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PRE-RADON MEASURES ON TOPOLOGICAL SPACES

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§ 1. Introduction.

There are two directions in the study of the measure theory on arbitrary topological spaces: the theory of Radon measures and the theory of Baire measures. The outline of the developments in these fields is referred to Bourbaki [2], Hirschfeld [8], Schwartz [11] and Varadarajan [13].

The purpose of this paper is to study infinite Borel measures.

Originally, in 1970, the first author has proposed the notion of a pre-Radon measure on a topological space, which is defined as a class of "measures determined by an open base with a smoothness condition" (Amemiya [1]). It seems to be of use for the study of infinite measures, especially Borel measures on a topological space. In this paper, we formulate a pre-Radon measure as a Borel measure (see Definition 2.2) and develop the topics in a survey of Amemiya [1] from a different viewpoint.

Finite pre-Radon measures are said to be τ-smooth Borel measures which have been investigated by many mathematicians. For infinite Borel measures with τ-smoothness, Fremlin [3] recently presented the class of quasi-Radon measures. Our pre-Radon measures are slightly different from quasi-Radon measures.

Main results of this paper are three constructions of pre-Radon measures given in Section 3. The fundamental idea is suggested by Kirk [9]. In Theorem 3.1, we extend a finitely additive set function satisfying some smoothness conditions defined on the ring generated by an open base to a pre-Radon measure. Similarly, in Theorem 3.2 we consider a set function defined on the algebra generated by an open base. In Theorem 3.4, an infinite Baire measure with τ-smoothness on a normal space is extended to a pre-Radon measure. For finite τ-smooth Baire measures, this extension is known (see for example Kirk [9]).

In Section 4, we give the decomposition theorem for σ-finite pre-Radon measures.

In Section 5, we deal with the restriction of pre-Radon measures. We present the several conditions that the restriction is a pre-Radon measure.

In Section 6, we prove the decomposability of pre-Radon measures. For Radon measures, the decomposability is given in [2, § 1, Proposition 9] and for quasi-Radon measures, Fremlin [2, Theorem 72B].

In section 7, we give some topological spaces with the (K)-property (for the

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definition, see Section 2). We prove if a topological space X is a Borel subset of its Stone-Čech compactification, then X has the (K)-property (Theorem 7.4). In particular, topologically complete spaces and σ-compact spaces have the (K)-property.

In Section 8, we prove that there exists a one-to-one correspondence between pre-Radon measures and smooth linear functionals.

In Section 9, we show the uncountable product of pre-Radon probability measures is uniquely extended to a pre-Radon measure on the product space (Theorem 9.9). In the countable product case, Tortrat [12] has proved the same result, still we show using a Fubini type theorem (Theorem 9.6) for the sake of completeness.

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§ 2. Preliminaries.

Let X be a set. A family $\mathcal{U}$ of subsets of X is said to be a *paving* if it satisfies the following conditions:

1) $\phi\in\mathcal{U}$;

2) $\bigcup_{U\in\mathcal{U}} U=X$;

3) If $U_1, U_2\in\mathcal{U}$, then $U_1\cap U_2\in\mathcal{U}$ and $U_1\cup U_2\in\mathcal{U}$.

We denote by $R[\mathcal{U}]$ the ring generated by a paving $\mathcal{U}$.

LEMMA 2.1 (Kirk and Crenshaw [10, Proposition 1.2]). *Let F be a subset of X, then F belongs to $R[\mathcal{U}]$ if and only if there are sets W_i, V_i in $\mathcal{U}$ $(i=1, 2, \cdots, n)$ such that the following conditions hold*:

1) $V_i\subset W_i$ $(i=1, 2, \cdots, n)$;

2) $(W_i-V_i)\cap(W_j-V_j)=\emptyset$ *for* $i\neq j$;

3) $F=\bigcup_{i=1}^{n}(W_i-V_i)$.

Let m be a non-negative, extended real valued set function on an algebra $\mathcal{A}$ of subsets of X. We say m is *σ-finite* if there exists a countable subfamily $\{A_n\in\mathcal{A}; m(A_n)<\infty, n=1, 2, \cdots\}$ such that $X=\bigcup_{n=1}^{\infty}A_n$, and m is *semi-finite if* m satisfies

$$m(A)=\sup\{m(B); \mathcal{A}\ni B\subset A, m(B)<\infty\}$$

for every A in $\mathcal{A}$. A *measure* μ is a non-negative, extended real valued and countably additive set function defined on a σ-algebra $\mathcal{B}$ such that $\mu(\phi)=0$.

Let $(X, \mathscr{B}, \mu)$ be a measure space and A be an element in $\mathscr{B}$. We denote by μ_A the measure on the measurable space $(A, A\cap\mathscr{B})$ defined by

$$\mu_A(A\cap B)=\mu(A\cap B)$$

for every B in $\mathscr{B}$. We call μ_A the *restriction* of μ to A.

Let $\{(X, \mathscr{B}_\lambda, \mu_\lambda)\,;\, \lambda\in\Lambda\}$ be a family of measure spaces such that $\mu_\lambda(X_\lambda)=1$. By $\bigotimes_{\lambda\in\Lambda}\mathscr{B}_\lambda$, we mean the *product σ-algebra*, that is, the smallest σ-algebra which makes each projection of $\prod_{\lambda\in\Lambda} X_\lambda$ onto X_λ measurable. Then there exists a unique probabilty measure $\bigotimes_{\lambda\in\Lambda}\mu_\lambda$ on $\bigotimes_{\lambda\in\Lambda}\mathscr{B}_\lambda$ such that

$$(\bigotimes_{\lambda\in\Lambda}\mu_\lambda)(A)=\mu_{\lambda_1}(A_{\lambda_1})\cdots\mu_{\lambda_n}(A_{\lambda_n})$$

for every set A of the form $A_{\lambda_1}\times\cdots\times A_{\lambda_n}\times\prod_{\lambda\neq\lambda_i} X_\lambda$ in $\bigotimes_{\lambda\in\Lambda}\mathscr{B}_\lambda$. This measure $\bigotimes_{\lambda\in\Lambda}\mu_\lambda$ is called the *product measure*.

Let X be a topological space. By the *Borel field* $\mathscr{B}(X)$, we mean the minimal σ-algebra generated by all open subsets of X. By $C(X)$, we denote the algebra of all real continuous functions on X. The *Baire field* $\mathscr{B}_a(X)$ is the minimal σ-algebra generated by the family of *zero sets*

$$Z(X)=\{f^{-1}(0)\,;\, f\in C(X)\}\,.$$

Now we define pre-Radon measures and Radon measures.

DEFINITION 2.2. Let X be a topological space. A *pre-Radon measure* μ is a Borel measure on $\mathscr{B}(X)$ such that:

1) For every x in X, there exists an open neighborhood O of x such that $\mu(O)<\infty$;

2) For every net $\{O_\alpha\}$ of open sets increasing to an open subset O, $\lim_\alpha \mu(O_\alpha)=\mu(O)$;

3) For every open subset O such that $\mu(O)<\infty$,

$$\mu(O)=\sup\{\mu(F)\,;\, F\subset O \text{ and } F \text{ is closed}\}\,;$$

4) For every A in $\mathscr{B}(X)$,

$$\mu(A)=\inf\{\mu(O)\,;\, O\supset A \text{ and } O \text{ is open}\}\,.$$

We say a Borel measure satisfying 3), 4) a *regular* Borel measure.

In the same manner as in the proof of Theorem (11.32) of Hewitt and Ross [6], it follows that the above conditions 3), 4) imply the following 3)′:

3)′ For every A in $\mathscr{B}(X)$ such that $\mu(A)<\infty$,

$$\mu(A)=\sup\{\mu(F)\,;\, F\subset A \text{ and } F \text{ is closed}\}\,.$$

Consequently, the conditions 3), 4) are equivalent to 3)′, 4).

Remark 2.3. In general, the above conditions 3)′ and 4) are not necessarily equivalent. If a Borel measure μ is σ-finite and satisfies 4). then 3)′ holds. For infinite Borel measures, the conditions which deduce 4) from 1) and 3)′ are not known except that X is locally compact and σ-compact, as far as the authors are concerned. We shall discuss this problem in Appendix.

Remark 2.4. There exists a non-regular Borel measure on a compact space (see Halmos [5, 52, Exercise (10)]). This also gives an example of a Borel measure which is not a pre-Radon measure.

DEFINITION 2.5. Let X be a topological space. A *Radon measure* μ is a Borel measure on $\mathscr{B}(X)$ such that

1) For every x in X, there exists an open neighborhood O of x such that $\mu(O)<\infty$;

2) For every open set O,

$$\mu(O)=\sup\{\mu(K)\,;\ K\subset O \text{ and } K \text{ is compact}\}\,;$$

3) For every A in $\mathscr{B}(X)$,

$$\mu(A)=\inf\{\mu(O)\,;\ O\supset A \text{ and } O \text{ is open}\}\,.$$

Our definition of a Radon measure is different from Bourbaki [2] whose "Radon measure" is a Borel measure satisfying 1) and 2) in Definition 2.5.

It follows that a Radon measure is a pre-Radon measure. Conversely it is easily verified that a pre-Radon measure on a locally compact space is a Radon measure. We say a topological space has the (K)-*property* if every pre-Radon measure is a Radon measure.

The *support* of a Borel measure μ on a topological space X is the set of all points x in X with the property that, for every open set O containing x, $\mu(O)>0$. We denote by $\operatorname{supp}\mu$ the support of μ. We have the following easy consequence.

THEOREM 2.6. *Every non-zero pre-Radon measure has the non-empty support.*

§ 3. Construction of pre-Radon measure.

In this section, we give three methods of constructions of pre-Radon measures.

Firstly, we discuss a set function defined on a ring.

THEOREM 3.1. *Let X be a topological space, $\mathcal{U}$ be a paving generated by an open base of X and m be a non-negative, real valued, finitely additive set function on $R[\mathcal{U}]$ such that*

1) *For any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to a set U in $\mathcal{U}$,*

$$\lim_{\alpha} m(U_\alpha)=m(U);$$

2) *For every U in $\mathcal{U}$,*

$$m(U)=\sup\{m(F);\ U\supset F\in R[\mathcal{U}] \text{ and } F \text{ is closed}\}.$$

Then m is uniquely extensible to a pre-Radon measure.

