and (3.6), the path C^* is contained entirely in the open half plane $\operatorname{Re} z>0$. On the other hand, it can be verified by the assumption $A=0$ that $G(z)$ must have at least one zero-point. So the relation (3.1) implies

$$\operatorname{Re}\frac{G'(z)}{G(z)}>0 \tag{3.8}$$

for values of z with $\operatorname{Re} z>0$. Thus C^* is a differentiable path and the identity (3.7) yields

$$G'(z(t))z'(t)=-1 \tag{3.9}$$

for $0\leqq t<x^*-r$. Taking into account of (3.7), (3.8) and (3.9), we thus find

$$\operatorname{Re} z'(t)<0$$

for $0\leqq t<x^*-r$, which means that the real part of $z(t)$ decreases as t varies from 0 to x^*-r. Hereby the path C^* must be contained in the open strip $0<\operatorname{Re} z<1$ save for the initial point $z(0)$. In particular, the continuation $G^{-1}(I_r)$ does not continue along the segment I to the point $t=1$. Therefore we may assume that this continuation $G^{-1}(I_r)$ defines a transcendental singularity at the point $t=r(1\leqq r<x^*)$. Accordingly, by Iversen's theorem [5], the path C^* must be an asymptotic path of $G(z)$ and as z tends to infinity along this path C^*, $G(z)$ approaches the value r. In addition to these facts, by virtue of (3.5) and (3.6), $G(z)$ omits the value $\exp(-\imath A')$ in the open half plane $\operatorname{Re} z>0$. Hence $G(z)$ omits the finite values 0 and $\exp(-\imath A')$ there. Thus using Lindelöf-Iversen-Gross' theorem [6], we obtain

$$G(z)\to r \tag{3.10}$$

as z tends to infinity along the positive real axis. However by the fact $B>0$ and by what mentioned at the end of the previous section 2, $G(z)$ must tend to infinity when z tends to infinity along the positive real axis. This contradicts (3.10). Consequently, if $G(1+\imath y^*)\geqq 1$ for some real number y^*, then $G(1+\imath y^*)=1$. Lemma A is thus proved.

By this Lemma A, using the exactly same argument developed in the proof of Lemma 10 of [4], we can obtain the next lemma.

LEMMA B. *Assume that* $A=0$. *Then* B *is positive and*

$$\lim_{r\to\infty}\frac{N(r,1,G)}{r}=\frac{2}{\pi}B,$$

where $N(r,1,G)$ *denotes the usual counting function for the one-points of* $G(z)$.

3.2. Now we are in a position to prove the following lemma.

LEMMA C. *The quantities* A *and* B *which appear in the relations* (3.1) *and* (3.2) *must be*

$$AB\neq 0.$$

Proof. Let us assume that $A=0$. Then by the above Lemma B, the quantity B must be positive and

$$\lim_{r\to\infty}\frac{N(r,1,G)}{r}=\frac{2}{\pi}B. \tag{3.11}$$

Further since $AB=0$, $G(z)$ has no finite deficient values. So (3.11) implies

$$\liminf_{r\to\infty}\frac{T(r,G)}{r}=\frac{2}{\pi}B. \tag{3.12}$$

On the other hand, by the fact $B>0$, $G(z)$ tends to infinity as z does to infinity along the positive real axis. Since $G(z)$ fails to take the two values 0 and 1 in the open half plane $\operatorname{Re} z>1$, Lindelöf-Iversen-Gross' theorem implies

$$\lim_{r\to+\infty}|G(re^{it}+1)|=+\infty \tag{3.13}$$

uniformly for $|t|\leqq t_*$, where t_* is an arbitrarily fixed number in $(0,\pi/2)$. Hence from (3.5) and (3.13), for an arbitrarily fixed number t^* with $0<t^*<\pi/2$, it is possible to find a positive number R such that

$$|G(-re^{it}+2)-1|\geqq 1 \tag{3.14}$$

for $r\geqq R$ and $|t|\leqq t^*$. Here recall the relation (3.4) again. Then after a slight modification, we have

$$\begin{aligned}|G(re^{-it})-1|=|G(-re^{it}+2)-1|\\ \exp(2Br\cos t-2B).\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we thus obtain

$$\log|G(re^{-it})-1|\geqq 2Br\cos t-2B,$$

so that

$$\log^{+}|G(re^{it})| \geqq 2Br\cos t - 2B - \log 2 \tag{3.16}$$

for $r \geqq R$ and $|t| \leqq t^{*}$. Therefore by means of the functional equation (3.5), this inequality (3.16) implies

$$\begin{aligned} T(r, G) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \log^{+}|G(re^{it})|\,dt \\ &\geqq \frac{1}{\pi}\int_{-t^{*}}^{t^{*}} \log^{+}|G(re^{it})|\,dt \\ &\geqq \frac{4}{\pi} Br\sin t^{*} + 0(1) \end{aligned}$$

for values of r with $r \geqq R$. Hereby

$$\liminf_{r\to\infty} \frac{T(r, G)}{r} \geqq \frac{4}{\pi} B\sin t^{*}$$

for an arbitrary number t^{*} with $0 < t^{*} < \pi/2$. Consequently,

$$\liminf_{r\to\infty} \frac{T(r, G)}{r} \geqq \frac{4}{\pi} B\,,$$

which contradicts (3.12). Hence the quantity A is never equal to zero.

Next assume that $B=0$. In this case we consider the auxiliary function defined by

$$G^{*}(z) = 1 - G(1-z)\,.$$

Evidently, all the zero-points and all the one-points of $G^{*}(z)$ lie on the lines $\operatorname{Re} z=0$ and $\operatorname{Re} z=1$, respectively. Further the relations (3.3) and (3.4) yield

$$\overline{G^{*}(\bar{z})} = G^{*}(-z)\exp(-2Bz + iB')$$

and

$$\overline{G^{*}(\bar{z}+1)} - 1 = (G^{*}(-z+1)-1)\exp(-2Az + iA')\,.$$

Hence we can apply the above result to this function $G^{*}(z)$, and arrive at a contradiction. Consequently, the quantity B must be a non-zero real constant. Lemma C is thus proved.

