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KODAI MATH. J.
28 (2005), 390–422

DIMENSIONS OF JULIA SETS OF EXPANDING RATIONAL SEMIGROUPS*

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Dedicated to the memory of Professor Nobuyuki Suita

Abstract

We estimate the upper box and Hausdorff dimensions of the Julia set of an expanding semigroup generated by finitely many rational functions, using the thermodynamic formalism in ergodic theory. Furthermore, we show Bowen's formula, and the existence and uniqueness of a conformal measure, for a finitely generated expanding semigroup satisfying the open set condition.

1. Introduction

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. This is a semigroup whose semigroup operation is the functional composition. A **rational semigroup** is a subsemigroup of $\mathrm{End}(\bar{\mathbf{C}})$ without any constant elements. We say that a rational semigroup G is a **polynomial semigroup** if each element of G is a polynomial. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([HM1]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and F. Ren's group ([ZR], [GR]). For references on research into rational semigroups, see [HM1], [HM2], [HM3], [ZR], [GR], [SSS], [Bo], [St1], [St2], [St3], [S1], [S2], [S3], [S4], [S5], [S6], and [S7]. The research on the dynamics of rational semigroups can be considered a generalization of studies of both the **iteration of rational functions** and **self-similar sets constructed using iterated function systems of some similarity transformations in $\mathbf{R}^2$** in fractal geometry. In both fields, the estimate of the upper (resp. lower) box dimension, which is denoted by $\overline{\dim}_B$ (resp. $\underline{\dim}_B$), and the Hausdorff dimension, which is denoted by $\dim_H$, of the invariant sets (Julia sets or attractors) has been of great interest and has been investigated for a long time. In this paper, we consider the following: For a rational semigroup G, We set

* Keywords: expanding rational semigroup, Hausdorff dimension of a Julia set, thermodynamic formalism

Received January 19, 2004; revised September 29, 2004.

$$F(G) = \{x \in \bar{\mathbf{C}} \mid G \text{ is normal in a neighborhood of } x\}, \quad J(G) = \bar{\mathbf{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** for G and $J(G)$ is called the **Julia set** for G. We use $\langle f_1, f_2, \dots \rangle$ to denote the rational semigroup generated by the family $\{f_i\}$. For a finitely generated rational semigroup $G = \langle f_1, \dots, f_m \rangle$, we set $\Sigma_m = \{1, \dots, m\}^{\mathbf{N}}$ (this is a compact metric space) and we use $\sigma : \Sigma_m \to \Sigma_m$ to denote the shift map, which is $(w_1, w_2, \dots) \mapsto (w_2, w_3, \dots)$ for $w = (w_1, w_2, w_3, \dots) \in \Sigma_m$. We define the map $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ using

$$\tilde{f}((w, x)) = (\sigma w, f_{w_1}(x)).$$

We call map $\tilde{f}$ the **skew product map associated with the generator system** $\{f_1, \dots, f_m\}$. For each $w \in \Sigma_m$, we use F_w to denote the set of all the points $x \in \bar{\mathbf{C}}$ that satisfy the fact that there exists an open neighborhood U of x such that the family $\{f_{w_n} \circ \cdots \circ f_{w_1}\}_n$ is normal in U. We set $J_w = \bar{\mathbf{C}} \setminus F_w$ and $\tilde{J}_w = \{w\} \times J_w$. Moreover, we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{w \in \Sigma_m} \tilde{J}_w}, \quad \tilde{F}(\tilde{f}) = (\Sigma_m \times \bar{\mathbf{C}}) \setminus \tilde{J}(\tilde{f}),$$

where the closure is taken in the product space $\Sigma_m \times \bar{\mathbf{C}}$ (this is a compact metric space). We call $\tilde{F}(\tilde{f})$ the **Fatou set** for $\tilde{f}$ and $\tilde{J}(\tilde{f})$ the **Julia set** for $\tilde{f}$. For each $(w, x) \in \Sigma_m \times \bar{\mathbf{C}}$ and $n \in \mathbf{N}$ we set

$$(\tilde{f}^n)'((w, x)) = (f_{w_n} \cdots f_{w_1})'(x).$$

Furthermore, we denote the first (resp. second) projection by $\pi : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m$ (resp. $\pi_{\bar{\mathbf{C}}} : \Sigma_m \times \bar{\mathbf{C}} \to \bar{\mathbf{C}}$). We say that a finitely generated rational semigroup $G = \langle f_1, \dots, f_m \rangle$ is an **expanding rational semigroup** if $J(G) \neq \emptyset$ and the skew product map $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ associated with the generator system $\{f_1, \dots, f_m\}$ is expanding along fibers, i.e., there exists a positive constant C and a constant $\lambda > 1$ such that for each $n \in \mathbf{N}$,

$$\inf_{z \in \tilde{J}(\tilde{f})} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we use $\| \cdot \|$ to denote the norm of the derivative with respect to the spherical metric.

For a general rational semigroup G and a non-negative number t, we say that a Borel probability measure τ on $\bar{\mathbf{C}}$ is t-**subconformal** (for G) if for each $g \in G$ and for each Borel measurable set A in $\bar{\mathbf{C}}$, $\tau(g(A)) \leq \int_A \|g'\|^t \, d\tau$. Moreover, we set $s(G) = \inf\{t \mid \exists \tau\text{: } t\text{-subconformal measure}\}$.

Furthermore, we say that a Borel probability measure τ on $J(G)$ is t-**conformal** (for G) if for any Borel set A and $g \in G$, if $A, g(A) \subset J(G)$ and $g : A \to g(A)$ is injective, then $\tau(g(A)) = \int_A \|g'\|^t \, d\tau$.

For any $s \geq 0$ and $x \in \bar{\mathbf{C}}$, we set $S(s, x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$. Moreover, we set $S(x) = \inf\{s \geq 0 \mid S(s, x) < \infty\}$ (If no s exists with $S(s, x) < \infty$, then we set $S(x) = \infty$). We set $s_0(G) = \inf\{S(x) \mid x \in \bar{\mathbf{C}}\}$. Note that if G has only countably many elements, then $s(G) \leq s_0(G)$ (Theorem 4.2 in [S2]).

Then, under the above notations, we show the following:

Theorem 1.1 (Main Theorem A). *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be the skew product map associated with $\{f_1, \ldots, f_m\}$. Then, there exists a unique zero δ of the function: $P(t) := P(\tilde{f}|_{\tilde{J}(\tilde{f})}, t\tilde{\varphi})$, where $\tilde{\varphi}$ is the function on $\tilde{J}(\tilde{f})$ defined by: $\tilde{\varphi}((w,x)) = -\log(\|(f_{w_1})'(x)\|)$ for $(w,x) = ((w_1, w_2, \ldots), x) \in \tilde{J}(\tilde{f})$ and $P(\ ,\ )$ denotes the pressure. Furthermore, δ satisfies the fact that there exists a unique probability measure $\tilde{\nu}$ on $\tilde{J}(\tilde{f})$ such that $M_\delta^* \tilde{\nu} = \tilde{\nu}$, where M_δ is an operator on $C(\tilde{J}(\tilde{f}))$ (the space of continuous functions on $\tilde{J}(\tilde{f})$) defined by*

$$M_\delta \psi((w,x)) = \sum_{\tilde{f}((w',y)) = (w,x)} \frac{\psi((w',y))}{\|(f_{w_1'})'(y)\|^\delta},$$

where $w' = (w_1', w_2', \ldots) \in \sum_m$. Moreover, δ satisfies

$$\overline{\dim}_B(J(G)) \le s(G) \le s_0(G) \le \delta = \frac{h_{\alpha\tilde{\nu}}(\tilde{f})}{-\int_{\tilde{J}(\tilde{f})} \tilde{\varphi}\alpha \, d\tilde{\nu}} \le \frac{\log(\sum_{j=1}^m \deg(f_j))}{-\int_{\tilde{J}(\tilde{f})} \tilde{\varphi}\alpha \, d\tilde{\nu}},$$

where $\alpha = \lim_{l \to \infty} M_\delta^l(1)$ and we denote the metric entropy of $(\tilde{f}, \alpha\tilde{\nu})$ by $h_{\alpha\tilde{\nu}}(\tilde{f})$. The support for $\nu := (\pi_{\bar{\mathbf{C}}})_* \tilde{\nu}$ equals $J(G)$.

Furthermore, let $A(G) = \overline{\bigcup_{g \in G} g(\{x \in \bar{\mathbf{C}} \mid \exists h \in G, h(x) = x, |h'(x)| < 1\})}$ and $P(G) = \overline{\bigcup_{g \in G} \{\text{all critical values of } g\}}$. *Then, $A(G) \cup P(G) \subset F(G)$ and for each $x \in \bar{\mathbf{C}} \setminus (A(G) \cup P(G))$, we have δ is equal to:*

$$\inf\left\{ t \ge 0 \,\middle|\, \sum_{n \in \mathbf{N}} \sum_{(w_1, \ldots, w_n) \in \{1, \ldots, m\}^n} \sum_{(f_{w_1} \cdots f_{w_n})(y) = x} \|(f_{w_1} \cdots f_{w_n})'(y)\|^{-t} < \infty \right\}.$$

Theorem 1.2 (Main Theorem B). *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated expanding rational semigroup. Suppose that there exists a non-empty open set U in $\bar{\mathbf{C}}$ such that $f_j^{-1}(U) \subset U$ for each $j = 1, \ldots, m$ and $\{f_j^{-1}(U)\}_{j=1}^m$ are mutually disjoint. Then, we have the following:*

1. $\dim_H(J(G)) = \overline{\dim}_B(J(G)) = s(G) = s_0(G) = \delta$, *where δ denotes the number in Theorem* 1.1.
2. $\nu := (\pi_{\bar{\mathbf{C}}})_* \tilde{\nu}$ *is the unique δ-conformal measure, where $\tilde{\nu}$ is the measure in Theorem* 1.1. *Furthermore, ν satisfies the fact that there exists a positive constant C such that for any $x \in J(G)$ and any positive number r with $r <$* diam $\bar{\mathbf{C}}$, *we have*
$$C^{-1} \le \frac{\nu(B(x,r))}{r^\delta} \le C.$$
3. ν *satisfies $\nu(f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))) = 0$, for each $i, j \in \{1, \ldots, m\}$ with $i \ne j$. Furthermore, for each (i,j) with $i \ne j$, we have $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ is nowhere dense in $f_j^{-1}(J(G))$.*
4. $0 < H^\delta(J(G)) < \infty$, *where H^δ denotes the δ-dimensional Hausdorff measure with respect to the spherical metric. Furthermore, we have* $\nu = \dfrac{H^\delta|_{J(G)}}{H^\delta(J(G))}$.

5. *If there exists a t-conformal measure* τ, *then* $t = \delta$ *and* $\tau = \nu$.
6. *For any* $x \in \bar{\mathbf{C}} \setminus (A(G) \cup P(G))$, *we have*

$$\dim_H(J(G)) = \delta = \inf\left\{ t \geq 0 \,\middle|\, \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-t} < \infty \right\}.$$

Remark 1. In [S6], it is shown that if $G = \langle f_1 \dots, f_m \rangle$ is expanding and there exists a non-empty open set U such that $f_j^{-1}(U) \subset U$ for each $j = 1, \dots, m$, $\{f_j(U)\}_j$ are mutually disjoint and $\overline{U} \neq J(G)$, then $J(G)$ is porous and $\overline{\dim}_B(J(G)) < 2$.

Remark 2. In addition to the assumption of Main Theorem B, if $J(G) \subset \mathbf{C}$, then we can also show a similar result for the Euclidean metric.

For the precise notation, see the following sections. The proof of Main Theorem A is given in section 3 and the proof of Main Theorem B is given in section 5. The existence of a subconformal or conformal measure is deduced by applying some of the results in [W1] and the thermodynamic formalism in ergodic theory to the skew product map associated with the generator system. Since generator maps are not injective in general and we do not assume the "cone condition" (the existence of uniform cones) for the boundary of the open set, much effort is needed to estimate $\nu(B(x,r))$ in Main Theorem B. Indeed, we cut the closure of the open set into small pieces $\{K_j\}$, and for a fixed $s \in \mathbf{N}$, let $\mathscr{K}$ be the set of all (γ, k_j) that satisfies that γ is a well defined inverse branch of $(f_{w_1} \circ \cdots f_{w_u})^{-1}$ defined on K_j for some $(w_1, \dots, w_u) \in \{1, \dots, m\}^u$ with $u \leq s$. Then we introduce an equivalence class "$\sim$" in a subset Γ of $\mathscr{K}$, and an order "$\leq$" in $\Gamma/\sim$. We obtain an upper estimate of the cardinality of the set of all minimal elements of $(\Gamma/\sim, \leq)$ by a constant independent of r and x, which gives us the key to estimate $\nu(B(x,r))$.

Note that in [MU1], it was discussed the case in which there are infinitely many injective generator maps and the boundary of the open set satisfies the cone condition.

The uniqueness of a conformal measure τ is deduced from some results in [W1] and an estimate of $\tau(B(x,r))$. Note that our definition of conformal measure differs from that of [MU1] and [MU2]. In this paper, we do not require the separating condition for the definition of conformal measure.

2. Preliminaries

In this section, we give the notation and definitions for rational semigroups and the associated skew products that we need to give our main result.

2.1. Rational semigroups

We use the definition in [S5].

Definition 2.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \bar{\mathbf{C}} \mid G \text{ is normal in a neighborhood of } z\}, \quad J(G) = \bar{\mathbf{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** for G and $J(G)$ is called the **Julia set** for G. The backward orbit $G^{-1}(z)$ of z and the **set of exceptional points** $E(G)$ are defined by: $G^{-1}(z) = \bigcup_{g \in G} g^{-1}(z)$ and $E(G) = \{z \in \bar{\mathbf{C}} \mid \# G^{-1}(z) \le 2\}$. For any subset A of $\bar{\mathbf{C}}$, we set $G^{-1}(A) = \bigcup_{g \in G} g^{-1}(A)$. We use $\langle f_1, f_2, \dots \rangle$ to denote the rational semigroup generated by the family $\{f_i\}$. For a rational map g, we use $J(g)$ to denote the Julia set of dynamics of g.

For a rational semigroup G, for each $f \in G$, we have $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that we do not have this equality hold in general. If $\# J(G) \ge 3$, then $J(G)$ is a perfect set, $\# E(G) \le 2$, $J(G)$ is the smallest closed backward invariant set containing at least three points, and $J(G)$ is the closure of the union of all repelling fixed points of elements of G, which implies that $J(G) = \overline{\bigcup_{g \in G} J(g)}$. If a point z is not in $E(G)$, then for every $x \in J(G)$, $x \in \overline{G^{-1}(z)}$. In particular, if $z \in J(G) \setminus E(G)$, then $\overline{G^{-1}(z)} = J(G)$. Further, for a finitely generated rational semigroup $G = \langle f_1, \dots, f_m \rangle$, if we use G_n to denote the subsemigroup of G that is generated by n-products of generators $\{f_j\}$, then $J(G_n) = J(G)$. For more precise statements, see Lemma 2.3 in [S5], for which the proofs are based on [HM1] and [GR]. Furthermore, if G is generated by a precompact subset Λ of $\mathrm{End}(\bar{\mathbf{C}})$, then $J(G) = \overline{\bigcup_{f \in \Lambda} f^{-1}(J(G))} = \bigcup_{h \in \bar{\Lambda}} h^{-1}(J(G))$. In particular, if Λ is compact, then we have $J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G))$ ([S3]). We call this property of a Julia set the **backward self-similarity**.

Remark 3. Using the backward self-similarity, research on the Julia sets of rational semigroups may be considered a generalization of research on self-similar sets constructed using some similarity transformations from $\mathbf{C}$ to itself, which can be regarded as the Julia sets of some rational semigroups. It is easily seen that the Sierpiński gasket is the Julia set of a rational semigroup $G = \langle f_1, f_2, f_3 \rangle$ where $f_i(z) = 2(z - p_i) + p_i$, $i = 1, 2, 3$ with $p_1 p_2 p_3$ being a regular triangle.

2.2. Associated skew products

We use the notation in [S5]. Let m be a positive integer. We use Σ_m to denote the one-sided wordspace that is $\Sigma_m = \{1, \dots, m\}^{\mathbf{N}}$ and use $\sigma : \Sigma_m \to \Sigma_m$ to denote the shift map, which is $(w_1, \dots) \mapsto (w_2, \dots)$ for $w = (w_1, w_2, w_3, \dots) \in \Sigma_m$. For any $w, w' \in \Sigma_m$, we set $d(w, w') := \sum_{n=1}^{\infty} (1/2^n) \cdot c(w_k, w'_k)$, where $c(w_k, w'_k) = 0$ if $w_k = w'_k$ and $c(w_k, w'_k) = 1$ if $w_k \ne w'_k$. Then, (Σ_m, d) is a compact metric space. Furthermore, the dynamics of $\sigma : \Sigma_m \to \Sigma_m$ is expanding with respect to this metric d. That is, each inverse branch σ_j^{-1} of σ^{-1} on Σ_m, which is defined by $\sigma_j^{-1}((w_1, w_2, \dots)) = (j, w_1, w_2, \dots)$ for $j = 1, \dots, m$, satisfies $d(\sigma_j^{-1}(w), \sigma_j^{-1}(w')) \le (1/2) \cdot d(w, w')$.

Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. We define the map $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ using

$$\tilde{f}((w,x)) = (\sigma w, f_{w_1}(x)).$$

We call map $\tilde{f}$ the **skew product map associated with the generator system** $\{f_1,\dots,f_m\}$. $\tilde{f}$ is finite-to-one and an open map. We hold that point $(w,x) \in \Sigma_m \times \bar{\mathbf{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w,x). Hence, the map $\tilde{f}$ has infinitely many critical points in general.

Definition 2.2. For each $w \in \Sigma_m$, we use F_w to denote the set of all the points $x \in \bar{\mathbf{C}}$ that satisfy the fact that there exists an open neighborhood U of x such that the family $\{f_{w_n} \circ \cdots \circ f_{w_1}\}_n$ is normal in U. We set $J_w = \bar{\mathbf{C}} \setminus F_w$ and $\tilde{J}_w = \{w\} \times J_w$. Moreover, we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{w \in \Sigma_m} \tilde{J}_w}, \quad \tilde{F}(\tilde{f}) = (\Sigma_m \times \bar{\mathbf{C}}) \setminus \tilde{J}(\tilde{f}),$$

where the closure is taken in the product space $\Sigma_m \times \bar{\mathbf{C}}$. We often write $\tilde{F}(\tilde{f})$ as $\tilde{F}$ and $\tilde{J}(\tilde{f})$ as $\tilde{J}$. We call $\tilde{F}(\tilde{f})$ the **Fatou set** for $\tilde{f}$ and $\tilde{J}(\tilde{f})$ the **Julia set** for $\tilde{f}$. Here, we remark that $\bigcup_{w \in \Sigma_m} \tilde{J}_w$ may not be compact in general. That is why we consider the closure of that set in $\Sigma_m \times \bar{\mathbf{C}}$ (this is a compact space) concerning the definition of the Julia set for $\tilde{f}$.

For each $(w,x) \in \Sigma_m \times \bar{\mathbf{C}}$ and $n \in \mathbf{N}$ we set

$$(\tilde{f}^n)'((w,x)) = (f_{w_n} \cdots f_{w_1})'(x).$$

Furthermore, we denote the first (resp. second) projection by $\pi : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m$ (resp. $\pi_{\bar{\mathbf{C}}} : \Sigma_m \times \bar{\mathbf{C}} \to \bar{\mathbf{C}}$). Note that we have $\tilde{f}(\tilde{F}(\tilde{f})) = \tilde{f}^{-1}(\tilde{F}(\tilde{f})) = \tilde{F}(\tilde{f})$, $\tilde{f}(\tilde{J}(\tilde{f})) = \tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$ and $\pi_{\bar{\mathbf{C}}}(\tilde{J}(\tilde{f})) = J(G)$. (For the fundamental properties of these sets, see Proposition 3.2 in [S5]. In addition, see [S3].)

Definition 2.3. Let $G = \langle f_1,\dots,f_m \rangle$ be a finitely generated rational semigroup. Let us fix the generator system $\{f_1,\dots,f_m\}$. We set $f_w := f_{w_1} \circ \cdots \circ f_{w_k}$ for any $w = (w_1,\dots,w_k) \in \{1,\dots,m\}^k$. We set $\mathscr{W} = \bigcup_{n \in \mathbf{N}} \{1,\dots,m\}^n \cup \Sigma_m$ and set $\mathscr{W}^* = \bigcup_{n \in \mathbf{N}} \{1,\dots,m\}^n$. For any $w = (w_1,w_2,\dots) \in \mathscr{W}$, we set $|w| = n$ if $w \in \{1,\dots,m\}^n$ and $|w| = \infty$ if $w \in \Sigma_m$. Furthermore, we set $w|k := (w_1,\dots,w_k)$, for any $k \in \mathbf{N}$ with $k \leq |w|$. Moreover, for any $w \in \mathscr{W}^*$, we set $\Sigma_m(w) := \{w' \in \Sigma_m \mid w'_j = w_j, j = 1,\dots,|w|\}$. For any $w^1 \in \mathscr{W}^*$ and $w^2 \in \mathscr{W}$, we set $w^1w^2 = (w^1_1,\dots,w^1_{|w^1|},w^2_1,w^2_2,\dots) \in \mathscr{W}$.

Notation. Let (X,d) be a metric space. For any subset A of X, we set $\operatorname{diam} A := \sup\{d(x,y) \mid x,y \in A\}$. Let μ be a Borel measure on X. We use $\operatorname{supp}\mu$ to denote the support of μ. For any Borel set A in X, we use $\mu|_A$ to denote the measure on A such that $\mu|_A(B) = \mu(B)$ for each Borel subset B of A. We set $L^1(\mu) = \{\varphi : X \to \mathbf{R} \mid \int_X |\varphi|\, d\mu < \infty\}$, with L^1 norm. For any $\varphi \in L^1(\mu)$, we sometimes use $\mu(\varphi)$ to mean $\int_X \varphi\, d\mu$. For any $\varphi \in L^1(\mu)$, we use $\varphi\mu$ to denote the measure such that $(\varphi\mu)(A) = \int_A \varphi\, d\mu$ for any Borel set A. We set $C(X) = \{\varphi : X \to \mathbf{R} \mid \text{continuous}\}$. (If X is compact, then $C(X)$ is the Banach space with the supremum norm.) For any subset A of X and any $r > 0$, we set $B(A, r) = \{y \in X \mid d(y, A) < r\}$. For any subset A of X, we use int A to denote the interior of A.

Remark 4. In this paper, we always use the spherical metric on $\bar{\mathbf{C}}$. However, we note that conjugating a rational semigroup G by a Möbius transformation, we may assume that $J(G) \subset \mathbf{C}$, and then for a neighborhood V of $J(G)$, the identity map $i : (V, d_s) \to (V, d_e)$ is a bi-Lipschitz map, where d_s and d_e denote the spherical and Euclidean distance, respectively. In what follows, we often use the above implicitly, especially when we need to use the facts in [F] and [Pe].

3. Main Theorem A

In this section, we show Main Theorem A. We investigate the estimate of the upper box and Hausdorff dimensions of Julia sets of expanding semigroups using thermodynamic formalism in ergodic theory. For the notation used in ergodic theory, see [DGS] and [W2].

Definition 3.1. Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. We say that G is an **expanding rational semigroup** if $J(G) \neq \emptyset$ and the skew product map $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ associated with the generator system $\{f_1, \dots, f_m\}$ is expanding along fibers, i.e., there exists a positive constant C and a constant $\lambda > 1$ such that for each $n \in \mathbf{N}$,

$$\inf_{z \in \tilde{J}(\tilde{f})} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we use $\|\cdot\|$ to denote the norm of the derivative with respect to the spherical metric.

Remark 5. By Theorem 2.6, Theorem 2.8, and Remark 4 in [S2], we see that if $G = \langle f_1, \dots, f_m \rangle$ contains an element of degree at least two, each Möbius transformation in G is neither the identity nor an elliptic element, and G is **hyperbolic**, i.e., the postcritical set $P(G)$ of G, which is defined as:

$$P(G) := \overline{\bigcup_{g \in G} \{\text{all critical values of } g\}},$$

is included in $F(G)$, then G is expanding. Conversely, if $G = \langle f_1, \dots, f_m \rangle$ is expanding, then G is hyperbolic and each Möbius transformation in G is loxodromic. Hence, the notion of expandingness does not depend on any choice of a generator system for a finitely generated rational semigroup.

Lemma 3.2. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup. Suppose $\#J(G) \leq 2$. Then, $\#J(G) = 1$ and $J(G)$ is a common repelling fixed point of any f_j.*

Proof. Suppose $\#J(G) = 2$ and let $J(G) = \{z_1, z_2\}$. Then, f_j is a Möbius transformation, for each $j = 1, \dots, m$. Since G is expanding, each f_j is loxodromic. We may assume that z_1 is a repelling fixed point of f_1. Then, since $f_1^{-1}(J(G)) \subset J(G)$, it follows that z_2 is an attracting fixed point of f_1. This is a contradiction, however, since G is expanding. Hence, $\#J(G) = 1$. $\square$

Definition 3.3. Let G be a rational semigroup and let t be a non-negative number. We say that a Borel probability measure τ on $\bar{\mathbf{C}}$ is **t-subconformal** (for G) if for each $g \in G$ and for each Borel measurable set A in $\bar{\mathbf{C}}$, $\tau(g(A)) \le \int_A \|g'\|^t \, d\tau$. Moreover, we set

$$s(G) = \inf\{t \mid \exists \tau\colon \ t\text{-subconformal measure}\}.$$

Definition 3.4. Let X be a compact metric space. Let $f : X \to X$ be a continuous map:

1. We use $h(f)$ to denote the **topological entropy** of f (see p83 in [DGS]). We use $h_\mu(f)$ to denote the **metric entropy** of f with respect to an invariant Borel probability measure μ (see p60 in [DGS]).
2. Furthermore, let $\varphi : X \to \mathbf{R}$ be a continuous function. Then, we use $P(f, \varphi)$ to denote the **pressure** for the dynamics of f and the function φ (see p141 in [DGS]). According to a well known fact: the variational principle (see p142 in [DGS]), we have
$$P(f, \varphi) = \sup\left\{h_\mu(f) + \int_X \varphi \, d\mu\right\},$$
where the supremum is taken over all f-invariant Borel probability measures μ on X. If an invariant probability measure μ attains the supremum in this manner, then μ is called an **equilibrium state** for (f, φ). For more details on this notation and the variational principle, see [DGS] and [W2].
3. For a real-valued continuous function φ on X and for each $n \in \mathbf{N}$, we define a continuous function $S_n\varphi$ on X as $(S_n\varphi)(z) = \sum_{j=0}^{n-1} \varphi(f^j(z))$. Note that $P(f^n, S_n\varphi) = nP(f, \varphi)$ (see Theorem 9.8 in [W2]).

Definition 3.5. Let X be a compact metric space and let $f : X \to X$ be a continuous map satisfying the fact that there exists a number $k \in \mathbf{N}$ such that $\#f^{-1}(z) = k$ for each $z \in X$. Let φ be a continuous function on X. We define an operator $L = L_\varphi$ on $C(X)$ using

$$L\psi(z) = \sum_{f(z')=z} \exp(\varphi(z'))\psi(z').$$

This is called the **transfer operator for** (f, φ). Note that L_φ^n equals the transfer operator for $(f^n, S_n\varphi)$, for each $n \in \mathbf{N}$.

Lemma 3.6. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be the skew product map associated* *with* $\{f_1, \dots, f_m\}$. *Then, for each Hölder continuous function* φ *on* $\tilde{J}(\tilde{f})$, *the transfer operator* L_φ *for* $(\tilde{f}|_{\tilde{J}(\tilde{f})}, \varphi)$ *on* $C(\tilde{J}(\tilde{f}))$ *satisfies the fact that there exists a unique probability measure* $\tilde{\nu} = \tilde{\nu}_\varphi$ *on* $\tilde{J}(\tilde{f})$ *satisfying all of the following*:

1. $L_\varphi^* \tilde{\nu} = \exp(P)\tilde{\nu}$, *where* $P = P(\tilde{f}|_{\tilde{J}(\tilde{f})}, \varphi)$ *is the pressure of* $(\tilde{f}|_{\tilde{J}(\tilde{f})}, \varphi)$.
2. *For each* $\psi \in C(\tilde{J}(\tilde{f}))$, $\left\| \frac{1}{(\exp(P))^n} L_\varphi^n \psi - \tilde{\nu}(\psi)\alpha_\varphi \right\|_{\tilde{J}(\tilde{f})} \to 0$, $n \to \infty$, *where we set* $\alpha_\varphi = \lim_{l\to\infty} \frac{1}{(\exp(P))^l} L_\varphi^l(1) \in C(\tilde{J}(\tilde{f}))$ *and we use* $\|\cdot\|_{\tilde{J}(\tilde{f})}$ *to denote the supremum norm on* $\tilde{J}(\tilde{f})$.
3. $\alpha_\varphi \tilde{\nu}$ *is* $\tilde{f}$*-invariant, exact* (*hence ergodic*) *and is an equilibrium state for* $(\tilde{f}|_{\tilde{J}(\tilde{f})}, \varphi)$.
4. $\alpha_\varphi(z) > 0$ *for each* $z \in \tilde{J}(\tilde{f})$.

Proof. According to the Koebe distortion theorem and since the dynamics of $\sigma : \Sigma_m \to \Sigma_m$ is expanding, there exists a number $s \in \mathbf{N}$ such that the map $\tilde{f}^s : \tilde{J}(\tilde{f}) \to \tilde{J}(\tilde{f})$ satisfies condition I on page 123 in [W1] (each of X_0, X, and $\overline{X}$ in [W1] corresponds to $\tilde{J}(\tilde{f})$). Furthermore, by Proposition 3.2 (f) in [S5] and Lemma 3.2, we have the fact that $\tilde{f}^s$ on $\tilde{J}(\tilde{f})$ satisfies condition II on page 125 in [W1]. The map $\mu \to L_\varphi^*\mu/(L_\varphi^*\mu)(1)$ is continuous on the space $M(\tilde{J}(\tilde{f}))$ of Borel probability measures on $\tilde{J}(\tilde{f})$. Hence, this map has a fixed point $\tilde{\nu}$ based on the Schauder-Tychonoff fixed point theorem. Let $\lambda = (L_\varphi^*\tilde{\nu})(1)$. Then, $L_\varphi^*\tilde{\nu} = \lambda\tilde{\nu}$. Hence, we have $(L_\varphi^s)^*\tilde{\nu} = \lambda^s\tilde{\nu}$. By Theorem 8, Corollary 12, and the statement on equilibrium states on page 140 in [W1], we get $\lambda^s = \exp(P(\tilde{f}^s|_{\tilde{J}(\tilde{f})}, S_s\varphi)) = \exp(sP(\tilde{f}|_{\tilde{J}(\tilde{f})}, \varphi))$. Hence, we obtain $\lambda = \exp(P)$. The other results also follow from Theorem 8, Corollary 12, and the statement on equilibrium states on page 140 in [W1]. □

Notation. Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be the skew product map associated with $\{f_1, \dots, f_m\}$. Suppose that no critical point of $\tilde{f}$ exists in $\tilde{J}(\tilde{f})$. Then, we define a function $\tilde{\varphi}$ on $\tilde{J}(\tilde{f})$ as: $\tilde{\varphi}((w, x)) := -\log\|(f_{w_1})'(x)\|$ for $(w, x) = ((w_1, w_2, \dots), x) \in \tilde{J}(\tilde{f})$.

Lemma 3.7. *Let* $G = \langle f_1, f_2, \dots f_m \rangle$ *be a finitely generated expanding rational semigroup. Then, using the above notation, we have the following*:

1. *The function* $P(t) = P(\tilde{f}|_{\tilde{J}(\tilde{f})}, t\tilde{\varphi})$ *on* $\mathbf{R}$ *is convex and strictly decreasing as* t *increases. Furthermore,* $P(t) \to -\infty$ *as* $t \to \infty$.
2. *There exists a unique zero* $\delta \geq 0$ *of* $P(t)$. *Furthermore, if* $h(\tilde{f}|_{\tilde{J}(\tilde{f})}) > 0$ *then* $\delta > 0$.
3. *There exists a unique probability measure* $\tilde{\nu} = \tilde{\nu}_{\delta\tilde{\varphi}}$ *on* $\tilde{J}(\tilde{f})$ *such that* $M_\delta^*\tilde{\nu} = \tilde{\nu}$, *where* M_δ *is an operator on* $C(\tilde{J}(\tilde{f}))$ *defined by*

$$M_\delta\psi((w, x)) = \sum_{\tilde{f}((w', y))=(w, x)} \frac{\psi((w', y))}{\|(f_{w_1'})'(y)\|^\delta}. \tag{1}$$

Note that $M_\delta = L_{\delta\tilde{\varphi}}$.

4. *δ satisfies the fact that*

$$\delta = \frac{h_{\alpha\tilde{\nu}}(\tilde{f})}{-\int_{\tilde{J}(\tilde{f})} \tilde{\varphi}\alpha \, d\tilde{\nu}} \leq \frac{\log(\sum_{j=1}^{m} \deg(f_j))}{-\int_{\tilde{J}(\tilde{f})} \tilde{\varphi}\alpha \, d\tilde{\nu}}, \tag{2}$$

where $\alpha = \lim_{l\to\infty} M_\delta^l(1) \in C(\tilde{J}(\tilde{f}))$.