Proof. If two nets $\{U_\alpha\}$ and $\{V_\tau\}$ increase to an open set O, then we have

$$\lim_{\alpha} m(U_\alpha)=\lim_{\tau} m(V_\tau).$$

For every open set O, we put

$$\lambda(O)=\sup\{m(U);\ O\supset U\in\mathcal{U}\}.$$

Then it follows that λ is a non-negative, monotone and subadditive set function on the family of open subsets of X. It can be easily shown that for any net $\{O_\alpha\}$ increasing to O,

$$\lim_{\alpha}\lambda(O_\alpha)=\lambda(O).$$

We define a set function μ^* as follows:

$$\mu^*(A)=\inf\{\lambda(O);\ O\supset A \text{ and } O \text{ is open}\}$$

for every subset A of X. It is evident that μ^* is an outer measure defined on all subsets of X. We shall prove that every open subset is μ^*-measurable by the way similar to Kirk [9, Lemma 1.12]. Let O be an open subset of X and A be a subset. It is sufficient to show

$$\mu^*(A)\geqq\mu^*(A\cap O)+\mu^*(A-O).$$

We may assume that $\mu^*(A)$ is finite. For arbitrary ε positive, there is an open subset O_1 containing A such that $\lambda(O_1)<\varepsilon+\mu^*(A)$. Let $\{U_\alpha\}$ be a net in $\mathcal{U}$ increasing to O_1 and V be a set in $\mathcal{U}$ contained in O. By the condition 2), there exists a closed set F in $R[\mathcal{U}]$ with $F\supset V$ such that $m(F)+\varepsilon>m(V)$. Then it holds

$$m(U_\alpha-F)-m(U_\alpha-V)\leqq m(V-F)<\varepsilon,$$

so that it follows

$$\varepsilon+\lim_{\alpha} m(U_\alpha-V)<\lim_{\alpha} m(U_\alpha-F)=\lambda(O_1-F)\geqq\mu^*(A-O).$$

Thus we have

$$\begin{aligned}\varepsilon+\mu^*(A)&>\lambda(O_1)=\lim_{\alpha} m(U_\alpha)\\&=\lim_{\alpha}(m(U_\alpha\cap V)+m(U_\alpha-V))>\lambda(O_1\cap V)+\mu^*(A-O)-\varepsilon.\end{aligned}$$

Since V is arbitrary, we have

$$2\varepsilon+\mu^*(A)\geqq\lambda(O_1\cap O)+\mu^*(A-O)\geqq\mu^*(A\cap O)+\mu^*(A-O),$$

which shows every open subset is μ^*-measurable. So the restriction μ of μ^* to the Borel field $\mathscr{B}(X)$ is a Borel measure. From the definition of μ, it is obvious that μ is a pre-Radon measure.

We show μ is an extension of m. By Lemma 2.1, every A in $R[\mathcal{U}]$ can be represented as a disjoint union $A=\bigcup_{i=1}^{n}(W_i-V_i)$. Thus we have

$$m(A)=\sum_{i=1}^{n}(m(W_i)-m(V_i))=\sum_{i=1}^{n}(\mu(W_i)-\mu(V_i))=\mu(A).$$

Finally we shall prove the uniqueness of μ. Let ν be another pre-Radon measure extending m. For any open set O, we can find a net $\{U_\alpha\}$ in $\mathcal{U}$ increasing to O. Then we have

$$\mu(O)=\lim_\alpha \mu(U_\alpha)=\lim_\alpha m(U_\alpha)=\lim_\alpha \nu(U_\alpha)=\nu(O).$$

By the regularity of μ and ν, we have

$$\begin{aligned}\mu(A)&=\inf\{\mu(O)\,;\ O\supset A \text{ and } O \text{ is open}\}\\&=\inf\{\nu(O)\,;\ O\supset A \text{ and } O \text{ is open}\}\\&=\nu(A).\end{aligned}$$

This completes the proof.

Secondly we deal with a set function on an algebra.

THEOREM 3.2. *Let X be a topological space, $\mathcal{U}$ be a paving generated by an open base of X and m be a non-negative, extended real valued, countably additive set function on the algebra $A[\mathcal{U}]$ generated by $\mathcal{U}$. If m satisfies the following conditions:*

1) *There exists an increasing sequence $\{U_n\}$ in $\mathcal{U}$ such that $m(U_n)$ is finite, and $X=\bigcup_{n=1}^{n}U_n$;*

2) *For any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to a set U in $\mathcal{U}$ such that $m(U)$ is finite,*

$$\lim_\alpha m(U_\alpha)=m(U);$$

3) *For every U in $\mathcal{U}$ such that $m(U)$ is finite,*

$$m(U)=\sup\{m(F)\,;\ U\supset F\in A[\mathcal{U}] \text{ and } F \text{ is closed}\},$$

then m is uniquely extended to a pre-Radon measure.

Proof. For every open set O, we set

$$\lambda(O)=\sup\{m(U)\,;\, O\supset U\in\mathcal{U} \text{ and } m(U) \text{ is finite}\}\,.$$

Furthermore we put

$$\mu^*(A)=\inf\{\lambda(O)\,;\, O\supset A \text{ and } O \text{ is open}\}$$

for any subset A of X. In the same manner as Theorem 3.1, every open set is μ^*-measurable. Moreover, the restriction μ of μ^* to $\mathcal{B}(X)$ is a pre-Radon measure.

We shall prove that μ is an extension of m. For each U in $\mathcal{U}$, the algebra $U\cap A[\mathcal{U}]$ is generated by $U\cap\mathcal{U}$. In fact the family $\{A\subset X\,;\, U\cap A\in A_U[U\cap\mathcal{U}]\}$ is an algebra containing $\mathcal{U}$, where $A_U[U\cap\mathcal{U}]$ denotes the algebra of subsets of U generated by $U\cap\mathcal{U}$. So this family contains $A[\mathcal{U}]$. By Lemma 2.1, for every A in $A[\mathcal{U}]$, we have

$$U\cap A=\bigcup_{i=1}^{n}(U\cap W_i-U\cap V_i) \qquad \text{(disjoint union)}\,,$$

where W_i and V_i are in $\mathcal{U}$. Particularly, if $m(U)=\mu(U)$ is finite, we have

$$\begin{aligned} m(U\cap A)&=\sum_{i=1}^{n}(m(U\cap W_i)-m(U\cap V_i)) \\ &=\sum_{i=1}^{n}(\mu(U\cap W_i)-\mu(U\cap V_i)) \\ &=\mu(U\cap A)\,. \end{aligned}$$

For every A in $A[\mathcal{U}]$, we have

$$\begin{aligned} m(A)&=\lim_n m(U_n\cap A) \\ &=\lim_n \mu(U_n\cap A) \\ &=\mu(A)\,. \end{aligned}$$

Consequently μ is an extension of m.

From the arguments in Theorem 3.1, the uniqueness of μ is clear. The proof is complete.

Remark 3.3. In Theorem 3.2, if m is totally finite, finitely additive set function on $A[\mathcal{U}]$ satisfying the conditions 2) and 3), then it is easy to verify that m is uniquely extended to a pre-Radon measure.

Lastly we consider a set function defined on the Baire field $\mathcal{B}_a(X)$. We recall that a cozero set is the complement of a zero set. We denote by $U(X)$ the family of all cozero sets of X.

THEOREM 3.4. *Let X be a normal topological space and m be a non-negative extended real valued, finitely additive set function on $\mathcal{B}_a(X)$ satisfying the follow-*

ing conditions:

1) *For any x in X, there exists a cozero set U containing x such that m(U) is finite*;

2) *For any net* $\{U_\alpha\}$ *of cozero sets increasing to a cozero set U*,

$$\lim_\alpha m(U_\alpha)=m(U);$$

3) *For every Baire set A in* $\mathcal{B}_a(X)$,

$$\begin{aligned} m(A)&=\sup\{m(Z);\ A\supset Z\in Z(X)\} \\ &=\inf\{m(U);\ A\subset U\in U(X)\}. \end{aligned}$$

Then m is uniquely extensible to a pre-Radon measure.

Proof. In the same manner as in the proofs of Theorem 3.1 and 3.2, we obtain a pre-Radon measure μ which coincides with m on $U(X)$. The uniqueness is trivial if μ is an extension of m. We only prove that μ is an extension of m. For every Z in $Z(X)$, we have

$$\begin{aligned} \mu(Z)&=\inf\{\mu(O);\ O\supset Z \text{ and } O \text{ is open}\} \\ &\leqq\inf\{\mu(U);\ Z\subset U\in U(X)\} \\ &=m(Z). \end{aligned}$$

Conversely, since X is normal, for any open set O containing the zero set Z, there exists a cozero set U such that $O\supset U\supset Z$. Consequently we have $\mu(Z)=m(Z)$ for every Z in $Z(X)$. Let A be any Baire set in $\mathcal{B}_a(X)$. Then we have

$$\begin{aligned} \mu(A)&=\inf\{\mu(O);\ O\supset A \text{ and } O \text{ is open}\} \\ &\leqq\inf\{\mu(U);\ A\subset U\in U(X)\} \\ &=\inf\{m(U);\ A\subset U\in U(X)\} \\ &=m(A) \\ &=\sup\{m(Z);\ A\supset Z\in Z(X)\} \\ &=\sup\{\mu(Z);\ A\supset Z\in Z(X)\} \\ &\leqq\mu(A). \end{aligned}$$

Thus μ is identical to m on $\mathcal{B}_a(X)$. This proves the theorem.

Remark 3.5. We can prove the same result as in Theorem 3.4 even if m is defined on the algebra generated by $Z(X)$.

§ 4. Decomposition theorem.

LEMMA 4.1. *Let μ be a pre-Radon measure on a regular space X. Then there exists a unique Radon measure ν such that ν is absolutely continuous with respect to μ and $\nu(K)=\mu(K)$ for every compact subset K.*

Proof. For any open subset O, put

$$m(O)=\sup\{\mu(K);\ K\subset O \text{ and } K \text{ is compact}\}.$$

Then we can easily prove that $\lim_{\alpha} m(O_\alpha)=m(O)$ for every net $\{O_\alpha\}$ of open subsets increasing to an open subset O. Let O_1 and O_2 be two open subsets, then we have

$$m(O_1\cup O_2)\leqq m(O_1)+m(O_2)$$

since μ is a regular Borel measure. Since X is a regular space, we have

$$m(O)=\sup\{m(W);\ W\subset \overline{W}\subset O \text{ and } W \text{ is open}\}$$

for every open set O, where $\overline{W}$ is the closure of W in X.

