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KODAI MATH. J.
25 (2002), 61–71

RELATIONS BETWEEN SYMMETRIC POWER L-FUNCTIONS AND SPINOR L-FUNCTIONS ATTACHED TO IKEDA LIFTS

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Abstract

We prove that the spinor L-functions attached to some Siegel cusp forms of degree ≥ 3, which are Ikeda lifts, have meromorphic continuations and the functional equations.

1. Introduction

For positive integers k and n, let $S_k(\Gamma_n)$ be the space of cusp forms with weight k and degree n, Γ_n denoting the Siegel modular group $Sp_n(\boldsymbol{Z})$ of degree n. Around 1996, Duke and Imamoḡlu conjectured the existence of the following lifting which is considered as a generalization of the Saito-Kurokawa lifting (cf. [3]); let f be a normalized Hecke eigenform in the space $S_{2k}(\Gamma_1)$ and $k \equiv n \pmod 2$, then there exists a Hecke eigenform F in $S_{k+n}(\Gamma_{2n})$. The standard L-function of F is the product of usual Hecke L-functions of f and the Riemann zeta function. In 1999 Ikeda [5] proved that the Duke-Imamoḡlu conjecture was true by using representation theoretic methods and techniques from the theory of Fourier-Jacobi expansions. We call such a Hecke eigenform F the Ikeda lift of f. In 2000 Ikeda [6] proved a generalization of the Miyawaki's conjecture [8]: if $g \in S_{k+n+r}(\Gamma_r)$ a Hecke eigenform, then Petterson inner product of g and the Ikeda lift F of f is the Hecke eigenform $\mathscr{F}_{f,g} \in S_{k+n+r}(\Gamma_{2n+r})$. The standard L-function of $\mathscr{F}_{f,g}$ is the product of usual Hecke L-function of f and the standard L-function of g. We also call such a Hecke eigenform $\mathscr{F}_{f,g}$ the Ikeda lift of (f,g).

We know that the spinor L-function attached to a Hecke eigenform in $S_k(\Gamma_n)$ continues meromorphically to all of $\boldsymbol{C}$ and has the functional equation for $n \leq 2$. It is proved for $n = 1$ by Hecke, and for $n = 2$ by Andrianov [1]. The case of $n \geq 3$ is still unknown.

The purpose of this paper is to give examples of the spinor L-function for $n \geq 3$ which has good analytic properties. Such spinor L-function is one attached to the Ikeda lift, which is that of even degree $2n$ expressed as the product of the m-th symmetric power L-functions for $m \leq n$. Since the m-th symmetric power L-function for $m \leq 5$ has good analytic properties, the spinor L-function

Received April 23, 2001; revised September 26, 2001.

attached to every Ikeda lift of even degree ≤ 10 is continued meromorphically to the whole complex s-plane. Especially the m-th symmetric power L-function for $m \leq 3$ has holomorphic continuation, so we have the location of the poles of some spinor L-functions in Section 6.

If we assume the meromorphy and the functional equation of the m-th symmetric power L-functions for all $m \geq 1$, then the spinor L-function attached to every Ikeda lift of f has similar analytic properties, and vice versa.

This paper is extracted from author's master thesis in Japanese (Tokyo Institute of Technology, March 2001).

2. Preliminaries

Let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform, and for a prime p, $\{\alpha_p, \alpha_p^{-1}\}$ be the Satake p-parameter of f such that

$$1 - a(p)X + p^{2k-1}X^2 = (1 - \alpha_p p^{k-1/2}X)(1 - \alpha_p^{-1} p^{k-1/2}X),$$

where $a(p)$ is the p-th Fourier coefficient of f. Then we define the m-th symmetric power L-function attached to f by

$$L_m(s, f) := \prod_{i=0}^{m} (1 - \alpha_p^{m-2i} p^{m(k-1/2)-s})^{-1}.$$

The following conjecture is due to Serre and Langlands;

Conjecture 2.1 (cf. [10], [13]). *Let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform. Put $\tilde{L}_m(s, f) := \gamma_m(s) L_m(s, f)$, where the gamma factor $\gamma_m(s)$ is defined by the following relation,*

$$\gamma_m(s) := \begin{cases} (2\pi)^{-rs} \displaystyle\prod_{j=0}^{r-1} \Gamma(s - j(k-1)) & (m = 2r - 1) \\ \pi^{-s/2} \Gamma\left(\dfrac{s}{2} - \left[\dfrac{r(k-1)}{2}\right]\right) \gamma_{2r-1}(s) & (m = 2r). \end{cases}$$

Then $\tilde{L}_m(s, f)$ can be continued meromorphically to the whole complex s-plane, has the functional equation $\tilde{L}_m(s, f) = \pm \tilde{L}_m((2k-1)m + 1 - s, f)$.

