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ENDPOINT ESTIMATES FOR COMMUTATORS OF CALDERÓN-ZYGMUND TYPE OPERATORS

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Abstract

In this paper, we establish some endpoint estimates for the commutator, $[b, T]$, of a class of Calderón-Zygmund type operator, such as the weak type $L \log L$ estimate, weak type (H^1, L^1) estimate and some estimates in the Hardy type spaces associated with b, where $b \in \mathrm{BMO}(\boldsymbol{R}^n)$.

1. Introduction

Calderón-Zygmund operators and their generalizations on Euclidean space $\boldsymbol{R}^n$ have been extensively studied [1–4]. In particular, Yabuta [3] introduced certain θ type Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operator. In this paper, we study the commutator of the following so-called θ type Calderón-Zygmund operator.

Definition 1. Let θ be a non-negative non-decreasing function on $\boldsymbol{R}^+$ with $\int_0^1 \theta(t)t^{-1}|\log t|\,dt < \infty$. A measurable function K on $\boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{(x,x) : x \in \boldsymbol{R}^n\}$ is said to be a θ type kernel if it satisfies

(i) $|K(x,y)| \le C|x-y|^{-n}$ for $x \ne y$;

(ii) $|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \dfrac{C\theta(|x-z|/|x-y|)}{|x-y|^n}$, for $|x-z| < |x-y|/2$.

Let T be a linear operator from $\mathscr{S}(\boldsymbol{R}^n)$ into its dual $\mathscr{S}'(\boldsymbol{R}^n)$. We say T is a θ type Calderón-Zygmund operator if

(1) T can be extended to be a bounded linear operator on $L^2(\boldsymbol{R}^n)$;

(2) There is a θ type kernel K such that $Tf(x) = \int_{\operatorname{supp} f} K(x,y)f(y)\,dy$ for all $f \in C_0^\infty(\boldsymbol{R}^n)$ and for all $x \notin \operatorname{supp} f$, where $C_0^\infty(\boldsymbol{R}^n)$ is the space of all infinitely differentiable functions on $\boldsymbol{R}^n$ with compact supports.

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Remark 1. The θ type Calderón-Zygmund operator introduced in Definition 1 is a special case of operator which is introduced by Yabuta, so the results for Yabuta's operator also hold for our operator. The following lemma is a result in [4].

LEMMA 1. *Let θ be a non-negative non-decreasing function on $\boldsymbol{R}^+$ with $\int_0^1 \theta(t)t^{-1}\,dt < \infty$. Let T be a θ type Calderón-Zygmund operator. Then the following conditions are equivalent*:

(1) $\int_Q |Ta(x)|\,dx \le C\|a\|_{L^\infty(\boldsymbol{R}^n)}$ *for* $a \in L^\infty(\boldsymbol{R}^n)$ *with* $\operatorname{supp} a \subset Q$, *a cube in* $\boldsymbol{R}^n$;
(2) *T is a bounded operator from $H^1(\boldsymbol{R}^n)$ to $L^1(\boldsymbol{R}^n)$*;
(3) *T is a bounded operator from $L_0^\infty(\boldsymbol{R}^n)$ to* $\mathrm{BMO}(\boldsymbol{R}^n)$;
(4) *T is a bounded operator from $L^q(\boldsymbol{R}^n)$ to $WL^q(\boldsymbol{R}^n)$ for some $q \in (1,\infty)$*;
(5) *T is a bounded operator on $L^q(\boldsymbol{R}^n)$ for some $q \in (1,\infty)$*;
(6) *T is weak type $(1,1)$.*

In this paper, we establish some endpoint estimates for commutator of the θ type Calderón-Zygmund operator:

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x),$$

where $b \in \mathrm{BMO}(\boldsymbol{R}^n)$.

Most the notation that we use is standard. Q will always denote a cube with sides parallel to the axes, λQ $(\lambda > 0)$ denotes the cube Q dilated by λ. For a locally integrable function f, f_Q denotes the average of f on $Q : f_Q = (1/|Q|)\int_Q f(y)\,dy$.

