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INFINITESIMAL VARIATIONS OF INVARIANT SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

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§ 0. Introduction.

Recently infinitesimal variations of submanifolds have been studied by Chen [1], Goldstein [2], Ryan [2], Tachibana [3, 4] and one of the present authors [1, 4].

The purpose of the present paper is to study infinitesimal variations of invariant submanifolds of a Kaehlerian manifold and to generalize some of recent results of Tachibana and one of the present authors.

In the preliminary § 1, we state some properties of invariant submanifolds of a Kaehlerian manifold.

In § 2 we prove fundamental formulas in the theory of infinitesimal variations and study complex variations, that is, infinitesimal variations which carry an invariant submanifold into an invariant submanifolds. In § 3, we study holomorphic variations, that is, complex variations which preserve complex structures induced on invariant submanifolds.

In § 4, we study complex conformal variations and prove that a complex conformal variation of a compact invariant submanifold of a Kaehlerian manifold is necessarily isometric and hence holomorphic, (Theorem 4.1). In the last § 5 we prove an integral formula and show some of its applications.

§ 1. Invariant submanifolds of a Kaehlerian manifold.

Let M^{2m} be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $F_i{}^h$ the almost complex structure tensor and g_{ji} the Hermitian metric tensor, where here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$.

Then we have

$$F_i{}^t F_t{}^h = -\delta_i^h, \qquad F_j{}^t F_i{}^s g_{ts} = g_{ji}, \tag{1.1}$$

$$\nabla_j F_i{}^h = 0, \tag{1.2}$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols $\Gamma_j{}^h{}_i$ formed with g_{ji}.

Let M^n be an n-dimensional Riemannian manifold covered by a system of

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coordinate neighborhoods $\{V; y^a\}$ and with the metric tensor g_{cb} where here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We assume that M^n is isometrically immersed in M^{2m} by the immersion $i: M^n \to M^{2m}$ and identify $i(M^n)$ with M^n itself. We represent the immersion i locally by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h$, $\partial_b = \partial/\partial y^b$, which are n linearly independent vectors of M^{2m} tangent to M^n. Since the immersion i is isometric, we have

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i . \tag{1.3}$$

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel, the indices $x, y, z, \cdots$ run over the range $\{n+1, n+2, \cdots, 2m\}$. Then the equation of Gauss are written as

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h , \tag{1.4}$$

where ∇_c denotes the operator of van der Waerden-Bortolotti covariant differentiation along M^n with respect the Christoffel symbols $\Gamma_j{}^h{}_i$ formed with g_{ji} and those $\Gamma_c{}^a{}_b$ formed with g_{cb} and $h_{cb}{}^x$ the second fundamental tensors of M^n with respect to the normals $C_x{}^h$, and those of Weingarten as

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h , \tag{1.5}$$

where

$$h_c{}^a{}_x = h_{cbx} g^{ba} = h_{cb}{}^z g^{ba} g_{zx} , \qquad (g^{ba}) = (g_{ba})^{-1} ,$$

and g_{zx} denotes the metric tensor of the normal bundle.

If the transform by F of any vector tangent to M^n is always tangent to M^n, that is, if there exists a tensor field $f_b{}^a$ of type (1, 1) of M^n such that

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h , \tag{1.6}$$

we say that M^n is *invariant* (or *complex*) in M^{2m}. (1.6) shows that $F_{ih} B_b{}^i C_x{}^h = 0$, where $F_{ih} = F_i{}^t g_{th}$.

For the transforms by F of normals $C_y{}^h$, we then have equations of the form

$$F_i{}^h C_y{}^i = f_y{}^x C_x{}^h . \tag{1.7}$$

If we put $f_{yx} = f_y{}^z g_{zx}$, then we have $f_{yx} = -f_{xy}$.

From (1.1), (1.3), (1.6) and (1.7), we easily see that

$$f_b{}^e f_e{}^a = -\delta_b^a , \qquad f_c{}^e f_b{}^d g_{ed} = g_{cb} , \tag{1.8}$$

$$f_y{}^z f_z{}^x = -\delta_y^x . \tag{1.9}$$

Differentiating (1.6) and (1.7) covariantly along M^n and using (1.2), (1.4) and (1.5), we find

$$\nabla_c f_b{}^a = 0 , \tag{1.10}$$

$$\nabla_c f_y{}^x = 0\,, \tag{1.11}$$

$$h_{cb}{}^y f_y{}^x = h_{ce}{}^x f_b{}^e\,. \tag{1.12}$$

Thus, equations (1.8) and (1.10) show that M^n is also Kaehlerian. Moreover it follows from (1.12) that

$$h_e{}^e{}_y = 0\,, \tag{1.13}$$

that is, M^n is minimal.