3.3. From this point on, we discuss the case $AB>0$. Assume that A and B are both positive. Then all the one-points of $G(z)$, if exist, are simple. Further if $G(z)$ does not approach the value 1 when z tends to infinity along the negative real axis, then the one-points of $G(z)$ which we write as $\{1+ic_n\}$, must satisfy

$$(m-n-1)\pi \leqq B(c_m - c_n) \leqq (m-n+1)\pi \tag{3.17}$$

for arbitrary integers m and n with $m\geqq n$. From these inequalities (3.17), we can assert that either

$$G(z)\to 1$$

as z tends to infinity along the negative real axis, or

$$\lim_{r\to\infty}\frac{N(r,1,G)}{r}=\frac{2}{\pi}B.$$

These facts are Lemmas 9 and 10 of [4].

In order to go further, we need the next lemma.

LEMMA D. *Let S be a positive real number and let s_n be*

$$s_n=\frac{n}{S}\pi \qquad (n=1,2,3,\cdots).$$

Then

$$\lim_{x\to+\infty}\sum_{n\geqq 1}\frac{x}{x^2+s_n{}^2}=\frac{S}{2}.$$

This Lemma D follows immediately from the identity

$$\cot z=\frac{1}{z}+\sum_{n\geqq 1}\frac{2z}{z^2-n^2\pi^2}.$$

However we shall give an alternative proof because it suggests us a method for proving our fundamental Lemma E.

Proof of Lemma D. Let us set

$$E(z)=\prod_{n\geqq 1}\left(1-\frac{z}{is_n}\right)\exp\left(\frac{z}{is_n}\right). \tag{3.18}$$

Then $E(z)$ is an entire function of order one and vanishes only at the points is_n $(n=1,2,3,\cdots)$. Further set

$$F(z)=zE(z)E(-z). \tag{3.19}$$

Then this entire function $F(z)$ satisfies

$$F(-z)=-F(z). \tag{3.20}$$

Since $F(z)$ and $\exp(2Sz)-1$ take the value 0 at the same points, $F(z)$ can be expressed as

$$F(z)=\exp(az+b)(\exp(2Sz)-1) \tag{3.21}$$

with suitable constants a and b. Combining (3.20) and (3.21), we easily find that $\exp(2Sz)=\exp(-2az)$, so that

$$a=-S. \tag{3.22}$$

On the other hand, it follows from (3.18), (3.19) and (3.21) that

$$\operatorname{Re}\frac{F'(z)}{F(z)}=\frac{\operatorname{Re} z}{|z|^2}+\sum_{n\geqq 1}\frac{\operatorname{Re} z}{|z-is_n|^2}+\sum_{n\geqq 1}\frac{\operatorname{Re} z}{|z+is_n|^2}$$

$$=a+\operatorname{Re}\left(\frac{2S\exp(2Sz)}{\exp(2Sz)-1}\right).$$

Hence

$$\frac{1}{x}+\sum_{n\geqq 1}\frac{2x}{x^2+s_n{}^2}=a+\frac{2S\exp(2Sx)}{\exp(2Sx)-1}$$

for real values of x. On letting $x\to+\infty$, we thus find

$$\sum_{n\geqq 1}\frac{2x}{x^2+s_n{}^2}\to a+2S. \tag{3.23}$$

Therefore, Lemma D follows from (3.22) and (3.23) at once.

Assume now that the quantities A and B are both positive. Further assume that $G(z)$ does not approach the value 1 as z tends to infinity along the negative real axis. Then the one-points of $G(z)$ satisfy the inequalities (3.17). Here let us set

$$d_n=c_n-c_0, \qquad h_n=\frac{n}{B}\pi$$

for each integer n. Then from (3.17),

$$h_{n-1}\leqq d_n\leqq h_{n+1}.$$

Here by we obtain

$$\sum_{n\geqq 1}\frac{x}{x^2+h_{n-1}^2}\leqq\sum_{n\geqq 1}\frac{x}{x^2+d_n{}^2}\leqq\sum_{n\geqq 1}\frac{x}{x^2+h_{n+1}^2}$$

for negative real values of x. By means of Lemma D, we thus find

$$\lim_{x\to-\infty}\sum_{n\geqq 1}\frac{x}{x^2+d_n{}^2}=-\frac{B}{2}. \tag{3.24}$$

Similarly, by means of Lemma D, we also obtain

$$\lim_{x\to-\infty}\sum_{n\leqq -1}\frac{x}{x^2+d_n{}^2}=-\frac{B}{2}. \tag{3.25}$$

On the other hand, from the relation (3.2),

$$\operatorname{Re}\frac{G'(x+ic_0)}{G(x+ic_0)-1}=B+\sum_{n}\frac{x-1}{|x-1-id_n|^2} \tag{3.26}$$

for real values of x. Combining (3.24), (3.25) and (3.26), we therefore have

$$\lim_{x\to-\infty} \mathrm{Re}\frac{G'(x+ic_0)}{G(x+ic_0)-1}=0\,.$$

In particular, it is possible to choose a negative number M such that

$$-1\leqq \mathrm{Re}\frac{G'(x+ic_0)}{G(x+ic_0)-1}\leqq 1 \tag{3.27}$$

for each negative number x with $x\leqq M$. So this (3.27) implies

$$\exp(t-s)\leqq\left|\frac{G(s+ic_0)-1}{G(t+ic_0)-1}\right|\leqq\exp(s-t) \tag{3.28}$$

for arbitrary negative numbers s and t with $t\leqq s\leqq M$.

For a moment, we assume that there exists a strictly decreasing sequence $\{r_n\}$ such that

$$r_n\to-\infty \tag{3.29}$$

and that

$$\lim_{n\to\infty}|G(r_n+ic_0)|=+\infty\,. \tag{3.30}$$

In view of (3.3) and (3.4), we easily deduce

$$\begin{aligned} &A_1^*G(z)\exp(2(B-A)z)\\ &\quad=A_2^*(G(z+2)-1)+\exp(2Bz)\,, \end{aligned} \tag{3.31}$$

where A_1^* and A_2^* are non-zero constants. Therefore it follows from (3.29), (3.30) and (3.31) that

$$\exp(2(\mathrm{B}-\mathrm{A})r_n)-K\frac{G(r_n+2+ic_0)-1}{G(r_n+ic_0)}\to 0 \tag{3.32}$$

as $n\to+\infty$, where K is a suitable non-zero constant. Further using (3.29) and (3.30), we find

$$\frac{2}{3}\leqq\left|\frac{G(r_n+ic_0)-1}{G(r_n+ic_0)}\right|\leqq\frac{3}{2} \tag{3.33}$$

for sufficiently large n. Combining (3.28), (3.29) and (3.33), we thus obtain

$$\frac{2}{3}e^{-2}\leqq\left|\frac{G(r_n+2+ic_0)-1}{G(r_n+ic_0)}\right|\leqq\frac{3}{2}e^2 \tag{3.34}$$

for sufficiently large n. Accordingly, from (3.29), (3.32) and (3.34), we conclude that $A=B$, so that

$$G(z)=P(\exp Az)$$

with a quadratic polynomial $P(z)$. However it is clear that

$$\lim_{x\to-\infty} G(x+ic_0)=P(0)\,,$$

which contradicts (3.29) and (3.30). Consequently, if $G(z)$ does not converge to 1 as z tends to infinity along the negative real axis, then $G(x+ic_0)$ is bounded for negative real values of x.