Remark. For an integer $m \leq 5$, it is known that above conjecture is true. Moreover in the case of $m \leq 3$, the function $\tilde{L}_m(s, f)$ is entire. It is proved for $m = 1$ by Hecke, for $m = 2$ by Shimura [12], and for $m = 3$ by Shahidi [11].

In general let $F \in S_k(\Gamma_n)$ be a Hecke eigenform, and $\{\alpha_{p,0}, \ldots, \alpha_{p,n}\}$ be the Satake p-parameters of F. Then the spinor L-function (resp. the standard L-function) is defined by $L(s, F, \mathrm{spin}) = \prod_{p:\ \mathrm{prime}} L_p(s, F, \mathrm{spin})$ (resp. $L(s, F, \mathrm{st}) = \prod_{p:\ \mathrm{prime}} L_p(s, F, \mathrm{st})$). Here

$$L_p(s, F, \mathrm{spin}) := \prod_p \left[(1 - \alpha_{p,0} p^{-s}) \prod_{r=1}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - \alpha_{p,0} \alpha_{p,i_1} \cdots \alpha_{p,i_r} p^{-s}) \right]^{-1},$$

$$L_p(s, F, \mathrm{st}) := \prod_p \left[(1 - p^{-s}) \prod_{j=1}^{n} (1 - \alpha_{p,j} p^{-s})(1 - \alpha_{p,j}^{-1} p^{-s}) \right]^{-1}.$$

Now we introduce Ikeda's result.

THEOREM 2.2 ([5]). *Let $n \in \boldsymbol{N}$ with $k \equiv n \pmod 2$, let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform. Then there exists a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ such that*

$$L(s, F, \mathrm{st}) = \zeta(s) \prod_{j=1}^{2n} L(s + k + n - j, f).$$

Here the Hecke L-function $L(s, f)$ is defined by $\sum_{n \ge 1} a(n) n^{-s}$ with $a(n)$ being the n-th Fourier coefficient of f.

We call this F the Ikeda lift of f.

For an integer $r \ge 1$ with $k \equiv n + r \pmod 2$, let $g \in S_{k+n+r}(\Gamma_r)$ be a Hecke eigenform. We put the integral

$$\mathscr{F}_{f,g}(Z) := \int_{\Gamma_r \backslash \mathscr{H}_r} F\left(\begin{pmatrix} Z & \\ & Z' \end{pmatrix} \right) \overline{g(Z')} (\det \operatorname{Im} Z')^{k+n-1} \, dZ' \quad (Z \in \mathscr{H}_{2n+r}).$$

Since the Ikeda lift $F(Z)$ is a cusp form, we have $\mathscr{F}_{f,g} \in S_{k+n+r}(\Gamma_{2n+r})$.

THEOREM 2.3 ([6]). *Assume that $\mathscr{F}_{f,g}(Z)$ is not identically zero. Then $\mathscr{F}_{f,g}(Z)$ is a Hecke eigenform whose standard L-function is equal to*

$$L(s, \mathscr{F}_{f,g}, \mathrm{st}) = L(s, g, \mathrm{st}) \prod_{j=1}^{2n} L(s + k + n - j, f).$$

This theorem shows that one of Miyawaki's conjectures [8] is true. We also call this $\mathscr{F}_{f,g}$ the Ikeda lift of (f, g).