As usual, a function $A : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex and increasing and satisfying $A(0) = 0$, $A(t) \to \infty$ as $t \to \infty$. We define the A-average of a function f over a cube Q by means of the following Luxemberg norm:

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\right\}.$$

The generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\,dy \le \|f\|_{A,Q}\|g\|_{\bar{A},Q}$$

holds, where $\bar{A}$ be the complementary Young function associated to A.

It is well known that $\bar{A}(t) \approx \exp t$ with $A(t) = t(1+\log^+ t)$. The maximal function associated to $A(t) = t(1+\log^+ t)$ was defined as

$$M_{L\log L}f(x) = \sup_{x \in Q}\|f\|_{A,Q}.$$

The maximal function associated to $A(t) = t$ is the well-known Hardy-Littlewood maximal function. For $\delta > 0$, we define the δ-maximal function as $M_\delta(f) = [M(|f|^\delta)]^{1/\delta}$ and the δ-Sharp maximal function as

$$M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{1/\delta},$$

where $M^\sharp$ be the well-known Fefferman-Stein's Sharp maximal function:

$$M^{\sharp}f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy.$$

Following the results in [6], the L^p $(1 < p < \infty)$ boundedness for commutator $[b, T]$ is the corollary of the following Sharp function estimates.

LEMMA 2. *Let T be a θ type Calderón-Zygmund operator and $1 < p < \infty$. Then*

$$M^{\sharp}(Tf)(x) \le CM_p f(x).$$

Proof. For any $x \in \boldsymbol{R}^n$ and any cube Q with $x \in Q$, let x_0 be the centre of Q, and

$$f = f\chi_{2Q} + f\chi_{\boldsymbol{R}^n \setminus 2Q} = f_1 + f_2.$$

By the L^p boundedness of T and the Hölder inequality, we have

$$\frac{1}{|Q|}\int_Q |Tf_1(y)|\, dy \le \left(\frac{1}{|Q|}\int_Q |Tf_1(y)|^p\, dy\right)^{1/p} \le C\left(\frac{1}{|Q|}\int_{2Q} |f(y)|^p\, dy\right)^{1/p} \le CM_p f(x).$$

When $y \in Q$, we apply the condition of θ to get that

$$\begin{aligned}
|Tf_2(y) - Tf_2(x_0)| &\le \int_{\boldsymbol{R}^n \setminus 2Q} |K(y,z) - K(x_0,z)|\,|f(z)|\, dz \\
&\le C\sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \frac{\theta(|y - x_0|/|z - x_0|)}{|z - x_0|^n} |f(z)|\, dz \\
&\le C\sum_{j=1}^{\infty} \theta(2^{-j}) \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(z)|\, dz \\
&\le C\sum_{j=1}^{\infty} \theta(2^{-j}) \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(z)|^p\, dz\right)^{1/p} \\
&\le C\int_0^1 \theta(t)t^{-1}\, dt M_p f(x) \le CM_p f(x).
\end{aligned}$$

This implies that

$$\frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x_0)|\, dy \le CM_p f(x).$$

Thus, we obtain that

$$M^{\sharp}(Tf)(x) \le CM_p f(x).$$

This completes the proof of Lemma 2.

2. Weak type $L \log L$ estimates and weak type (H^1, L^1) estimates

In this section, we establish firstly the weak type $L \log L$ estimates for $[b, T]$ by the method of the Sharp function estimates. Then we get the weak type (H^1, L^1) estimates. Our main results are the following theorems

Theorem 1. *Let $b \in \mathrm{BMO}(\boldsymbol{R}^n)$ and T be a θ type Calderón-Zygmund operator. Then, there exists a positive constant C such that for each smooth function f with compact support and for all $\lambda > 0$,*

$$\left|\{x \in \boldsymbol{R}^n : |[b,T]f(x)| > \lambda\}\right| \leq C\|b\|_* \int_{\boldsymbol{R}^n} \frac{|f(y)|}{\lambda}\left(1 + \log^+ \frac{|f(y)|}{\lambda}\right) dy.$$

Following the ideas of Pérez [9], we only need prove the following two Sharp function estimates.