Using (1.8), (1.9) and (1.12) we easily verify that

$$h_{cb}{}^x = -h_{ed}{}^x f_c{}^e f_b{}^d\,. \tag{1.14}$$

Equations of Gauss and Codazzi of the submanifold M^n are respectively given by

$$K_{dcb}{}^a = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i B^a{}_h + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x\,, \tag{1.15}$$

$$K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x) = 0\,, \tag{1.16}$$

where $K_{dcb}{}^a$ is the curvature tensor of M^n.

Finally, we prepare an useful indentity on a Kaehlerian manifold M^n for later use (See [6]);

$$\frac{1}{2} f^{ce} f_b{}^d K_{ceda} = K_{ab}\,. \tag{1.17}$$

§2. Infinitesimal variations of invariant submanifolds.

We consider an infinitesimal variation of invariant submanifold M^n of a Kaehlerian manifold M^{2m} given by

$$\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon\,, \tag{2.1}$$

where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

$$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon\,, \tag{2.2}$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the origin point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_j{}^h{}_i(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon$$

at the point (x^h), or

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon\,, \tag{2.3}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_j{}^h{}_i B_b{}^j \xi^i . \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h , \tag{2.5}$$

we have from (2.3)

$$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon . \tag{2.6}$$

Putting

$$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h , \tag{2.7}$$

we have

$$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h \tag{2.8}$$

because of (1.4) and (1.5).

Now we denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_j{}^h{}_i (x + \xi\varepsilon) \xi^j \bar{C}_y{}^i \varepsilon . \tag{2.9}$$

We put

$$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h \tag{2.10}$$

and assume that $\delta C_y{}^h$ is of the form

$$\delta C_y{}^h = \eta_y{}^h \varepsilon = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon . \tag{2.11}$$

Then, from (2.9), (2.10) and (2.11), we have

$$\bar{C}_y{}^h = C_y{}^h - \Gamma_j{}^h{}_i \xi^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon . \tag{2.12}$$

Applying the operator δ to $B_b{}^j C_y{}^i g_{ji} = 0$ and using (2.6), (2.8), (2.11) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi_y + h_{bay} \xi^a) + \eta_{yb} = 0 ,$$

where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, or

$$\eta_y{}^a = -(\nabla^a \xi_y + h_b{}^a{}_y \xi^b) , \tag{2.13}$$

∇^a being defined to be $\nabla^a = g^{ac} \nabla_c$. Applying also the operator δ to $C_y{}^j C_x{}^i g_{ji} = g_{yx}$, and using (2.11) and $\delta g_{ji} = 0$, we find

$$\eta_{yx} + \eta_{xy} = 0 , \tag{2.14}$$

where $\eta_{yx} = \eta_y{}^z g_{zx}$.

We assume that the infinitesimal variation (2.1) carries an invariant submanifold into an invariant submanifold, that is,

(2.15) $$F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i \text{ are linear combinations of } \bar{B}_b{}^h .$$

Then, using $\nabla_j F_i{}^h=0$ and (1.6), we see that

$$\begin{aligned} F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i &= (F_i{}^h+\xi^j\partial_j F_i{}^h\varepsilon)(B_b{}^i+\partial_b\xi^i\varepsilon) \\ &= [F_i{}^h-\xi^j(\Gamma_j{}^h{}_t F_i{}^t-\Gamma_j{}^t{}_i F_t{}^h)\varepsilon](B_b{}^i+\partial_b\xi^i\varepsilon) \\ &= F_i{}^h B_b{}^i+(F_i{}^h\nabla_b\xi^i-f_b{}^a\Gamma_j{}^h{}_i B_a{}^j\xi^i)\varepsilon , \end{aligned}$$

that is, by (2.2)

(2.16) $$F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i=f_b{}^a\bar{B}_a{}^h+[F_i{}^h\nabla_b\xi^i-f_b{}^a\nabla_a\xi^h]\varepsilon ,$$

or, using (2.8),

(2.17) $$\begin{aligned} F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i=f_b{}^a\bar{B}_a{}^h &+f_e{}^a(\nabla_b\xi^e+h_b{}^e{}_x\xi^x)\bar{B}_a{}^h\varepsilon \\ &+(\nabla_b\xi^y+h_{ba}{}^y\xi^a)f_y{}^x\bar{C}_x{}^h\varepsilon \\ &-f_b{}^a(\nabla_a\xi^e-h_a{}^e{}_x\xi^x)\bar{B}_e{}^h\varepsilon \\ &-f_b{}^e(\nabla_e\xi^x+h_{ec}{}^x\xi^c)\bar{C}_x{}^h\varepsilon . \end{aligned}$$