3. Main result

For $m \in \boldsymbol{Z}_{\ge 0}$, $n \in \boldsymbol{Z}_{\ge 1}$, and $i_{m,n} \in \boldsymbol{Z}$, the function $r_{m,n}(i_{m,n})$ with values in $\boldsymbol{Z}_{\ge 0}$ is defined by

(3.1)
$$\begin{aligned} r_{m,n+2}(i_{m,n+2}) &= r_{m-2,n}(i_{m,n+2}) + (1 + \delta_m) \cdot r_{m,n}(i_{m,n+2} - 1) + r_{m+2,n}(i_{m,n+2} - 2) \\ &\quad + \sum_{y = -n-2, -n-1, n, n+1} \{ r_{m-1,n}(i_{m,n+2} + y) + r_{m+1,n}(i_{m,n+2} + y - 1) \} \\ &\quad + \sum_{x = -2n-1, 0, 2, 2n+3} r_{m,n}(i_{m,n+2} - x), \end{aligned}$$

$$r_{0,1}(i_{0,1}) = 1 \quad \text{if} \quad i_{0,1} = 0, 1, \qquad r_{0,2}(i_{0,2}) = 1 \quad \text{if} \quad i_{0,2} = -1, 0, 1, 2, 3,$$

$$r_{1,1}(i_{1,1}) = 1 \quad \text{if} \quad i_{1,1} = 0, \qquad r_{1,2}(i_{1,2}) = 1 \quad \text{if} \quad i_{1,2} = -1, 0, 1, 2,$$

$$r_{m,1}(i_{m,1}) = 0 \quad \text{if otherwise}, \qquad r_{2,2}(i_{2,2}) = 1 \quad \text{if} \quad i_{2,2} = 0,$$

$$r_{m,2}(i_{m,2}) = 0 \quad \text{if otherwise}.$$

Here δ_m is equal to 0 for $m = 0$, and is equal to 1 for $m > 0$.

The main result of this paper is the following

THEOREM 3.1. *Assume* $k \equiv n \pmod 2$. *Let* $F_{2n} \in S_{k+n}(\Gamma_{2n})$ *be the Ikeda lift of a normalized Hecke eigenform* $f \in S_{2k}(\Gamma_1)$. *Then we have*

$$L(s, F_{2n}, \mathrm{spin}) = \prod_{m=0}^{n} \prod_{i_{m,n}} L_m(s - (n-m)k + i_{m,n}, f)^{r_{m,n}(i_{m,n})}, \tag{3.2}$$

where the product is taken over $m(m-1)/2 - n(n-1)/2 \le i_{m,n} \le n(n+1)/2 - m(m+1)/2$, *we regard* $L_0(s, f)$ *as the Riemann zeta function* $\zeta(s)$, *and* $r_{m,n}(i_{m,n})$ *satisfies the above relations* (3.1).

For such a spinor L-function $L(s, F_{2n}, \mathrm{spin})$, we put

$$\tilde{L}(s, F_{2n}, \mathrm{spin}) = \prod_{m=0}^{n} \prod_{i_{m,n}} \tilde{L}_m(s - (n-m)k + i_{m,n}, f)^{r_{m,n}(i_{m,n})}.$$

Since Conjecture 2.1 is true for $m \le 5$, by using Theorem 3.1 we obtain

COROLLARY 3.2. *Suppose that* $n \le 5$ *with the same notation in Theorem* 3.1. *Then* $\tilde{L}(s, F_{2n}, \mathrm{spin})$ *has meromorphic continuation to the whole* s*-plane and satisfies the functional equation*

$$\tilde{L}\left(nk - \frac{n(n+1)}{2} + 1 - s, F_{2n}, \mathrm{spin}\right) = (-1)^{k \cdot 2^{n-2}} \tilde{L}(s, F_{2n}, \mathrm{spin}). \tag{3.3}$$

Remark. We calculated some poles of $\tilde{L}(s, F_{2n}, \mathrm{spin})$ in Section 6 for $n = 2, 3$.

Conversely suppose the spinor L-function attached to $F_n \in S_k(\Gamma_n)$ of arbitrary even degree $n \in \boldsymbol{N}$ has meromorphic continuation to the whole s-plane and the functional equation (3.3), then from Theorem 3.1 it follows that Conjecture 2.1 is true for every cusp form f of weight $2k$.

4. Proof of Theorem 3.1

First, we show the following two lemmas. Let $F_{2n} \in S_{k+n}(\Gamma_{2n})$ be the Ikeda lift of a normalized Hecke eigenform $f \in S_{2k}(\Gamma_1)$.