We define a set function on the family of all subsets of X as follows:

$$\nu^*(A)=\inf\{m(O);\ O\supset A \text{ and } O \text{ is open}\}.$$

Then it follows that ν^* is an outer measure. In the same manner as in the proof of Theorem (11.30) of Hewitt and Ross [6], we can show every Borel subset of X is ν^*-measurable. We denote by ν the restriction of ν^* to $\mathscr{B}(X)$.

For any compact subset K, we have $\nu(K)=\mu(K)$. In fact, we have

$$\begin{aligned}\nu(K)&=\inf\{m(O);\ O\supset K \text{ and } O \text{ is open}\}\\&\leqq\inf\{\mu(O);\ O\supset K \text{ and } O \text{ is open}\}\\&=\mu(K).\end{aligned}$$

On the other hand, for any open subset O containing K, we have $m(O)\geqq\mu(K)$. Thus we have $\nu(K)\geqq\mu(K)$.

It is obvious that ν is Radon measure and absolutely continuous with respect to μ. The uniqueness of ν is obvious from the definition of Radon measure. This completes the proof.

We shall prove the following decomposition theorem.

THEOREM 4.2. *Let X be a regular space and μ be a pre-Radon measure on X. Then there uniquely exist a Radon measure ν and a pre-Radon measure ρ such that*

1) $\mu=\nu+\rho$;
2) $\rho(K)=0$ *for every compact subset K.*

Furthermore if μ is σ-finite, then ρ is singular with repect to ν.

Proof. If we put

$$\mathcal{U}=\{U\,;\ U \text{ is open and } \mu(U)<\infty\}\,,$$

then $\mathcal{U}$ is an open base of X. We define a set function m on $R[\mathcal{U}]$ by

$$m(A)=\mu(A)-\nu(A)$$

for every A in $R[\mathcal{U}]$, where ν is a Radon measure obtained in Lemma 4.1. By Theorem 3.1, m is uniquely extensible to a pre-Radon measure ρ. Then it is clear that $\mu(O)=\nu(O)+\rho(O)$ for every open subset O. If we remark that both μ and $\nu+\rho$ are pre-Radon measures, then we have $\mu=\nu+\rho$. For every compact subset K we have $\rho(K)=\mu(K)-\nu(K)=0$. The uniqueness of the decomposition is obvious.

Assume that μ is σ-finite, then ν is also σ-finite, which implies for a σ-compact subset L, $\nu(X-L)=0$. On the other hand we have $\rho(L)=0$. Hence ρ is singular with respect to ν. The theorem is proved.

Remark 4.3. In our original version, we assumed that μ is σ-finite. The improvement of the theorem is based on a suggestion of Fremlin (personal communication).

§ 5. Restriction of pre-Radon measure.

In this section we consider the restriction of pre-Radon meaures to subsets.

Let $(X, \mathcal{B}, \mu)$ be a measure space. We denote by $(X, \overline{\mathcal{B}}, \bar{\mu})$ the completion of $(X, \mathcal{B}, \mu)$

LEMMA 5.1. *Let μ be a regular Borel measure on a topological space X and A be a subset in $\overline{\mathcal{B}(X)}$. Then the restriction $\bar{\mu}_A$ of $\bar{\mu}$ to A is a regular Borel measure on A.*

Proof. It is obvious from the definition of the completion.

By Lemma 5.1, it is easy to verify the following theorem.

THEOREM 5.2. *Let μ be a pre-Radon measure on a topological space X and O be an open subset of X. Then the restriction μ_0 of μ to O is a pre-Rondon measure.*

If μ is semi-finite, then the restriction of μ to any Borel subset is a pre-Radon measure. In general we have the following theorem.

THEOREM 5.3. *Let μ be a pre-Radon measure on a topological space X and A be a subset in $\overline{\mathcal{B}(X)}$ such that $\bar{\mu}_A$ is semi-finite on $(A, A\cap\overline{\mathcal{B}(X)})$. Then $\bar{\mu}_A$ is a pre-Radon measure.*

Proof. At first, we shall prove in the case that $\bar{\mu}(A)$ is finite. Let $\{O_\alpha\}$ be a net of open subsets of A increasing to an open subset O of A. Since A belongs to $\overline{\mathscr{B}(X)}$, there exist a set A_0 in $B(X)$ and a set N such that

$$A=A_0\cup N \quad \text{and} \quad \mu^*(N)=0\,,$$

where μ^* denotes the outer measure induced by μ. There exists an open subset $\tilde{O}_\alpha$ of X such that $\tilde{O}_\alpha\cap A_0=O_\alpha\cap A_0$ for every α. Since μ is a regular measure, there exists an open subset $\tilde{O}$ of finite measure such that $\tilde{O}\cap A_0=O\cap A_0$ and $\tilde{O}\subset\bigcup_\alpha O_\alpha$. We put

$$\tilde{U}_\alpha=(\bigcup_{\beta\leqq\alpha}\tilde{O}_\beta)\cap\tilde{O}\,,$$

then this net $\{\tilde{U}_\alpha\}$ of subsets of X increases to the open set $\tilde{O}$. Thus we have

$$\begin{aligned}\lim_\alpha \bar{\mu}_A(O-O_\alpha)&=\lim_\alpha \mu_{A_0}(\tilde{O}\cap A_0-\tilde{U}_\alpha\cap A_0)\\ &\leqq\lim_\alpha \mu(\tilde{O}-\tilde{U}_\alpha)=0\,.\end{aligned}$$

We consider the general case that $\bar{\mu}_A$ is semi-finite. Let $\{O_\alpha\}$ be a net of open subsets of A increasing to an open subset O of A. If $\bar{\mu}_A(O)$ is finite, then from the first step we have

$$\begin{aligned}\bar{\mu}_A(O)=\bar{\mu}_o(O)&=\lim_\alpha \bar{\mu}_o(O_\alpha)\\ &=\lim_\alpha \bar{\mu}_A(O_\alpha)\,.\end{aligned}$$

If $\bar{\mu}_A(O)$ is infinite, for any natural number N there exists a set B in $A\cap\overline{\mathscr{B}(X)}$ such that $B\subset O$ and $N<\bar{\mu}_A(B)<\infty$. Since the net $\{O_\alpha\cap B\}$ increases to B, we have

$$\begin{aligned}N<\bar{\mu}_A(B)&=\bar{\mu}_B(B)\\ &=\lim_\alpha \bar{\mu}_B(O_\alpha\cap B)\\ &\leqq\lim_\alpha \bar{\mu}_A(O_\alpha)\,.\end{aligned}$$

Thus we have $\lim_\alpha \bar{\mu}_A(O_\alpha)=\bar{\mu}_A(O)$. By Lemma 5.1, $\bar{\mu}_A$ is a pre-Radon measure on $\mathscr{B}(A)$. This completes the proof.

Let $(X, \mathscr{B}, \mu)$ be a measure space and Y be a μ-thick subset of X. Then there exists a measure μ_Y on $(Y, \mathscr{B}\cap Y)$ such that

$$\mu(B\cap Y)=\mu_Y(B)$$

for every set B in $\mathscr{B}$ by Hylmos [5, § 17, Theorem A].

THEOREM 5.4. *Let μ be a pre-Radon measure on a topological space X and Y be a μ-thick subset of X. Then μ_Y is a pre-Radon measure on Y.*

Proof. Let $\{O_\alpha\}$ be a net of open subsets of Y increasing to an open subset O of Y. There exists an open subset $\tilde{O}_\alpha$ of X such that $\tilde{O}_\alpha \cap Y = O_\alpha$. Putting $\tilde{U}_\alpha = \bigcup_{\beta \leqq \alpha} \tilde{O}_\beta$, $\{\tilde{U}_\alpha\}$ is a net of open subsets of X increasing to $\bigcup_\alpha \tilde{U}_\alpha$ such that $\tilde{U}_\alpha \cap Y = O_\alpha$. Since μ is a pre-Radon measure, we have

$$\begin{aligned} \mu_Y(O) &= \mu_Y((\bigcup_\alpha \tilde{U}_\alpha) \cap Y) \\ &= \mu(\bigcup_\alpha \tilde{U}_\alpha) \\ &= \lim_\alpha \mu(\tilde{U}_\alpha) \\ &= \lim_\alpha \mu_Y(\tilde{U}_\alpha \cap Y) \\ &= \lim_\alpha \mu_Y(O_\alpha) . \end{aligned}$$

Since μ is a regular Borel measure, it is easy to verify that μ_Y is a regular Borel measure. Therefore μ_Y is a pre-Radon measure, which completes the proof.

Let $(X, \mathcal{B}, \mu)$ be a measure space and A be any subset of X. We say a set B in $\mathcal{B}$ is a minimal measurable cover of A if A is μ_B-thick in B, that is, $(\mu_B)_*(B-A)=0$, where $(\mu_B)_*$ is the inner measure induced by μ_B. If μ is σ-finite, then there exists a minimal measurable cover of every subset. We define the restriction of μ to A. Since A is μ_B-thick in B, $(\mu_B)_A$ exists. It is clear that $(\mu_B)_A$ is identical to $(\mu_{B'})_A$ for another minimal measurable cover B' of A. Putting $\mu_A=(\mu_B)_A$, we call μ_A the restriction of μ to A.

Under the above preparations we have the following final result in this section.

THEOREM 5.5. *Let μ be a pre-Radon measure on a topological space X and A be subset of X. If A has a minimal measurable cover B in $\overline{\mathcal{B}(X)}$ such that the restriction $\bar{\mu}_B$ of $\bar{\mu}$ is semi-finite on $(B, B \cap \overline{\mathcal{B}(X)})$, then the restriction μ_A of μ to A is a pre-Radon measure.*

Proof. It follows from Theorem 5.3 and 5.4.

COROLLARY 5.6. *Let μ be a σ-finite pre-Radon measure on a topological space X. Then for any subset A of X, μ_A is a pre-Radon measure.*

Proof. Since μ is σ-finite, A has a minimal measurable cover.

Remark 5.7. Fremlin has pointed out the followings:

1) Let μ be a pre-Radon measure on X. If A has a minimal measurable cover, then it holds $\mu_A(B)=\inf\{\mu(C);\ C \in \mathcal{B}(X)$ and $C \supset B\}$ for every B in $A \cap \overline{\mathcal{B}(X)}$.

2) If μ is a quasi-Radon measure, then every subset A has a minimal measurable cover. Particularly μ can be restricted to A and the restriction μ_A is a

quasi-Radon measure on A.