Lemma 3. *Let $b \in \mathrm{BMO}(\boldsymbol{R}^n)$, T be a θ type Calderón-Zygmund operator and $0 < \delta < \varepsilon < 1$. Then, there exists a positive constant $C = C_{\delta,\varepsilon} > 0$ such that for each smooth function f with compact support,*

$$M_\delta^\sharp([b,T]f)(x) \leq C\|b\|_*(M_\varepsilon(Tf)(x) + M_{L\log L}f(x)).$$

Proof. Let $Q = Q(x,r)$ be an arbitrary cube. Since $0 < \delta < \varepsilon < 1$ implies $\left||\alpha|^\delta - |\beta|^\delta\right| \leq |\alpha - \beta|^\delta$ for $\alpha, \beta \in \boldsymbol{R}$, it is enough to show for some complex constant $c = c_Q$ that there exists $C = C_\delta > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q |[b,T]f(y) - c|^\delta\, dy\right)^{1/\delta} \leq C\|b\|_*(M_\varepsilon(Tf)(x) + M_{L\log L}f(x)).$$

Let $f = f\chi_{2Q} + f\chi_{\boldsymbol{R}^n\setminus 2Q} = f_1 + f_2$. We write

$$[b,T]f = (b - b_{2Q})Tf - T((b - b_{2Q})f_1) - T((b - b_{2Q})f_2).$$

If we pick $c = c_Q = (T((b - b_{2Q})f_2))_Q$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |[b,T]f(y) - c|^\delta\, dy\right)^{1/\delta} \\
&\quad\leq C\left(\frac{1}{|Q|}\int_Q |b(y) - b_{2Q}|^\delta |Tf(y)|^\delta\, dy\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_1)(y)|^\delta\, dy\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_2)(y) - (T((b - b_{2Q})f_2))_Q|^\delta\, dy\right)^{1/\delta} \\
&\quad= I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1, we use the Hölder inequality with exponents r and r' where $1 < r < \varepsilon/\delta$:

$$\begin{aligned}
I_1 &\leq C\left(\frac{1}{|Q|}\int_Q |b(y) - b_{2Q}|^{\delta r'}\, dy\right)^{1/\delta r'} \left(\frac{1}{|Q|}\int_Q |Tf(y)|^{\delta r}\, dy\right)^{1/\delta r} \\
&\leq C\|b\|_* M_{\delta r}(Tf)(x) \leq C\|b\|_* M_\varepsilon(Tf)(x).
\end{aligned}$$

Since T is of weak type $(1,1)$ and $0<\delta<1$, the Kolmogorov inequality implies

$$
\begin{aligned}
I_2 &\le C|Q|^{-1}\frac{\|T((b-b_{2Q})f_1)\chi_Q\|_{L^\delta(\boldsymbol{R}^n)}}{|Q|^{1/\delta-1}} \le C|Q|^{-1}\|T((b-b_{2Q})f_1)\chi_Q\|_{WL^1(\boldsymbol{R}^n)}\\
&\le C|Q|^{-1}\|(b-b_{2Q})f_1\|_{L^1(\boldsymbol{R}^n)} \le C\|b-b_{2Q}\|_{\exp L,2Q}\|f\|_{L\log L,2Q}\\
&\le C\|b\|_* M_{L\log L}f(x).
\end{aligned}
$$

In the last inequality, we use the estimate that $\|b-b_Q\|_{\exp L,Q}\le C\|b\|_*$, it is equivalent to the inequality

$$
\frac{1}{|Q|}\int_Q \exp\left(\frac{|b(y)-b_Q|}{C\|b\|_*}\right)dy \le C_0,
$$

it is just a corollary of the well-known John-Nirenberg inequality. Then the Jensen inequality and the Fubini theorem yield