Thus (2.15) is equivalent to

(2.18) $$(\nabla_b\xi^y+h_{bc}{}^y\xi^c)f_y{}^x=f_b{}^e(\nabla_e\xi^x+h_{ce}{}^x\xi^c) ,$$

or, by (1.12), to

(2.19) $$(\nabla_b\xi^y)f_y{}^x=f_b{}^e(\nabla_e\xi^x) .$$

An infinitesimal variation given by (2.1) is called an *complex variation* if it carries an invariant submanifold into an invariant submanifold. Thus we have

THEOREM 2.1. *In order for an infinitesimal variation to be complex, it is necessary and sufficient that the variation vector* ξ^h *satisfies* (2.19).

COROLLARY 2.2. *If a vector field* ξ^h *defines a complex variation, then another vector field* ξ'^h *which has the same normal part as* ξ^h *has the same property.*

Suppose that an infinitesimal variation given by (2.1) carries a submanifold $x^h=x^h(y)$ into another submanifold $\bar{x}^h=\bar{x}^h(y)$ and the tangent space of the original submanifold at (x^h) and that of the varied submanifold at the corresponding point $(\bar{x}^h)$ are parallel. Then we say that the variation is *parallel.*

Since we have from (2.5), (2.6) and (2.8),

(2.20) $$\tilde{B}_b{}^h=[\delta_b{}^a+(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)\varepsilon]B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h\varepsilon ,$$

we have the following proposition [5]:

In order for an infinitesimal variation to be parallel, it is necessary and sufficent that

$$\nabla_b \xi^x + h_{ba}{}^x \xi^a = 0 . \tag{2.21}$$

If (2.21) is satisfied, then (2.19) is satisfied. Thus we have

THEOREM 2.3. *A parallel variation is a complex variation.*

§ 3. Holomorphic variations.

Suppose that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ carries an invariant submanifold into an invariant submanifold, that is, it is a complex variation. Then putting

$$F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i - (f_b{}^a + \delta f_b{}^a)\bar{B}_a{}^h , \tag{3.1}$$

we have from (2.17) and (2.18)

$$\delta f_b{}^a = [(\nabla_b \xi^e - h_b{}^e{}_x \xi^x) f_e{}^a - f_b{}^e (\nabla_e \xi^a - h_e{}^a{}_x \xi^x)]\varepsilon . \tag{3.2}$$

From this fact we conclude

PROPOSITION 3.1. *Suppose that an infinitesimal variation is complex. Then the variation of $f_b{}^a$ is given by* (3.2).

We define T_{cb} by

$$T_{cb} = \nabla_c \xi_b - f_c{}^e f_b{}^d \nabla_e \xi_d - 2h_{cbx} \xi^x . \tag{3.3}$$

Equations (3.2) and (3.3) imply that $\delta f_b{}^a = 0$ is equivalent to $T_{cb} = 0$ because of (1.8) and (1.14).

If a complex variation preserves $f_b{}^a$, then we say that it is *holomorphic* [3].

According to (3.2), (3.3) and remark above, we have

PROPOSITION 3.2. *A complex variation is holomorphic if and only if $\nabla_b \xi^a - h_b{}^a{}_x \xi^x$ commutes with $f_b{}^a$, that is,*

$$(\nabla_b \xi^e - h_b{}^e{}_x \xi^x) f_e{}^a - f_b{}^e (\nabla_e \xi^a - h_e{}^a{}_x \xi^x) = 0 ,$$

or, equivalently $T_{cb} = 0$.