§ 6. Decomposability.

Let X be a topological space and μ be a pre-Radon measure on X. A subset A of X is called *locally negligible* if $\mu^*(O\cap A)=0$ for every open set O such that $\mu(O)$ is finite, where μ^* denotes the outer measure derived from μ.

For pre-Radon measures, we give the following decomposition theorem which is similar to Bourbaki [2, § 1, Proposition 9].

THEOREM 6.1. *Let μ be a pre-Radon measure on a topological space X. Then there exists a family $\{B_\alpha\}$ of closed sets satisfying the following*:

1) *Each $\mu(B_\alpha)$ is finite and* $\operatorname{supp}\mu_{B_\alpha}=B_\alpha$;
2) *The family $\{B_\alpha\}$ is pairwise disjoint*;
3) $X-\bigcup_\alpha B_\alpha$ *is locally negligible*;
4) *If A is a Borel set of finite measure, the cardinal of $\{\alpha\,;B_\alpha\cap A\neq\emptyset\}$ is at most countable and it holds*

$$\mu(A)=\sum_\alpha \mu(A\cap B_\alpha)\,.$$

Proof. Let $\mathcal{A}$ be the collection of all disjoint families $\{C_\lambda\}$ of closed sets of finite measure satisfying $\operatorname{supp}\mu_{C_\lambda}=C_\lambda$.

Since the family $\{\phi\}$ satisfies these conditions, the collection $\mathcal{A}$ is non-void. By Zorn's lemma, $\mathcal{A}$ has a maximal family $\{B_\alpha\}$. We shall show the family $\{B_\alpha\}$ satisfies the conditions 1), 2), 3) and 4). Let O be any open subset of finite measure. If $B_\alpha\cap O$ is non-void, then we have

$$0<\mu_{B_\alpha}(B_\alpha\cap O)=\mu(B_\alpha\cap O)<\mu(O)<\infty\,,$$

for $\operatorname{supp}\mu_{B_\alpha}$ equals B_α. Consequently the cardinal of $\{\alpha\,;B_\alpha\cap O\neq\emptyset\}$ is at most countable. Hence $O\cap(X-\bigcup_\alpha B_\alpha)$ is a Borel set. Assume $\mu(O\cap(X-\bigcup_\alpha B_\alpha))$ is positive. Since μ is regular, there exists a closed subset F of X contained in $O\cap(X-\bigcup_\alpha B_\alpha)$ such that $\mu(F)$ is positive. Since μ_F is a pre-Radon measure on F by Theorem 5.3, the set $B=\operatorname{supp}\mu_F$ is closed in X and we have

$$\mu_F(B)=\mu(B)>0\,.$$

For every x in B and any open neighborhood V of x in B, there is an open subset $\tilde{V}$ of F such that $\tilde{V}\cap B=V$. Thus we have

$$\begin{aligned}\mu_B(V)=\mu_F(V)&=\mu_F(V\cup(B^c\cap F))\\&\geqq\mu_F(\tilde{V})>0\,,\end{aligned}$$

for $\operatorname{supp}\mu_F$ is equal to B. Hence we obtain

$$\operatorname{supp} \mu_B = B .$$

Consequently the family $\{B_\alpha\} \cup \{B\}$ belongs to $\mathcal{A}$, which contradicts to the maximality of $\{B_\alpha\}$. Therefore we have

$$\mu(O \cap (X - \bigcup_\alpha B_\alpha)) = 0 .$$

This shows that $X - \bigcup_\alpha B$ is locally negligible.

Let A be a Borel set in $\mathcal{B}(X)$ of finite measure. Then it follows that the cardinal of $\{\alpha ; A \cap B_\alpha \neq \emptyset\}$ is at most countable. In fact, there exists an open subset O containing A such that $\mu(O)$ is finite. Hence the set $A \cap (X - \bigcup_\alpha B_\alpha)$ belongs to $\mathcal{B}(X)$ and it holds that

$$\mu(A \cap (X - \bigcup_\alpha B_\alpha)) = 0 .$$

Thus we have

$$\mu(A) = \sum_\alpha (A \cap B_\alpha) .$$

This completes the proof.

For a set $\{a_\lambda ; \lambda \in \Lambda\}$ of non-negative numbers, we define the sum of $\{a_\lambda ; \lambda \in \Lambda\}$ by

$$\sum_{\lambda \in \Lambda} a_\lambda = \sup \{ \sum_{\lambda \in \Lambda_0} a_\lambda ; \ \Lambda_0 \text{ is a finite subset of } \Lambda\} .$$

In the semi-finite case the following corollary holds.

COROLLARY 6.2. *Let μ be a semi-finite pre-Radon measure on a topological space X. Then for the family $\{B_\alpha\}$ obtained in Theorem* 6.1 *we have*

4)′ *For every A in $\mathcal{B}(X)$;*

$$\mu(A) = \sum_\alpha \mu(A \cap B_\alpha) ;$$

5) $\mu_*(X - \bigcup_\alpha B_\alpha) = 0$,

where μ_* denotes the inner measure defined from μ.

Proof. Since μ is semi-finite, we have

$$\begin{aligned} \mu(A) &= \sup \{\mu(B) ; B \subset A \quad \text{and} \quad \mu(B) \text{ is finite}\} \\ &= \sup \{\sum_\alpha \mu(B \cap B_\alpha) ; B \subset A \quad \text{and} \quad \mu(B) \text{ is finite}\} \\ &\leqq \sum_\alpha \mu(A \cap B_\alpha) \leqq \mu(A) , \end{aligned}$$

which shows 4)′.

Let C be any Borel set in $\mathcal{B}(X)$ contained in $X - \bigcup_\alpha B_\alpha$. For any Borel set

B of finite measure contained in $X-\bigcup_{\alpha} B_{\alpha}$ we have $\mu(B)=0$ by Theorem 6.1. Thus we have

$$\begin{aligned}\mu(C)&=\sup\{\mu(B)\,;\ B\subset C \ \text{ and } \ \mu(B) \text{ is finite}\}\\ &\leqq\sup\{\mu(B)\,;\ B\subset X-\bigcup_{\alpha} B_{\alpha} \ \text{ and } \ \mu(B) \text{ is finite}\}\\ &=0\,.\end{aligned}$$

§ 7. (K)-property.

Let μ be a pre-Radon measure on a topological space X and f be a continuous mapping of X into another topological space Y. We denote by $f(\mu)$ the image measure of μ defined by

$$f(\mu)(A)=\mu(f^{-1}(A))$$

for every Borel set A in $\mathcal{B}(Y)$. In general, it is not true that the image measure $f(\mu)$ is a pre-Radon measure. But the following theorem holds.

THEOREM 7.1. *Let μ be a pre-Radon measure on a topological space X and f be a continuous mapping of X into a regular space Y. If we put*

$$\begin{aligned}Y_0=\{y\in Y\,;\ &\textit{there exists an open neighborhood } U \textit{ of } y\\ &\textit{such that } \mu(f^{-1}(U))<\infty\}\,,\end{aligned}$$

then there uniquely exists a pre-Radon measure ν on Y_0 such that

$$\nu(O)=\mu(f^{-1}(O))$$

for every open subset O of Y_0.

Proof. We remark that Y_0 is an open subset of Y. If we put

$$\mathcal{U}=\{U\subset Y_0\,;\ U \text{ is open in } Y_0 \text{ and } \mu(f^{-1}(U)) \text{ is finite}\}\,,$$

then $\mathcal{U}$ is an open base of Y_0. If we define a set function m on $R[\mathcal{U}]$ by

$$m(A)=\mu(f^{-1}(A))$$

for every A in $R[\mathcal{U}]$, then for any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to U in $\mathcal{U}$ it follows that

$$\begin{aligned}\lim_{\alpha} m(U_\alpha)&=\lim_{\alpha} \mu(f^{-1}(U_\alpha))\\ &=\mu(f^{-1}(U))=m(U)\,.\end{aligned}$$

Since Y is regular, we have

$$m(U)=\sup\,\{m(V)\,;\ V\subset \bar{V}\subset U \text{ and } V \text{ is open}\}$$

for every U in $\mathcal{U}$, where $\bar{V}$ is the closure of V in Y. By Theorem 3.1, there exists a pre-Radon measure ν on Y_0 extending m. For each open subset O of Y there exists a net $\{U_\alpha\}$ in $\mathcal{U}$ increasing to O. Thus we have

$$\begin{aligned}\nu(O)&=\sup_\alpha \nu(U_\alpha)\\&=\sup_\alpha m(U_\alpha)\\&=\sup_\alpha \mu(f^{-1}(U_\alpha))\\&=\mu(f^{-1}(O)).\end{aligned}$$

The uniqueness of ν is clear, which completes the proof.

COROLLARY 7.2. *In the above theorem, if μ is finite, then Y_0 equals Y and $f(\mu)$ is a pre-Radon measure. Therefore ν is identical with $f(\mu)$.*

If X is a Borel subset of Y, then the restriction of ν to $\mathcal{B}(X)$ is identical to μ on $\mathcal{B}(X)$.

LEMMA 7.3. *Let X be a Borel subset of a regular space Y, μ be a pre-Radon measure on X and ν be the pre-Radon measure obtained in Theorem 7.1. Then the restriction of ν to $\mathcal{B}(X)$ is equal to μ on $\mathcal{B}(X)$.*

Proof. Let A be any Borel set in $\mathcal{B}(X)$ and W be an open subset of X containing A. Then there exists an open subset $\widetilde{W}$ of Y_0 such that $W=\widetilde{W}\cap X$. Hence we have

$$\begin{aligned}\mu(W)=\nu(\widetilde{W})&\geqq\inf\{\nu(\tilde{O})\,;\ \tilde{O}\supset A \text{ and } \tilde{O} \text{ is open in } Y_0\}\\&=\nu(A)\end{aligned}$$

by Theorem 7.1. Since μ is regular, we have

$$\begin{aligned}\mu(A)&=\inf\{\nu(W)\,;\ W\supset A \text{ and } W \text{ is open in } X\}\\&\geqq\nu(A).\end{aligned}$$

Conversely, we have

$$\begin{aligned}\nu(A)&=\inf\{\nu(\tilde{O})\,;\ \tilde{O}\supset A \text{ and } \tilde{O} \text{ is open in } Y_0\}\\&=\inf\{\mu(\tilde{O}\cap X)\,;\ \tilde{O}\supset A \text{ and } \tilde{O} \text{ is open in } Y_0\}\\&\geqq\inf\{\mu(O)\,;\ O\supset A \text{ and } O \text{ is open in } X\}\\&=\mu(A).\end{aligned}$$

This proves the lemma.