LEMMA 4.1. *Let* $\{\alpha_p, \alpha_p^{-1}\}$ *be the Satake* p*-parameters of* f, *and* $\{\beta_{p,0}, \beta_{p,1}, \dots, \beta_{p,2n}\}$ *be the Satake* p*-parameters of* F_{2n}. *If* $k \equiv n \pmod 2$, *the Satake* p*-parameters, those are uniquely determined up to the action of the Weyl group, are given for* $1 \le i \le n$ *by*

$$\begin{cases} \beta_{p,0} = p^{nk-n(n+1)/2}, \\ \beta_{p,i} = \alpha_p p^{i-1/2}, \\ \beta_{p,n+i} = \alpha_p^{-1} p^{i-1/2}. \end{cases}$$

Proof. Since F_{2n} is the Ikeda lift of f, the standard L-function of F_{2n} satisfies

$$L(s, F_{2n}, \mathrm{st}) = \zeta(s) \prod_{i=1}^{2n} L(s+k+n-i, f).$$

From the definition of the standard L-function, for $1 \le i \le n$ we obtain the relations

$$\begin{cases} \beta_{p,i} = \alpha_p p^{i-1/2}, \\ \beta_{p,n+i} = \alpha_p^{-1} p^{i-1/2}. \end{cases}$$

According to [5], there exists the polynomial $\Phi(X) \in \boldsymbol{C}[X + X^{-1}]$ such that the eigenvalue for Siegel's Eisenstein series $E_{k+n}^{(2n)}(Z)$ by Hecke operator $T(p)$ is $\Phi(p^{k-1/2})$. Then the eigenvalue for F_{2n} by Hecke operator $T(p)$ is equal to $\Phi(\alpha_p)$. The eigenvalue $t(p)$ under the Hecke operator $T(p)$ for some prime p (cf. [4], note the different normalization) is

$$t(p) = \beta_{p,0} \sum_{i=0}^{2n} \phi_i(\beta_{p,1}, \dots, \beta_{p,2n}), \tag{4.1}$$

where ϕ_i is the i-th elementary symmetric polynomial. By substituting the Satake p-parameters of Siegel's Eisenstein seris $\{1, p^{k+n-1}, p^{k+n-2}, \dots, p^{k-n}\}$ for (4.1), the eigenvalue of $T(p)$ on $E_{k+n}^{(2n)}(Z)$ is equal to

$$\begin{aligned} \prod_{i=1}^{2n}(1+p^{k+n-i}) &= \prod_{i=1}^{n}(1+p^{k-i})(1+p^{k+n-i}) \\ &= (p^{k-1/2})^n \prod_{i=1}^{n}(p^{k-1/2} + p^{-k+1/2} + p^{i-i/2} + p^{-i+1/2}). \end{aligned}$$

Thus we obtain

$$\Phi(X) = (p^{k-1/2})^n \prod_{i=1}^{n}(X + X^{-1} + p^{i-1/2} + p^{-i+1/2}).$$

On the other hand by (4.1) the eigenvalue of $T(p)$ on $F_{2n}(Z)$ is equal to

$$\begin{aligned} t(p, F_{2n}) &= \beta_{p,0} \sum_{i=0}^{2n} \phi_i(\alpha_p p^{1/2}, \dots, \alpha_p^{-1} p^{n-1/2}) \\ &= \beta_{p,0} \prod_{i=1}^{n} (1 + \alpha_p p^{i-1/2})(1 + \alpha_p^{-1} p^{i-1/2}) \\ &= \beta_{p,0} p^{n^2/2} (p^{k-1/2})^{-n} \Phi(\alpha_p). \end{aligned}$$

Hence it follows from $t(p, F_{2n}) = \Phi(\alpha_p)$ that $\beta_{p,0} = p^{kn-n(n+1)/2}$. ∎

We have the following lemma by a direct computation.

Lemma 4.2. *For* $m \geq 0$

$$L_m^{(p)}(\alpha_p X, f) L_m^{(p)}(\alpha_p^{-1} X, f) = L_{m+1}^{(p)}(p^{-k+1/2} X, f) L_{m-1}^{(p)}(p^{k-1/2} X, f).$$

For $m \geq 1$

$$L_m^{(p)}(\alpha^2 X, f) L_m^{(p)}(\alpha^{-2} X, f) = L_{m+2}^{(p)}(p^{-2k+1} X, f) L_{m-2}^{(p)}(p^{2k-1} X, f).$$

where $L_m^{(p)}(X, f) = 1$ $(m = -1)$.