Proof. Using the variational principle, we have $P(t) = \sup\{h_\mu(\tilde{f}|_{\tilde{J}(\tilde{f})}) + \int_{\tilde{J}(\tilde{f})} t\tilde{\varphi} \, d\mu\}$, where the supremum is taken over all $\tilde{f}$-invariant Borel probability measures μ on $\tilde{J}(\tilde{f})$. In addition, note that by Theorem 6.1 in [S5], we determine that the topological entropy $h(\tilde{f})$ of $\tilde{f}$ on $\Sigma_m \times \bar{\mathbf{C}}$ is less than or equal to $\log(\sum_{j=1}^{m} \deg(f_j))$. By the variational principle: $h(\tilde{f}) = \sup\{h_\mu(\tilde{f}) \mid \tilde{f}_*\mu = \mu\}$ (see p138 in [DGS] or Theorem 8.6 in [W2]); it follows that

$$h_\mu(\tilde{f}) \leq \log\left(\sum_{j=1}^{m} (\deg(f_j))\right)$$

for any $\tilde{f}$-invariant Borel probability measure μ on $\tilde{J}(\tilde{f})$. Combining this with the fact that the dynamics of $\tilde{f}$ on $\tilde{J}(\tilde{f})$ is expanding, we see that the function $P(t)$ on $\mathbf{R}$ is convex, strictly decreasing as t increases, and $P(t) \to -\infty$ as $t \to \infty$. Hence, there exists a unique number $\delta \in \mathbf{R}$ satisfying $P(\delta) = 0$. Since $P(0) = h(\tilde{f}|_{\tilde{J}(\tilde{f})})$, we have $\delta > 0$ if $h(\tilde{f}|_{\tilde{J}(\tilde{f})}) > 0$. The statements 3 and 4 follow from Lemma 3.6 and this argument. □

Definition 3.8. We define an operator $\hat{M}_\delta$ acting on the space of all Borel measurable functions on $\tilde{J}(\tilde{f})$ using the same formula as that for M_δ. (See (1)).

We now show that $\hat{M}_\delta$ acts on $L^1(\tilde{\nu})$ and that $\hat{M}_\delta$ on $L^1(\tilde{\nu})$ is a bounded operator, where $\tilde{\nu} = \tilde{\nu}_{\delta\tilde{\varphi}}$.

Lemma 3.9. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated expanding rational semigroup. Using the above notation, we have the following*:

1. *Let A be a Borel set in $\tilde{J}(\tilde{f})$. If $\tilde{\nu}(A) = 0$, then $\tilde{\nu}(\tilde{f}^{-1}(A)) = 0$.*
2. *Let ψ be a Borel measurable function on $\tilde{J}(\tilde{f})$. Let $\{\psi_n\}_n$ be a sequence of Borel measurable functions on $\tilde{J}(\tilde{f})$. Suppose $\psi_n(z) \to \psi(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{\nu}$. Then, we have $(\hat{M}_\delta\psi_n)(z) \to (\hat{M}_\delta\psi)(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{\nu}$.*
3. *If $\psi \in L^1(\tilde{\nu})$, then $\hat{M}_\delta\psi \in L^1(\tilde{\nu})$. Furthermore, $\hat{M}_\delta$ is a bounded operator on $L^1(\tilde{\nu})$ and the operator norm $\|\hat{M}_\delta\|$ is equal to 1.*

Proof. Let $\mu = \alpha\tilde{\nu}$, where $\alpha = \lim_{l\to\infty} M_\delta^l 1$. Then, by Lemma 3.6-3, we have $\tilde{f}_*\mu = \mu$. Furthermore, by Lemma 3.6-4, μ and $\tilde{\nu}$ are absolutely continuous with respect to each other. Hence, we obtain the statement 1, and the statement 2 follows easily from this.

We now show the statement 3. First, we show the following claim:

Claim. For any $\psi \in C(\tilde{J}(\tilde{f}))$, we have $\int |\hat{M}_\delta \psi|\, d\tilde{\nu} \le \int |\psi|\, d\tilde{\nu}$.

To show this claim, let $\psi \in C(\tilde{J}(\tilde{f}))$. Let $\psi^+ = \max\{\psi, 0\}$ and $\psi^- = -\min\{\psi, 0\}$. Then, we have $\psi = \psi^+ - \psi^-$ and $|\psi| = \psi^+ + \psi^-$. Since $M_\delta^* \tilde{\nu} = \tilde{\nu}$, we obtain $\int |\hat{M}_\delta \psi|\, d\tilde{\nu} = \int |M_\delta \psi^+ - M_\delta \psi^-|\, d\tilde{\nu} \le \int M_\delta \psi^+\, d\tilde{\nu} + \int M_\delta \psi^-\, d\tilde{\nu} = \int \psi^+ + \psi^-\, d\tilde{\nu} = \int |\psi|\, d\tilde{\nu}$. Hence, the above claim holds.

Now, let ψ be a general element of $L^1(\tilde{\nu})$. Let $\{\psi_n\}_n$ be a sequence in $C(\tilde{J}(\tilde{f}))$ such that $\psi_n \to \psi$ in $L^1(\tilde{\nu})$. We may assume that $\psi_n(z) \to \psi(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{\nu}$. Then, according to the statement 2, we have $(\hat{M}_\delta \psi_n)(z) \to (\hat{M}_\delta \psi)(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{\nu}$. Using this claim, $\{\hat{M}_\delta \psi_n\}_n$ is a Cauchy sequence in $L^1(\tilde{\nu})$. Hence, it follows that $\hat{M}_\delta \psi \in L^1(\tilde{\nu})$. Furthermore, we have $\int |\hat{M}_\delta \psi|\, d\tilde{\nu} = \lim_{n\to\infty} \int |\hat{M}_\delta \psi_n|\, d\tilde{\nu} \le \lim_{n\to\infty} \int |\psi_n|\, d\tilde{\nu} = \int |\psi|\, d\tilde{\nu}$. Hence, $\|\hat{M}_\delta\| \le 1$. Since $\int \hat{M}_\delta 1\, d\tilde{\nu} = \int 1\, d\tilde{\nu} = 1$, we obtain $\|\hat{M}_\delta\| = 1$. □

We now show that the measure $\tilde{\nu} = \tilde{\nu}_{\delta\tilde{\varphi}}$ is "conformal".

Lemma 3.10. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let $k \in \mathbf{N}$ and let A be a Borel set in $\tilde{J}(\tilde{f})$ such that $\tilde{f}^k : A \to \tilde{f}^k(A)$ is injective. Then, using the above notation, we have $\tilde{\nu}(\tilde{f}^k(A)) = \int_A \|(\tilde{f}^k)'\|^\delta\, d\tilde{\nu}$.*

Proof. We have $M_\delta^k \tilde{\nu} = \tilde{\nu}$ and M_δ^k is a transfer operator for $(\tilde{f}^k, \delta S_k \tilde{\varphi})$. By Proposition 2.2 in [DU] and Lemma 3.9-3, we obtain the statement. □

Lemma 3.11. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated expanding rational semigroup. Then, with our notation, the probability measure $\nu := (\pi_{\bar{\mathbf{C}}})_*(\tilde{\nu})$ is δ-subconformal.*

Proof. First, note that by Lemma 3.10, it follows that for any Borel set B in $\Sigma_m \times \bar{\mathbf{C}}$ we have $\tilde{\nu}(\tilde{f}^k(B)) \le \sum_j \tilde{\nu}(\tilde{f}^k(B_j)) = \sum_j \int_{B_j} \|(\tilde{f}^k)'\|^\delta\, d\tilde{\nu} = \int_B \|(\tilde{f}^k)'\|^\delta\, d\tilde{\nu}$, where $B = \sum_j B_j$ is a measurable partition such that $\tilde{f}^k|_{B_j}$ is injective for each j. Hence, for any Borel set A in $\bar{\mathbf{C}}$ and any $w \in \mathscr{W}^*$ with $|w| = k$, it follows that $\nu(f_w(A)) = \tilde{\nu}(\pi_{\bar{\mathbf{C}}}^{-1}(f_w(A))) = \tilde{\nu}(\tilde{f}^k(\Sigma_m(w) \times A)) \le \int_{\Sigma_m(w)\times A} \|(\tilde{f}^k)'\|^\delta\, d\tilde{\nu} \le \int_A \|(f_w)'\|^\delta\, d\nu$. □

We now consider the Poincaré series and critical exponent for a rational semigroup.

Definition 3.12. Let G be a rational semigroup. We set

$$A(G) = \overline{\bigcup_{g \in G} g(\{z \in \bar{\mathbf{C}} \mid \exists h \in G, h(z) = z, |h'(z)| < 1\})}.$$

For any $s \geq 0$ and $x \in \bar{\mathbf{C}}$, we set $S(s,x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$. Furthermore, we set $S(x) = \inf\{s \geq 0 \mid S(s,x) < \infty\}$ (If no s exists with $S(s,x) < \infty$, then we set $S(x) = \infty$). We set $s_0(G) = \inf\{S(x) \mid x \in \bar{\mathbf{C}}\}$.

If G is generated by finite elements $\{f_1, \dots, f_m\}$, then for any $x \in \bar{\mathbf{C}}$ and $t \geq 0$, we set $T(t,x) = \sum_{w \in \mathscr{W}^*} \sum_{f_w(y)=x} \|(f_w)'(y)\|^{-t}$ and $T(x) = \inf\{t \geq 0 \mid T(t,x) < \infty\}$ (If no t exists with $T(t,x) < \infty$, then we set $T(x) = \infty$). Note that $S(x) \leq T(x)$.

Lemma 3.13. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be a skew product map associated with $\{f_1, \dots, f_m\}$. Let $z \in \tilde{F}(\tilde{f})$ be a point. Then, there exists a number $n \in \mathbf{N}$ such that $\pi_{\bar{\mathbf{C}}}(\tilde{f}^n(z)) \in F(G)$.*

Proof. Let $z \in \tilde{F}(\tilde{f})$ be a point. Then, there exists a word $w \in \mathscr{W}^*$ and an open neighborhood V of $\pi_{\bar{\mathbf{C}}}(z)$ in $\bar{\mathbf{C}}$ such that $z \in \Sigma_m(w) \times V \subset \tilde{F}(\tilde{f})$. Let $n = |w|$. Then, $\tilde{F}(\tilde{f}) \supset \tilde{f}^n(\Sigma_m(w) \times V) = \Sigma_m \times f_w(V)$. Since $\pi_{\bar{\mathbf{C}}} \tilde{J}(\tilde{f}) = J(G)$ (Proposition 3.2 in [S5]), it follows that $f_w(V) \subset F(G)$. Hence, $\pi_{\bar{\mathbf{C}}} \tilde{f}^n(z) = f_w(\pi_{\bar{\mathbf{C}}}(z)) \in f_w(V) \subset F(G)$. $\square$

Lemma 3.14. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be a skew product map associated with $\{f_1, \dots, f_m\}$. Let $z \in \Sigma_m \times \bar{\mathbf{C}}$ be a point with $x := \pi_{\bar{\mathbf{C}}}(z) \in \bar{\mathbf{C}} \backslash A(G)$. Then, for each open neighborhood V of $\tilde{J}(\tilde{f})$ in $\Sigma_m \times \bar{\mathbf{C}}$, there exists a number $l \in \mathbf{N}$ such that $\bigcup_{n \geq l} (\tilde{f}^n)^{-1}(z) \subset V$. Furthermore, we have $A(G) \cup P(G) \subset F(G)$ and if $x \in \bar{\mathbf{C}} \backslash (A(G) \cup P(G))$, then $T(x) < \infty$.*

Proof. By Remark 5, we have $P(G) \subset F(G)$. Next we show $A(G) \subset F(G)$. Since G is expanding, then using the Koebe distortion theorem and $\pi_{\bar{\mathbf{C}}}(\tilde{J}(\tilde{f})) = J(G)$ (Proposition 3.2 in [S5]), we obtain that there exist an $n \in \mathbf{N}$ and a number $\varepsilon > 0$ such that for each $a \in J(G)$ and each $w \in \mathscr{W}^*$ with $|w| = n$, we can take well-defined inverse branches of f_w^{-1} on $B(a, \varepsilon)$ and any inverse branch γ of f_w^{-1} on $B(a, \varepsilon)$ satisfies $\gamma(B(a, \varepsilon)) \subset B\big(\gamma(a), \frac{1}{2}\varepsilon\big)$ and $\|\gamma'(y)\| \leq \frac{1}{2}$ for each $y \in B(a, \varepsilon)$. Taking a small enough ε, it follows that for each $a \in J(G)$ and each $w \in \mathscr{W}^*$, we can take well-defined inverse branches γ of f_w^{-1} on $B(a, \varepsilon)$ and we have

$$\sup\{\|\gamma'(y)\| \mid y \in B(a, \varepsilon), a \in J(G), \gamma\text{: a branch of } f_w^{-1}, |w| = n\} \to 0$$

as $n \to \infty$. Let $y \in \bar{\mathbf{C}}$ be a point such that $g(y) = y$ and $|g'(y)| < 1$ for some $g \in G$. Suppose that there exist an element $h \in G$ and a point $a \in J(G)$ such that $h(y) \in B(a, \varepsilon)$. Let γ_n be a well-defined inverse branch of $(hg^n)^{-1}$ on $B(a, \varepsilon)$ such that $\gamma_n(hg^n(y)) = \gamma_n(h(y)) = y$. Then $|\gamma_n'(y)| \to \infty$ as $n \to \infty$. This contradicts the previous argument. Hence $A(G) \subset \bar{\mathbf{C}} \backslash B(J(G), \varepsilon) \subset F(G)$.

Next, suppose that there exists a sequence (z_j) in $\tilde{F}(\tilde{f})$ such that $\tilde{f}^{n_j}(z_j) = z$ and $z_j \to z_\infty \in \tilde{F}(\tilde{f})$ where $n_j \in \mathbf{N}$ with $n_j \to \infty$ as $j \to \infty$. Then, by Lemma 3.13, there exists a number $n \in \mathbf{N}$ such that $\pi_{\bar{\mathbf{C}}}(\tilde{f}^n(z_\infty)) \in F(G)$. Let $x_j = \pi_{\bar{\mathbf{C}}}(\tilde{f}^n(z_j))$ for each $j \in \mathbf{N}$ and let $x_\infty = \pi_{\bar{\mathbf{C}}} \tilde{f}^n(z_\infty)$. Then, for each j with $n_j > n$, there exists an element $g_j \in G$ such that $g_j(x_j) = x$. Let $\alpha = d(x, A(G))$. Since $x_j \to x_\infty \in F(G)$, we have $\#\left\{ j \mid d(g_j(x_\infty), x) < \dfrac{\alpha}{2} \right\} = \infty$.

By contrast, we have $\sup\{d(f_w(x_\infty), A(G)) \mid |w| = n\} \to 0$ as $n \to \infty$. For, if $P(G) \neq \emptyset$, the above follows from Theorem 1.34 in [S3]. Even if $P(G) = \emptyset$, since G is expanding, by the Koebe distortion theorem, then for each $z \in F(G)$, $\bigcup_{g \in G} g(z) \subset F(G)$. Using the same argument as in the proof of Theorem 1.34 in [S3], we obtain the above.

Hence, we obtain a contradiction. Therefore, we have shown that for each open neighborhood V of $\tilde{J}(\tilde{f})$ in $\Sigma_m \times \bar{\mathbf{C}}$, there exists a number $l \in \mathbf{N}$ such that $\bigcup_{n \geq l} (\tilde{f}^n)^{-1}(z) \subset V$. If $x \in \bar{\mathbf{C}} \backslash (A(G) \cup P(G))$, then since G is expanding, combining the above result with the Koebe distortion theorem, we obtain $T(x) < \infty$. $\square$

Definition 3.15. Let E be a subset of $\bar{\mathbf{C}}$, $t \geq 0$ a number and $\beta > 0$ a number. We set

$$H_\beta^t(E) := \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(U_i))^t \mid \mathrm{diam}(U_i) \leq \beta, E \subset \bigcup_{i=1}^{\infty} U_i \right\}$$

and $H^t(E) = \lim_{\beta \to 0} H_\beta^t(E)$ with respect to the spherical metric on $\bar{\mathbf{C}}$. $H^t(E)$ is called the t-dimensional (outer) Hausdorff measure of E with respect to the spherical metric. Note that $H^t(E)$ is a Borel regular measure on $\bar{\mathbf{C}}$ (see [R]). We set $\dim_H(E) := \sup\{t \geq 0 \mid H^t(E) = \infty\} = \inf\{t \geq 0 \mid H^t(E) = 0\}$. $\dim_H(E)$ is called the Hausdorff dimension of E. Furthermore, let $N_r(E)$ be the smallest number of sets of spherical diameter r that can cover E. We set $\overline{\dim}_B(E) = \limsup_{r \to 0} \dfrac{\log N_r(E)}{-\log r}$. $\overline{\dim}_B(E)$ is called the upper box dimension of E.