LEMMA E. *If* $A>B>0$, *then*

$$\lim_{r\to+\infty} G(-re^{it})=0$$

uniformly for $|t|\leqq t^*$, *where* t^* *is an arbitrarily fixed number in* $(0, \pi/2)$. *Further if* $B>A>0$, *then*

$$\lim_{r\to+\infty} G(-re^{it})=1$$

uniformly for $|t|\leqq t^*$.

Proof. Assume that $A>B>0$. If $G(z)$ tends to 1 as z approaches infinity along the negative real axis, then the functional equation (3.31) implies

$$\lim_{x\to-\infty} A_1^* G(x)\exp(2(B-A)x)=0\,.$$

However since $B-A<0$ and $A_1^*\neq 0$, this is clearly impossible. Therefore $G(z)$ does not approach the value 1 as z tends to infinity along the negative real axis. Thus by what mentioned just above, $G(x+ic_0)$ is bounded on the negative real axis. Recall the functional equation (3.31) again. Then

$$G(x+ic_0)\exp(2(B-A)x)$$

must be bounded for negative real values of x. Since $B-A<0$, we hence conclude that

$$\lim_{x\to-\infty} G(x+ic_0)=0\,. \tag{3.35}$$

By means of Lindelöf-Iversen-Gross' theorem, it follows from (3.35) that

$$\lim_{r\to+\infty} G(-re^{it})=0$$

uniformly for $|t|\leqq t^*$, where t^* is an arbitrarily fixed number with $0<t^*<\pi/2$.

Next let us consider the case where $B>A>0$. Assume that $G(z)$ does not converge to 1 as z tends to infinity along the negative real axis. Then $G(x+ic_0)$ is bounded for negative real values of x. As above, it thus follows from (3.31) that

$$\lim_{x\to-\infty} G(x+2+ic_0)=1\,, \tag{3.36}$$

since $B-A>0$ and $A_2^* \neq 0$. By this (3.36), using Lindelöf-Iversen-Gross' theorem, we obtain

$$\lim_{x\to-\infty} G(x)=1,$$

which contradicts the assumption. Consequently, $G(z)$ must approach the value 1 as z tends to infinity along the negative real axis. By this fact, Lindelöf-Iversen-Gross' theorem gives the desired result. This completes the proof of Lemma E.

4. Proof of Proposition 1. Let $f(z)$ be a transcendental entire function having three distinct finite linearly distributed values c_1, c_2 and c_3 which do not lie on any straight line of the complex plane. By L_j, we denote the straight line on which all the c_j-points of $f(z)$ lie ($j=1, 2, 3$). Our goal of this section is to show that if L_1 and L_2 coincide with each other, then $f(z)$ must be a function of the form

$$f(z)=A_*+B_* \exp(C_* z) \tag{4.1}$$

with constants A_*, B_* and C_*.

We may assume, as we may do without loss of generality, that the lines L_1 and L_2 coincide with the imaginary axis and that $c_1=0$, $c_2=1$ and $c_3=c$, where c is a non-real complex number.

By Theorems 4 and 6 of [4], $f(z)$ has at most order one, mean type. Hence we find

$$\overline{f(\bar{z})}=f(-z)\exp(-A^*z+iB^*),$$

$$\overline{f(\bar{z})}-1=(f(-z)-1)\exp(-C^*z+iD^*)$$

with suitable real constants A^*, B^*, C^* and D^*. Therefore

$$\begin{aligned} &f(z)\exp(A^*z+iB^*)-1\\ &\quad=(f(z)-1)\exp(C^*z+iD^*), \end{aligned}$$

so that

$$\begin{aligned} &f(z)(\exp(A^*z+iB^*)-\exp(C^*z+iD^*))\\ &\quad=1-\exp(C^*z+iD^*). \end{aligned} \tag{4.2}$$

If $\exp(A^*z+iB^*)-\exp(C^*z+iD^*)$ is identically equal to 0, then

$$\exp(A^*z+iB^*)=\exp(C^*z+iD^*)=1$$

for values of z. Thus $A^*=C^*=0$ and $\exp(iB^*)=\exp(iD^*)=1$, so that all the zero-points of $f'(z)$ lie on the imaginary axis only and $f(z)$ satisfies

$$\overline{f(\bar{z})}=f(-z). \tag{4.3}$$

If $\exp(A^*z+iB^*)-\exp(C^*z+iD^*)$ is not identically equal to 0, then (4.2) implies

$$f(z)=\frac{1-\exp(C^*z+iD^*)}{\exp(A^*z+iB^*)-\exp(C^*z+iD^*)}.$$

Since $f(z)$ is entire, using an elementary calculation, we easily conclude that either $f(z)$ is a function of the form (4.1), or $f(z)$ can be expressed as

$$f(z)=\frac{1-\exp(qEz+iqF)}{1-\exp(Ez+iF)}, \tag{4.4}$$

where E and F are non-zero real constants and q is some integer with $q\neq 0, 1$. Here let us set

$$Q_N(z)=\sum_{k=1}^{N} z^k, \tag{4.5}$$

where N is a natural number. Then (4.4) can be rewritten as

$$f(z)=Q_{q-1}(\exp(Ez+iF))+1 \tag{4.6}$$

when $q\geqq 2$, and

$$f(z)=-Q_{-q}(\exp(-Ez-iF)) \tag{4.7}$$

when $q\leqq -1$.

Hereafter, our proof is divided into the consideration of the following two cases.

1) $f(z)$ satisfies the functional equation (4.3).
2) $f(z)$ is a function of the form (4.6) or else (4.7).