Now, applying the operator δ to (1.3) and using (2.6), (2.8) and $\delta g_{ji} = 0$, we find (cf. [5])

$$\delta g_{cb} = (\nabla_b \xi_c + \nabla_c \xi_b - 2h_{cbx} \xi^x)\varepsilon , \tag{3.4}$$

from which,

$$\delta g^{ba} = -(\nabla^b \xi^a + \nabla^a \xi^b - 2h^{ba}{}_x \xi^x)\varepsilon . \tag{3.5}$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric* and that for which δg_{cb} proportional to g_{cb} is said to be *conformal*. Thus we have

PROPOSITION 3.3 ([5]). *In order for a variation of a submanifold to be isometric or conformal, it is necessary and sufficient that*

$$\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x=0, \tag{3.6}$$

or

$$\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x=2\lambda g_{cb}, \tag{3.7}$$

respectively λ being a certain function given by

$$\lambda=\frac{1}{n}(\nabla_c\xi^c-h_e{}^e{}_x\xi^x). \tag{3.8}$$

We now put

$$\bar{\Gamma}_c{}^a{}_b=(\partial_c\bar{B}_b{}^h+\Gamma_j{}^h{}_i(\bar{x})\bar{B}_c{}^j\bar{B}_b{}^i)\bar{B}^a{}_h \tag{3.9}$$

and

$$\delta\Gamma_c{}^a{}_b=\bar{\Gamma}_c{}^a{}_b-\Gamma_c{}^a{}_b,$$

where $\bar{\Gamma}_c{}^a{}_b$ are Christoffel symbols of the deformed submanifold.

Substituting (2.2) and (2.20) into (3.9), we obtain by a straightforward computation,

$$\delta\Gamma_c{}^a{}_b=[(\nabla_c\nabla_b\xi^h+K_{kji}{}^h\xi^kB_c{}^jB_b{}^i)B^a{}_h+h_{cb}{}^x(\nabla^a\xi_x+h_d{}^a{}_x\xi^d)]\varepsilon, \tag{3.10}$$

from which, using equations (1.15) of Gauss and those (1.16) of Codazzi of the submanifolds (cf. [5]), we have

$$\begin{aligned}\delta\Gamma_c{}^a{}_b=&(\nabla_c\nabla_b\xi^a+K_{dcb}{}^a\xi^d)\varepsilon\\&-[\nabla_c(h_{bex}\xi^x)+\nabla_b(h_{cex}\xi^x)-\nabla_e(h_{cbx}\xi^x)]g^{ea}\varepsilon\end{aligned} \tag{3.11}$$

because of (2.8).

A variation of a submanifold for which $\delta\Gamma_c{}^a{}_b=0$ is said to be *affine*.

We now prove

THEOREM 3.4. *A complex isometric variation of a compact invariant submanifold M^n of a Kaehlerian manifold is necessarily holomorphic.*

Proof. If we take account of (1.14), (3.3) and (3.6), we get the following relations:

$$T_{cb}+f_c{}^ef_b{}^dT_{ed}=0, \tag{3.12}$$

$$T_{cb}+T_{bc}=0 \tag{3.13}$$

and hence

(3.14) $$h^{cb}{}_x\xi^x T_{cb}=0\,.$$

We now calculate $T_{cb}T^{cb}$:

(3.15) $$T_{cb}T^{cb}=\frac{1}{2}T^{cb}(T_{cb}-T_{bc}) \qquad \text{(by (3.13))}$$

$$=\frac{1}{2}T^{cb}[\nabla_c\xi_b-\nabla_b\xi_c-f_c{}^e f_b{}^d(\nabla_e\xi_d-\nabla_d\xi_e)]$$
(by (3.3) and (3.14))

$$=2T^{cb}\nabla_c\xi_b\,. \qquad \text{(by (3.12))}$$

On the other hand, applying the operator ∇^c to (3.3) and using $\nabla_c f_b{}^a=0$, we find

$$\nabla^c T_{cb}=\nabla^c\nabla_c\xi_b-\frac{1}{2}f^{ce}f_b{}^d(\nabla_c\nabla_e\xi_d-\nabla_e\nabla_c\xi_d)-2\nabla^c(h_{cbx}\xi^x)\,,$$

from which, using the Ricci-identity,

$$\nabla^c T_{cb}=\nabla^c\nabla_c\xi_b+\frac{1}{2}f^{ce}f_b{}^d K_{ced}{}^a\xi_a-2\nabla^c(h_{cbx}\xi^x)\,,$$

or, using (1.17)

(3.16) $$\nabla^c T_{cb}=\nabla^c\nabla_c\xi_b+K_b{}^a\xi_a-2\nabla^c(h_{cbx}\xi^x)\,.$$

An isometric variation is affine and hence we have

$$\nabla_c\nabla_b\xi^a+K_{dcb}{}^a\xi^d-[\nabla_c(h_b{}^a{}_x\xi^x)+\nabla_b(h_c{}^a{}_x\xi^x)-\nabla^a(h_{cbx}\xi^x)]=0$$

because of (3.11), from which

$$\nabla^c\nabla_c\xi^a+K_d{}^a\xi^d-2\nabla^c(h_c{}^a{}_x\xi^x)=0$$

because of (1.13). Therefore $\nabla^c T_{cb}=0$. From this fact and (3.15), we get

$$\nabla^c(T_{cb}\xi^b)=\frac{1}{2}T_{cb}T^{cb}\,.$$

Thus, integrating this over M^n, we see that $T_{cb}=0$ and consequently the variation is holomorphic by Proposition 3.2. This completes the proof.