Lemma 3.16. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let τ be a t-subconformal measure. Then, there exists a positive constant c such that for each r with $0 < r < \mathrm{diam}\, \bar{\mathbf{C}}$ and each $x \in J(G)$, we have $\tau(B(x, r)) \geq cr^t$. Furthermore, $H^t|_{J(G)}$ is absolutely continuous with respect to τ, $H^t(J(G)) < \infty$ and $\overline{\dim}_B(J(G)) \leq t$.*

Proof. Let τ be a t-subconformal measure. Using the argument in the proof of Theorem 3.4 in [S2], we find that there exists a positive constant c such that for each r with $0 < r < \mathrm{diam}\, \bar{\mathbf{C}}$ and each $x \in J(G)$, $\tau(B(x, r)) \geq cr^t$. (Note that for an estimate of this type, we need only expandingness and we do not need the strong open set condition used in [S2].) By Proposition 2.2 in [F], we find that $H^t|_{J(G)}$ is absolutely continuous with respect to τ. In particular, $H^t(J(G)) < \infty$. Furthermore, by Theorem 7.1 in [Pe], we get $\overline{\dim}_B(J(G)) \leq t$. $\square$

Using these arguments, we now demonstrate Main Theorem A.

Proof of Main Theorem A. By Lemma 3.7, we have that the function $P(t)$ has a unique zero δ, there exists a unique probability measure $\tilde{\nu}$ on $\tilde{J}(f)$ such that $M_\delta^* \tilde{\nu} = \tilde{\nu}$, and δ satisfies (2).

By Lemma 3.16, $\overline{\dim}_B(J(G)) \le s(G)$. By Lemma 3.14, we have $A(G) \cup P(G) \subset F(G)$. Let $x \in \bar{\mathbf{C}} \setminus (A(G) \cup P(G))$ be a point. According to Theorem 4.2 in [S2] and the fact that $S(x) \le T(x) < \infty$ (Lemma 3.14), we obtain $s(G) \le s_0(G) \le S(x) \le T(x)$.

We show $\delta = T(x)$. We consider the following two cases:

Case 1. $T(T(x), x) = \infty$.

Case 2. $T(T(x), x) < \infty$.

Suppose we have Case 1. Let $z \in \Sigma_m \times \bar{\mathbf{C}}$ be a point with $\pi_{\bar{\mathbf{C}}}(z) = x$. Let t_n be a sequence of real numbers such that $t_n > T(x)$ for each $n \in \mathbf{N}$ and $t_n \to T(x)$. For each $n \in \mathbf{N}$, let μ_n be a Borel probability measure on $\Sigma_m \times \bar{\mathbf{C}}$ defined by:

$$\mu_n = \frac{1}{T(t_n, x)} \sum_{p \in \mathbf{N}} \sum_{\tilde{f}^p(z') = z} \|(\tilde{f}^p)'(z')\|^{-t_n} \delta_{z'},$$

where $\delta_{z'}$ denotes the Dirac measure concentrated at z'. Since the space of Borel probability measures on $\Sigma_m \times \bar{\mathbf{C}}$ is compact, we may assume that there exists a Borel probability measure μ_∞ on $\Sigma_m \times \bar{\mathbf{C}}$ such that $\mu_n \to \mu_\infty$ as $n \to \infty$, with respect to the weak topology. Then, by Lemma 3.14, we have supp $\mu_\infty \subset \tilde{J}(\tilde{f})$. We now show the following claim:

Claim 1. For any Borel set A in $\tilde{J}(\tilde{f})$ such that $\tilde{f} : A \to \tilde{f}(A)$ is injective, we have $\mu_\infty(\tilde{f}(A)) = \int_A \|(\tilde{f})'\|^{T(x)} \, d\mu_\infty$.

To show this claim, let A be a Borel set in $\Sigma_m \times \bar{\mathbf{C}}$ such that $\tilde{f} : A \to \tilde{f}(A)$ is injective. Then, $\mu_n(\tilde{f}(A)) = \int_A \|(\tilde{f})'\|^{t_n} \, d\mu_n - \dfrac{1}{T(t_n, x)} \#(\tilde{f}^{-1}(z) \cap A)$. If A satisfies that $\mu_\infty(\partial \tilde{f}(A)) = \mu_\infty(\partial A) = 0$, then letting $n \to \infty$ in the above, it follows that $\mu_\infty(\tilde{f}(A)) = \int_A \|(\tilde{f})'\|^{T(x)} \, d\mu_\infty$.

Now let B be a general Borel set in $\tilde{J}(\tilde{f})$ such that $\tilde{f} : B \to \tilde{f}(B)$ is injective. Then, let $B = \sum_{j \in \mathbf{N}} B_j$ be a countable disjoint union of Borel sets B_j satisfying the fact that for each $j \in \mathbf{N}$, there exists an open neighborhood W_j of $\overline{B_j}$ in $\Sigma_m \times \bar{\mathbf{C}}$ such that $\tilde{f} : W_j \to \tilde{f}(W_j)$ is a homeomorphism. Let j be a fixed number and K a fixed compact subset of W_j. Then, for each $n \in \mathbf{N}$, there exists a number $\varepsilon_n > 0$ such that the set $V_n := \{z \in \Sigma_m \times \bar{\mathbf{C}} \mid d(z, K) < \varepsilon_n\}$ satisfies $V_n \subset W_j$, $\mu_\infty(\partial V_n) = \mu_\infty(\partial(\tilde{f}(V_n))) = 0$, $\mu_\infty(\tilde{f}(V_n) \setminus \tilde{f}(K)) < \dfrac{1}{n}$ and $\mu_\infty(V_n \setminus K) < \dfrac{1}{n}$. For these sets V_n, by the previous argument, we have $\mu_\infty(\tilde{f}(V_n)) = \int_{V_n} \|(\tilde{f})'\|^{T(x)} \, d\mu_\infty$. Letting $n \to \infty$, we obtain $\mu_\infty(\tilde{f}(K)) = \int_K \|(\tilde{f})'\|^{T(x)} \, d\mu_\infty$.

Next, for each $l \in \mathbf{N}$, we can take a compact subset K_l of B_j such that $\mu_\infty(B_j \backslash K_l) < \frac{1}{l}$ and $\mu_\infty(\tilde{f}(B_j) \backslash \tilde{f}(K_l)) < \frac{1}{l}$. For these sets K_l, using the above argument, we have $\mu_\infty(\tilde{f}(K_l)) = \int_{K_l} \|(\tilde{f})'\|^{T(x)}\, d\mu_\infty$. Letting $l \to \infty$, we obtain $\mu_\infty(\tilde{f}(B_j)) = \int_{B_j} \|(\tilde{f})'\|^{T(x)}\, d\mu_\infty$. Since $B = \sum_j B_j$ and $\tilde{f}$ is injective on B, we obtain $\mu_\infty(\tilde{f}(B)) = \int_B \|(\tilde{f})'\|^{T(x)}\, d\mu_\infty$. Hence, we have shown Claim 1.

Using Claim 1 and Proposition 2.2 in [DU], it follows that $L^*_{T(x)\tilde{\varphi}}\mu_\infty = \mu_\infty$.

We now show that $\delta = T(x)$. Suppose $\delta < T(x)$. Then, by Lemma 3.7-1, we have $P(T(x)) < 0$. Then, for each $\psi \in C(\tilde{J}(\tilde{f}))$, we have $\mu_\infty(\psi) = (\exp P(T(x)))^l \cdot \mu_\infty\left(\frac{L_{T(x)\tilde{\varphi}}\psi}{(\exp(P(T(x))))^l}\right) \to 0$ as $l \to \infty$, by Lemma 3.6-2. Hence, $\mu_\infty(\psi) = 0$ and this implies a contradiction. Suppose $T(x) < \delta$. Then, by a similar argument to the one above, we get a contradiction. Hence, $T(x) = \delta$.

We now consider Case 2: $T(T(x), x) < \infty$. Let $z \in \Sigma_m \times \bar{\mathbf{C}}$ be a point with $x = \pi_{\bar{\mathbf{C}}}(z)$. Then, we take Patterson's function ([Pa]) Φ: i.e., Φ is a continuous, non-decreasing function from $\mathbf{R}_+ := \{t \in \mathbf{R} \mid t \geq 0\}$ to $\mathbf{R}_+$ that satisfies the following:

1. $Q(t) := \sum_n \sum_{\tilde{f}^n(z')=z} \Phi(\|(\tilde{f}^n)'(z')\|)\|(\tilde{f}^n)'(z')\|^{-t}$ converges for each $t > T(x)$ and does not converge for each $t \leq T(x)$.
2. For each $\varepsilon > 0$, there is a number $r_0 \in \mathbf{R}_+$ such that $\Phi(rs) \leq s^\varepsilon \Phi(r)$ for each $r > r_0$ and each $s > 1$.

Let t_n be a sequence of $\mathbf{R}$ such that $t_n > T(x)$ for each $n \in \mathbf{N}$, $t_n \to T(x)$ as $n \to \infty$ and the measures:

$$\tau_n := \frac{1}{Q(t_n)} \sum_p \sum_{\tilde{f}^p(z')=z} \Phi(\|(\tilde{f}^p)'(z')\|)\|(\tilde{f}^p)'(z')\|^{-t_n}\delta_{z'}$$

tend to a Borel probability measure τ_∞ on $\Sigma_m \times \bar{\mathbf{C}}$ as $n \to \infty$. Then, by Lemma 3.14, we have $\operatorname{supp} \tau_\infty \subset \tilde{J}(\tilde{f})$. Furthermore, combining the argument in the proof of Claim 1 in Case 1 with the properties of Φ, we find that for each Borel set A in $\tilde{J}(\tilde{f})$ such that $\tilde{f} : A \to \tilde{f}(A)$ is injective, $\tau_\infty(\tilde{f}(A)) = \int_A \|(\tilde{f})'\|^{T(x)}\, d\tau_\infty$. Combining this with the argument used in Case 1, we obtain $\delta = T(x)$.

Since G is expanding and ν is δ-subconformal (Lemma 3.11), using an argument in the proof of Theorem 4.4 in [S2], we obtain $\operatorname{supp} \nu \supset J(G)$. Hence, $\operatorname{supp} \nu = J(G)$.

Hence, we have shown Main Theorem A. □

Corollary 3.17. *Let $G = \langle f_1, f_2, \ldots f_m \rangle$ be a finitely generated expanding rational semigroup. Then,* $\overline{\dim}_B(J(G)) \leq \delta \leq \dfrac{\log(\sum_{j=1}^m \deg(f_j))}{\log \lambda}$, *where λ denotes the number in Definition* 3.1.

Proof. By Main Theorem A and (2), we have

$$\begin{aligned}\overline{\dim}_B(J(G)) \le \delta &\le \frac{\log(\sum_{j=1}^m \deg(f_j))}{-\int_{\tilde{J}(\tilde{f})} \tilde{\varphi}\alpha \, d\tilde{\nu}} \\ &= \frac{n \log(\sum_{j=1}^m \deg(f_j))}{-\int_{\tilde{J}(\tilde{f})} S_n\tilde{\varphi}\alpha \, d\tilde{\nu}} \\ &\le \frac{n \log(\sum_{j=1}^m \deg(f_j))}{\log C + n \log \lambda},\end{aligned}$$

for each $n \in \mathbf{N}$. Letting $n \to \infty$, we obtain the result. □

4. Conformal measure

In this section we introduce the notion of "conformal measure", which is needed in Main Theorem B.

Definition 4.1. 1. Let G be a rational semigroup. Let $t \in \mathbf{R}$ with $t \ge 0$. We say that a Borel probability measure τ on $J(G)$ is t-**conformal** (for G) if for any Borel set A and $g \in G$, if $A, g(A) \subset J(G)$ and $g : A \to g(A)$ is injective, then

$$\tau(g(A)) = \int_A \|g'\|^t \, d\tau.$$

2. Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated rational semigroup. We say that a Borel probability measure μ on $J(G)$ satisfies the **separating condition for** $\{f_1, \ldots, f_m\}$ if $\mu(f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))) = 0$ for any (i, j) with $i, j \in \{1, \ldots, m\}$ and $i \ne j$.

We show some fundamental properties of conformal measures.

Lemma 4.2. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Let τ be a t-conformal measure. Then, τ is a t-subconformal measure.*

Proof. Let A be a Borel set in $\bar{\mathbf{C}}$ and g an element of G. Let $J(G) = \sum B_i$ be a measurable partition of $J(G)$ such that we can take the well-defined inverse branches of g^{-1} on B_i, for each i (we divide $J(G)$ into $\{B_i\}$ so that for a critical value $c \in J(G)$ of g, there exists an i such that $B_i = \{c\}$). Let $\{C_{i,j}\}_j$ be the images of B_i using the inverse branches of g^{-1} so that $g : C_{i,j} \to B_i$ is bijective for each j. Then, we have

$$\begin{aligned}\tau(g(A)) = \tau(g(A) \cap J(G)) &= \sum_i \tau(g(A) \cap B_i) \le \sum_{i,j} \tau(g(A \cap C_{i,j})) \\ &= \sum_{i,j} \int_{A \cap C_{i,j}} \|g'\|^t \, d\tau = \int_{A \cap \cup_{i,j} C_{i,j}} \|g'\|^t \, d\tau \le \int_A \|g'\|^t \, d\tau.\end{aligned}$$

Hence, τ is t-subconformal. □

Lemma 4.3. *Let G be a rational semigroup. Let τ be a Borel probability measure on $J(G)$, $g \in G$ an element, and V an open set in $\bar{\mathbf{C}}$ with $V \cap g^{-1}(J(G)) \neq \emptyset$. Suppose that $g : V \to g(V)$ is a homeomorphism and that for any Borel set A in $V \cap g^{-1}(J(G))$, $\tau(g(A)) = \int_A \|g'\|^t \, d\tau$. Let $h := (g|_V)^{-1} : g(V) \to V$. Then, we find that for any Borel set B in $g(V) \cap J(G)$, $\tau(h(B)) = \int_B \|h'\|^t \, d\tau$.*

Proof. Let $\mu := h_*(\tau|_{g(V)\cap J(G)})$. Then, by the assumption, $d\mu = \|g'\|^t \, d\tau'$, where $\tau' = \tau|_{V \cap g^{-1} J(G)}$. Let B be a Borel set in $g(V) \cap J(G)$. Then, $\tau(h(B)) = \int_{h(B)} \|g'\|^{-t} \cdot \|g'\|^t \, d\tau = \int_{h(B)} \|g'\|^{-t} \, d\mu = \int_B \|g'\|^{-t} \circ h \, d\tau = \int_B \|h'\|^t \, d\tau$. □

Lemma 4.4. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be the skew product map associated with $\{f_1, \dots, f_m\}$. Let $\tilde{\tau}$ be a Borel probability measure on $\tilde{J}(\tilde{f})$, $n \in \mathbf{N}$ an integer, and V an open set in $\Sigma_m \times \bar{\mathbf{C}}$ such that $V \cap \tilde{J}(\tilde{f}) \neq \emptyset$. Suppose that $\tilde{f}^n : V \to \tilde{f}^n(V)$ is a homeomorphism and that for any Borel set A in $\tilde{J}(\tilde{f})$, $\tilde{\tau}(\tilde{f}^n(A)) = \int_A \|(\tilde{f}^n)'\|^t \, d\tilde{\tau}$. Let $h = (\tilde{f}^n|_V)^{-1} : \tilde{f}^n(V) \to V$. Then, we obtain the result that for any Borel set B in $\tilde{f}^n(V) \cap \tilde{J}(\tilde{f})$, $\tilde{\tau}(h(B)) = \int_B \|(\tilde{f}^n)'(h)\|^{-t} \, d\tilde{\tau}$.*

Proof. This lemma can be shown using the same method as in the proof of Lemma 4.3. □

Lemma 4.5. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let τ be a t-conformal measure satisfying the separating condition for $\{f_1, \dots, f_m\}$. Suppose that for any $g \in G$, if c is a critical point of g with $g(c) \in J(G)$, then $\tau(\{c\}) = 0$. Then, for any $k \in \mathbf{N}$, $\tau(f_w^{-1}(J(G)) \cap f_{w'}^{-1}(J(G))) = 0$ for any $w = (w_1, \dots, w_k)$, $w' = (w'_1, \dots, w'_k) \in \{1, \dots, m\}^k$ with $w \neq w'$.*

Proof. Let $w = (w_1, \dots, w_k)$, $w' = (w'_1, \dots, w'_k) \in \{1, \dots, m\}^k$ with $w \neq w'$. Let $1 \le u \le k$ be the maximum such that $w_u \neq w'_u$. If $u = k$, then $\tau(f_w^{-1}(J(G)) \cap f_{w'}^{-1}(J(G))) \le \tau(f_{w_k}^{-1}(J(G)) \cap f_{w'_k}^{-1}(J(G))) = 0$. Suppose that $u < k$. Let $g = f_{w_{u+1}} \cdots f_{w_k} = f_{w'_{u+1}} \cdots f_{w'_k}$. Then, $f_w^{-1}(J(G)) \cap f_{w'}^{-1}(J(G)) \subset g^{-1}(f_{w_u}^{-1}(J(G)) \cap f_{w'_u}^{-1}(J(G)))$. By Lemma 4.3, we have $\tau(g^{-1}(f_{w_u}^{-1}(J(G)) \cap f_{w'_u}^{-1}(J(G)))) = 0$. Hence, we obtain $\tau(f_w^{-1}(J(G)) \cap f_{w'}^{-1}(J(G))) = 0$. □