4.1. Firstly, we consider the case 1). From (4.3), $f(z)$ takes only real values on the imaginary axis. Hence if the line $L=L_3$ on which all the c-points of $f(z)$ lie coincides with the imaginary axis, then $f(z)$ never takes the value c. Thus by means of (4.3), $f(z)$ omits the two finite values c and $\bar{c}$. This is absurd. Accordingly, the line L is distinct from the imaginary axis. Further it is clear by the functional equation (4.3) that $\delta(c, f)=\delta(\bar{c}, f)$. So Lemma 2.2 implies

$$\delta(c, f)=\delta(\bar{c}, f)=0\,. \tag{4.8}$$

In addition to these facts, the characteristic function of $f(z)$ satisfies

$$\liminf_{r\to\infty}\frac{T(r, f)}{r}\neq 0\,. \tag{4.9}$$

On the contrary, if (4.9) is false, by virtue of Theorem D of [4], all the zero-points of $f'(z)$ lie on the imaginary axis and the line L, simultaneously. Thus $f'(z)$ has at most one zero-point and satisfies

$$\liminf_{r\to\infty}\frac{T(r, f')}{r}=0 .$$

By these facts, $f'(z)$ reduces to a polynomial. This is impossible. Hence (4.9) is true. In particular, $f(z)$ is a function of order one, mean type.

Express the line L as

$$L=\{z : \mathrm{Re}(uz)=r\}, \tag{4.10}$$

where u is a complex number with $|u|=1$, and r is a real number. Then with this representation (4.10), we obtain

$$\overline{f(\bar{u}\bar{z}+\bar{u}r)-c}=(f(-\bar{u}z+\bar{u}r)-c)\exp(2Kz+iK'), \tag{4.11}$$

$$\mathrm{Re}\frac{\bar{u}f'(z)}{f(z)-c}=K+\sum_{n}\frac{\mathrm{Re}(uz)-r}{|z-c_n^*|^2}, \tag{4.12}$$

where K and K' are suitable real constants and $\{c_n^*\}$ denote the c-points of $f(z)$.

We now have the following three subcases.

1.1) L runs parallel with the imaginary axis.

1.2) L is orthogonal to the imaginary axis.

1.3) Neither the imaginary axis nor the real axis runs parallel with the line L.

4.2. Assume that the subcase 1.1) occurs. Then we may assume that the quantities u and r of (4.10) satisfy $u=1$ and $r\neq 0$, respectively. Here let us consider the auxiliary function defined by

$$G_*(z)=\frac{f(rz)}{c}. \tag{4.13}$$

Then it is clear that all the zero-points and all the one-points of $G_*(z)$ lie on the lines $\mathrm{Re}\, z=0$ and $\mathrm{Re}\, z=1$, respectively. Further by referring to (4.3), (4.11) and (4.13), we find

$$\overline{G_*(\bar{z})}=\frac{c}{\bar{c}}G_*(-z) \tag{4.14}$$

and

$$\overline{G_*(\bar{z}+1)}-1=\frac{c}{\bar{c}}(G_*(-z+1)-1)\exp(2Krz+iK').$$

However, since we can apply the results of the previous section 3 to this function $G_*(z)$, Lemma C and (4.14) give us a contradiction. Hence the subcase 1.1) never occurs.

Let us discuss the subcase 1.2). In this case, the complex number u is equal to $\imath$ or $-\imath$. If the constant K which appears in the relation (4.12) is equal to 0, then $f'(z)$ never takes the value 0 except at the point $-\imath r$ or else at the point $\imath r$. Hence

$$f(z)=Q_*(z)\exp(a^*z)+b^*, \tag{4.15}$$

where $a^*(\neq 0)$ and b^* are constants and $Q_*(z)$ is a polynomial. From this representation (4.15), using asymptotic properties of $\exp(a^*z)$, we easily conclude that either 0 and 1, or c and $\bar{c}$ must be radially distributed values of $f(z)$. This is absurd by (4.9) and Lemma 2.4. If the constant K is not equal to 0, then we can easily see that either $f(it)$ or $f(-\imath t)$ tends to infinity as the positive real variable t tends to infinity. Thus the values 0 and 1 are both radially distributed values of $f(z)$. This is absurd again. Hereby the subcase 1.2) does not occur, either.

Assume now that the subcase 1.3) occurs. It follows from (4.3) and (4.10) that all the $\bar{c}$-points of $f(z)$ lie on the line which is the symmetry of the line L relative to the imaginary axis. If $K=0$, then by the same reason as above, $f(z)$ must be a function of the form (4.15). Hence we arrive at a contradiction. If $K>0$, then $f(z)$ approaches infinity when z tends to infinity along the half line $J=\{z: z=\bar{u}t, t>0\}$, which does not run parallel with the imaginary axis. Since $f(z)$ omits the values 0 and 1 in the open half planes $\mathrm{Re}\, z>0$ and $\mathrm{Re}\, z<0$, using Lindelöf-Iversen-Gross' theorem and the functional equation (4.3), we thus find that $f(z)$ tends to infinity as z does to infinity along an arbitrary half line which is not parallel with the imaginary axis. By this fact, $f(z)$ takes the two values c and $\bar{c}$ only a finite number of times. This is clearly untenable. In the case where $K<0$, by the same fashion as above, we also arrive at a contradiction. Hence the subcase 1.3) never happens.

Consequently, all the three subcases do not occur. Therefore the case 1) never happens.

4.3. Secondly, we discuss the case 2). Assume that $f(z)$ has the form (4.6). By $w_1, w_2, \cdots, w_{q-1}$, let us denote all the roots of the algebraic equation $Q_{q-1}(z)=c-1$. Since $Q_{q-1}(0)=0$, $w_k\neq 0$ $(k=1, 2, \cdots, q-1)$. Then from the representation (4.6), $f(z)$ takes the value c only at the points of the form

$$\frac{1}{E}\log|w_k|-\frac{i}{E}(F-\arg w_k+2n\pi) \qquad (k=1, 2, \cdots, q-1,\ n=0, \pm 1, \pm 2, \cdots). \tag{4.16}$$

On the other hand, by the assumptions, all the c-points of $f(z)$, that is, all the points (4.16) must lie on the line L. It thus follows that

$$|w_1|=|w_2|=\cdots=|w_{q-1}|=R\neq 0\,. \tag{4.17}$$

Similarly, if $f(z)$ is a function of the form (4.7), then

$$|w_1^*|=|w_2^*|=\cdots=|w_{-q}^*|=R^*\neq 0\,, \tag{4.18}$$

where $w_1^*, w_2^*, \cdots, w_{-q}^*$ are all the solutions of the algebraic equation $Q_{-q}(z)=-c$.

Here, let us observe the following fact.