§ 4. Conformal variations.

In this section, we prove the following theorem as a generalization of Theorem 3.4.

THEOREM 4.1. *A complex conformal variation of a compact invariant submanifold M^n of a Kaehlerian manifold is necessarily isometric and hence holomorphic.*

Proof. Differentiating (3.7) covariantly along M^n, we find

$$\nabla_c\nabla_b\xi_a+\nabla_c\nabla_a\xi_b=2\nabla_c(h_{bax}\xi^x+\lambda g_{ba}), \tag{4.1}$$

from which, using the Ricci-identity

$$\nabla_c\nabla_b\xi_a+\nabla_a\nabla_c\xi_b-K_{cabd}\xi^d=2\nabla_c(h_{bax}\xi^x+\lambda g_{ba}),$$

or, substituting (4.1) into this,

$$\begin{aligned}\nabla_c\nabla_b\xi_a-\nabla_a\nabla_b\xi_c-K_{cabd}\xi^d\\ =2\nabla_c(h_{bax}\xi^x+\lambda g_{ba})-2\nabla_a(h_{bcx}\xi^x+\lambda g_{bc}).\end{aligned}$$

If we take the skew-symmetric part of this with respect to a and b and make use of the Ricci-identity, then we have

$$\begin{aligned}\nabla_c\nabla_b\xi_a-\nabla_c\nabla_a\xi_b+K_{abcd}\xi^d+K_{cabd}\xi^d+K_{cbad}\xi^d\\ =-2\nabla_a(h_{cbx}\xi^x+\lambda g_{cb})+2\nabla_b(h_{cax}\xi^x+\lambda g_{ca}),\end{aligned}$$

or, using (4.1) and the first Bianchis identity,

$$\begin{aligned}\nabla_c\nabla_b\xi_a-K_{bacd}\xi^d=\nabla_c(h_{bax}\xi^x+\lambda g_{ba})+\nabla_b(h_{cax}\xi^x+\lambda g_{ca})\\ -\nabla_a(h_{cbx}\xi^x+\lambda g_{cb}).\end{aligned} \tag{4.2}$$

Transvecting (4.2) with g^{cb} and using (1.13), we have

$$\nabla^c\nabla_c\xi_a+K_{ad}\xi^d-2\nabla^c(h_{cax}\xi^x)+(n-2)\nabla_a\lambda=0. \tag{4.3}$$

As in the proof of Theorem 3.4, we also have (3.12)~(3.16) under the conformal variation because of (1.14), (1.17), (3.3) and (3.7).

Comparing (3.16) with (4.3), we obtain

$$\nabla^c(T_{cb}+(n-2)\lambda g_{cb})=0. \tag{4.4}$$

Thus, we have

$$\begin{aligned}\nabla^c[(T_{cb}+(n-2)\lambda g_{cb})\xi^b]&=(T_{cb}+(n-2)\lambda g_{cb})\nabla^c\xi^b\\ &=\frac{1}{2}T_{cb}T^{cb}+n(n-2)\lambda^2\end{aligned}$$

because of (3.8) and (3.15). Since M^n is compact, by Green's theorem, it follows that $T_{cb}=0$ and $\lambda=0$ on M^n and consequently the variation is isometric and holomorphic. Hence the theorem is proved.

§ 5. An integral formula.

In section 3, we found that a variation of invariant submanifold of a Kaehlerian manifold is holomorphic if and only if $T_{cb}=0$.

In this section, we find some integral formulas involving $T_{cb}T^{cb}$ and prove theorems on holomorphic variations.