Definition 4.6. Let $G = \langle f_1, \dots, f_m \rangle$ be a rational semigroup. Suppose that for each $g \in G$, no critical value of g exists in $J(G)$. Let $t \in \mathbf{R}$. We define an operator $N_t : C(J(G)) \to C(J(G))$ as follows:

$$(N_t \psi)(z) = \sum_{j=1}^{m} \sum_{f_j(y)=z} \|f_j'(y)\|^{-t} \psi(y) \quad \text{for each } \psi \in C(J(G)).$$

Lemma 4.7. *Let $G = \langle f_1, \dots, f_m \rangle$ be a rational semigroup. Suppose that for each $g \in G$, no critical value of g exists in $J(G)$. Let $\tilde{f} : \Sigma_m \times \bar{\mathbf{C}} \to \Sigma_m \times \bar{\mathbf{C}}$ be* *the skew product map associated with* $\{f_1,\dots,f_m\}$. *Then, we have the following commutative diagram*:

$$\begin{array}{ccc} C(J(G)) & \xrightarrow{N_t} & C(J(G)) \\ {\scriptstyle (\pi_{\bar{\mathbb{C}}})^*}\big\downarrow & & \big\downarrow{\scriptstyle (\pi_{\bar{\mathbb{C}}})^*} \\ C(\tilde{J}(\tilde{f})) & \xrightarrow[L_{t\tilde{\varphi}}]{} & C(\tilde{J}(\tilde{f})). \end{array}$$

Proof. Let $\psi \in C(\tilde{J}(\tilde{f}))$ and $(w,x) \in \tilde{J}(\tilde{f})$. Then, $((\pi_{\bar{\mathbb{C}}})^* N_t\psi)((w,x)) = (N_t\psi)(x) = \sum_{j=1}^m \sum_{f_j(y)=x} \|f_j'(y)\|^{-t}\psi(y)$. Conversely, $(L_{t\tilde{\varphi}}(\pi_{\bar{\mathbb{C}}})^*\psi)((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \|f_{w_1'}'(y)\|^{-t}((\pi_{\bar{\mathbb{C}}})^*\psi)((w',y)) = \sum_{j=1}^m \sum_{f_j(y)=x} \|f_j'(y)\|^{-t}\psi(y)$. □

LEMMA 4.8. *Let* $G = \langle f_1,\dots,f_m\rangle$ *be a rational semigroup. Suppose that for each* $g \in G$, *no critical value of* g *exists in* $J(G)$. *Then, we have the following*:

1. *Let* τ *be a* t*-conformal measure. Then, we have* $N_t^*\tau \geq \tau$; *i.e., for each* $\psi \in C(J(G))$ *such that* $0 \leq \psi(z)$ *for each* $z \in J(G)$, *we have* $(N_t^*\tau)(\psi) \geq \tau(\psi)$.
2. *If* τ *is a* t*-conformal measure satisfying the separating condition for* $\{f_1,\dots,f_m\}$, *then* $N_t^*\tau = \tau$.
3. *If* τ *is a* t*-conformal measure satisfying* $N_t^*\tau = \tau$, *then* τ *satisfies the separating condition for* $\{f_1,\dots,f_m\}$.

Proof. Let $J(G) = \sum_{i=1}^u B_i$ be a measurable partition of $J(G)$ such that for each $j = 1,\dots,m$ and $i = 1,\dots,u$, we can take the well-defined inverse branches of f_j^{-1} on B_i. Then, for any Borel probability measure τ on $J(G)$ and any $\psi \in C(J(G))$, we have

$$\begin{aligned}\int N_t\psi\, d\tau &= \int \sum_{j=1}^m \sum_{f_j(y)=z} \|f_j'(y)\|^{-t}\psi(y)\, d\tau(z)\\ &= \sum_j \sum_i \sum_\gamma \int_{B_i} \|f_j'(\gamma(z))\|^{-t}\psi(\gamma(z))\, d\tau(z),\end{aligned}$$

where γ runs over all inverse branches of f_j^{-1} on B_i. Suppose that τ is t-conformal. Then, we have

$$\begin{aligned}\int_{B_i} \|f_j'(\gamma(z))\|^{-t}\psi(\gamma(z))\, d\tau(z) &= \int_{\gamma(B_i)} \|f_j'(x)\|^{-t}\psi(x)\, d(\gamma_*(\tau|_{B_i}))(x)\\ &= \int_{\gamma(B_i)} \|f_j'(x)\|^{-t}\psi(x)\cdot \|f_j'(x)\|^{t}\, d\tau(x)\\ &= \int_{\gamma(B_i)} \psi(x)\, d\tau(x).\end{aligned}$$

Hence, $\int N_t\psi\, d\tau = \sum_j \sum_i \sum_\gamma \int_{\gamma(B_i)} \psi\, d\tau$, which is larger than or equal to $\int_{J(G)} \psi\, d\tau$ if $0 \le \psi(z)$ for each $z \in J(G)$, since $J(G) = \bigcup_{j=1}^m f_j^{-1}(J(G))$ (Lemma 1.1.4 in [S1]). Furthermore, if τ is a t-conformal measure satisfying the separating condition for $\{f_1, \dots, f_m\}$, then for each $\psi \in C(J(G))$, we have $\int N_t\psi\, d\tau = \sum_j \sum_i \sum_\gamma \int_{\gamma(B_i)} \psi\, d\tau = \int_{J(G)} \psi\, d\tau$, by $J(G) = \bigcup_{j=1}^m f_j^{-1}(J(G))$.

We now show the statement 3. Let τ be a t-conformal measure satisfying $N_t^*\tau = \tau$. Let $\psi \in C(J(G))$ be an element with $\psi(x) \ge 0$ for each $x \in J(G)$. Then, by the above argument, it follows that $\int_{J(G)} N_t\psi\, d\tau = \sum_j \sum_i \sum_\gamma \int_{\gamma(B_i)} \psi\, d\tau \ge \int_{J(G)} \psi\, d\tau$, where γ runs over all inverse branches of f_j^{-1} on B_i. Since $N_t^*\tau = \tau$, we have the equality shown above. Hence, τ satisfies the separating condition for $\{f_1, \dots, f_m\}$. $\square$

LEMMA 4.9. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let δ be the number in Lemma 3.7, $t \ge 0$ a number, and $\tilde{\nu}_{t\tilde{\varphi}}$ the Borel probability measure on $\tilde{J}(\tilde{f})$ that is obtained in Lemma 3.6 (the unique fixed point of $L_{t\tilde{\varphi}}^*$). Let $\nu_t := (\pi_{\bar{\mathbf{C}}})_* \tilde{\nu}_{t\tilde{\varphi}}$. Then, we have the following:*

1. *$\nu := \nu_\delta$ satisfies $N_\delta^* \nu = \nu$.*
2. *$\dfrac{1}{(\exp(P(t)))^l} N_t^l \psi \to \nu_t(\psi) \cdot \lim_{l\to\infty} N_t^l 1$ in $C(J(G))$, where $P(t) = P(\tilde{f}|_{\tilde{J}(\tilde{f})}, t\tilde{\varphi})$.*
3. *If τ is a Borel probability measure on $J(G)$ such that $N_t^*\tau = \tau$, then $t = \delta$ and $\tau = \nu$.*

Proof. By Lemma 4.7, we obtain the statement 1. Since $\pi_{\bar{\mathbf{C}}}(\tilde{J}(\tilde{f})) = J(G)$ (Proposition 3.2 in [S5]), we find that $(\pi_{\bar{\mathbf{C}}})^* : C(J(G)) \to C(\tilde{J}(\tilde{f}))$ is an isometry with respect to the supremum norms. Hence, by Lemma 3.6 and Lemma 4.7, we find that $\left\{ \dfrac{1}{(\exp(P(t)))^l} N_t^l \psi \right\}_{l \in \mathbf{N}}$ is a Cauchy sequence in $C(J(G))$. Let $\psi_0 = \lim_{l\to\infty} \dfrac{1}{(\exp(P(t)))^l} N_t^l \psi$. Then, by Lemma 3.6, we obtain

$$
\begin{aligned}
(\pi_{\bar{\mathbf{C}}})^* \psi_0 &= \lim_{l\to\infty} \frac{1}{(\exp(P(t)))^l} L_{t\tilde{\varphi}}^l (\pi_{\bar{\mathbf{C}}})^* \psi \\
&= \tilde{\nu}_{t\tilde{\varphi}}((\pi_{\bar{\mathbf{C}}})^* \psi) \cdot \alpha_{t\tilde{\varphi}} \\
&= \nu_t(\psi) \cdot \lim_{l\to\infty} L_{t\tilde{\varphi}}^l (\pi_{\bar{\mathbf{C}}})^* 1 \\
&= (\pi_{\bar{\mathbf{C}}})^* \left(\nu_t(\psi) \cdot \lim_{l\to\infty} N_t^l 1 \right).
\end{aligned}
$$

Hence, we obtain $\psi_0 = \nu_t(\psi) \cdot \lim_{l\to\infty} N_t^l 1$.

Now, let τ be a Borel probability measure on $J(G)$ such that $N_t^*\tau = \tau$. Then for any $\psi \in C(J(G))$, we have $\tau(\psi) = ((N_t^l)^*\tau)(\psi) = \tau(N_t^l\psi) =$

$((\exp(P(t)))^l) \cdot \tau\left(\frac{1}{(\exp(P(t)))^l} N_t^l \psi\right)$ for any $l \in \mathbf{N}$. Since $\frac{1}{(\exp(P(t)))^l} N_t^l \psi \to$ $\nu_t(\psi) \cdot \lim_{l\to\infty} N_t^l 1$ and $N_t^* \tau = \tau$, we have $\tau\left(\frac{1}{(\exp(P(t)))^l} N_t^l \psi\right) \to \nu_t(\psi)$ as $l \to \infty$. Hence, it must be true that $P(t) = 0$, otherwise, we have $\tau(\psi) = 0$ for all ψ or $\tau(\psi)$ is not bounded, both of which produce a contradiction. Hence, it follows that $t = \delta$. Further, by the above argument, we obtain $\tau(\psi) = \nu_\delta(\psi)$ for any $\psi \in C(J(G))$. $\square$

LEMMA 4.10. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then, under the notation in Lemma* 4.9, *we have the following*:

1. *If there exists a t-conformal measure τ, then $s(G) \le t \le \delta$.*
2. *If there exists a t-conformal measure τ satisfying the separating condition for $\{f_1, \dots, f_m\}$, then $t = \delta$ and $\tau = \nu$.*

Proof. First, we show the statement 1. By Lemma 4.8, we have $N_t^* \tau \ge \tau$. Hence, for each $\psi \in C(J(G))$ such that $0 \le \psi(z)$ for each $z \in J(G)$, we have $\tau(N_t^l \psi) \ge \tau(\psi)$ for each $l \in \mathbf{N}$. Suppose that $t > \delta$. Then, by Lemma 3.7-1, $P(t) < 0$. Hence, we obtain $\tau(N_t^l \psi) = (\exp(P(t)))^l \cdot \tau\left(\frac{N_t^l \tau}{(\exp(P(t)))^l}\right) \to 0$ as $l \to \infty$, by Lemma 4.9-2. Hence, $\tau(\psi) = 0$ for each $\psi \in C(J(G))$ such that $0 \le \psi(z)$ for each $z \in J(G)$. This is a contradiction, since $\tau(1) = 1$. Hence, $t \le \delta$ must hold. By Lemma 4.2, we have $s(G) \le t$. Hence, the statement 1 holds.

Next, we show the statement 2. By Lemma 4.8-2, we have $N_t^* \tau = \tau$. Hence, by Lemma 4.9-3, it follows that $t = \delta$ and $\tau = \nu$. $\square$

LEMMA 4.11. *Let G be a rational semigroup and $t \ge 0$ a number. Suppose that $0 < H^t(J(G)) < \infty$. Let $\tau = \frac{H^t|_{J(G)}}{H^t(J(G))}$. Then, τ is a t-conformal measure.*

Proof. Suppose that $t = 0$. Then, each point $z \in \overline{\mathbf{C}}$ satisfies $H^0(\{z\}) = 1$. Since we assume $0 < H^t(J(G)) < \infty$, it follows that $1 \le \#(J(G)) < \infty$. Then, G consists of degree 1 maps and it is easy to see that τ is 0-conformal.

Suppose that $t > 0$. Then, H^t has no point mass. Let $g \in G$ be an element.

Step 1: For a critical point c of g in $J(G)$, we have $0 = \tau(g(\{c\})) = \int_{\{c\}} \|g'\|^t \, d\tau$.

Step 2: Let W be a non-empty open set in $\overline{\mathbf{C}}$ such that $g : W \to g(W)$ is a diffeomorphism. Let K be a compact subset of W and $c > 0$ a number. Let A be a Borel set such that $A \subset \{z \in g^{-1}(J(G)) \mid d(z, A) < c\} \subset K \cap g^{-1}(J(G))$. Then, we show the following claim:

CLAIM 1. We have $H^t(g(A)) = \int_A \|g'\|^t \, dH^t$.

To show this claim, let $\varepsilon > 0$ be a given number. Let $K = \sum_{i=1}^{l} K_i$ be a disjoint union of Borel sets K_i, $\{z_i\}_{i=1}^{l}$ a set with $z_i \in K_i$ for each i, and $\xi > 0$ a real number, such that:

1. $1 - \varepsilon \leq \dfrac{\|g'(z)\|}{\|g'(z_i)\|} \leq 1 + \varepsilon$, for each $z \in B(K_i, \xi)$, and
2. $(1 - \varepsilon)\|g'(z_i)\| \text{ diam } C \leq \text{diam } g(C) \leq (1 + \varepsilon)\|g'(z_i)\| \text{ diam } C$, for each subset C of $B(K_i, \xi)$.

Let $i \in \mathbf{N}$ $(1 \leq i \leq l)$ be a fixed number. Let β be a number with $0 < \beta < \xi$. Let $\{U_p\}_{p=1}^{\infty}$ be a sequence of sets such that $A \cap K_i \subset \bigcup_{p=1}^{\infty} U_p$, $A \cap K_i \cap U_p \neq \emptyset$ for each $p \in \mathbf{N}$ and diam $U_p \leq \beta$ for each $p \in \mathbf{N}$. Then, since $\beta < \xi$, we have $U_p \subset B(K_i, \xi)$ for each $p \in \mathbf{N}$. Hence, we obtain $g(A \cap K_i) \subset \bigcup_{p=1}^{\infty} g(U_p)$, $\text{diam } g(U_p) \leq (1 + \varepsilon)\|g'(z_i)\| \text{ diam } U_p$ for each $p \in \mathbf{N}$ and $\sum_{p=1}^{\infty}(\text{diam } g(U_p))^t \leq (1 + \varepsilon)^t \|g'(z_i)\|^t \sum_{p=1}^{\infty}(\text{diam } U_p)^t$. This implies that $H^t_{(1+\varepsilon)\|g'(z_i)\|\beta}(g(A \cap K_i)) \leq (1 + \varepsilon)^t \|g'(z_i)\|^t \sum_{p=1}^{\infty}(\text{diam } U_p)^t$. Hence, we obtain $H^t_{(1+\varepsilon)\|g'(z_i)\|\beta}(g(A \cap K_i)) \leq (1 + \varepsilon)^t \|g'(z_i)\|^t H^t_\beta(A \cap K_i)$. Then, we obtain $H^t(g(A \cap K_i)) \leq (1 + \varepsilon)^t \|g'(z_i)\|^t \cdot H^t(A \cap K_i)$, letting $\beta \to 0$. Similarly, we obtain $H^t(A \cap K_i) \leq (1 - \varepsilon)^{-t}\|g'(z_i)\|^{-t} \cdot H^t(g(A \cap K_i))$. Hence, it follows that $(1 - \varepsilon)^t \|g'(z_i)\|^t H^t(A \cap K_i) \leq H^t(g(A \cap K_i)) \leq (1 + \varepsilon)^t \|g'(z_i)\|^t H^t(A \cap K_i)$. Moreover, $(1 - \varepsilon)^t \|g'(z_i)\|^t \cdot H^t(A \cap K_i) \leq \int_{A \cap K_i} \|g'\|^t \, dH^t \leq (1 + \varepsilon)^t \|g'(z_i)\|^t H^t(A \cap K_i)$. Hence, we obtain

$$\left| H^t(g(A \cap K_i)) - \int_{A \cap K_i} \|g'\|^t \, dH^t \right| \leq ((1 + \varepsilon)^t - (1 - \varepsilon)^t)\|g'(z_i)\|^t H^t(A \cap K_i).$$

This implies that $|H^t(g(A)) - \int_A \|g'\|^t \, dH^t| \leq ((1 + \varepsilon)^t - (1 - \varepsilon)^t) \cdot \max_{z \in K} \|g'(z)\|^t \cdot \sum_{i=1}^{l} H^t(A \cap K_i)$. Since this inequality holds for each $\varepsilon > 0$, it follows that $H^t(g(A)) = \int_A \|g'\|^t \, dH^t$. Hence, we have shown Claim 1.