Proof of Theorem 3.1. We prove the main theorem by induction on the degree n. From the assumption, we put $k \equiv n \pmod 2$. For $n = 1$ an integer k is odd, and the Ikeda lift F_2 satisfies (3.2) by the Saito-Kurokawa lifting, whose spinor L-function is

$$L(s, F_2, \mathrm{spin}) = L(s, f)\zeta(s-k)\zeta(s-k+1).$$

When $n = 2$, the integer k is even, by using the definition of the spinor L-function and Lemma 4.1, the spinor L-function of the Ikeda lift F_4 is

$$L(s, F_4, \mathrm{spin}) = L_2(s, f) \prod_{-1 \leq i \leq 2} L(s-k+i, f) \prod_{-1 \leq j \leq 3} \zeta(s-2k+j). \tag{4.2}$$

Let $F_{2n} \in S_{k+n}(\Gamma_{2n})$, and $F_{2(n+2)} \in S_{k+n+2}(\Gamma_{2(n+2)})$ be the Ikeda lifts of a normalized Hecke eigenform $f \in S_{2k}(\Gamma_1)$. Suppose the Ikeda lift F_{2n} satisfies (3.2), then by Lemma 4.1 the Satake p-parameters of F_{2n} are given by

$$\beta_{p,0} = p^{nk-n(n+1)/2},$$
$$\{\beta_{p,1}, \dots, \beta_{p,2n}\} = \{\alpha_p p^{1-1/2}, \dots, \alpha_p p^{n-1/2}, \alpha_p^{-1} p^{1-1/2}, \dots, \alpha_p^{-1} p^{n-1/2}\},$$

and the Satake p-parameters of $F_{2(n+2)}$ are given by

$$\gamma_{p,0} = p^{nk-n(n+1)/2} p^{2k-2n-3},$$

$$\{\gamma_{p,1}, \ldots, \gamma_{p,2(n+2)}\} = \{\alpha_p p^{1-1/2}, \ldots, \alpha_p^{-1} p^{n-1/2}, \alpha_p p^{n+1/2}, \alpha_p p^{n+3/2}, \alpha_p^{-1} p^{n+1/2}, \alpha_p^{-1} p^{n+3/2}\}.$$

From that, we obtain the spinor L-function of $F_{2(n+2)}$:

$$\begin{aligned} L_p(X, F_{2(n+2)}, \mathrm{spin}) &= \prod_{x=-2n-1,0,1,2,2n+3} L_p(p^{2k-x}X, F_{2n}, \mathrm{spin}) \\ &\times \prod_{y=-n-2,-n-1,n,n+1} L_p(p^{2k+y-1/2}\alpha_p X, F_{2n}, \mathrm{spin}) L_p(p^{2k+y-1/2}\alpha_p^{-1} X, F_{2n}, \mathrm{spin}) \\ &\times L_p(\alpha_p^2 p^{2k-1} X, F_{2n}, \mathrm{spin}) L_p(p^{2k-1} X, F_{2n}, \mathrm{spin}) L_p(\alpha_p^{-2} p^{2k-1} X, F_{2n}, \mathrm{spin}). \end{aligned}$$