We present a sufficient condition under which a topological space has the

(K)-property.

THEOREM 7.4. *Let X be a completely regular Hausdorff space such that X is a Borel subset of its Stone-Čech compactification βX. Then every pre-Radon measure μ on X is a Radon measure, that is, X has the* (K)*-property.*

Proof. Let ι be the natural embedding of X into βX, $(\beta X)_0$ be the open subset of βX obtained in Theorem 7.1 and ν be the pre-Radon measure on $(\beta X)_0$ in Theorem 7.1. Since $(\beta X)_0$ is locally compact, ν is a Radon measure. If we remark that the restriction of the Radon measure ν to the Borel set X is a Radon measure on X, μ is a Radon measure by Lemma 7.3. The proof is complete.

We recall that a completely regular Hausdorff space is *topologically complete* if it is a G_δ-subset of its Stone-Čech compactification.

COROLLARY 7.5. *Every topologically complete space has the* (K)*-property. Particularly, a complete metric space has the* (K)*-property.*

COROLLARY 7.6. *Every completely regular Haussdorff, σ-compact space has the* (K)*-property.*

§ 8. Smooth linear functional.

In this section, we show that there is a one-to-one correspondence between pre-Radon measures and smooth linear functionals.

Let $C(X)$ be the Riesz space of all real continuous functions on a completely regular Hausdorff space X. A Riesz subspace J of $C(X)$ is said to be *order-dense* if for every x in X, there exists f in J such that $f(x)\neq 0$. A positive linear functional Φ on a Riesz subspace J of $C(X)$ is called smooth if for every net $\{f_\alpha\}$ in J decreasing to 0,

$$\lim_\alpha \Phi(f_\alpha)=0\,.$$

THEOREM 8.1. *Let μ be a pre-Radon measure on a completely regular Hausdorff space X and J_μ be the Riesz subspace of all μ-integrable continuous functions. Then J_μ is an order-dense ideal and the functional Φ_μ on J_μ defined by*

$$\Phi_\mu(f)=\int_X f\,d\mu$$

is a smooth linear functional.

Proof. For every x in X, there exists an open neighborhood U of x such that $\mu(U)$ is finite. There exists f in $C(X)$ such that

1) $0\leqq f\leqq 1$;

2) $f(x)=1$;

3) $f=0$ on U^c,

where U^c means the complement of U. Since it holds

$$\Phi_\mu(f)=\int_X f d\mu \leqq \mu(U)<\infty\,,$$

f belongs to J_μ, which shows J_μ is an order-dense ideal.

Let $\{f_\alpha\}$ be a net of non-negative functions in J_μ increasing to f in J_μ. Since f is non-negative, we have

$$\Phi_\mu(f)=\int_X f\, d\mu=\int_{\underset{n}{\cup}\{x;\, f(x)>\frac{1}{n}\}} f\, d\mu$$

$$=\lim_{n\to\infty}\int_{\{x;\, f(x)>\frac{1}{n}\}} f\, d\mu\,.$$

For any positive ε, there exists n such that

$$\Phi_\mu(f)<\int_{\{x;\, f(x)>\frac{1}{n}\}} f\, d\mu+\varepsilon\,.$$

Since the sequence $\{f\wedge m\}$ converges to f, there exists m such that

$$\int_{\{x;\, f(x)>\frac{1}{n}\}} f\, d\mu<\int_{\{x;\, f(x)>\frac{1}{n}\}} f\wedge m\, d\mu+\varepsilon\,.$$

If we remark that $\mu(\{x\,;f(x)>1/n\})$ is finite and $f\wedge m$ is bounded, in the same manner as in the proof of Theorem 24 of Varadrajan [13, Part I] we obtain

$$\int_{\{x;\, f(x)>\frac{1}{n}\}} f\wedge m\, d\mu=\lim_\alpha\int_{\{x;\, f(x)>\frac{1}{n}\}} f_\alpha\wedge m\, d\mu$$

$$=\lim_\alpha\int_X f_\alpha d\mu\,.$$

Thus we have

$$\Phi_\mu(f)=\lim_\alpha \Phi_\mu(f_\alpha)\,.$$

Let $\{f_\alpha\}$ be a net in J_μ decreasing to 0. If we fix α_0, the net $\{f_{\alpha_0}-f_\alpha\}$ increases to f_{α_0}. Then we have

$$\lim_\alpha \Phi_\mu(f_{\alpha_0}-f_\alpha)=\Phi_\mu(f_{\alpha_0})\,.$$

Hence we have

$$\lim_\alpha \Phi_\mu(f_\alpha)=0\,.$$

This completes the proof.

The idea of the proof of the following theorem is essentially due to Fremlin [3], Hewitt and Ross [6] and Kirk [9].

PROPOSITION 8.2. *Let J be an order-dense ideal of $C(X)$ and Φ be a non-*

negative smooth linear functional on J. Then there exists a unique pre-Radon measure μ such that J is contained in J_μ and

$$\Phi(f)=\int_X f\,d\mu$$

for every f in J.

Proof. Let M^+ be the set of functions $\{\lim_\alpha f_\alpha;\ \{f_\alpha\}$ is an increasing net, each f_α belongs to $J^+\}$, where J^+ is the subspace of all non-negative functions in J. We define a functional $\bar{\Phi}$ on M^+, by

$$\bar{\Phi}(\lim_\alpha f_\alpha)=\lim_\alpha \Phi(f_\alpha).$$

This definition of $\bar{\Phi}$ is well-defined, in fact, if $\lim_\alpha f_\alpha=\lim_\lambda f'_\lambda$ in M^+ it follows that

$$\lim_\alpha \Phi(f_\alpha)=\lim_\lambda \Phi(f'_\lambda).$$

From the definition, we have

$$\bar{\Phi}(cg)=c\bar{\Phi}(g)$$

for every g in M^+ and any positive number c.

For every net $\{g_\alpha\}$ in M^+ increasing to g in M^+, we obtain

$$\lim_\alpha \bar{\Phi}(g_\alpha)=\bar{\Phi}(g)$$

in the same manner as in the proof of Theorem 11.13 of Hewitt and Ross [6]. Therefore we have

$$\bar{\Phi}(g_1+g_2)=\bar{\Phi}(g_1)+\bar{\Phi}(g_2)$$

for g_1, g_2 in M^+.

Since J is an order-dense ideal and X is completely regular, the characteristic function χ_o of an open subset O of X belongs to M^+. We put

$$m(O)=\bar{\Phi}(\chi_o)$$

for every open subset O. Then for any net $\{O_\alpha\}$ of open subsets increasing to an open subset O, we have

$$\lim_\alpha m(O_\alpha)=m(O).$$

Moreover for open subsets O_1 and O_2, we obtain

$$m(O_1\cup O_2)\leqq m(O_1)+m(O_2)$$

in the same manner as in the proof of Lemma 1.10 of Kirk [9].

If we set

$$\mu^*(A)=\inf\{m(O);\ O\supset A \text{ and } O \text{ is open}\}$$

for every subset A of X, then μ^* is an outer measure. Since X is a regular space, it holds

$$m(O)=\sup\{m(W);\ W\subset\overline{W}\subset O \text{ and } W \text{ is open}\}$$

for every open subset O. Hence it follows that every Borel subset is μ^*-measurable (see for example, Hewitt and Ross [6, Theorem (11.30)]). If we denote by μ the restriction of μ^* to the Borel field $\mathscr{B}(X)$, μ is a pre-Radon measure.

We show it holds that

$$\Phi(f)=\int_X f\,d\mu$$

for each f in J^+. In the same manner as in the proof (a) of Lemma 71 F of Fremlin [3], we have

$$\Phi(f)\geqq\int f\,d\mu$$

for every f in J^+.

In order to prove the converse, we slightly modify the proof (b) of Lemma 71 F of Fremlin [3]. For any f in J^+, we put

$$f_n=f\wedge 2^n-f\wedge 2^{-n},$$

then the sequence $\{f-f_n;\ n=1, 2, \cdots\}$ in J^+ decreases to 0. Therefore for any positive number ε, there exists n such that

$$\Phi(f)\leqq\Phi(f_n)+\varepsilon.$$

If we set

$$H=\{x;\ f(x)\geqq 2^{-n}\},$$

$\mu(H)$ is finite. There exists a positive number c such that

$$\mu(U)-\mu(H)<\varepsilon/2^n,$$

where $U=\{x;\ f(x)>2^{-n}-c\}$. By Varadarajan [13, Part I, Theorem 10], there exists g in J^+ such that

1) $0\leqq g\leqq 1$;
2) $g=1$ on H;
3) $g=0$ on U^c.

Then we have

$$0<\Phi(g)-\mu(H)<\varepsilon/2^n.$$

Thus we obtain

$$\begin{aligned}\Phi(f)&\leqq\Phi(f_n)+\varepsilon\\ &=\Phi(f_n)+\bar{\Phi}(2^n g)-\bar{\Phi}(2^n g)+\varepsilon\\ &=2^n\bar{\Phi}(g)-\bar{\Phi}(2^n g-f_n)+\varepsilon\end{aligned}$$

$$\leqq 2^n\Phi(g)-\int_X(2^n g-f_n)d\mu+\varepsilon$$

$$\leqq 2^n\mu(H)+\varepsilon-\int_X 2^n g d\mu+\int_X f_n d\mu+\varepsilon$$

$$\leqq 2^n\mu(H)-2^n\mu(H)+\int_X f_n d\mu+2\varepsilon$$

$$\leqq \int_X f d\mu+2\varepsilon\,.$$

Hence we have

$$\Phi(f)=\int_X f d\mu$$

for every f in J^+.