Step 3: Let B be a general Borel subset of $g^{-1}(J(G))$ such that $g : B \to g(B)$ is injective. Let $B = \sum_{u=1}^{q} \{c_u\} \amalg \sum_{v=1}^{\infty} B_v$ be a disjoint union of Borel sets such that each c_u is a critical point of g (if one exists) and for each B_v there exists an open set W_v in $\overline{\mathbf{C}}$ such that $\overline{B_v} \subset W_v$ and $g : W_v \to g(W_v)$ is a diffeomorphism. Then, by Steps 1 and 2, we obtain $0 = \tau(\{g(c_u)\}) = \int_{\{c_u\}} \|g'\|^t \, d\tau$ for each u, and $\tau(g(B_v)) = \int_{B_v} \|g'\|^t \, d\tau$ for each v. Combining this result with the fact that $g : B \to g(B)$ is injective, it follows that $\tau(g(B)) = \sum_{u=1}^{q} \tau(\{g(c_u)\}) + \sum_{v=1}^{\infty} \tau(g(B_v)) = \sum_{v=1}^{\infty} \int_{B_v} \|g'\|^t \, d\tau = \int_B \|g'\|^t \, d\tau$.

Hence, we have shown Lemma 4.11. □

Lemma 4.12. *Let G be a rational semigroup. Let τ be a t-subconformal measure for some $t \in \mathbf{R}$. Suppose that supp $\tau = J(G)$. Let $g \in G$. Then, each Borel subset A of $g^{-1}(J(G))$ with $\tau(A) = 0$ has no interior points with respect to the induced topology on $g^{-1}(J(G))$.*

Proof. Suppose there exists an open set U of $\overline{\mathbf{C}}$ such that $A \supset U \cap g^{-1}(J(G)) \neq \emptyset$. Then, it follows that $\tau(g(U)) = \tau(g(U) \cap J(G)) = \tau(g(U \cap g^{-1}(J(G)))) \leq \int_{U \cap g^{-1}(J(G))} \|g'\|^t \, d\tau = 0$. This is a contradiction because we assume supp $\tau = J(G)$. □

The following proposition is needed to show Main Theorem B.

Proposition 4.13. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbf{C}} \to \Sigma_m \times \overline{\mathbf{C}}$ be the skew product map associated with $\{f_1, \dots, f_m\}$. Let δ be a number in Lemma 3.7. Let $v := (\pi_{\overline{\mathbf{C}}})_*(\tilde{v}_{\delta\tilde{\varphi}})$. Suppose that $0 < H^{\delta}(J(G))$. Then, we have $H^{\delta}(J(G)) < \infty$, $v = \dfrac{H^{\delta}|_{J(G)}}{H^{\delta}(J(G))}$, and v is a δ-conformal measure satisfying the separating condition with respect to $\{f_1, \dots, f_m\}$. Furthermore, $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ is nowhere dense in $f_j^{-1}(J(G))$, for each (i, j) with $i \neq j$.*

Proof. By Lemma 3.11 and Lemma 3.16, we obtain that v is a δ-subconformal measure, $H^{\delta}|_{J(G)}$ is absolutely continuous with respect to v, and $H^{t}(J(G)) < \infty$. Let $\tau := \dfrac{H^{\delta}|_{J(G)}}{H^{\delta}(J(G))}$. Let $\varphi \in L^1(v)$ be the density function such that $\tau(A) = \int_A \varphi \, dv$ for any Borel subset A of $J(G)$. We show the following claim:

Claim 1. We have $(\varphi \circ \pi_{\overline{\mathbf{C}}} \circ \tilde{f})(z) \geq (\varphi \circ \pi_{\overline{\mathbf{C}}})(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{v} := \tilde{v}_{\delta\varphi}$.

To show this claim, let j $(1 \leq j \leq m)$ be a number and A an open subset of $J(G)$ such that we can take a well-defined inverse branch γ of f_j^{-1} on A. By Lemma 4.11, τ is δ-conformal. Hence, for each Borel subset B of A, we have $\tau(B) = \int_{\gamma(B)} \|f_j'\|^{\delta} \, d\tau = \int_{\gamma(B)} \|f_j'\|^{\delta} \varphi \, dv$. Moreover, by Lemma 3.11, we have v is δ-subconformal. Hence, we obtain $\tau(B) = \int_B \varphi \, dv = \int_A (\varphi \circ f_j \circ \gamma) \cdot (1_{\gamma(B)} \circ \gamma) \, dv = \int_A (\varphi \circ f_j) \cdot 1_{\gamma(B)} \, d(\gamma_*(v|_A)) \leq \int_{\gamma(B)} \|f_j'\|^{\delta} ((\varphi \circ f_j)) \, dv$. Hence, we obtain $\varphi(x) \leq (\varphi \circ f_j)(x)$ for almost every $x \in \gamma(A)$ with respect to v. It follows that for each $j = 1, \dots, m$, we have $\varphi(x) \leq (\varphi \circ f_j)(x)$ for almost every $x \in f_j^{-1}(J(G))$ with respect to v. This implies that for each $j = 1, \dots, m$, we have $(\varphi \circ \pi_{\overline{\mathbf{C}}})(z) \leq (\varphi \circ f_j \circ \pi_{\overline{\mathbf{C}}})(z)$ for almost every $z \in \pi_{\overline{\mathbf{C}}}^{-1} f_j^{-1}(J(G))$ with respect to $\tilde{v}$. Since $\tilde{J}(\tilde{f}) = \bigcup_{j=1}^{m} \Sigma_m(j) \cap \tilde{J}(\tilde{f})$ and $\Sigma_m(j) \cap \tilde{J}(\tilde{f}) \subset \pi_{\overline{\mathbf{C}}}^{-1}(f_j^{-1}(J(G)))$ (the latter follows from $\pi_{\overline{\mathbf{C}}} \tilde{f}((w, x)) = f_j(x) = f_j(\pi_{\overline{\mathbf{C}}}((w, x)))$ for each $(w, x) \in \Sigma_m(j) \cap \tilde{J}(\tilde{f})$), it follows that $(\varphi \circ \pi_{\overline{\mathbf{C}}}(z)) \leq (\varphi \circ \pi_{\overline{\mathbf{C}}} \circ \tilde{f})(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\tilde{v}$. Hence, we have shown Claim 1.

By Claim 1, we have $(\varphi \circ \pi_{\overline{\mathbf{C}}}(z)) \leq (\varphi \circ \pi_{\overline{\mathbf{C}}} \circ \tilde{f})(z)$ for almost every $z \in \tilde{J}(\tilde{f})$ with respect to $\alpha\tilde{v}$, where α is the function in Lemma 3.7. Let $\psi = \varphi \circ \pi_{\overline{\mathbf{C}}}$. Then, we obtain for each $n \in \mathbf{N}$, $\psi(z) \leq \dfrac{1}{n} \sum_{j=0}^{n-1} \psi \circ \tilde{f}^j(z)$ for almost every z with respect to $\alpha\tilde{v}$. Note that by Lemma 3.6-3, the measure $\alpha\tilde{v}$ is $\tilde{f}$-invariant. Hence, by Birkhoff's ergodic theorem (see [DGS]), we have $\psi(z) \leq \lim_{n \to \infty} \dfrac{1}{n} \sum_{j=0}^{n-1} (\psi \circ \tilde{f}^j)(z)$ for almost every z with respect to $\alpha\tilde{v}$. Since $\int \psi\alpha \, d\tilde{v} =$

$\int \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ \tilde{f}^j)(z) \alpha \, d\tilde{\nu}(z)$, which follows from Birkhoff's ergodic theorem again, it follows that $\psi(z) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ \tilde{f}^j)(z)$ for almost every z with respect to $\alpha\tilde{\nu}$. Since $\alpha\tilde{\nu}$ is ergodic (Lemma 3.6-3), then there exists a constant c such that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ \tilde{f}^j)(z) = c$ for almost every z with respect to $\alpha\tilde{\nu}$. Hence, it follows that $\psi(z) = c$ for almost every z with respect to $\tilde{\nu}$. Since τ and ν are probability measures, it follows that $c = 1$. Hence, $\tau = \nu$. Since $N_\delta^* \nu = \nu$ (Lemma 4.9-1) and τ is δ-conformal (Lemma 4.11), by Lemma 4.8-3 it follows that $\nu = \tau$ is a δ-conformal measure satisfying the separating condition with respect to $\{f_1, \dots, f_m\}$. Since supp $\nu = J(G)$ (Main Theorem A), by Lemma 4.12, it follows that $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ is nowhere dense in $f_j^{-1}(J(G))$ for each (i, j) with $i \neq j$.

Hence, we have shown Proposition 4.13. □

Example 4.14. Let $f_1(z) = z^2$, $f_2(z) = \frac{z^2}{4}$ and $f_3(z) = \frac{z^2}{3}$. Let $G = \langle f_1, f_2, f_3 \rangle$ and $\tilde{f} : \Sigma_3 \times \overline{\mathbf{C}} \to \Sigma_3 \times \overline{\mathbf{C}}$ be the skew product map with respect to $\{f_1, f_2, f_3\}$. Then, it is easy to see $J(\langle f_1, f_2 \rangle) = \{z \mid 1 \leq |z| \leq 4\}$. Since $f_3^{-1}(J(\langle f_1, f_2 \rangle)) = \{z \mid \sqrt{3} \leq |z| \leq 2\sqrt{3}\} \subset J(\langle f_1, f_2 \rangle)$, we have $J(G) = \{z \mid 1 \leq |z| \leq 4\}$. Then, $P(G) = \{0, \infty\} \subset F(G)$. By Theorem 2.6 in [S2], we find that G is expanding. Furthermore, we have $0 < H^2(J(G)) < \infty$ and $H^2(f_1^{-1}(J(G)) \cap f_3^{-1}(J(G))) > 0$. Hence, by Proposition 4.13, the number δ in Lemma 3.7 for $\tilde{f}$ satisfies $\delta > 2$.

5. Main Theorem B

In this section, we demonstrate Main Theorem B. First, we need the following notation.

Definition 5.1. Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let U be a non-empty open set in $\overline{\mathbf{C}}$. We say that G satisfies the **open set condition** with U with respect to the generator system $\{f_1, \dots, f_m\}$ if $f_j^{-1}(U) \subset U$ for each $j = 1, \dots, m$ and $\{f_j^{-1}(U)\}_{j=1}^m$ are mutually disjoint.

Lemma 5.2. 1. *If a rational semigroup $G = \langle f_1, \dots, f_m \rangle$ satisfies the open set condition with U and $\# J(G) \geq 3$, then $J(G) \subset \overline{U}$.*

2. *If a rational semigroup $G = \langle f_1, \dots, f_m \rangle$ is expanding and if G satisfies the open set condition with U, then $J(G) \subset \overline{U}$.*

Proof. By Lemma 2.3 (f) in [S5] and Lemma 3.2, it is easy to see the statement. □

To show Main Theorem B, we need the following key lemma.

LEMMA 5.3. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated expanding rational semigroup satisfying the open set condition with an open set U with respect to $\{f_1, \dots, f_m\}$. Then, we have the following.*

1. *There exists a positive constant C such that for each r with $0 < r < \operatorname{diam} \overline{\mathbf{C}}$ and each $x \in J(G)$, we have $C^{-1} r^{\delta} \le \nu(B(x,r)) \le C r^{\delta}$. Furthermore, $0 < H^{\delta}(J(G)) < \infty$ and $\dim_H(J(G)) = \overline{\dim}_B(J(G)) = \delta$.*
2. *Suppose that there exists a t-conformal measure τ. Then, there exists a positive constant C_0 such that for any r with $0 < r < \operatorname{diam} \overline{\mathbf{C}}$ and any $x \in J(G)$, we have $C_0^{-1} r^t \le \tau(B(x,r)) \le C_0 r^t$. Furthermore, we have $0 < H^t(J(G)) < \infty$ and $\dim_H(J(G)) = t = \delta$. Moreover, ν and τ are absolutely continuous with respect to each other.*

To show this lemma, we need several other lemmas (Lemma 5.4–Lemma 5.15). We suppose the assumption of Lemma 5.3, until the end of the proof of Lemma 5.3.

Preparation to show Lemma 5.3.

1. First, we may assume that $\overline{U} \cap P(G) = \emptyset$. For, let V be a ε_0-neighborhood of $P(G)$ with respect to the hyperbolic metric on $F(G)$. Then, for each $g \in G$, we have $g(V) \subset V$, which implies that $W := U \setminus \overline{V}$ satisfies $f_j^{-1}(W) \subset W$, for each $j = 1, \dots, m$, and $\{f_j^{-1}(W)\}_j$ are mutually disjoint. Hence we may assume the above.

   Assuming that $\overline{U} \cap P(G) = \emptyset$, take a number $\varepsilon > 0$ such that $B(\overline{U}, 2\varepsilon) \cap P(G) = \emptyset$. Then for each $y \in \overline{U}$ and any $g \in G$, we can take well-defined inverse branches of g^{-1} on $B(y, 2\varepsilon)$.
2. Let $\overline{U} = \sum_{j=1}^{k} K_j$ be a measurable partition such that for each $j = 1, \dots, k$, we have $\operatorname{int} K_j \neq \emptyset$ and $\operatorname{diam} K_j \le \frac{1}{10}\varepsilon$. We take a point $z_j \in K_j$, for each $j = 1, \dots, k$.
3. To show Lemma 5.3, we may assume that: for each $j = 1, \dots, k$ and each $w \in \mathscr{W}^*$, if γ is an inverse branch of f_w^{-1} on $B(z_j, 2\varepsilon)$, then we have

$$\operatorname{diam} \gamma(A) \le \left(\frac{1}{10}\right)^{|w|} \cdot \operatorname{diam} A, \tag{3}$$

   for each subset A of $B(z_j, 2\varepsilon)$. For, for each $n \in \mathbf{N}$, G_n (see the notation in section 2.1) satisfies $J(G_n) = J(G)$. Further, if we use ν^n to denote the Borel probability measure on $J(G_n) = J(G)$ constructed by the generator system $\{f_w \mid |w| = n\}$ of G_n, for which the construction method is the same as that for ν from $\{f_j\}$, then ν^n satisfies $(N_{\delta_n}^n)^* \nu^n = \nu^n$ for some $\delta_n \in \mathbf{R}$. Since ν satisfies $(N_{\delta}^n)^* \nu = \nu$, by Lemma 4.9 we obtain $\delta_n = \delta$ and $\nu^n = \nu$. Moreover, since G is expanding, by the Koebe distortion theorem there exist numbers $\varepsilon' > 0$ and $n \in \mathbf{N}$ such that if γ is a well-defined inverse branch of f_w^{-1} on $B(z, 2\varepsilon')$, where $|w| = n$ and $z \in J(G)$, then for any subset A of $B(z, 2\varepsilon')$, $\operatorname{diam} \gamma(A) \le \frac{1}{10} \operatorname{diam} A$. Let $U' := U \cap B(J(G), \varepsilon')$. Then, for each $w \in \{1, \dots, m\}^n$, $f_w^{-1}(U') \subset U'$ and $\{f_w^{-1}(U')\}_{w:|w|=n}$ are mutually disjoint. Hence, we may assume the above.

4. Let $r > 0$ be fixed. There exists a number $s \in \mathbf{N}$ with $s \geq 3$ such that for each $j = 1, \dots, k$ and each $w \in \mathscr{W}^*$ with $|w| \geq s - 1$, we have $\operatorname{diam} \gamma(K_j) \leq r$, for each well-defined inverse branch γ of f_w^{-1} on $B(z_j, 2\varepsilon)$. We fix such an s. Let $\mathscr{A}$ be the set of all (γ, K_j) that satisfies $j \in \{1, \dots, k\}$, γ is a well-defined inverse branch of f_w^{-1} on $B(z_j, 2\varepsilon)$ for some $w \in \{1, \dots, m\}^s$, and $\gamma(K_j) \cap B(x, r) \neq \emptyset$.

Then, we have the following:

Lemma 5.4. $B(x, r) \cap J(G) = B(x, r) \cap \bigcup_{(\gamma, K_j) \in \mathscr{A}} \gamma(J(G) \cap K_j)$.

Proof. Since $J(G) = \bigcup_{j=1}^m f_j^{-1}(J(G))$ (Lemma 2.4 in [S5]) and $J(G) \subset \overline{U}$ (Lemma 5.2), it is easy to see the statement. □

Definition 5.5. 1. Let $(\gamma, K_j) \in \mathscr{A}$ be any element such that γ is an inverse branch of f_w^{-1}, where $w = (w_1, \dots, w_s) \in \{1, \dots, m\}^s$. Then, we decompose γ as $\gamma = \gamma_1 \cdots \gamma_s$, where, for each $i = 1, \dots, s$, we use γ_i to denote the inverse branch of $f_{w_i}^{-1}$ on $B(\gamma_{i+1} \cdots \gamma_s(z_j), 2\varepsilon)$.
2. For each $A = (\gamma, K_j) \in \mathscr{A}$, let $l(A)$ be the minimum of $l \in \mathbf{N}$ that satisfies $3 \leq l \leq s$ and if $\gamma_l \cdots \gamma_s(K_j) \cap K_i \neq \emptyset$, then $\operatorname{diam} \gamma_1 \cdots \gamma_{l-1}(K_i) \leq r$. Note that by (3), we have $\gamma_1 \cdots \gamma_{l-1}$ is defined on K_i with $\gamma_l \cdots \gamma_s(K_j) \cap K_i \neq \emptyset$. Moreover, note that according to the choice of s, $l(A)$ exists, for each $A \in \mathscr{A}$.