$$
\begin{aligned}
I_3 &\le \frac{C}{|Q|}\int_Q |T((b-b_{2Q})f_2)(y)-(T((b-b_{2Q})f_2))_Q|\,dy\\
&\le \frac{C}{|Q|^2}\int_Q\int_Q\int_{\boldsymbol{R}^n\setminus 2Q}|K(y,w)-K(z,w)|\,|(b(w)-b_{2Q})f(w)|\,dwdzdy\\
&\le \frac{C}{|Q|^2}\int_Q\int_Q\sum_{j=1}^\infty\int_{2^{j+1}Q\setminus 2^jQ}\frac{\theta(|y-z|/|x-w|)}{|x-w|^n}|b(w)-b_{2Q}|\,|f(w)|\,dwdzdy\\
&\le C\sum_{j=1}^\infty \theta(2^{-j})\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(w)-b_{2Q}|\,|f(w)|\,dw\\
&\le C\sum_{j=1}^\infty \theta(2^{-j})\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(w)-b_{2^{j+1}Q}|\,|f(w)|\,dw\\
&\quad + C\sum_{j=1}^\infty \theta(2^{-j})|b_{2^{j+1}Q}-b_{2^jQ}|\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|f(w)|\,dw\\
&\le C\sum_{j=1}^\infty \theta(2^{-j})\|b-b_{2^{j+1}Q}\|_{\exp L,2^{j+1}Q}\|f\|_{L\log L,2^{j+1}Q} + C\sum_{j=1}^\infty j\theta(2^{-j})\|b\|_* Mf(x)\\
&\le C\sum_{j=1}^\infty j\theta(2^{-j})\|b\|_* M_{L\log L}f(x) \le C\int_0^1 \theta(t)t^{-1}|\log t|\,dt\|b\|_* M_{L\log L}f(x)\\
&\le C\|b\|_* M_{L\log L}f(x).
\end{aligned}
$$

This completes the proof of Lemma 3.

Using a similar method, we can establish the following Sharp function estimate and omit the details.

Lemma 4. *Let $0 < \alpha < 1$ and T be a θ type Calderón-Zygmund operator. Then, for any $f \in C_0^\infty(\boldsymbol{R}^n)$ and $x \in \boldsymbol{R}^n$, there exists a constant $C = C_\alpha > 0$, such that*

$$M_\alpha^\sharp(Tf)(x) \le CMf(x).$$

Now, we establish the weak type (H^1, L^1) estimate for $[b, T]$.

Theorem 2. *Let $b \in \mathrm{BMO}(\boldsymbol{R}^n)$ and T be a θ type Calderón-Zygmund operator. Then the commutator $[b, T]$ is a bounded operator from $H^1(\boldsymbol{R}^n)$ to weak $L^1(\boldsymbol{R}^n)$, i.e. for any $\lambda > 0$, there exists a constant $C > 0$, such that*

$$\left|\{x \in \boldsymbol{R}^n : |[b, T]f(x)| > \lambda\}\right| \le \frac{C}{\lambda}\|f\|_{H^1(\boldsymbol{R}^n)}.$$

Proof. For any given $f \in H^1(\boldsymbol{R}^n)$, by atomic decomposition we get $f = \sum_{j=1}^\infty \lambda_j a_j$, where each a_j be a $(1, \infty, 0)$ atom with $\|f\|_{H^1(\boldsymbol{R}^n)} = \inf(\sum_{j=1}^\infty |\lambda_j|)$. We may assume that f is a finite sum $\sum_Q \lambda_Q a_Q$ with $\sum_Q |\lambda_Q| \le 2\|f\|_{H^1(\boldsymbol{R}^n)}$. Once Theorem 2 is proven for such f, for general f is the limit of this kind of f_k (in H^1 norm or almost everywhere sense) where f_k are finite sums having forms of $\sum_Q \lambda_Q a_Q$, Theorem 2 follows by a limiting argument, using the L^2-boundedness of $[b, T]$. It is convenient for us to assume that each Q (the supporting cube of a_Q) in the given atomic decomposition of f is dyadic and $\lambda_Q > 0$.

For fixed $\lambda > 0$ and the finite collection of dyadic cube Q and associated positive scalars $\lambda_Q > 0$ in the given atomic decomposition of f, by Lemma 4.1 in [5], there exists a collection of pairwise disjoint dyadic cubes S such that

(1) $\displaystyle\sum_{Q \subset S} \lambda_Q \le 2^n \lambda |S|$, for all S; (2) $\displaystyle\sum_S |S| \le \lambda^{-1} \sum_Q \lambda_Q$;

(3) $\displaystyle\left\|\sum_{Q \not\subset \text{any } S} \lambda_Q |Q|^{-1} \chi_Q\right\|_{L^\infty(\boldsymbol{R}^n)} \le \lambda.$

Denote $E = \bigcup_S 2S$, then $|E| \le C\lambda^{-1}\|f\|_{H^1(\boldsymbol{R}^n)}$.