The assumption and Lemma 4.2 lead that

$$\begin{aligned} L(s, F_{2(n+2)}, \mathrm{spin}) &= \prod_{x=-2n-1,0,1,2,2n+3} \prod_{m=0}^{n} \prod_{i_{m,n}} L_m(s-(n+2-m)k+i_{m,n}+x, f)^{r_{m,n}(i_{m,n})} \\ &\times \prod_{y=-n-2,-n-1,n,n+1} \prod_{m=1}^{n+1} \prod_{i_{m-1,n}} L_m(s-(n+2-m)k+i_{m-1,n}-y, f)^{r_{m-1,n}(i_{m-1,n})} \\ &\times \prod_{y=-n-2,-n-1,n,n+1} \prod_{m=0}^{n-1} \prod_{i_{m+1,n}} L_m(s-(n+2-m)k+i_{m+1,n}-y+1, f)^{r_{m+1,n}(i_{m+1,n})} \\ &\times \prod_{m=2}^{n+2} \prod_{i_{m-2,n}} L_m(s-(n+2-m)k+i_{m-2,n}, f)^{r_{m-2,n}(i_{m-2,n})} \\ &\times \prod_{m=1}^{n} \prod_{i_{m,n}} L_m(s-(n+2-m)k+i_{m,n}+1, f)^{r_{m,n}(i_{m,n})} \\ &\times \prod_{m=0}^{n-2} \prod_{i_{m+2,n}} L_m(s-(n+2-m)k+i_{m+2,n}+2, f)^{r_{m+2,n}(i_{m+2,n})} \\ &= \prod_{m=0}^{n+2} \prod_{i_{m,n+2}} L_m(s-(n+2-m)k+i_{m,n+2}, f)^{r_{m,n+2}(i_{m,n+2})}. \end{aligned}$$

Here the sum is taken over interval $m(m-1)/2-(n+2)(n+1)/2 \le i_{m,n+2} \le (n+2)(n+3)/2-m(m+1)/2$, and $r_{m,n+2}(i_{m,n+2})$ satisfies (3.1). This proves Theorem 3.1. ■

5. The Ikeda lift of odd degree

For $m \in \mathbf{Z}_{\geq 0}$, $n \in \mathbf{Z}_{\geq 1}$, and $i_{m,n} \in \mathbf{Z}$, the function $t_{m,n}(i_{m,n})$ with values in $\mathbf{Z}_{\geq 0}$ is defined by

$$(5.1)\qquad \begin{aligned} t_{m,n+2}(i_{m,n+2}) = & \sum_{x=-2n-1,0,2,2n+3} t_{m,n}(i_{m,n+2}-x) \\ & + \sum_{y=-n-2,-n-1,n,n+1} t_{m-1,n}(i_{m,n+2}+y) \\ & + t_{m-2,n}(i_{m,n+2}), \end{aligned}$$

$$\begin{aligned} t_{0,1}(i_{0,1}) &= 1 \quad \text{if } i_{0,1} = 0, 1, & t_{2,2}(i_{2,2}) &= 1 \quad \text{if } i_{2,2} = 0, \\ t_{1,1}(i_{1,1}) &= 1 \quad \text{if } i_{1,1} = 0, & t_{1,2}(i_{1,2}) &= 1 \quad \text{if } i_{1,2} = -1, 0, 1, 2, \\ t_{m,1}(i_{m,1}) &= 0 \quad \text{if otherwise}, & t_{0,2}(i_{0,2}) &= 1 \quad \text{if } i_{0,2} = -1, 0, 2, 3, \\ & & t_{m,2}(i_{m,2}) &= 0 \quad \text{if otherwise}. \end{aligned}$$

For Siegel's Eisenstein series $E_{k+n+r}^{(2n+2r)}$, the integral

$$\int_{\Gamma_r \backslash \mathscr{H}_r} E_{k+n+r}^{(2n+2r)}\left(\begin{pmatrix} Z & \\ & Z' \end{pmatrix}\right) \overline{g(Z')} (\det \operatorname{Im} Z')^{k+n-1}\, dZ'$$

is a scalar multiple of the Klingen's Eisenstein series $[g]_r^{2n+r}(Z)$ (cf. [2]).

If r is odd in Theorem 2.3, the Ikeda lift $\mathscr{F}_{f,g}(Z)$ of (f,g) is a Hecke eigenform with odd degree $2n+r$ and weight $k+n+r$. Let $r=1$, and we argue in the same way as the proof of Theorem 3.1 and replacing Siegel's Eisenstein series by Klingen's Eisenstein series. Then we have the following theorem.

Theorem 5.1. *Assume $k \equiv n+1 \pmod 2$. Let $f \in S_{2k}(\Gamma_1)$, $g \in S_{k+n+1}(\Gamma_1)$ be normalized Hecke eigenforms, and $F_{2n+1} \in S_{k+n+1}(\Gamma_{2n+1})$ be the Ikeda lift of (f,g). Then we have*

$$(5.2)\qquad L(s, F_{2n+1}, \text{spin}) = \prod_{m=0}^{n} \prod_{i_{m,n}} L(s-(n-m)k + i_{m,n}, g \times \underbrace{f \times \cdots \times f}_{m \text{ times}})^{t_{m,n}(i_{m,n})},$$

where the product is taken over $m(m-1)/2 - n(n-1)/2 \leq i_{m,n} \leq n(n+1)/2 - m(m+1)/2$, and $t_{m,n}(i_{m,n})$ satisfies the above relations (5.1).