Lemma 5.6. *Let* $A = (\gamma, K_j) \in \mathscr{A}$. *If*

$$r < \min\{\operatorname{diam} \gamma_1'(K_i) \mid (\gamma', K_t) \in \mathscr{A}, i \in \{1, \dots, k\}, K_i \subset B(\gamma_2' \cdots \gamma_s'(z_t), 2\varepsilon)\}, \tag{4}$$

then there exists an element K_i *such that* $\gamma_{l(A)-1} \cdots \gamma_s(K_j) \cap K_i \neq \emptyset$ *and* $\operatorname{diam} \gamma_1 \cdots \gamma_{l(A)-2}(K_i) > r$.

Proof. If $l(A) \geq 4$, then it is trivial. If $l(A) = 3$, then by (4), the above is true. □

Remark 6. For the rest, we assume (4). To show Lemma 5.3, we may make this assumption.

Definition 5.7. For any $A = (\gamma, K_j) \in \mathscr{A}$, we set

$$\Gamma_A := \{(\gamma_1 \cdots \gamma_{l(A)-1}, K_i) \mid \gamma_{l(A)} \cdots \gamma_s(K_j) \cap K_i \neq \emptyset\}.$$

Further, we set $\Gamma = \bigcup_{A \in \mathscr{A}} \Gamma_A$ (disjoint union).

Let B_1 and B_2 be two elements in Γ with $B_1 = (\gamma_1 \cdots \gamma_{l(A)-1}, K_{i_1}) \in \Gamma_A$ and $B_2 = (\gamma_1' \cdots \gamma_{l(A')-1}', K_{i_2}) \in \Gamma_{A'}$, where $A = (\gamma, K_{j(A)}) \in \mathscr{A}$ and $A' = (\gamma', K_{j(A')}) \in \mathscr{A}$. Then,

1. We write $B_1 \sim B_2$ if and only if $K_{i_1} = K_{i_2}$ and $\gamma_1 \cdots \gamma_{l(A)-1} = \gamma_1' \cdots \gamma_{l(A')-1}'$ on $B(\gamma_{l(A)} \cdots \gamma_s(z_{j(A)}), 2\varepsilon) \cap B(\gamma_{l(A')}' \cdots \gamma_s'(z_{j(A')}), 2\varepsilon)$. Note that this $\sim$ is an equivalence relation on Γ, by (3) and the uniqueness theorem.

2. We write $B_1 \preccurlyeq B_2$ if and only if

$$\gamma_1 \cdots \gamma_{l(A)-1}(\operatorname{int} K_{i_1}) \cap \gamma'_1 \cdots \gamma'_{l(A')-1}(\operatorname{int} K_{i_2}) \neq \emptyset$$

and $l(A) \leq l(A')$.

For any two elements B and B' in Γ, we write $B \preccurlyeq\preccurlyeq B'$ if and only if there exists a sequence $\{B_l\}_{l=1}^{v}$ in Γ such that $B = B_1 \preccurlyeq \cdots \preccurlyeq B_v = B'$.

Lemma 5.8. *Let B_1 and B_2 be two elements in Γ with $B_1 = (\gamma_1 \cdots \gamma_{l(A)-1}, K_{i_1}) \in \Gamma_A$ and $B_2 = (\gamma'_1 \cdots \gamma'_{l(A')-1}, K_{i_2}) \in \Gamma_{A'}$, where $A = (\gamma, K_{j(A)}) \in \mathscr{A}$ and $A' = (\gamma', K_{j(A')}) \in \mathscr{A}$. Suppose that $B_1 \preccurlyeq B_2$. Then, we have the following.*
1. *If $l(A) = l(A')$, then $B_1 \sim B_2$.*
2. *If $l(A) < l(A')$, then*
   (a) $\operatorname{int} K_{i_1} \cap \gamma'_{l(A)} \cdots \gamma'_{l(A')-1}(\operatorname{int} K_{i_2}) \neq \emptyset$ *and*
   (b) $\gamma_1 \cdots \gamma_{l(A)-1} = \gamma'_1 \cdots \gamma'_{l(A)-1}$ *on* $B(\gamma_{l(A)} \cdots \gamma_s(z_{j(A)}), 2\varepsilon) \cap B(\gamma'_{l(A)} \cdots \gamma'_s(z_{j(A')}), 2\varepsilon)$.

Proof. First, we show 2. Under the assumption of 2, suppose that $\gamma_1 \cdots \gamma_{l(A)-1}$ is an inverse branch of $f_{w_1}^{-1} \cdots f_{w_{l(A)-1}}^{-1}$ and that $\gamma'_1 \cdots \gamma'_{l(A')-1}$ is an inverse branch of $f_{w'_1}^{-1} \cdots f_{w'_{l(A')-1}}^{-1}$. By the open set condition, it follows that $w_j = w'_j$, for each $j = 1, \ldots, l(A) - 1$. Hence, 2a holds.

Next, take a point $z \in \gamma_1 \cdots \gamma_{l(A)-1}(\operatorname{int} K_{i_1}) \cap \gamma'_1 \cdots \gamma'_{l(A')-1}(\operatorname{int} K_{i_2})$. Let $a := f_{w_{l(A)-1}} \cdots f_{w_1}(z)$. Then, we have $a \in \operatorname{int} K_{i_1} \cap \gamma'_{l(A)} \cdots \gamma'_{l(A')-1}(\operatorname{int} K_{i_2})$. Furthermore, each of $\gamma_1 \cdots \gamma_{l(A)-1}$ and $\gamma'_1 \cdots \gamma'_{l(A)-1}$ is a well-defined inverse branch of $(f_{w_{l(A)-1}} \cdots f_{w_1})^{-1}$ on $B(a, \varepsilon)$ and maps a to z. Hence, they are equal on $B(a, \varepsilon)$. By the uniqueness theorem, we obtain 2b.

We can show 1 using the same method as above. □

Lemma 5.9. *If B and B' are two elements of Γ such that $B \preccurlyeq\preccurlyeq B'$ and $B' \preccurlyeq\preccurlyeq B$, then $B \sim B'$.*

Proof. There exists a sequence $\{B_j\}_{j=1}^{v}$ in Γ such that $B = B_1 \preccurlyeq \cdots \preccurlyeq B_u = B' \preccurlyeq \cdots \preccurlyeq B_v = B$. Suppose $B_j \in \Gamma_{A_j}$, for each $j = 1, \ldots, v$. Then we have $l(A_1) \leq \cdots \leq l(A_v) = l(A_1)$. By Lemma 5.8, we obtain $B_j \sim B_{j+1}$, for each $j = 1, \ldots, u - 1$. □

Lemma 5.10. *If $B_1 \sim B_2$, $B_3 \sim B_4$ and $B_1 \preccurlyeq B_3$, then $B_2 \preccurlyeq B_4$.*

Proof. This is easy to see, from the definitions of "$\sim$" and "$\preccurlyeq$", by using (3). □

Definition 5.11. For any $B \in \Gamma$, we use $[B] \in \Gamma/\sim$ to denote the equivalence class of B, with respect to the equivalence relation $\sim$ in Γ.

Let $[B_1]$ and $[B_2]$ be two elements of $\Gamma/\sim$, where $B_1, B_2 \in \Gamma$. We write $[B_1] \preccurlyeq [B_2]$ if and only if $B_1 \preccurlyeq B_2$. Note that this is well defined by Lemma

5.10. Furthermore, we write $[B_1] \le [B_2]$ if and only if $B_1 \preccurlyeq\preccurlyeq B_2$. Note that this is also well defined by Lemma 5.10 and that the "$\le$" determines a partial order in $\Gamma/\sim$, by Lemma 5.9.

Lemma 5.12. *Let $q \in \mathbf{N}$ be an integer with $q \ge 2$. Let $\{B_j\}_{j=1}^q$ be a sequence in Γ such that $B_1 \preccurlyeq \cdots \preccurlyeq B_q$ and $B_j \not\sim B_{j+1}$, for each $j = 1, \dots, q-1$. Suppose that for each $j = 1, \dots, q$, we have $B_j \in \Gamma_{A_j}$, $A_j = (\gamma^j, K_{t_j}) \in \mathscr{A}$ and $B_j = (\gamma_1^j \cdots \gamma_{l(A_j)-1}^j, K_{i_j})$. Then, we have the following.*

1. $\gamma_{l(A_1)}^q \cdots \gamma_{l(A_q)-1}^q(K_{i_q}) \subset B\big(K_{i_1}, \sum_{j=1}^{q-1} \big(\frac{1}{10}\big)^j \frac{1}{10}\varepsilon\big)$
2. $\gamma_1^1 \cdots \gamma_{l(A_1)-1}^1 = \gamma_1^q \cdots \gamma_{l(A_1)-1}^q$ *on* $V := B(\gamma_{l(A_1)}^1 \cdots \gamma_s^1(z_{t_1}), 2\varepsilon) \cap B(\gamma_{l(A_1)}^q \cdots \gamma_s^q(z_{t_q}), 2\varepsilon)$. *(Note that by* 1, *we have* $V \neq \emptyset$.*)*

Proof. We will show the statement by induction on q. If $q = 2$, then the statement follows from Lemma 5.8 and (3). Let $q \ge 3$. Suppose that the statement holds for each q' with $2 \le q' \le q-1$. By Lemma 5.8, we have $l(A_j) < l(A_{j+1})$, for each $j = 1, \dots, q-1$. By the hypothesis of induction, we have the following claim.

Claim 1.
1. $\gamma_{l(A_2)}^q \cdots \gamma_{l(A_q)-1}^q(K_{i_q}) \subset B\big(K_{i_2}, \sum_{j=1}^{q-2} \big(\frac{1}{10}\big)^j \frac{1}{10}\varepsilon\big)$.
2. $\gamma_1^2 \cdots \gamma_{l(A_2)-1}^2 = \gamma_1^q \cdots \gamma_{l(A_2)-1}^q$ on $B(\gamma_{l(A_2)}^2 \cdots \gamma_s^2(z_{t_2}), 2\varepsilon) \cap B(\gamma_{l(A_2)}^q \cdots \gamma_s^q(z_{t_q}), 2\varepsilon)$.

Combining Claim 1 with (3), we obtain

$$\gamma_{l(A_1)}^q \cdots \gamma_{l(A_q)-1}^q(K_{i_q}) \subset B\left(\gamma_{l(A_1)}^2 \cdots \gamma_{l(A_2)-1}^2(K_{i_2}), \sum_{j=2}^{q-1} \left(\frac{1}{10}\right)^j \frac{1}{10}\varepsilon\right). \tag{5}$$

Moreover, by Lemma 5.8 and (3), we have $\gamma_{l(A_1)}^2 \cdots \gamma_{l(A_2)-1}^2(K_{i_2}) \subset B\big(K_{i_1}, \frac{1}{10}\frac{1}{10}\varepsilon\big)$. Hence, we obtain

$$\gamma_{l(A_1)}^q \cdots \gamma_{l(A_q)-1}^q(K_{i_q}) \subset B\left(K_{i_1}, \sum_{j=1}^{q-1} \left(\frac{1}{10}\right)^j \frac{1}{10}\varepsilon\right). \tag{6}$$

Hence, the statement 1 in our lemma holds for q.

Next, we will show that the statement 2 in our lemma holds for q. Let us consider 2 in Claim 1. By the open set condition, for each $j = 1, \dots, l(A_1)-1$, there exists a number $\alpha_j \in \{1, \dots, m\}$ such that each of γ_j^2 and γ_j^q is an inverse branch of $f_{\alpha_j}^{-1}$. Hence, we obtain

$$\gamma_{l(A_1)}^2 \cdots \gamma_{l(A_2)-1}^2 = \gamma_{l(A_1)}^q \cdots \gamma_{l(A_2)-1}^q \tag{7}$$

on $V_0 := B(\gamma_{l(A_2)}^2 \cdots \gamma_s^2(z_{t_2}), 2\varepsilon) \cap B(\gamma_{l(A_2)}^q \cdots \gamma_s^q(z_{t_q}), 2\varepsilon)$.

Let $\beta := \gamma_{l(A_1)}^2 \cdots \gamma_{l(A_2)-1}^2 = \gamma_{l(A_1)}^q \cdots \gamma_{l(A_2)-1}^q$ on V_0. Then by 2 in Claim 1, we obtain $\gamma_1^2 \cdots \gamma_{l(A_1)-1}^2 = \gamma_1^q \cdots \gamma_{l(A_1)-1}^q$ on $\beta(V_0)$. Hence, by the uniqueness theorem, we get

$$(8) \qquad \gamma_1^2 \cdots \gamma_{l(A_1)-1}^2 = \gamma_1^q \cdots \gamma_{l(A_1)-1}^q$$

on $B(\gamma_{l(A_1)}^2 \cdots \gamma_s^2(z_{t_2}), 2\varepsilon) \cap B(\gamma_{l(A_1)}^q \cdots \gamma_s^q(z_{t_q}), 2\varepsilon)$.

Moreover, by Lemma 5.8, we have the following claim.

Claim 2.
1. $\operatorname{int} K_{i_1} \cap \gamma_{l(A_1)}^2 \cdots \gamma_{l(A_2)-1}^2(\operatorname{int} K_{i_2}) \neq \emptyset$.
2. $\gamma_1^1 \cdots \gamma_{l(A_1)-1}^1 = \gamma_1^2 \cdots \gamma_{l(A_1)-1}^2$ on $B(\gamma_{l(A_1)}^1 \cdots \gamma_s^1(z_{t_1}), 2\varepsilon) \cap B(\gamma_{l(A_1)}^2 \cdots \gamma_s^2(z_{t_2}), 2\varepsilon)$.

Combining 1 in Claim 2 with (3), we obtain

$$(9) \qquad d(\gamma_{l(A_1)}^1 \cdots \gamma_s^1(z_{t_1}), \gamma_{l(A_1)}^2 \cdots \gamma_s^2(z_{t_2})) \leq \frac{1}{5}\varepsilon.$$

Furthermore, by (6) and (3), we obtain

$$(10) \quad d(\gamma_{l(A_1)}^q \cdots \gamma_s^q(z_{t_q}), \gamma_{l(A_1)}^1 \cdots \gamma_s^1(z_{t_1})) \leq \frac{1}{10}\varepsilon + \frac{1}{10}\varepsilon + \sum_{j=1}^{q-1}\left(\frac{1}{10}\right)^j \frac{1}{10}\varepsilon \leq \frac{3}{10}\varepsilon.$$

Hence, by (9) and (10), we get $W := \bigcap_{j=1,2,q} B(\gamma_{l(A_1)}^j \cdots \gamma_s^j(z_{t_j}), 2\varepsilon) \neq \emptyset$. By 2 in Claim 2 and (8), then on W, $\gamma_1^1 \cdots \gamma_{l(A_1)-1}^1 = \gamma_1^q \cdots \gamma_{l(A_1)-1}^q$. Hence, by the uniqueness theorem, it follows that $\gamma_1^1 \cdots \gamma_{l(A_1)-1}^1 = \gamma_1^q \cdots \gamma_{l(A_1)-1}^q$ on $B(\gamma_{l(A_1)}^1 \cdots \gamma_s^1(z_{t_1}), 2\varepsilon) \cap B(\gamma_{l(A_1)}^q \cdots \gamma_s^q(z_{t_q}), 2\varepsilon)$. Hence, the statement 2 in our lemma holds for q. Hence, the induction is completed. □

Lemma 5.13. *Using the same assumption as for Lemma* 5.12, *it holds that* $\gamma_{l(A_1)}^q \cdots \gamma_s^q(K_{t_q}) \subset B\big(K_{i_1}, \frac{1}{5}\varepsilon\big)$.