Set $h(x) = \sum_S \sum_{Q \subset S} \lambda_Q a_Q$ and $g(x) = f(x) - h(x)$. By (3), $\|g\|_{L^\infty(\boldsymbol{R}^n)} \le \lambda$ and the $L^2(\boldsymbol{R}^n)$ boundedness of $[b, T]$ implies

$$\begin{aligned}\left|\{x \in \boldsymbol{R}^n \backslash E : |[b, T]g(x)| > \lambda/4\}\right| &\le \frac{C}{\lambda^2}\|[b, T]g\|_{L^2(\boldsymbol{R}^n)}^2 \le \frac{C}{\lambda^2}\|g\|_{L^2(\boldsymbol{R}^n)}^2 \\ &\le \frac{C}{\lambda}\|g\|_{L^1(\boldsymbol{R}^n)} \le \frac{C}{\lambda}\|f\|_{L^1(\boldsymbol{R}^n)} \le \frac{C}{\lambda}\|f\|_{H^1(\boldsymbol{R}^n)}.\end{aligned}$$

Thus, we only need prove following inequality

$$\left|\{x \in \boldsymbol{R}^n \backslash E : |[b, T]h(x)| > \lambda/4\}\right| \le \frac{C}{\lambda}\|f\|_{H^1(\boldsymbol{R}^n)}.$$

For any fixed cube $Q = Q(x_Q, r_Q)$, by moments condition for a_Q we have

$$\begin{aligned}[b,T]a_Q(x) &= \int_{\boldsymbol{R}^n}(b(x)-b(y))K(x,y)a_Q(y)\,dy\\ &= \int_{\boldsymbol{R}^n}(b(x)-b_Q)[K(x,y)-K(x,x_Q)]a_Q(y)\,dy\\ &\quad+\int_{\boldsymbol{R}^n}K(x,y)(b_Q-b(y))a_Q(y)\,dy.\end{aligned}$$

Since $x\in\boldsymbol{R}^n\backslash E$ implies that $x\in\boldsymbol{R}^n\backslash 2Q$ for any cube Q in the given atomic decomposition, by the smoothness condition of θ we get

$$\begin{aligned}|[b,T]h(x)| &\le C\sum_S\sum_{Q\subset S}\lambda_Q\frac{|b(x)-b_Q|\theta(r_Q/|x-x_Q|)}{|x-x_Q|^n}+\left|\sum_S\sum_{Q\subset S}T((b_Q-b)a_Q)(x)\right|\\ &= I_1(x)+I_2(x).\end{aligned}$$

By the condition of $\theta(t)$, we obtain

$$\begin{aligned}&\left|\{x\in\boldsymbol{R}^n\backslash E : I_1(x)>\lambda/8\}\right|\\ &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\int_{\boldsymbol{R}^n\backslash 2Q}\frac{|b(x)-b_Q|\theta(r_Q/|x-x_Q|)}{|x-x_Q|^n}\,dx\\ &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\sum_{l=1}^{\infty}\int_{2^{l+1}Q\backslash 2^lQ}\frac{|b(x)-b_Q|\theta(r_Q/|x-x_Q|)}{|x-x_Q|^n}\,dx\\ &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\sum_{l=1}^{\infty}\theta(2^{-l})\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|b(x)-b_Q|\,dx\\ &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\sum_{l=1}^{\infty}l\theta(2^{-l})\|b\|_*\le\frac{C\|b\|_*}{\lambda}\int_0^1\theta(t)t^{-1}|\log t|\,dt\sum_S\sum_{Q\subset S}\lambda_Q\\ &\le\frac{C\|b\|_*}{\lambda}\|f\|_{H^1(\boldsymbol{R}^n)}.\end{aligned}$$

The weak type $(1,1)$ boundedness of T implies the following estimate

$$\begin{aligned}\left|\{x\in\boldsymbol{R}^n\backslash E : I_2(x)>\lambda/8\}\right| &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\|(b-b_Q)a_Q\|_{L^1(\boldsymbol{R}^n)}\\ &\le\frac{C}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\frac{1}{|Q|}\int_Q|b(y)-b_Q|\,dy\\ &\le\frac{C\|b\|_*}{\lambda}\sum_S\sum_{Q\subset S}\lambda_Q\le\frac{C\|b\|_*}{\lambda}\|f\|_{H^1(\boldsymbol{R}^n)}.\end{aligned}$$

This finishes the proof of Theorem 2.