LEMMA F. *Let $Q_N(z)$ be the polynomial* (4.5), *and let S^* be a finite complex number. Assume that all the roots of the algebraic equation $Q_N(z)=S^*$ lie on a circle with center at the origin. Then if $N\geqq 3$, $S^*=-1$. If $N=2$, then S^* must be a negative real number.*

With the help of this Lemma F, we obtain the desired result immediately. In fact, if $q\geqq 4$, then (4.17) and Lemma F imply $c-1=-1$. Hence $c=0$, which is a contradiction. If $q=3$, then the value c must be real. This is absurd. If $q\leqq -3$, then (4.18) and Lemma F yield $c=1$. This is also untenable. Further if $q=-2$, then c must be a positive real number. This is a contradiction again. Consequently, the integer q is either 2 or -1. By referring to (4.5), (4.6) and (4.7), we conclude that $f(z)$ must have the form (4.1).

4.4. It remains to prove Lemma F. Assume that $N\geqq 3$. By $t_1, t_2, \cdots, t_N$, we denote all the roots of the equation $Q_N(z)=S^*$. Then

$$\begin{aligned} Q_N(z)-S^* &= z^N+z^{N-1}+\cdots+z-S^* \\ &= (z-t_1)(z-t_2)\cdots(z-t_N)\,, \end{aligned} \tag{4.19}$$

and the assumption implies

$$|t_1|=|t_2|=\cdots=|t_N|=r\neq 0\,. \tag{4.20}$$

From (4.19), it is clear that

$$t_1+t_2+\cdots+t_N=-1\,, \tag{4.21}$$

$$\sum_{k=1}^{N}\frac{t_1t_2\cdots t_N}{t_k}=(-1)^{N-1}\,, \tag{4.22}$$

and

$$t_1t_2\cdots t_N=(-1)^{N-1}S^*\,. \tag{4.23}$$

It follows from (4.20) and (4.21) that

$$1=\Big(\sum_{j=1}^{N}t_j\Big)\Big(\sum_{k=1}^{N}\bar{t}_k\Big)=Nr^2+\sum_{j\neq k}t_j\bar{t}_k\,. \tag{4.24}$$

Similarly, it follows from (4.20) and (4.22) that

$$1=\sum_{k=1}^{N}\sum_{j=1}^{N}r^{2N-4}t_j\bar{t}_k=r^{2N-4}\Big(Nr^2+\sum_{j\neq k}t_j\bar{t}_k\Big)\,. \tag{4.25}$$

Hence on combining (4.24) and (4.25), we have $r^{2N-4}=1$. Since $N\geqq 3$, $r=1$, so that

(4.26) $$|t_1|=|t_2|=\cdots=|t_N|=1\,.$$

Therefore from (4.23) and (4.26), $|S^*|=1$. Further since $Q_N(1)=N\geqq 3$, $t_k\neq 1$ ($k=1, 2, \cdots, N$).

Here let us set
$$P_{N+1}(z)=z^{N+1}-(1+S^*)z+S^*.$$
Then from
$$(t_k-1)Q_N(t_k)=t_k^{N+1}-t_k=(t_k-1)S^*,$$
the numbers $t_1, t_2, \cdots, t_N$ must be also roots of $P_{N+1}(z)=0$. Clearly, $P_{N+1}(1)=0$. Therefore

(4.27) $$\begin{aligned}P_{N+1}(z)&=z^{N+1}-(1+S^*)z+S^*\\&=(z-t_1)(z-t_2)\cdots(z-t_N)(z-1)\,.\end{aligned}$$

By referring to this relation (4.27), we thus find

(4.28) $$t_1+t_2+\cdots+t_N+t_{N+1}=0\,,$$

and

(4.29) $$\sum_{k=1}^{N+1}\frac{t_1t_2\cdots t_Nt_{N+1}}{t_k}=(-1)^N(1+S^*)\,,$$

where $t_{N+1}=1$. Hence by the same fashion as above, (4.26), (4.28) and (4.29) imply
$$|1+S^*|^2=N+1+\sum_{j\neq k}t_j\bar{t}_k=0\,,$$
which gives $S^*=-1$.

Next assume that $N=2$. By a and b, let us denote the roots of the equation $Q_2(z)=S^*$. Then clearly $a+b=-1$, $ab=-S^*$. Thus $\mathrm{Re}\,a+\mathrm{Re}\,b=-1$, $\mathrm{Im}\,a=-\mathrm{Im}\,b$. Further since $|a|=|b|$, $(\mathrm{Re}\,a)^2=(\mathrm{Re}\,b)^2$, so that $\mathrm{Re}\,a=\mathrm{Re}\,b$. Consequently, $\bar{a}=b$. This implies $a\bar{a}=-S^*>0$. Lemma F is hereby proved.

5. Proof of Proposition 2. Let $f(z)$ be a non-constant entire function satisfying the hypotheses of Proposition 2. With suitable real numbers r_j and suitable complex numbers u_j with $|u_j|=1$, we can express the straight lines L_j as

(5.1) $$L_j=\{z:\mathrm{Re}(u_jz)=r_j\}\qquad (j=1, 2, 3)\,.$$

By Lemma 2.5 and (5.1),

(5.2) $$\begin{aligned}\overline{f(\bar{u}_j\bar{z}+\bar{u}_jr_j)-c_j}&=(f(-\bar{u}_jz+\bar{u}_jr_j)-c_j)\\&\qquad\exp(2A_jz+iB_j)\,,\end{aligned}$$

(5.3) $$\mathrm{Re}\frac{\bar{u}_jf'(z)}{f(z)-c_j}=A_j+\sum_n\frac{\mathrm{Re}(u_jz)-r_j}{|z-a_{nj}^*|^2}\,,$$

where A_j and B_j are real constants, and $\{a^*_{nj}\}$ denote the c_j-points of $f(z)$ $(j=1, 2, 3)$.

For a moment, let us assume that the characteristic function of $f(z)$ satisfies

$$\liminf_{r\to\infty} \frac{T(r, f)}{r} = 0 . \tag{5.4}$$

Then by means of Theorem D of [4], all the zero-points of $f'(z)$ must lie on the lines L_j, simultaneously. Hence $f'(z)$ has at most one zero-point, since these lines L_j are distinct from one another. On the other hand, it follows from (5.4) that

$$\liminf_{r\to\infty} \frac{T(r, f')}{r} = 0 .$$

By these facts, $f'(z)$ must reduce to a polynomial. Consequently, $f(z)$ reduces to a polynomial. Thus our proof of Proposition 2 will be complete when we have proved (5.4).