6. Examples

We calculate $r_{m,n}(i_{m,n})$ for the spinor L-function appearing in Theorem 3.1, and compare with the following Andrianov's conjecture about the spinor L-function.

CONJECTURE 6.1 (cf. [1]). *Let $F \in S_k(\Gamma_n)$ be a Hecke eigenform. We set*

$$\hat{L}(s, F, \mathrm{spin}) := (2\pi)^{-2^{n-1}s}\gamma_{n,k}(s)L(s, F, \mathrm{spin}),$$

where the $\gamma_{n,k}(s)$ are defined by the relations

$$\gamma_{1,k}(s) = \Gamma(s), \quad \gamma_{n,k}(s) = \gamma_{n-1,k}(s-k+n)\gamma_{n-1,k}(s) \quad (n > 1).$$

Then $\hat{L}(s, F, \mathrm{spin})$ can be meromorphically continued to the whole complex s-plane, and the following functional equation holds

$$\hat{L}\left(nk - \frac{n(n+1)}{2} + 1 - s, F, \mathrm{spin}\right) = (-1)^{k\cdot 2^{n-2}}\hat{L}(s, F, \mathrm{spin}).$$

By calculating $r_{m,n}(i_{m,n})$, we have the following analytic properties of the spinor L-function attached to the Ikeda lift with degree 3, 4, 5, 6 and 8.

The case of degree 4. Let k be even, and $F_4 \in S_{k+2}(\Gamma_4)$ be the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. We have the spinor L-function of F_4 by (4.2) in the previous section. The function $\tilde{L}(s, F_4, \mathrm{spin})$ is meromorphically continued to the whole complex s-plane, and has at most simple poles at the points $s = 2k+2, 2k+1, 2k-1, 2k-2$, and satisfies the functional equation

$$\tilde{L}(s, F_4, \mathrm{spin}) = \tilde{L}(4k-1-s, F_4, \mathrm{spin}).$$

The case of degree 6. Let k be odd, and $F_6 \in S_{k+3}(\Gamma_6)$ be the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. Then the spinor L-function of F_6 satisfies

$$L(s, F_6, \mathrm{spin}) = L_3(s, f) \prod_{-2 \le i_{2,3} \le 3} L_2(s-k+i_{2,3}, f)$$
$$\times \prod_{-3 \le i_{1,3} \le 5} L(s-2k+i_{1,3}, f)^{r_{1,3}(i_{1,3})} \prod_{-3 \le i_{0,3} \le 6} \zeta(s-3k+i_{0,3})^{r_{0,3}(i_{0,3})},$$

where the exponent $r_{m,3}(i_{m,3})$ are the following value;

$$r_{1,3}(i_{1,3}) = \begin{cases} 1 & i_{1,3} = -3, -2, 4, 5 \\ 2 & i_{1,3} = -1, 0, 1, 2, 3 \end{cases}, \quad r_{0,3}(i_{0,3}) = \begin{cases} 1 & i_{0,3} = -3, -2, -1, 4, 5, 6 \\ 2 & i_{0,3} = 0, 1, 2, 3 \end{cases}.$$

The function $\tilde{L}(s, F_6, \mathrm{spin})$ is meromorphically continued to the whole complex s-plane, and has at most simple poles at the point $s = 3k+4$. The functional equation of $\tilde{L}(s, F_6, \mathrm{spin})$ is equal to

$$\tilde{L}(s, F_6, \mathrm{spin}) = \tilde{L}(6k-2-s, F_6, \mathrm{spin}).$$

The case of degree 8. Let k be even, and $F_8 \in S_{k+4}(\Gamma_8)$ be the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. Then the spinor L-function of F_8 satisfies