We put

$$f=T_{cb}T^{cb}\,. \tag{5.1}$$

If we take account of (1.1) and (1.14), then (5.1) reduces to

$$\begin{aligned}\frac{1}{2}f=&(\nabla_c\xi_b)(\nabla^c\xi^b)+f_a{}^c f^{be}(\nabla_c\xi_b)(\nabla^a\xi_e)\\ &-4(h_{cbx}\xi^x)\nabla^c\xi^b+2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)\,.\end{aligned} \tag{5.2}$$

On the other hand, we have

$$\begin{aligned}\nabla_b w^b=&(\nabla^b\nabla_b\xi^c)\xi_c+(\nabla_c\xi_b)(\nabla^c\xi^b)-f^{be}f^{ac}(\nabla_b\nabla_e\xi_c)\xi_a\\ &-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a)\end{aligned} \tag{5.3}$$

because of (1.10), where we have put

$$w^b=(\nabla^b\xi^c)\xi_c-f^{be}f^{ac}(\nabla_e\xi_c)\xi_a\,,$$

from which, using the Ricci-identity and (1.17),

$$\nabla^b w_b=\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)+(\nabla_c\xi_b)(\nabla^c\xi^b)-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a)\,. \tag{5.4}$$

Comparing (5.2) with (5.4), we have

$$\begin{aligned}\frac{1}{2}f=&\nabla^b w_b-\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)-4(h_{cbx}\xi^x)\nabla^c\xi^b\\ &+2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)\,,\end{aligned}$$

or, equivalently

$$\begin{aligned}\frac{1}{2}f=&\nabla^b w_b-\xi^c[\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)]\\ &-2\nabla^b(h_{cbx}\xi^x\xi^c)-2(h_{cbx}\xi^x)\nabla^c\xi^b+2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)\\ =&\nabla^b(w_b-2h_{cbx}\xi^x\xi^c)-\xi^c[\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)]\\ &-(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\,.\end{aligned}$$

Thus, assuming the submanifold M^n to be compact, we apply Green's theorem and obain

$$\int\Big[\frac{1}{2}f+\xi^c\{\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)\}$$
$$+(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\Big]dV=0\,, \tag{5.5}$$

dV being the volume element of M^n.

From (3.4) and (3.5), the variation of dV is given by (cf. [5])

$$\delta dV=(\nabla_a\xi^a-h_a{}^a{}_x\xi^x)dV\varepsilon\,. \tag{5.6}$$

For a compact orientable submanifold M^n, we know the following integral formula:

$$\int\Big[\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c)(\nabla^c\xi^b+\nabla^b\xi^c)-(\nabla_b\xi^b)^2\Big]dV=0\,,$$

which is valid for any vector ξ^c in M^n ([6]), from which

$$\int\Big[\xi^c\{(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)-2\nabla^b(h_{cbx}\xi^x)+\nabla_c(h_b{}^b{}_x\xi^x)\}$$
$$+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)$$
$$-(\nabla_c\xi^c-h_c{}^c{}_x\xi^x)(\nabla_b\xi^b)$$
$$+(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\Big]dV=0\,, \tag{5.7}$$

or, using (1.13), (5.1) and (5.5)

$$\int[-T_{cb}T^{cb}+(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)$$
$$-2(\nabla_c\xi^c)^2]dV=0\,, \tag{5.8}$$

or, using $T_c{}^c=0$ which is obtained from (1.13) and (3.3),

$$\int\Big[(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_y\xi^y)$$
$$-\Big(T_{cb}+\sqrt{\frac{2}{n}}(\nabla_e\xi^e)g_{cb}\Big)\Big(T^{cb}+\sqrt{\frac{2}{n}}(\nabla_d\xi^d)g^{cb}\Big)\Big]dV=0\,, \tag{5.9}$$

that is, we get an integral formula for an invariant submanifold M^n in a Kaehlerian manifold. Thus we have

PROPOSITION 5.1. *In order for a complex variation of a compact invariant submanifold of a Kaehlerian manifold to be isometric it is necessary and sufficient that the variation is volume-preserving and holomorphic.*

Now, if a variation of the submanifold is affine, we have from (3.11)

$$\nabla_c\nabla_b\xi_a+K_{dcba}\xi^d-\nabla_c(h_{bax}\xi^x)-\nabla_b(h_{cax}\xi^x)+\nabla_a(h_{cbx}\xi^x)=0\,,$$

from which, using (1.13)

$$\nabla_c(\nabla_a\xi^a)=0\,,$$

that is, $\nabla_a\xi^a=$const. Thus, assuming the submanifold to be compact, we have $\nabla_a\xi^a=0$. From this fact and Proposition 5.1, we obtain

THEOREM 5.2. *A complex variation of a compact invariant submanifold of a Kaehlerian manifold is isometric if and only if the variation is affine and holomorphic.*

Remark. From (5.9), we immediately see that if a variation of submanifold is conformal, then $\lambda=0$ and $T_{cb}=0$. This gives another proof of Theorem 4.1.

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