In order to prove (5.4), we shall make use of the relations (5.2) and (5.3). Hereafter, we divide our consideration into three parts.

5.1. Assume that at least two of the three constants A_j which appear in (5.2) and (5.3), are equal to 0. Then we may assume that $A_1=A_2=0$. By what mentioned at the end of the section 2, all the zero-points of $f'(z)$ lie on the two distinct lines L_1 and L_2. Hence $f'(z)$ has at most one zero-point. Here remark that $T(r, f)=o(r^2)$. This fact is an immediate consequence of Lemma 2.5. Thus $T(r, f')=o(r^2)$. Therefore, $f'(z)$ can be expressed as

$$f'(z)=(z-a)^n \exp(A_* z+B_*)$$

with a non-negative integer n and constants a, A_* and B_*. Hereby

$$f(z)=Q^*(z) \exp(A_* z)+C_* , \tag{5.5}$$

where $Q^*(z)$ is a polynomial and C_* is a constant.

Assume now that our desired (5.4) is false. Then $A_* \neq 0$. While if $A_* \neq 0$, it is easily verified by asymptotic properties of $\exp(A_* z)$ that any function of the form (5.5) never fulfills the hypotheses of our Proposition 2. This is a contradiction. Therefore (5.4) is true.

5.2. Next, let us discuss the case where one of the three constants A_j is equal to 0 and the others are not. We may assume that $A_1=0$, $A_2\neq 0$ and $A_3\neq 0$. Further we may assume that the line L_2 is not orthogonal to the line L_1, since no two of the three lines L_j are parallel with each other. Let us consider the function defined by

$$F(z)=f(\bar{u}_1 z+\bar{u}_1 r_1)-c_1 . \tag{5.6}$$

Then all the zero-points of $F(z)$ are distributed on the imaginary axis only, and it follows from (5.2) and (5.6) that

$$\overline{F(\bar{z})}=F(-z)\exp(iB_1)\,. \tag{5.7}$$

Since L_1 and L_2 are not orthogonal and not parallel, the complex number $u=\bar{u}_1u_2$ is neither real nor pure imaginary. Here as a matter of convenience, let us set

$$c=c_2-c_1\,,\quad s^*=r_2-\mathrm{Re}(ur_1)\,,\quad t^*=\frac{1}{2}(u-\bar{u})r_1\,. \tag{5.8}$$

Then by a computation, all the roots of the equation $F(z)=c$ lie on the line

$$L^*=\{z:\mathrm{Re}(uz)=s^*\}\,, \tag{5.9}$$

which is neither parallel nor orthogonal to the imaginary axis. While by referring to the functional equation (5.7), we can easily assure that all the solutions of $F(z)=\bar{c}\exp(-iB_1)$ are distributed on the line

$$L_*=\{z:\mathrm{Re}(\bar{u}z)=-s^*\}\,, \tag{5.10}$$

which is the symmetry of the line L^* relative to the imaginary axis.

If $c=\bar{c}\exp(-iB_1)$, then by the above consideration, all the c-points of $F(z)$ must lie on the lines L^* and L_*. From their representations (5.9) and (5.10), it is clear that L^* and L_* are different from each other. Hence $F(z)$ never takes the value c except at the intersecting point of L^* and L_*. Since the genus of $F(z)-c$ is at most one, we can thus express $F(z)$ as

$$F(z)=c+(z-z^*)^n\exp(A^*z+B^*)\,,$$

where n is a non-negative integer and z^*, A^* and B^* are constants. Evidently, the auxiliary function $F(z)$ has the form (5.5). Hence by the same reason as before, we can conclude that $A^*=0$, so that $F(z)$ reduces to a polynomial. Turning back to the original function $f(z)$, we obtain the desired result, at once.

Assume now that $c\neq\bar{c}\exp(-iB_1)$. On taking account of (5.2), (5.6) and (5.8), we find

$$\begin{aligned}\overline{F(\bar{u}\bar{z}+\bar{u}s^*)-c}=&(F(-\bar{u}z+\bar{u}s^*)-c)\\ &\exp(2A_2z+iB_2-2A_2t^*)\,.\end{aligned} \tag{5.11}$$

Here let us further assume that $A_2>0$ and $0<\arg u<\pi/2$. Then from (5.11) and $A_2>0$, $F(z)$ must approach infinity when z tends to infinity along the half line

$$J^*=\{z:\arg(z-z^*)=-\arg u\}\,,$$

where $z^*=-is^*/\mathrm{Im}\,u$ is the intersecting point of the lines L^* and L_*. Since $F(z)$ omits the two values 0 and c in the open angular domain

$$D=\{z:\mathrm{Re}(uz)>s^*\quad\text{and}\quad\mathrm{Re}\,z>0\}\,,$$

and since the half line J^* is contained entirely in this domain D, Lindelöf-Iversen-Gross' theorem implies that $F(z)$ tends to infinity as z tends to infinity along an arbitrary half line which is contained in the domain D and parallels neither the line L^* nor the imaginary axis. On the other hand, from (5.10) and $0<\arg u<\pi/2$, the half line

$$J_*=\{z : \arg(z-z^*)=\arg u+3\pi/2\}$$

is contained entirely in the domain D. This half line J_* is the intersection of the line L_* and the open half plane $\mathrm{Re}\, z>0$. By these facts, we can see that the value $\bar{c}\exp(-iB_1)$ is a radially distributed value of $F(z)$. Hence by virtue of (5.7), the value c is also radially distributed. It therefore follows from Lemma 2.4 that the characteristic function of $F(z)$ must satisfy

$$\liminf_{r\to\infty}\frac{T(r, F)}{r}=0,$$

so that we have the desired (5.4), again. All other cases, say $A_2<0$ and $\pi<\arg u<3\pi/2$, can be also treated by the same way as above. Consequently, our desired (5.4) is true in the case where one of the three constants A_j is equal to 0 and the others are not.

5.3. Finally, we assume that no one of the three constants A_j is equal to 0. Assume further that $A_1>0$. Then it follows that $f(\bar{u}_1 t)$ tends to infinity when t does to infinity along the positive real axis. Without loss of generality, we can assume that

$$\arg u_1<\arg u_2<\arg u_3<\arg u_1+\pi .$$

There occur two cases. Either (I) $\arg u_3\leqq\arg u_1+\pi/2$, or (II) $\arg u_1+\pi/2<\arg u_3$.