3. The estimates on Hardy type spaces

It is well known that the commutator $[b, T]$ isn't a bounded operator from H^1 to L^1 even when T is a usual Calderón-Zygmund operator, but it is a bounded operator from H_b^1 to L^1, where H_b^1 is a Hardy type space associated with $b \in \mathrm{BMO}(\boldsymbol{R}^n)$. In this section, we discuss this problem when T is a θ type Calderón-Zygmund operator. Let us give some notations.

Definition 2. Let b be a locally integrable function, $0 < p \leq 1$. It is said that a bounded function a is a $H_b^p(\boldsymbol{R}^n)$ atom if it satisfies

(1) $\operatorname{supp} a \subset Q = Q(x_0, r)$ for some $r > 0$; (2) $\|a\|_{L^\infty(\boldsymbol{R}^n)} \leq |Q|^{-1/p}$;

(3) $\displaystyle\int_{\boldsymbol{R}^n} x^\beta a(x)\, dx = \int_{\boldsymbol{R}^n} x^\beta a(x) b(x)\, dx = 0$ for any $|\beta| \leq [1/p - 1]$.

It is said that a temperate distribution f belongs to $H_b^p(\boldsymbol{R}^n)$ if, in the $\mathscr{S}'(\boldsymbol{R}^n)$ sense, it can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j is a $H_b^p(\boldsymbol{R}^n)$ atom and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. We define on $H_b^p(\boldsymbol{R}^n)$ the quasinorm

$$\|f\|_{H_b^p(\boldsymbol{R}^n)} = \inf\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p}.$$

Theorem 3. *Let $b \in \mathrm{BMO}(\boldsymbol{R}^n)$ and T be a θ type Calderón-Zygmund operator, $0 < p \leq 1$ and $\displaystyle\int_0^1 \frac{\theta^p(t)|\log t|^p}{t^{(1-p)n+1}}\, dt < \infty$. Then the commutator $[b, T]$ is a bounded operator from $H_b^p(\boldsymbol{R}^n)$ to $L^p(\boldsymbol{R}^n)$.*

Proof. By the condition of $\theta(t)$, it is easy to see that $\theta(t) < \theta^p(t)$ where $t \in (0, \varepsilon)$ for some $\varepsilon \in (0, 1)$. This implies that

$$\int_0^1 \theta(t) t^{-1} |\log t|\, dt < C \int_0^1 \frac{\theta(t)}{t^{(1-p)n+1}}\, dt \leq C \int_0^1 \frac{\theta^p(t)|\log t|^p}{t^{(1-p)n+1}}\, dt.$$

Thus we can use the results in Section 1, and get the L^q $(1 < q < \infty)$ boundedness of $[b, T]$. Hence, as in the proof of Theorem 2, we only need to prove that, for any $H_b^p(\boldsymbol{R}^n)$ atom a, there exists a constant $C > 0$ independent of a, such that $\int_{\boldsymbol{R}^n} |[b, T]a(x)|^p\, dx \leq C$.

Let $\operatorname{supp} a \subset Q = Q(x_0, r)$ and write

$$\begin{aligned}\int_{\boldsymbol{R}^n} |[b, T]a(x)|^p\, dx &\leq \int_{2Q} |[b, T]a(x)|^p\, dx + \int_{\boldsymbol{R}^n \setminus 2Q} |[b, T]a(x)|^p\, dx \\ &= J_1 + J_2.\end{aligned}$$