Lastly we prove the uniqueness. Let O be any open subset of X and $\{f_\alpha\}$ be any net in J^+ increasing to χ_o, we have

$$\mu(O)=\lim_\alpha \int_X f_\alpha d\mu\,.$$

In fact, for $0<\delta<1$, putting

$$U_\alpha=\{x\,;\ f_\alpha(x)>\delta\}\ ,$$

$\{U_\alpha\}$ increases to O. Since it follows

$$\lim_\alpha \mu(U_\alpha)=\mu(O)\,,$$

we have

$$\int_X f_\alpha d\mu \geqq \int_{U_\alpha} f_\alpha d\mu \geqq \delta\mu(U_\alpha)\,.$$

Thus we have

$$\frac{1}{\delta}\lim_\alpha \int_X f_\alpha d\mu \geqq \mu(O)\,.$$

Since δ is arbitrary, we have

$$\mu(O)\geqq \lim_\alpha \int_X f_\alpha d\mu \geqq \mu(O)\,.$$

Let ν be another pre-Radon measure satisfying

$$\Phi(f)=\int_X f\, d\mu=\int_X f\, d\nu$$

for every f in J^+. By the preceeding argument, we have

$$\mu(O)=\nu(O)$$

for each open subset O. Since μ and ν are regular measures, ν is identical with μ. The theorem is proved.

§ 9. Product measure.

The purpose of this section is to study the product of pre-Radon measures. The proof of the following lemma is easy.

LEMMA 9.1. *Let X be a topological space, $\mathcal{U}$ be a paving generated by an open base containing X and m be a non-negative, totally finite real valued finitely additive set function on $R[\mathcal{U}]$ such that*

1) *For any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to X,*

$$\lim_\alpha m(U_\alpha)=m(X);$$

2) *For every U in $\mathcal{U}$,*

$$m(U)=\sup\{m(F);\ U\supset F\in R[\mathcal{U}] \text{ and } F \text{ is closed}\}.$$

Then we have for any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to a set U in $\mathcal{U}$,

$$\lim_\alpha m(U_\alpha)=m(U).$$

To begin with, we investigate the finite product case.

THEOREM 9.2. *Let μ, ν be totally finite pre-Radon measures on topological spaces X, Y respectively. Then the product measure $\rho=\mu\otimes\nu$ on $(X\times Y, \mathcal{B}(X)\otimes\mathcal{B}(Y))$ is uniquely extensible to a pre-Radon measure on the product space $X\times Y$.*

Proof. Let $\mathcal{U}$ be the paving generated by

$$\{U\times V;\ U(\text{resp. } V) \text{ is open in } X(\text{resp. } Y)\}$$

and $\{W_\alpha\}$ be a net of subsets in $\mathcal{U}$ increasing to $X\times Y$. If $\{U_\gamma\times V_\gamma\}$ is the collection of open subsets of $X\times Y$ such that each $U_\gamma\times V_\gamma$ is contained in W_α for some α, we have $\bigcup_\gamma(U_\gamma\times V_\gamma)=X\times Y$. For every x in X, the family $\{V_\gamma;\ x$ belongs to $U_\gamma\}$ covers Y. Since ν is a pre-Radon measure, for any positive number ε, there exists $\{\gamma_1^x, \gamma_2^x, \cdots, \gamma_{n(x)}^x\}$ such that

$$\nu\Big(\bigcup_{i=1}^{n(x)} V_{\gamma_i^x}\Big)>\nu(Y)-\varepsilon.$$

We set $U(x)=\bigcap_{i=1}^{n(x)} U_{\gamma_i^x}$. Since μ is a pre-Radon measure, there exists $\{x_1, \cdots, x_k\}$ such that

$$\mu\Big(\bigcup_{j=1}^{k} U(x_j)\Big)>\mu(X)-\varepsilon.$$

From the definition of product measures, it follows

$$\rho(\bigcup_{j=1}^{k}\bigcup_{i=1}^{n(x_j)}(U(x_j)\times V_{\underset{i}{\gamma}x_j}))=\rho(\bigcup_{j=1}^{k}[(U(x_j)-\bigcup_{p=1}^{j-1}U(x_p))\times\bigcup_{i=1}^{n(x_j)}V_{\underset{i}{\gamma}x_j}])$$

$$=\sum_{j=1}^{k}\rho((U(x_j)-\bigcup_{p=1}^{j-1}U(x_p))\times\bigcup_{i=1}^{n(x_j)}V_{\underset{i}{\gamma}x_j})$$

$$>\sum_{j=1}^{k}\mu(U(x_j)-\bigcup_{p=1}^{j-1}U(x_p))(\nu(Y)-\varepsilon)$$

$$=\mu(\bigcup_{j=1}^{k}U(x_j))(\nu(Y)-\varepsilon)>(\mu(X)-\varepsilon)(\nu(Y)-\varepsilon)\,.$$

Since $\{W_\alpha\}$ is directed, there exists α_0 such that

$$W_{\alpha_0}\supset\bigcup_{j=1}^{k}\bigcup_{i=1}^{n(x_j)}U(x_j)\times V_{\underset{i}{\gamma}x_j}\,.$$

Therefore we have

$$\lim_{\alpha}\rho(W_\alpha)=\rho(X\times Y)\,.$$

By Lemma 9.1 and Theorem 3.2, the restriction ρ_0 of ρ to $R[\mathcal{U}]=A[\mathcal{U}]$ is uniquely extended to a pre-Radon measure $\bar{\rho}_0$ on $X\times Y$. By Halmos [5, § 13, Theorem A], ρ coincides with $\bar{\rho}_0$ on the product σ-algebra $\mathcal{B}(X)\otimes\mathcal{B}(Y)$. This completes the proof.

Now we argue σ-finite product measures on finite product spaces in the following theorem.

THEOREM 9.3. *Let μ, ν be σ-finite pre-Radon measures on topological spaces X, Y respectively. Then the product measure $\rho=\mu\otimes\nu$ on $(X\times Y, \mathcal{B}(X)\otimes\mathcal{B}(Y))$ is uniquely extensible to a pre-Radon measure on $X\times Y$.*

Proof. Let $\mathcal{U}$ be the paving generated by

$$\{U\times V\,;\ U(\text{resp. } V) \text{ is open in } X(\text{resp. } Y) \text{ and } \mu(U)<\infty,\ \nu(V)<\infty\}$$

and $\{W_\alpha\}$ be a net in $\mathcal{U}$ increasing to W in $\mathcal{U}$. If we write $W=\bigcup_{i=1}^{n}(U_i\times V_i)$, we have $W\subset U_0\times V_0$, where $U_0=\bigcup_{i=1}^{n}U_i$ and $V_0=\bigcup_{i=1}^{n}V_i$. By Theorem 9.2 $\mu_{U_0}\otimes\nu_{V_0}$ is extensible to a pre-Radon measure on $U_0\times V_0$. Thus we have

$$\lim_{\alpha}(\mu\otimes\nu)(W_\alpha)=\lim_{\alpha}(\mu_{U_0}\otimes\nu_{V_0})(W_\alpha)$$

$$=(\mu_{U_0}\otimes\nu_{V_0})(W)$$

$$=\mu\otimes\nu(W)\,.$$

By Theorem 3.2 the restriction ρ_0 of ρ to $A[\mathcal{V}]$ is uniquely extensible to a pre-Radon measure $\bar{\rho}_0$. Since ρ is σ-finite, $\bar{\rho}_0$ is an extension of ρ, which proves the theorem.

Next we investigate a Fubini type theorem.

LEMMA 9.4. *Let μ be a semi-finite pre-Radon measure on a topological space X and $M^+(X)$ be the set of all non-negative, extended real valued lower semi-continuous functions on X. If a net $\{f_\alpha\}$ in $M^+(X)$ increases to f in $M^+(X)$, then we have*

$$\lim_\alpha \int_X f_\alpha d\mu = \int_X f\, d\mu .$$

Proof. Firstly we prove the case that f is a simple function $\sum_{i=1}^n a_i \chi_{E_i}$, where $\{E_i\}$ is disjoint. Since μ_{E_i} is a pre-Radon measure (Theorem 5.3), by the way similar to the proof of Theorem 8.2 we have

$$\lim_\alpha \int_{E_i} f_\alpha |_{E_i} d\mu_{E_i} = a_i \mu(E_i) .$$

Thus we obtain

$$\begin{aligned}\lim_\alpha \int_X f_\alpha d\mu &= \lim_\alpha \sum_{i=1}^n \int_X f_\alpha \chi_{E_i} d\mu \\ &= \sum_{i=1}^n \lim_\alpha \int_{E_i} f_\alpha |_{E_i} d\mu_{E_i} \\ &= \sum_{i=1}^n a_i \mu(E_i) \\ &= \int_X f\, d\mu .\end{aligned}$$

Next we prove the general case. If we put

$$g_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \chi\left\{x ; \frac{k-1}{2^n} < f(x) \leqq \frac{k}{2^n}\right\} + n \chi_{\{x, f(x) > n\}} ,$$

then we have

$$\lim_n \int_X g_n d\mu = \int_X f\, d\mu .$$

If we remark that g_n is lower semi-continuous, we have

$$\begin{aligned}\int_X f\, d\mu &= \lim_n \int_X g_n d\mu = \lim_n \lim_\alpha \int_X f_\alpha \wedge g_n\, d\mu \\ &= \lim_\alpha \lim_n \int_X f_\alpha \wedge g_n d\mu\end{aligned}$$

$$=\lim_{\alpha}\int_X \lim_n f_\alpha\wedge g_n d\mu$$

$$=\lim_{\alpha}\int_X f_\alpha\wedge f\, d\mu$$

$$=\lim_{\alpha}\int_X f_\alpha d\mu\,.$$

The lemma is proved.

LEMMA 9.5. *Let μ, ν be σ-finite pre-Radon measures on topological spaces X, Y respectively and $\overline{\mu\otimes\nu}$ be the pre-Radon extension of $\mu\otimes\nu$. Then for every open subset W of $X\times Y$, we have*

$$\overline{\mu\otimes\nu}(W)=\mu_x\nu_y(\chi_W(x, y))\,.$$

Proof. Let $\mathcal{U}$ be the paving generated by

$$\{U\times V\,;\,U(\text{resp. } V) \text{ is open in } X(\text{resp. } Y)\}$$

and $\{W_\alpha\}$ be a net in $\mathcal{U}$ increasing to W. By the way similar to Bourbaki [2, § 2, n°6, Prop. 11] the function

$$x\longmapsto \nu_y(\chi_{W_\alpha}(x, y))$$

is lower semi-continuous on X. By Lemma 9.4 it follows that

$$\lim_{\alpha}\nu_y(\chi_{W_\alpha}(x, y))=\nu_y(\chi_W(x, y))\,,$$

which shows that the function

$$x\longmapsto \nu_y(\chi_W(x, y))$$

is lower semi-continuous. Hence we have

$$\overline{\mu\otimes\nu}(W)=\lim_{\alpha}\mu\otimes\nu(W_\alpha)$$

$$=\lim_{\alpha}\mu_x\nu_y(\chi_{W_\alpha}(x, y))$$

$$=\mu_x\nu_y(\chi_W(x, y))\,.$$

The proof is complete.