In the case (I), since $\arg u_3-\arg u_1\leqq\pi/2$, the variable $t\bar{u}_1$ must be contained in the open angular domain

$$D^*=\{z : \mathrm{Re}(u_1 z)>r_1 \quad\text{and}\quad \mathrm{Re}(u_2 z)>r_2\}$$

for sufficiently large positive values of t. Here, observe that $f(z)$ fails to take the two values c_1 and c_2 in the domain D^*. Then using Lindelöf-Iversen-Gross' theorem, we find that $f(z)$ approaches infinity when z tends to infinity along an arbitrary half straight line of the form

$$\{z : z=z_*+t\exp(is),\ t>0\}, \tag{5.12}$$

where z_* is an arbitrary point and s is an arbitrary real number with $-\arg u_1-\pi/2<s<-\arg u_2+\pi/2$. On the other hand, from $\arg u_3-\arg u_1<\pi$, the intersection of the line L_3 and the domain D^* is a half straight line of the form (5.12). It thus follows that $f(z)$ approaches infinity as z does infinity along the unbounded part of L_3 which is contained in the domain D^*. Therefore the value

c_3 must be a radially distributed value of $f(z)$.

In the case (II), for sufficiently large positive values of t, the variable $t\bar{u}_1$ must lie in the open angular domain

$$D_* = \{z : \mathrm{Re}(u_1 z) > r_1 \quad \text{and} \quad \mathrm{Re}(u_3 z) < r_3\}.$$

Hence using Lindelöf-Iversen-Gross' theorem again, we can conclude that $f(z)$ tends to infinity as z does to infinity along an arbitrary half straight line of the form

$$\{z : z = z_* + t \exp(\imath v),\ t > 0\}, \tag{5.13}$$

where z_* is an arbitrary point and v is an arbitrary real number with $-\arg u_3 + \pi/2 < v < -\arg u_1 + \pi/2$. Further in this case, the intersection of the line L_2 and the domain D_* is a half straight line of the form (5.13). By these facts, the value c_2 must be a radially distributed value of $f(z)$.

Consequently, if $A_1 > 0$, then at least one of the two values c_2 and c_3 is a radially distributed value of $f(z)$. In the case $A_1 < 0$, we can also obtain the same conclusion. Similarly, it follows from the fact $A_2 \neq 0$ that either c_1 or c_3 is a radially distributed value of $f(z)$. Further the fact $A_3 \neq 0$ implies that either c_1 or c_2 is a radially distributed value of $f(z)$. Therefore $f(z)$ has at least two finite radially distributed values. Hereby Lemma 2.4 yields the desired (5.4). The proof of Proposition 2 is now complete.

6. Proof of Proposition 3. Let $f(z)$ be a transcendental entire function which satisfies the hypotheses of Proposition 3. Then $f(z)$ is a function of order one and mean type, and of regular growth. We may assume, as we may do without loss of generality, that $c_1 = 0$, $c_2 = 1$, $L_1 : \mathrm{Re}\, z = 0$ and $L_2 : \mathrm{Re}\, z = 1$. As before, we have

$$\overline{f(\bar{z})} = f(-z) \exp(2Az + \imath A'), \tag{6.1}$$

$$\overline{f(\bar{z}+1)} - 1 = (f(-z+1) - 1) \exp(2Bz + \imath B'), \tag{6.2}$$

where A, A', B and B' are real constants. From the results of the section 3, there may occur the following three cases only.

1) A and B are both positive.
2) A is negative and B is positive.
3) A and B are both negative.

Firstly, let us assume that the case 2) occurs. Then it follows that $f(z)$ approaches infinity when z does infinity along an arbitrary half straight line which is not parallel with the imaginary axis. Hence $f(z)$ takes the value c_3 only a finite number of times, since the line L_3 on which all the c_3-points of $f(z)$ lie, is not parallel with the imaginary axis. Thus $f(z)$ must have the value c_3 as a deficient value. However, since $AB < 0$, $f(z)$ has no finite deficient values.

This is clearly untenable. Therefore the case 2) never occurs.

Secondly, let us discuss the case 1). In this case, Lemma E implies $A=B$. In fact, if $A>B$, then by means of Lemma E, for an arbitrarily fixed number t^* in $(0, \pi/2)$, it is possible to choose a positive real number R_1 such that

$$|f(re^{it})-c_3| \geqq \frac{|c_3|}{2} \tag{6.3}$$

for real values of r and t with $r \geqq R_1$ and $|t-\pi| \leqq t^*$. Further from $A>0$, we can find a positive real number R_2 such that

$$|f(re^{it})-c_3| \geqq \frac{|c_3|}{2} \tag{6.4}$$

for real values of r and t with $r \geqq R_2$ and $|t| \leqq t^*$. On combining (6.3) and (6.4), we thus obtain

$$\begin{aligned} m(r, c_3, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{it})-c_3|}\,dt \\ &\leqq \left(1-\frac{2t^*}{\pi}\right)\log^+ M(r, c_3, f) + \frac{2t^*}{\pi}\log^+\frac{2}{|c_3|} \end{aligned} \tag{6.5}$$

for real values of r with $r \geqq \max(R_1, R_2)$, where

$$M(r, c_3, f) = \sup_{|z|=r} \frac{1}{|f(z)-c_3|}.$$

Therefore by virtue of Petrenko's result [7], the inequality (6.5) implies

$$\begin{aligned} \liminf_{r\to\infty} \frac{m(r, c_3, f)}{T(r, f)} &\leqq \left(1-\frac{2t^*}{\pi}\right) \liminf_{r\to\infty} \frac{\log^+ M(r, c_3, f)}{T(r, f)} \\ &\leqq \pi - 2t^* \end{aligned} \tag{6.6}$$

for every real number t^* with $0<t^*<\pi/2$. Hence from this (6.6), we have $\delta(c_3, f)=0$. On the other hand, from Lemma E and from the fact $A>0$, using Lindelöf-Iversen-Gross' theorem, we can see that $f(z)$ takes the value c_3 only a finite number of times. This is clearly absurd. Quite similarly, if $B>A$, then Lemma E leads us to a contradiction. Hereby we have $A=B$, as we claimed. Consequently,

$$f(z) = P^*(\exp Az)$$

with a quadratic polynomial $P^*(z)=az^2+bz+c$. Here let us assume that $a\neq 0$ and $b\neq 0$. Then at least one of the two roots of the equation $P^*(z)=c_3$ is not equal to 0. Since the constant A is real, we can thus conclude that the line L_3 on which all the c_3-points of $f(z)$ lie must be parallel with the imaginary axis. This contradicts the assumptions. Therefore either a or b is equal to 0. Accordingly, $f(z)$ is a function of the form

$$f(z)=A_*+B_*\exp(C_*z)\,, \tag{6.7}$$

where A_*, B_* and C_* are constants.