Proof. By Lemma 5.12 and (3), we obtain

$$\gamma_{l(A_1)}^q \cdots \gamma_s^q(K_{t_q}) \subset B\left(K_{i_1}, \left(\sum_{j=1}^{\infty}\left(\frac{1}{10}\right)^j\right)\frac{1}{10}\varepsilon + \frac{1}{10}\varepsilon\right) \subset B\left(K_{i_1}, \frac{1}{5}\varepsilon\right). \qquad \square$$

Definition 5.14. Let $\{[m_1], \ldots, [m_p]\}$ be the set of all minimal elements of $(\Gamma/\sim, \leq)$, where, for each $i = 1, \ldots, p$, $m_i \in \Gamma_{R_i}$, $R_i = (\gamma^i, K_{u_i}) \in \mathscr{A}$ and $m_i = (\gamma_1^i \cdots \gamma_{l(R_i)-1}^i, K_{v_i})$. Furthermore, for any $i = 1, \ldots, p$, we use $\eta^i : \pi_{\overline{\mathbf{C}}}^{-1}(B(\gamma_{l(R_i)}^i \cdots \gamma_s^i(z_{u_i}), 2\varepsilon)) \to \Sigma_m \times \overline{\mathbf{C}}$ to denote the inverse branch of $(\tilde{f}^{\,l(R_i)-1})^{-1}$ such that $\eta^i((w, y)) = (w^i w, \gamma_1^i \cdots \gamma_{l(R_i)-1}^i(y))$ for each $(w, y) \in \pi_{\overline{\mathbf{C}}}^{-1}(B(\gamma_{l(R_i)}^i \cdots \gamma_s^i(z_{u_i}), 2\varepsilon))$, where $w^i \in \mathscr{W}^*$ is a word satisfying $|w^i| = l(R_i) - 1$ and $\gamma_1^i \cdots \gamma_{l(R_i)-1}^i$ is an inverse branch of $f_{w^i}^{-1}$.

Lemma 5.15. 1. $\pi_{\overline{\mathbf{C}}}^{-1}(B(x, r)) \cap \tilde{J}(\tilde{f}) \subset \bigcup_{i=1}^p \eta^i\big(\pi_{\overline{\mathbf{C}}}^{-1}\big(B\big(K_{v_i}, \frac{1}{5}\varepsilon\big)\big) \cap \tilde{J}(\tilde{f})\big)$.
2. $B(x, r) \cap J(G) \subset \bigcup_{i=1}^p \gamma_1^i \cdots \gamma_{l(R_i)-1}^i\big(B\big(K_{v_i}, \frac{1}{5}\varepsilon\big) \cap J(G)\big)$.

Proof. Let $(w,z) \in \pi_{\bar{\mathbf{C}}}^{-1}(B(x,r)) \cap \tilde{J}(\tilde{f})$ be a point. By Lemma 5.2, there exists a number j such that $\pi_{\bar{\mathbf{C}}}\tilde{f}^s((w,z)) \in K_j$. Let $\eta : \pi_{\bar{\mathbf{C}}}^{-1}B(z_j, 2\varepsilon) \to \Sigma_m \times \bar{\mathbf{C}}$ be an inverse branch of $(\tilde{f}^s)^{-1}$ such that $\eta((w',x')) = ((w|s)\cdot w', \gamma(x'))$ where γ is an inverse branch of $f_{w|s}^{-1}$. Then, we have $(w,z) \in \eta(\pi_{\bar{\mathbf{C}}}^{-1}(B(z_j, 2\varepsilon)))$ and $A := (\gamma, K_j) \in \mathscr{A}$. Let $B = (\gamma_1 \cdots \gamma_{l(A)-1}, K_{i_1}) \in \Gamma_A$ be an element. Then, there exists a number i with $1 \le i \le p$ such that $[m_i] \le [B]$. We will show the following claim:

Claim 1. $(w,z) \in \eta^i\big(\pi_{\bar{\mathbf{C}}}^{-1}\big(B\big(K_{v_i}, \tfrac{1}{5}\varepsilon\big)\big) \cap \tilde{J}(\tilde{f})\big)$.

To show this claim, we consider the following two cases:

Case 1. $B \sim m_i$

Case 2. There exists a sequence $(B_j)_{j=1}^q$ in Γ such that $m_i = B_1 \preccurlyeq B_2 \preccurlyeq \cdots \preccurlyeq B_q = B$ and $B_j \not\sim B_{j+1}$ for each $j = 1, \dots q-1$.

Suppose that we have Case 2. Let $y = \pi_{\bar{\mathbf{C}}}(\tilde{f}^s((w,z))) \in K_j \cap J(G)$. Then, we have $z = \gamma(y) = \gamma_1 \cdots \gamma_{l(R_i)-1} \cdot \gamma_{l(R_i)} \cdots \gamma_s(y)$. By Lemma 5.13, we have $\gamma_{l(R_i)} \cdots \gamma_s(y) \in B\big(K_{v_i}, \tfrac{1}{5}\varepsilon\big) \cap J(G)$. Furthermore, by Lemma 5.12-2, we have $\gamma_1 \cdots \gamma_{l(R_i)-1} = \gamma_1^i \cdots \gamma_{l(R_i)-1}^i$ on $B\big(K_{v_i}, \tfrac{1}{5}\varepsilon\big)$. Combining this with $B\big(K_{v_i}, \tfrac{1}{5}\varepsilon\big) \cap U \neq \emptyset$ and the open set condition, we get $w \,|\, (l(R_i)-1) = w^i$. By these arguments, we obtain $(w,z) = \eta^i(\tilde{f}^{l(R_i)-1}((w,z))) \in \eta^i\big(\pi_{\bar{\mathbf{C}}}^{-1}\big(B\big(K_{v_i}, \tfrac{1}{5}\varepsilon\big) \cap \tilde{J}(\tilde{f})\big)$.

Suppose that we have Case 1. Then, by the open set condition, the statement in Claim 1 is true. Hence, we have shown Claim 1.

By Claim 1, it follows that the statement of our lemma is true. $\square$

We now demonstrate Lemma 5.3.

Proof of Lemma 5.3. Let $A = (\gamma, K_j) \in \mathscr{A}$ and $B = (\gamma_1 \cdots \gamma_{l(A)-1}, K_i) \in \Gamma_A$. By Lemma 5.6 and Remark 6, there exists a number $u \in \mathbf{N}$ with $1 \le u \le k$ such that $\gamma_{l(A)-1} \cdots \gamma_s(K_j) \cap K_u \neq \emptyset$ and $\operatorname{diam} \gamma_1 \cdots \gamma_{l(A)-2}(K_u) > r$. Then, by the Koebe distortion theorem, there exists a positive constant $C_1 = C_1(\min_j \operatorname{diam} K_j, \varepsilon)$, which is independent of r, s and $x \in J(G)$, such that $\|(\gamma_1 \cdots \gamma_{l(A)-2})'(z)\| \ge C_1 r$ for each $z \in B(\gamma_{l(A)-1} \cdots \gamma_s(z_j), \varepsilon)$. Hence, there exists a positive constant $C_2 = C_2(C_1, G)$ such that

$$\|(\gamma_1 \cdots \gamma_{l(A)-1})'(z)\| \ge C_2 r,$$

for each $z \in B(\gamma_{l(A)} \cdots \gamma_s(z_j), \varepsilon)$. Combining this with

$$K_i \subset B\left(\gamma_{l(A)} \cdots \gamma_s(z_j), \frac{1}{5}\varepsilon\right),$$

which follows from (3), we obtain $\|(\gamma_1 \cdots \gamma_{l(A)-1})'(z)\| \ge C_2 r$, for each $z \in K_i$. Hence, it follows that there exists positive constant C_3, which is independent of r, s and $x \in J(G)$, such that

$$\mathrm{meas}_2(\gamma_1 \cdots \gamma_{l(A)-1}(\mathrm{int}\, K_i)) \ge C_3 r^2, \tag{11}$$

where meas_2 denotes the 2-dimensional Lebesgue measure. We now show the following claim:

Claim. $\gamma_1^i \cdots \gamma_{l(R_i)-1}^i(\mathrm{int}\, K_{v_i}) \subset B(x, 3r)$, for each $i = 1, \dots, p$.

To show this claim, since $\gamma_1^i \cdots \gamma_s^i(K_{u_i}) \cap B(x, r) \ne \emptyset$ and $\gamma_{l(R_i)}^i \cdots \gamma_s^i(K_{u_i}) \cap K_{v_i} \ne \emptyset$, we obtain $\gamma_1^i \cdots \gamma_{l(R_i)-1}^i(K_{v_i}) \cap B(x, 2r) \ne \emptyset$. Combining this with the fact that $\mathrm{diam}(\gamma_1^i \cdots \gamma_{l(R_i)-1}^i(K_{v_i})) \le r$, it follows that the above claim holds.

Since $\{[m_1], \dots, [m_p]\}$ is the set of minimal elements of $(\Gamma/\!\sim, \le)$, we find that $\{\gamma_1^i \cdots \gamma_{l(R_i)-1}^i(\mathrm{int}\, K_{v_i})\}_{i=1}^p$ are mutually disjoint. Hence, by (11) and the claim, we obtain

$$p \le \frac{\mathrm{meas}_2(B(x, 3r))}{C_3 r^2} \le C_4, \tag{12}$$

where, C_4 is a positive constant independent of r, s and $x \in J(G)$. Furthermore, by the definition of $l(A)$, we have $\mathrm{diam}\, \gamma_1 \cdots \gamma_{l(A)-1}(K_i) \le r$. Hence, by the Koebe distortion theorem, there exists a positive constant C_5, which is independent of r and $x \in J(G)$, such that

$$\|(\gamma_1 \cdots \gamma_{l(A)-1})'(z)\| \le C_5 r, \tag{13}$$

for each $z \in B\big(K_i, \frac{1}{5}\varepsilon\big)$. Hence, by Lemma 5.15, Lemma 3.10, Lemma 4.4, (12) and (13), we obtain

$$\begin{aligned}
\nu(B(x,r)) &= \tilde{\nu}(\pi_{\bar{\mathbf{C}}}^{-1}(B(x,r)) \cap \tilde{J}(\tilde{f})) \\
&\le \sum_{i=1}^{p} \tilde{\nu}\left(\eta^i\left(\pi_{\bar{\mathbf{C}}}^{-1}\left(B\left(K_{v_i}, \frac{1}{5}\varepsilon\right)\right) \cap \tilde{J}(\tilde{f})\right)\right) \\
&= \sum_{i=1}^{p} \int_{\pi_{\bar{\mathbf{C}}}^{-1}(B(K_{v_i},(1/5)\varepsilon)) \cap \tilde{J}(\tilde{f})} \|(\gamma_1^i \cdots \gamma_{l(R_i)-1}^i)'(\pi_{\bar{\mathbf{C}}}(z))\|^{\delta}\, d\tilde{\nu}(z) \\
&\le C_4 C_5^{\delta} r^{\delta}.
\end{aligned}$$

Similarly, if τ is a t-conformal measure, then by Lemma 5.15, Lemma 4.3, (12), and (13), we obtain

$$\begin{aligned}
\tau(B(x,r)) &= \tau(B(x,r) \cap J(G)) \\
&\le \sum_{i=1}^{p} \tau\left(\gamma_1^i \cdots \gamma_{l(R_i)-1}^i\left(B\left(K_{v_i}, \frac{1}{5}\varepsilon\right) \cap J(G)\right)\right) \\
&= \sum_{i=1}^{p} \int_{B(K_{v_i},(1/5)\varepsilon) \cap J(G)} \|(\gamma_1^i \cdots \gamma_{l(R_i)-1}^i)'\|^t\, d\tau \\
&\le C_4 \cdot C_5^t \cdot r^t.
\end{aligned}$$

By Lemma 3.11 and Lemma 3.16, we find that a positive constant C' exists such that for each r with $0 < r < \operatorname{diam} \overline{\mathbf{C}}$ and $x \in J(G)$, we have $\nu(B(x,r)) \geq C' r^{\delta}$. Hence, it follows that a positive constant C_6 exists such that for each r with $0 < r < \operatorname{diam} \overline{\mathbf{C}}$ and each $x \in J(G)$, we have $C_6^{-1} r^{\delta} \leq \nu(B(x,r)) \leq C_6 r^{\delta}$. Hence, by Proposition 2.2 in [F] and Main Theorem A, we obtain $0 < H^{\delta}(J(G)) < \infty$ and $\dim_H(J(G)) = \overline{\dim}_B(J(G)) = \delta$.

Similarly, if τ is a t-conformal measure, then by Lemma 4.2, τ is t-subconformal. By Lemma 3.16, we find that a positive constant C_7 exists such that for each $r > 0$ and $x \in J(G)$, we have $\tau(B(x,r)) \geq C_7 r^t$. Hence, it follows that a positive constant C_8 exists such that for each $r > 0$ and $x \in J(G)$, we have $C_8^{-1} r^t \leq \tau(B(x,r)) \leq C_8 r^t$. Hence, by Proposition 2.2 in [F], we obtain $0 < H^t(J(G)) < \infty$ and $\dim_H(J(G)) = t = \delta$. Then, we find that a positive constant C_9 exists such that for each $x \in J(G)$ and each $r > 0$, we have $C_9^{-1} \tau(B(x,r)) \leq \nu(B(x,r)) \leq C_9 \tau(B(x,r))$. Hence, by the Besicovitch covering lemma (p294 in [Pe]), we find that ν and τ are absolutely continuous with respect to each other. Hence, we have shown Lemma 5.3. □

We now demonstrate Main Theorem B.

Proof of Main Theorem B. By Lemma 5.3, we find a positive constant C exists such that for each r with $0 < r < \operatorname{diam} \overline{\mathbf{C}}$ and each $x \in J(G)$, we have $C^{-1} r^{\delta} \leq \nu(B(x,r)) \leq C r^{\delta}$. Furthermore, $\dim_H(J(G)) = \overline{\dim}_B(J(G)) = s(G) = s_0(G) = \delta$. Combining this with Main Theorem A, we see that for each $x \in \overline{\mathbf{C}} \setminus (A(G) \cup P(G))$, we have $\dim_H(J(G)) = S(x) = T(x) = \delta$.

By Lemma 5.3 and Proposition 4.13, we obtain $\nu = \dfrac{H^{\delta}|_{J(G)}}{H^{\delta}(J(G))}$, ν is a δ-conformal measure satisfying the separating condition for $\{f_1, \dots, f_m\}$, and $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ is nowhere dense in $f_j^{-1}(J(G))$ for each (i,j) with $i \neq j$.

Let τ be a t-conformal measure. Then, by Lemma 5.3, we have $t = \delta$ and τ is absolutely continuous with respect to ν. Since ν satisfies the separating condition for $\{f_1, \dots, f_m\}$, it follows that τ also satisfies the separating condition for $\{f_1, \dots, f_m\}$. Combining this with Lemma 4.10-2, we obtain $\tau = \nu$.

Hence, we have shown Main Theorem B. □

6. Examples

Example 6.1. 1. Let $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2$ and $f_2(z) = 2.3(z-3) + 3$. Then, we can see easily that $\{|z| < 0.9\} \subset F(G)$ and G is expanding. By the corollary 3.17, we get

$$\overline{\dim}_B(J(G)) \leq \frac{\log 3}{\log 1.8} < 2.$$

In particular, $J(G)$ has no interior points. In [S3], it was shown that if a finitely generated rational semigroup satisfies the open set condition with an open set U, then the Julia set is equal to the closure of the open set

U or has no interior points. Note that the fact that the Julia set of the above semigroup G has no interior points was shown by using analytic quantity only. It appears to be true that G does not satisfy the open set condition.

2. Let $G = \left\langle \frac{z^3}{4}, z^2 + 8 \right\rangle$. Then, we can easily see that $\{|z| < 2\} \subset F(G)$ and G is expanding. Hence, we have

$$\overline{\dim}_B J(G) \le \frac{\log 5}{\log 3} < 2.$$

In particular, $J(G)$ has no interior points.

Example 6.2. Let p_1, p_2 and $p_3 \in \mathbf{C}$ be mutually distinct points such that $p_1p_2p_3$ makes a regular triangle. Let U be the inside part of the regular triangle. Let $f_i(z) = 2(z - p_i) + p_i$ for each $i = 1, 2, 3$. Let $D(x, r)$ be a Euclidean disk with radius r in $U \backslash \bigcup_{i=1}^{3} f_i^{-1}(\overline{U})$, where x denotes the barycenter of the regular triangle $p_1p_2p_3$. Let g be a polynomial such that $J(g) = \partial D(x, r)$. Let $f_4(z) = g^s(z)$, where $s \in \mathbf{N}$ is a large number such that $f_4^{-1}(U) \subset U \backslash \bigcup_{i=1}^{3} f_i^{-1}(U)$. Let $G = \langle f_1, f_2, f_3, f_4 \rangle$. Then, G satisfies the open set condition with U with respect to $\{f_i\}$. Furthermore, G is hyperbolic. Hence, G is expanding, by Theorem 2.6 in [S2]. Hence, G satisfies the assumption in Main Theorem B. (Note that $J(\langle f_1, f_2, f_3 \rangle)$ is the Sierpiński gasket.)

Example 6.3. For any b with $0 < b \le 0.1$, there exists an a with $0.2 < a \le 1$ such that $G = \langle a(z - b)^3 + b, z^2 \rangle$ satisfies that (1)G is expanding, (2)G satisfies the open set condition with $U = \left\{ z \in \mathbf{C} \,\middle|\, |z - b| < \frac{1}{\sqrt{a}}, |z| > 1 \right\}$, (3)$J(G)$ is connected, and (4)$J(G)$ is porous (hence $\delta = \dim_H(J(G)) = \overline{\dim}_B(J(G)) < 2$).

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