Under the above preparations, we present a Fubini type theorem.

THEOREM 9.6. *Let μ, ν be σ-finite pre-Radon measures on topological spaces X, Y respectively and $\overline{\mu\otimes\nu}$ be the pre-Radon extension of $\mu\otimes\nu$. Then for every Borel subset B of $X\times Y$, we have*

$$\overline{\mu\otimes\nu}(B)=\mu_x\nu_y(\chi_B(x, y))\,.$$

Proof. We recall that for every Borel set B, $\chi_B(x, y)$ is separately Borel measurable function on $X\times Y$. Let $\mathcal{M}_1$ be the class $\{E\subset\mathcal{B}(X\times Y)\,;\ \nu_y(\chi_E(x, y))$ is $\mathcal{B}(X)$-measurable$\}$. Since ν is σ-finite, $\mathcal{M}_1$ is a monotone class. Furthermore we can easily prove that $\mathcal{M}_1$ contains the algebra generated by all open subsets of $X\times Y$. Thus $\mathcal{M}_1$ equals $\mathcal{B}(X\times Y)$.

Let $\mathcal{M}_2$ be the class

$$\{E\subset\mathcal{B}(X\times Y)\,;\ \overline{\mu\otimes\nu}(E)=\mu_x\nu_y(\chi_E(x, y))\}\ .$$

Since μ and ν are σ-finite, $\mathcal{M}_2$ is a monotone class. By Lemma 9.5, $\mathcal{M}_2$ includes the algebra generated by all open subsets of $X\times Y$. Thus $\mathcal{M}_2$ is equal to $\mathcal{B}(X\times Y)$. This completes the proof.

COROLLARY 9.7. *Let f be a non-negative, extended real valued Borel measurable function on $X\times Y$, then we have*

1) $x\longmapsto\int_Y f(x, y)d\nu(y)$ is $\mathcal{B}(X)$-measurable;

2) $y\longmapsto\int_Y f(x, y)d\mu(x)$ is $\mathcal{B}(Y)$-measurable;

3) $$\int_X d\mu(x)\int_Y f(x, y)d\nu(y)=\int_Y d\nu(y)\int_X f(x, y)d\mu(x)$$
$$=\int_{X\times Y} f(x, y)\overline{d\mu\otimes\nu}\ .$$

Next we consider the countable product of pre-Radon measures.

THEOREM 9.8. *Let μ_n be a pre-Radon probability measure on a regular space X_n $(n=1, 2, \cdots)$. Then the product measure $\mu=\bigotimes_{n=1}^{\infty}\mu_n$ on $(\prod_{n=1}^{\infty}X_n, \bigotimes_{n=1}^{\infty}\mathcal{B}(X_n))$ is uniquely extensible to a pre-Radon measure on the product space $\prod_{n=1}^{\infty}X_n$.*

Proof. Let $\mathcal{U}$ be the paving generated by

$$\mathcal{U}_0=\{\prod_{n=1}^{\infty}U_n\,;\ U_n \text{ is open in } X_n,\ U_n=X_n \text{ except finitely many } n\}$$

and $\{W_\alpha\}$ be a net in $\mathcal{U}$ increasing to $\prod_{n=1}^{\infty}X_n$. If $\{\prod_{n=1}^{\infty}U_n^\gamma\}$ is the collection of open subsets in $\mathcal{U}_0$ such that each $\prod_{n=1}^{\infty}U_n^\gamma$ is contained in W_α for some α, we have $\bigcup_\gamma\prod_{n=1}^{\infty}U_n^\gamma=\prod_{n=1}^{\infty}X_n$. We put

$$\Gamma(k)=\{\gamma\,;\ U_n^\gamma=X_n \text{ for all } n>k\}$$

and put

$$U(k)=\bigcup_{\gamma\in\Gamma(k)}\prod_{n=1}^{\infty}U_n^\gamma\ .$$

We define a finitely additive set function ν on the algebra $\mathcal{A}=\bigcup_{n=1}^{\infty}(\mathcal{B}(\prod_{i=1}^{n}X_i)$

$\times \prod_{p>n} X_p)$ as follows:

$$\nu(B_n \times \prod_{p>n} X_p) = \overline{\bigotimes_{i=1}^{n}} \mu_i(B_n)$$

for every B_n in $\mathscr{B}(\prod_{i=1}^{n} X_i)$, where $\overline{\bigotimes_{i=1}^{n} \mu_i}$ denotes the pre-Rodon extension of $\bigotimes_{i=1}^{n} \mu_i$. From Corollary 7.2 ν is well-defined. We shall show ν is countably additive by the way similar to Halmos [5, § 38, Theorem B]. Let $\{E_k\}$ be a decreasing sequence in $\mathscr{A}$. Suppose there exists $\varepsilon>0$ such that $\nu(E_k) \geqq \varepsilon$ for every k. We put for every $E=B_{N(E)} \times \prod_{P>N(E)} X_p$ in $\mathscr{A}$

$$E(x_1, \cdots, x_n) = \{(x_{n+1}, x_{n+2}, \cdots);\ (x_1, \cdots, x_n, x_{n+1}, \cdots) \in E\},$$

and

$$\mu^{(n)}(E(x_1, \cdots, x_n)) = \overline{\bigotimes_{i=n+1}^{N(E)}} \mu_i(B_{N(E)}(x_1, \cdots, x_n)).$$

If we put

$$F_k = \left\{x_1 \in X_1;\ \mu^{(1)}(E_k(x_1)) \geqq \frac{\varepsilon}{2}\right\}.$$

then it follows by Theorem 9.6

$$\begin{aligned}
\nu(E_k) &= \overline{\bigotimes_{i=1}^{N(E_k)}} \mu_i(E_k) \\
&= \overline{\mu_1 \otimes (\overline{\bigotimes_{i=2}^{N(E_k)}} \mu_i)}(E_k) \\
&= \int_{X_1} \mu^{(1)}(E_k(x_1)) d\mu_1(x_1) \\
&= \int_{F_k} \mu^{(1)}(E_k(x_1)) d\mu_1(x_1) + \int_{F_k^c} \mu^{(1)}(E_k(x_1)) d\mu_1(x_1) \\
&\leqq \mu_1(F_k) + \frac{\varepsilon}{2}.
\end{aligned}$$

Thus we have

$$\mu_1(F_k) \geqq \frac{\varepsilon}{2},$$

which implies there exists $\bar{x}_1$ in X_1 such that

$$\mu^{(1)}(E_k(\bar{x}_1)) \geqq \frac{\varepsilon}{2}$$

for every k. Similarly there exists $(\bar{x}_n)$ in $\prod_{n=1}^{\infty} X_n$ such that

$$\mu^{(n)}(E_k(\bar{x}_1, \cdots, \bar{x}_n)) \geqq \frac{\varepsilon}{2^n}$$

for every k. If we remark that $(\bar{x}_n)$ is in $\bigcap_{k=1}^{\infty} E_k$, ν is countably additive.

Since ν is countably additive, for arbitrary $\varepsilon>0$ there exists k_0 such that

$$1-\varepsilon<\nu(U(k_0))=\overline{\bigotimes_{i=1}^{k_0}} \mu_i(\bigcup_{\gamma\in\Gamma(k_0)} \prod_{i=1}^{k_0} U_i^{\gamma}).$$

By Theorem 9.2 there exists $\{\gamma_1, \cdots, \gamma_p\}$ in $\Gamma(k_0)$ such that

$$\bigotimes_{i=1}^{k_0} \mu_i(\bigcup_{q=1}^{p} \prod_{i=1}^{k_0} U_i^{\gamma_q})>1-\varepsilon.$$

Since $\{W_\alpha\}$ is directed, there exists α_0 such that W_{α_0} contains $\bigcup_{q=1}^{p} \prod_{i=1}^{k_0} U_i^{\gamma_q} \times \prod_{m>k_0} X_m$,

$$\lim_{\alpha} \mu(W_\alpha)=1.$$

By Lemma 9.1 and Theorem 3.2, the restriction μ_0 of μ to $R[\mathcal{U}]=A[\mathcal{U}]$ is uniquely extended to a pre-Radon measure $\bar{\mu}$ on $\prod_{n=1}^{\infty} X_n$. By Halmos [5, § 13, Theorem A], μ is identical to $\bar{\mu}$ on the product σ-algebra $\bigotimes_{n=1}^{\infty} \mathcal{B}(X_n)$. Thus the theorem is proved.

Now we discuss the uncountable product of pre-Radon measures.

THEOREM 9.9. *Let μ_λ be a pre-Radon probability measure on a regular space X_λ $(\lambda\in\Lambda)$. Then the product measure $\mu=\bigotimes_{\lambda\in\Lambda} \xi_\lambda$ on $(\prod_{\lambda\in\Lambda} X_\lambda, \bigotimes_{\lambda\in\Lambda} \mathcal{B}(X_\lambda))$ is extended to a unique pre-Radon measure on $\prod_{\lambda\in\Lambda} X_\lambda$,*

Proof. Let $\mathcal{U}$ be the paving generated by

$$\{\prod_{\lambda\in\Lambda} U_\lambda;\ U_\lambda \text{ is open in } X_\lambda,\ U_\lambda=X_\lambda \text{ except finitely many } \lambda\}$$

and $\{W_\alpha\}$ be a net in $\mathcal{U}$ increasing to $\prod_{\lambda\in\Lambda} X_\lambda$. We put $c=\sup_\alpha \mu(W_\alpha)$. Then there exists $\{\alpha_n\}$ such that $c=\lim_n \mu(W_{\alpha_n})=\mu(\bigcup_{n=1}^{\infty} W_{\alpha_n})$. For simplicity we set $W=\bigcup_{n=1}^{\infty} W_{\alpha_n}$. We can write $W_\alpha=\bigcup_{n=1}^{N(\alpha)} \prod_{\lambda\in\Lambda} U_\lambda^{\alpha,i}$. If we set

$$\Lambda_0=\{\lambda\in\Lambda\,;\ U_n^{\alpha_n,i}=X_\lambda \text{ for every } n \text{ and } i=1, 2, \cdots, N(\alpha_n)\},$$

then $\Lambda_1=\Lambda-\Lambda_0$ is a countable set. Since W equals $q_1(W)\times \prod_{\lambda\in\Lambda_0} X_\lambda$, we have $\mu(W)=(\bigotimes_{\lambda\in\Lambda_1} \mu_\lambda)(q_1(W))$, where q_1 is the projection of $\prod_{\lambda\in\Lambda} X_\lambda$ onto $\prod_{\lambda\in\Lambda_1} X_\lambda$.