$$L(s, F_8, \mathrm{spin}) = L_4(s, f) \prod_{-3 \le i_{3,4} \le 4} L_3(s-k+i_{3,4}, f) \prod_{-5 \le i_{2,4} \le 7} L_2(s-2k+i_{2,4})^{r_{2,4}(i_{2,4})}$$
$$\times \prod_{-6 \le i_{1,4} \le 9} L(s-3k+i_{1,4})^{r_{1,4}(i_{1,4})} \prod_{-6 \le i_{0,4} \le 10} \zeta(s-4k+i_{0,4})^{r_{0,4}(i_{0,4})},$$

where the exponent $r_{m,4}(i_{m,4})$ are the following value;

$$r_{2,4}(i_{2,4}) = \begin{cases} 1 & i_{2,4} = -5, -4, 6, 7 \\ 2 & i_{2,4} = -3, -2, 4, 5 \\ 3 & i_{2,4} = -1, 0, 1, 2, 3 \end{cases}, \quad r_{1,4}(i_{1,4}) = \begin{cases} 1 & i_{1,4} = -6, -5, 8, 9 \\ 2 & i_{1,4} = -4, 7 \\ 3 & i_{1,4} = -3, -2, 5, 6 \\ 4 & i_{1,4} = -1, 4 \\ 5 & i_{1,4} = 0, 1, 2, 3 \end{cases}$$

$$r_{0,4}(i_{0,4}) = \begin{cases} 1 & i_{0,4} = -6, -5, -4, 8, 9, 10 \\ 2 & i_{0,4} = -3, 7 \\ 3 & i_{0,4} = -2, -1, 5, 6 \\ 4 & i_{0,4} = 0, 1, 2, 3, 4 \end{cases}.$$

The function $\tilde{L}(s, F_8, \mathrm{spin})$ is meromorphically continued to the whole complex s-plane and has the functional equation

$$\tilde{L}(s, F_8, \mathrm{spin}) = \tilde{L}(8k - 3 - s, F_8, \mathrm{spin}).$$

The case of degree 3. Let k be even, and $F_3 \in S_{k+2}(\Gamma_3)$ be the Ikeda lift of $(f, g) \in S_{2k}(\Gamma_1) \times S_{k+2}(\Gamma_1)$. Then the spinor L-function of F_3 satisfies

$$L(s, F_3, \mathrm{spin}) = L(s, f \times g) L(s - k, g) L(s - k + 1, g).$$

The function $\tilde{L}(s, F_3, \mathrm{spin})$ is holomorphically continued to the whole complex s-plane, and satisfies the functional equation

$$\tilde{L}(s, F_3, \mathrm{spin}) = \tilde{L}(3k + 1 - s, F_3, \mathrm{spin}).$$

The case of degree 5. Let k be odd, and $F_5 \in S_{k+3}(\Gamma_5)$ be the Ikeda lift of $(f, g) \in S_{2k}(\Gamma_1) \times S_{k+3}(\Gamma_1)$. Then the spinor L-function of F_5 satisfies

$$L(s, F_5, \mathrm{spin}) = L(s, f \times f \times g) \prod_{-1 \le i_{1,2} \le 2} L(s - k + i_{1,2}, f \times g) \prod_{i_{0,2} = -1, 0, 2, 3} L(s - 2k + i_{0,2}, g).$$

The function $\tilde{L}(s, F_5, \mathrm{spin})$ is holomorphically continued to the whole complex s-plane, and satisfies the functional equation

$$\tilde{L}(s, F_5, \mathrm{spin}) = \tilde{L}(5k + 1 - s, F_5, \mathrm{spin}).$$

For $n = 3, 4, 5, 6, 8$, we put

$$R_n(s) := \frac{\tilde{L}(s, F_n, \mathrm{spin})}{\hat{L}(s, F_n, \mathrm{spin})},$$

then $R_n(s)$ is the ratio of gamma function, and also satisfies the functional equation $R_n(s) = R_n(nk - n(n+1)/2 + 1 - s)$. This shows that our theorem satisfies Andrianov's conjecture [1, p. 115] on the spinor L-function.

Acknowledgments. I am thankful to Professor S. Mizumoto for his advice, encouragement and patient support in all the process. Also I am thankful to Professor T. Ikeda for his valuable comments.

I am grateful to my family and all of my friends for their constant encouragement.

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