Finally, we must consider the case 3). This case reduces to the case 1) as follows. Let us set

$$F(z)=1-f(1-z)\,.$$

Then this entire function $F(z)$ also satisfies the hypotheses of our Proposition 3. Indeed, all the zero-points and all the one-points of $F(z)$ lie on the lines $\operatorname{Re} z=0$ and $\operatorname{Re} z=1$, respectively. Further all the solutions of $F(z)=1-c_3$ lie on a straight line which is not parallel with the imaginary axis. Here, recall the relations (6.1) and (6.2). Then we find

$$\overline{F(\bar{z})}=F(-z)\exp(-2Bz+iB')\,,$$

$$\overline{F(\bar{z}+1)}-1=(F(-z+1)-1)\exp(-2Az+iA')\,.$$

Hence $F(z)$ is such a function considered in the above case 1), since the constants A and B both negative. It thus follows that this auxiliary function $F(z)$ must have the form (6.7). Turning back to the original function $f(z)$, we obtain the desired result, immediately. The proof of Proposition 3 is now complete.

7. Proof of Proposition 4. In this final section, we shall prove our Proposition 4. Let $f(z)$ be a transcendental entire function which satisfies the hypotheses of Proposition 4. Without loss of generality, we can assume that all the three lines L_j which are distinct from one another, run parallel with the imaginary axis. Let us set

$$L_j=\{z:\operatorname{Re} z=r_j\} \qquad (j=1, 2, 3)\,.$$

Then we can further assume that $r_1<r_2<r_3$. By the same reason as before, we have

$$\overline{f(\bar{z}+r_j)-c_j}=(f(-z+r_j)-c_j)\exp(2A_jz+iB_j) \tag{7.1}$$

with real constants A_j and B_j $(j=1, 2, 3)$. In order to obtain our desired result, let us consider the auxiliary functions defined by

$$F_{jk}(z)=\frac{f((r_k-r_j)z+r_j)-c_j}{c_k-c_j} \tag{7.2}$$

for each pair j and k with $1\leqq j<k\leqq 3$. Then these functions $F_{jk}(z)$ take the value 0 only on the imaginary axis, and take the value 1 only on the line $\operatorname{Re} z=1$. Further from (7.1),

$$\overline{F_{jk}(\bar{z})}=F_{jk}(-z)\exp(2(r_k-r_j)A_jz+iB'_{jk})\,, \tag{7.3}$$

$$\overline{F_{jk}(\bar{z}+1)}-1=(F_{jk}(-z+1)-1)$$
$$\exp(2(r_k-r_j)A_kz+iB'_{kj})$$

with suitable real constants B'_{jk} and B'_{kj}. It is thus clear that these $F_{jk}(z)$ are such functions treated in the section 3. Hence for the three real constants A_j, there may occur the following four cases only.

1) A_1, A_2 and A_3 are all positive.
2) A_1 is negative and the other two are positive.
3) A_3 is positive and the other two are negative.
4) The three are all negative.

In the first place, we shall show the impossibility of the cases 2) and 3). Assume now that the case 2) occurs. Then from $A_2>0$ and $A_3>0$, by virtue of Lemma E, when z tends to infinity along the negative real axis, $F_{23}(z)$ converges to 0 or 1 according to whether $A_2>A_3$ or $A_3>A_2$. Further if $A_2=A_3$, then with a quadratic polynomial $P(z)$, $F_{23}(z)=P(\exp(r_3-r_2)A_2z)$, so that $F_{23}(z)$ converges to $P(0)$ as z tends to infinity along the negative real axis. Turn back to the original function $f(z)$. Then it is clear from (7.2) that $f(z)$ must approach some finite value when z tends to infinity along the negative real axis. On the other hand, it follows from the fact $A_1<0$ that $f(z)$ tends to infinity as z does to infinity along the negative real axis. This is clearly absurd. Hence the case 2) never occurs. Similarly, we can also prove the impossibility of the case 3) from that of the case 2) by using the same arguments developed at the end of the proof of Proposition 3. Consequently, the two cases 2) and 3) never happen.

In the second place, let us consider the case 1). If we can prove that $A_j=A_k$ for some pair j and k with $1\leqq j<k\leqq 3$, then by making use of (7.3), we have

$$F_{jk}(z)=P(\exp(r_k-r_j)A_jz)$$

with a quadratic polynomial $P(z)$. Returning to the original function $f(z)$, we at once obtain the desired result.

Here, let us assume that $A_1>A_2$. Then as above, by means of Lemma E, the auxiliary function $F_{12}(z)$ must tend to 0 when z tends to infinity along the negative real axis. Turning back to $f(z)$, we thus find that $f(z)$ converges to the value c_1 as z approaches infinity along the negative real axis. Further assume for a moment that $A_2\neq A_3$. Then considering $F_{23}(z)$, we can easily conclude that as z approaches infinity along the negative real axis, $f(z)$ converges to c_2 or c_3 according to whether $A_2>A_3$ or $A_3>A_2$. This is impossible. Therefore if $A_1>A_2$, then $A_2=A_3$. Quite similarly, if $A_2>A_1$, then $A_1=A_3$. Hereby, we have the desired result.

For the case 4), by making use of Lemma E and the auxiliary functions $F_{jk}(z)$, we can also conclude that at least two of the three real constants A_j are equal to each other. Hence we can also obtain our desired result in this case 4). This completes the proof of Proposition 4.

References

[1] BAKER, I. N., Entire functions with linearly distributed values, Math. Zeitschr., **86** (1964), 263-267.
[2] EDREI, A., Meromorphic functions with three radially distributed values, Trans. Amer. Math. Soc., **78** (1955), 276-293.
[3] EDREI, A. AND W. H. J. FUCHS, Bounds for the number of deficient values of certain classes of meromorphic functions, Proc. London Math. Soc. 3. Ser., **12** (1962), 315-344.
[4] KOBAYASHI, T., On a characteristic property of the exponential function, Kōdai Math. Sem. Rep., 29 (1977), 130-156.
[5] NEVANLINNA, R., Analytic functions, Springer-Verlag, Berlin, 1970.
[6] NOSHIRO, K., Cluster sets, Springer-Verlag, Berlin, 1960.
[7] PETRENKO, V. P., Growth of meromorphic functions of finite lower order, Math. USSR Izv., **3** (1969), 391-432.

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