Then, by the L^q boundedness of $[b, T]$ and by the Hölder inequality, we have

$$\begin{aligned} J_1 &\le |2Q|^{1-p/q}\left(\int_{\boldsymbol{R}^n} |[b,T]a(x)|^q \, dx\right)^{p/q} \\ &\le C|Q|^{1-p/q}\|a\|_{L^q(\boldsymbol{R}^n)}^p \le C|Q|^{1-p/q}|Q|^{-1}|Q|^{p/q} = C. \end{aligned}$$

and

$$J_2 \le \sum_{j=1}^{\infty} \int_{2^{j+1}Q\backslash 2^jQ} |[b,T]a(x)|^p \, dx \le \sum_{j=1}^{\infty} |2^{j+1}Q|^{1-p}\left(\int_{2^{j+1}Q\backslash 2^jQ} |[b,T]a(x)| \, dx\right)^p.$$

We write

$$\begin{aligned} \int_{2^{j+1}Q\backslash 2^jQ} |[b,T]a(x)| \, dx &\le \int_{2^{j+1}Q\backslash 2^jQ} |b(x)-b_Q| \, |Ta(x)| \, dx \\ &\quad + \int_{2^{j+1}Q\backslash 2^jQ} |T((b-b_Q)a)(x)| \, dx \\ &= J_{21} + J_{22}. \end{aligned}$$

Since $2|y-x_0| < |x-x_0|$ when $y \in Q$ and $x \in 2^{j+1}Q\backslash 2^jQ$ with $j = 1, 2, \ldots,$ we get

$$\begin{aligned} J_{21} &\le \int_{2^{j+1}Q\backslash 2^jQ} |b(x)-b_Q| \int_Q |K(x,y)-K(x,x_0)| \, |a(y)| \, dydx \\ &\le \int_{2^{j+1}Q\backslash 2^jQ} |b(x)-b_Q| \int_Q \frac{\theta(|y-x_0|/|x-x_0|)}{|x-x_0|^n} |a(y)| \, dydx \\ &\le C\theta(2^{-j})|Q|^{1-1/p}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} |b(x)-b_Q| \, dx \\ &\le Cj\theta(2^{-j})|Q|^{1-1/p}\|b\|_*. \end{aligned}$$

Using the moment vanishing condition of a, we have

$$\begin{aligned} J_{22} &\le \int_{2^{j+1}Q\backslash 2^jQ} \int_Q |K(x,y)-K(x,x_0)| \, |b(y)-b_Q| \, |a(y)| \, dydx \\ &\le C\int_{2^{j+1}Q\backslash 2^jQ} \int_Q \frac{\theta(|y-x_0|/|x-x_0|)}{|x-x_0|^n} |b(y)-b_Q| \, |a(y)| \, dydx \\ &\le C\theta(2^{-j})\int_{2^{j+1}Q\backslash 2^jQ} \frac{dx}{|x-x_0|^n}\int_Q |b(y)-b_Q| \, |a(y)| \, dy \\ &\le C\theta(2^{-j})|Q|^{1-1/p}\frac{1}{|Q|}\int_Q |b(x)-b_Q| \, dx \\ &\le C\theta(2^{-j})|Q|^{1-1/p}\|b\|_*. \end{aligned}$$

Thus we obtain

$$J_2 \le C \sum_{j=1}^{\infty} |2^{j+1}Q|^{1-p}(j\theta(2^{-j})|Q|^{1-1/p}\|b\|_*)^p$$
$$\le C\|b\|_*^p \sum_{j=1}^{\infty} j^p 2^{j(1-p)n}\theta^p(2^{-j}) \le C\|b\|_*^p \int_0^1 \frac{\theta^p(t)|\log t|^p}{t^{(1-p)n+1}}\,dt \le C\|b\|_*^p.$$

This finishes the proof of Theorem 3.

Remark 2. In the case that T is a usual Calderón-Zygmund operator, $\theta(t) = t^{\varepsilon}$ for some $\varepsilon > 0$, we can see that the conditions of Theorem 3 hold with $n/(n+\varepsilon) < p \le 1$. Thus the commutator $[b, T]$ is a bounded operator from $H_b^p(\boldsymbol{R}^n)$ to $L^p(\boldsymbol{R}^n)$ whenever $n/(n+\varepsilon) < p \le 1$. This is the main result in [10].

Remark 3. When $p = 1$, the conditions of $\theta(t)$ in Theorem 3 coincide with those in Theorem 2.

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