In the first step, we assume supp μ_λ is equal to X_λ for every λ in Λ. Suppose $c<1$. Then there exists α_0 such that $(\bigotimes_{\lambda\in\Lambda_1}\mu_\lambda)(q_1(W_{\alpha_0})-q_1(W))>0$, for by Theorem 9.8 it holds $\sup(\bigotimes_{\lambda\in\Lambda_1}\mu_\lambda)(q_1(W_\alpha))=1$. Therefore, for some i, $1\leqq i\leqq N(\alpha_0)$ it follows $(\bigotimes_{\lambda\in\Lambda_1}\mu_\lambda)(\prod_{\lambda\in\Lambda_1}U_\lambda^{\alpha_0,i}-q_1(W))>0$. If we remark that $(\bigotimes_{\lambda\in\Lambda_0}\mu_\lambda)(\prod_{\lambda\in\Lambda_0}U_\lambda^{\alpha_0,i})>0$, we have

$$\begin{aligned}c&=\mu(W\cup W_{\alpha_0})\geqq\mu(W\cup\prod_{\lambda\in\Lambda}U_\lambda^{\alpha_0,i})\\&=\mu(W)+\mu(\prod_{\lambda\in\Lambda}U_\lambda^{\alpha_0,i}-W)\\&=\mu(W)+\mu((\prod_{\lambda\in\Lambda_1}U_\lambda^{\alpha_0,i}-q_1(W))\times\prod_{\lambda\in\Lambda_0}U_\lambda^{\alpha_0,i})\\&=\mu(W)+(\bigotimes_{\lambda\in\Lambda_1}\mu_\lambda)(\prod_{\lambda\in\Lambda_1}U_\lambda^{\alpha_0,i}-q_1(W))(\bigotimes_{\lambda\in\Lambda_0}\mu_\lambda)(\prod_{\lambda\in\Lambda_0}U_\lambda^{\alpha_0,i})\\&>c\,,\end{aligned}$$

which is a contradiction. Hence we obtain $\mu(W)=1$. If follows that

$$\lim_\alpha\mu(W_\alpha)=1=\mu(\prod_{\lambda\in\Lambda}X_\lambda)\,.$$

We shall prove the general case. By ν_λ we denote the restriction of μ_λ to $Y_\lambda=\operatorname{supp}\mu_\lambda$. Since the net $\{W_\alpha\cap\prod_{\lambda\in\Lambda}Y_\lambda\}$ increases to $\prod_{\lambda\in\Lambda}Y_\lambda$, from the first step we have

$$\begin{aligned}\lim_\alpha(\bigotimes_{\lambda\in\Lambda}\mu_\lambda)(W_\alpha)&=\lim_\alpha(\bigotimes_{\lambda\in\Lambda}\nu_\lambda)(W_\alpha\cap\prod_{\lambda\in\Lambda}Y_\lambda)\\&=(\bigotimes_{\lambda\in\Lambda}\nu_\lambda)(\prod_{\lambda\in\Lambda}Y_\lambda)\\&=(\bigotimes_{\lambda\in\Lambda}\mu_\lambda)(\prod_{\lambda\in\Lambda}X_\lambda)\,.\end{aligned}$$

By Lemma 8.1 and Theorem 3.2, the restriction μ_0 of μ to $R[\mathcal{U}]=A[\mathcal{U}]$ is uniquely extensible to a pre-Radon measure $\bar{\mu}$ on $\prod_{\lambda\in\Lambda}X_\lambda$. By Halmos [5, §13, Theorem A], μ is identical with $\bar{\mu}$ on the product σ-algebra $\bigotimes_{\lambda\in\Lambda}\mathcal{B}(X_\lambda)$. This proves Theorem 9.9.

We denote by $\overline{\bigotimes_{\lambda\in\Lambda}\mu_\lambda}$ the pre-Radon extension of $\bigotimes_{\lambda\in\Lambda}\mu_\lambda$ obtained in Theorem 9.9.

Lastly we consider the product measure of Radon measures. The following lemma is well-known (for example see Bourbaki [2, §4, Théorème 2]).

LEMMA 9.10. *Let μ_n be a Radon probability measure on a regular space X_n $(n=1, 2, \cdots)$. Tnen the pre-Radon extension $\overline{\bigotimes_{n=1}^{\infty}\mu_n}$ of $\bigotimes_{n=1}^{\infty}\mu_n$ is a Radon measure on $\prod_{n=1}^{\infty}X_n$.*

THEOREM 9.11. *Let μ_λ be a Radon probability measure on a regular space X_λ ($\lambda \in \Lambda$). Then the pre-Radon extension $\overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}$ is a Radon measure if and only if* supp μ_λ *is a compact subset of X_λ except countably many λ in Λ.*

Proof. Let $\overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}$ be a Radon measure. Suppose that there exists some uncountable subset Λ_0 of Λ such that supp μ_λ is not compact for every λ in Λ_0. Without loss of generality we may assume Λ_0 is equal to Λ. For each compact subset K of $\prod_{\lambda \in \Lambda} X_\lambda$, we shall show $\overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}(K)=0$. Putting

$$\Lambda_n = \left\{\lambda \in \Lambda \,;\ \mu_\lambda(p_\lambda(K)) < 1 - \frac{1}{n+1}\right\},$$

we have $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$, where p_λ is the projection of $\prod_{\lambda \in \Lambda} X_\lambda$ onto X_λ. Since Λ is uncountable, there exists n such that Λ_n is infinite. Hence Λ_n has an infinitely countable subset $\{\lambda_i\,;\ i=1, 2, \cdots\}$. Thus we have

$$\begin{aligned}
\overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}(K) &\leqq \overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}\Big(\prod_{\lambda \in \Lambda} p_\lambda(K)\Big) \\
&\leqq \overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}\Big(\prod_{i=1}^{\infty} p_{\lambda_i}(K) \times \prod_{\lambda \neq \lambda_i} X_\lambda\Big) \\
&= \lim_k \Big(\bigotimes_{\lambda \in \Lambda} \mu_\lambda\Big)\Big(\prod_{i=1}^{k} p_{\lambda_i}(K) \times \prod_{\lambda \neq \lambda_i} X_\lambda\Big) \\
&= \lim_k \mu_{\lambda_1}(p_{\lambda_1}(K)) \cdots \mu_{\lambda_k}(p_{\lambda_k}(K)) \\
&\leqq \lim_k \left(1 - \frac{1}{n+1}\right)^k = 0\,.
\end{aligned}$$

Therefore we have

$$\overline{\bigotimes_{\lambda \in \Lambda} \mu_\lambda}\Big(\prod_{\lambda \in \Lambda} X_\lambda\Big) = 0\,.$$

This is a contradiction.

The converse follows by Lemma 9.10. Thus we have proved Theorem 9.11.

Appendix

We shall examine the relation between inner *regularity* (*) and *outer regularity* (**):

(*) For every A in $\mathcal{B}(X)$ such that $\mu(A) < \infty$,

$$\mu(A) = \text{supp}\,\{\mu(F)\,;\ F \subset A \text{ and } F \text{ is closed}\}\,;$$

(**) For every A in $\mathcal{B}(X)$,

$$\mu(A)=\inf\{\mu(O)\,;\,O\supset A \text{ and } O \text{ is open}\}\,.$$

THEOREM A. *Let μ be a Borel measure on a topological space X such that $X=\bigcup_{n=1}^{\infty}U_n$ and $\mu(U_n)$ is finite for countable open subsets $\{U_n\}$. Then (*) implies (**).*

Proof. Let A be a Borel subset. Then for every positive ε, there exists a closed subset F_n contained in $A^c\cap U_n$ such that $\mu(A^c\cap U_n-F_n)<\varepsilon/2^n$. If we put $V_n=U_n\cap F_n^c$, then we have

$$\begin{aligned}\mu\Big(\bigcup_{n=1}^{\infty}V_n-A\Big)&\leqq\mu\Big(\bigcup_{n=1}^{\infty}V_n-\bigcup_{n=1}^{\infty}(U_n\cap A)\Big)\\&\leqq\sum_{n=1}^{\infty}\mu(V_n-U_n\cap A)\\&<\varepsilon\,.\end{aligned}$$

This completes the proof.

COROLLARY. *Let be a σ-compact locally compact space and μ be a Borel measure on X such that $\mu(K)$ is finite for every compact set K. Then (*) implies (**).*

THEOREM B. *Let μ be a Borel measure on a topological space X. If μ is σ-finite and satisfies (**), then (*) holds.*

Proof. Let E_n be a Borel subset of finite measure such that $X=\bigcup_{n=1}^{\infty}E_n$. For every A in $\mathscr{B}(X)$ and every positive ε, there exists and open subset U_n containing $A^c\cap E_n$ such that $\mu(U_n-A^c\cap E_n)<\varepsilon/2^n$. Then we have

$$\begin{aligned}\mu(A)-\mu\Big(\bigcap_{n=1}^{\infty}U_n^c\Big)&=\mu\Big(A\cap\Big(\bigcup_{n=1}^{\infty}U_n\Big)\Big)\\&=\mu\Big(\bigcup_{n=1}^{\infty}U_n-\bigcup_{n=1}^{\infty}(A^c\cap E_n)\Big)\\&\leqq\sum_{n=1}^{\infty}\mu(U_n-A^c\cap E_n)\\&<\varepsilon\,,\end{aligned}$$

which proves the theorem.

Lastly according to the comments of Fremlin we note the relation between pre-Radon measures and quasi-Radon measures.

1) Let μ be a pre-Radon measure on a topological space X. Then there exists a unique quasi-Radon measure ν on (X,Σ_{μ^*}) such that for any B in Σ_{μ^*} with $\mu^*(B)<\infty$, $\nu(B)=\mu^*(B)$, where μ^* is the outer measure derived from μ and Σ_{μ^*} is the family of all μ^*-measurable sets. Moreover it holds that $\nu(O)=\mu(O)$

for every open set O.

2) Let ν be a locally finite quasi-Radon measure on (X, Σ). Then there uniquely exists a pre-Radon measure μ on X such that $\mu(O)=\nu(O)$ for every open set O.

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