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WEIGHTED SHARING OF THREE VALUES AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract

Using the idea of weighted sharing we prove a result on uniqueness of meromorphic functions sharing three values which improve some results of Ueda, Yi and Ye.

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathscr{C}$. For $b \in \mathscr{C} \cup \{\infty\}$ we say that f and g share the value b CM (counting multiplicities) if $f - b$ and $g - b$ have the same zeros with the same multiplicities. If we do not take multiplicities into account, we say that f and g share the value b IM (ignoring multiplicities). For standard notations and definitions of the value distribution theory we refer [1].

H. Ueda [6] proved the following result.

Theorem A [6]. *Let f and g be two distinct nonconstant entire functions sharing $0, 1$ CM and let a $(\neq 0, 1)$ be a finite complex number. If a is lacunary for f then $1 - a$ is lacunary for g and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

Improving Theorem A H. X. Yi [8] proved the following theorem.

Theorem B [8]. *Let f and g be two distinct nonconstant entire functions sharing $0, 1$ CM and let a $(\neq 0, 1)$ be a finite complex number. If $\delta(a; f) > 1/3$ then a and $1 - a$ are Picard exceptional values of f and g respectively and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

Extending Theorem B to meromorphic functions S. Z. Ye [7] proved the following results.

Theorem C [7]. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty$ CM. Let a $(\neq 0, 1)$ be a finite complex*

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number. *If* $\delta(a; f) + \delta(\infty; f) > 4/3$ *then* a *and* $1 - a$ *are Picard exceptional values of* f *and* g *respectively and also* ∞ *is so and* $(f - a)(g + a - 1) \equiv a(1 - a)$.

Theorem D [7]. *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ CM. *Let* $a_1, a_2, \dots, a_p$ *be* p (≥ 1) *distinct finite complex numbers and* $a_j \neq 0, 1$ *for* $j = 1, 2, 3 \dots p$. *If* $\sum_{j=1}^{p} \delta(a_j; f) + \delta(\infty; f) > 2(p+1)/(p+2)$ *then there exist one and only one* a_k *in* $a_1, a_2, \dots, a_p$ *such that* a_k *and* $1 - a_k$ *are Picard exceptional values of* f *and* g *respectively and also* ∞ *is so and* $(f - a_k)(g + a_k - 1) \equiv a_k(1 - a_k)$.

Improving above results H. X. Yi [10] proved the following theorem.

Theorem E [10]. *Let* f *and* g *be two distinct nonconstant meromorphic functions such that* f *and* g *share* $0, 1, \infty$ CM. *Let* a $(\neq 0, 1)$ *be a finite complex number. If* $N(r, a; f) \neq T(r, f) + S(r, f)$ *and* $N(r, f) \neq T(r, f) + S(r, f)$ *then* a *and* $1 - a$ *are Picard exceptional values of* f *and* g *respectively and also* ∞ *is so and* $(f - a)(g + a - 1) \equiv a(1 - a)$.

Definition 1. Let p be a positive integer and $b \in \mathscr{C} \cup \{\infty\}$. Then by $N(r, b; f \mid\leq p)$ we denote the counting function of those zeros of $f - b$ (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, b; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, b; f \mid\geq p)$ and $\overline{N}(r, b; f \mid\geq p)$.

Hua and Fang [2] proved that if two nonconstant distinct meromorphic functions f and g share $0, 1, \infty$ CM then $N(r, a; f \mid\geq 3) = S(r, f)$ for any complex number a $(\neq 0, 1, \infty)$.

Also Yi [10] proved that if two nonconstant distinct meromorphic functions f and g share $0, 1, \infty$ CM then $N(r, \infty; f \mid\geq 2) = S(r, f)$.

Therefore Theorem E of Yi can easily be improved to the following result.

Theorem 1. *Let* f *and* g *be distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ CM. *If* a $(\neq 0, 1)$ *is a finite complex number such that* $N(r, a; f \mid\leq 2) \neq T(r, f) + S(r, f)$ *and* $N(r, \infty; f \mid\leq 1) \neq T(r, f) + S(r, f)$ *then* a *and* $1 - a$ *are Picard exceptional values of* f *and* g *respectively and also* ∞ *is so and* $(f - a)(g + a - 1) \equiv a(1 - a)$.

Following examples show that Theorem 1 is sharp.

Example 1. Let $f = (e^z - 1)/(e^z + 1)$, $g = (1 - e^z)/(1 + e^z)$, $a_1 = -1$ and $a_2 = 2$. Then f, g share $0, 1, \infty$ CM. Also $N(r, \infty; f \mid\leq 1) = T(r, f) + S(r, f)$, $N(r, a_1; f \mid \leq 2) \neq T(r, f) + S(r, f)$ and $N(r, a_2; f \mid\leq 2) = T(r, f) + S(r, f)$. Clearly $(f - a_i)(g + a_i - 1) \not\equiv a_i(1 - a_i)$ for $i = 1, 2$.

Example 2. Let $f = e^z$, $g = e^{-z}$ and $a = 2$. Then f, g share $0, 1, \infty$ CM.

Also $N(r,\infty;f\,|\leq 1) \neq T(r,f)+S(r,f)$, $N(r,a;f\,|\leq 2) = T(r,f)+S(r,f)$. Clearly $(f-a)(g+a-1) \not\equiv a(1-a)$.

Now one may ask the following question: *Is it possible to replace the hypothesis* $N(r,a;f\,|\leq 2) \neq T(r,f)+S(r,f)$ *of Theorem* 1 *by any one of the following*?
(i) $N(r,a;f\,|\leq 1) \neq T(r,f)+S(r,f)$,
(ii) $\overline{N}(r,a;f\,|\leq 2) \neq T(r,f)+S(r,f)$.

We can answer this question in the negative by the following example.

Example 3. Let $f = e^z(1-e^z)$, $g = e^{-z}(1-e^{-z})$ and $a = 1/4$. Then f,g share $0,1,\infty$ CM. Also $N(r,\infty;f\,|\leq 1) \neq T(r,f)+S(r,f)$. Since $f-a = -(e^z-2a)^2$, we see the following
(i) $N(r,a;f\,|\leq 1) \equiv 0$,
(ii) $\overline{N}(r,a;f\,|\leq 2) = N(r,2a;e^z) = (1/2)T(r,f)+S(r,f)$ and
(iii) $N(r,a;f\,|\leq 2) = 2N(r,2a;e^z) = T(r,f)+S(r,f)$.
Also clearly $(f-a)(g+a-1) \not\equiv a(1-a)$.

First we note that if f,g satisfy the conclusion of the theorems as stated above then f,g must share ∞ CM because in this case ∞ becomes lacunary for f and g and so the question of sharing ∞ IM does not arise.

Now the following two examples show that in the above theorems the sharing of 0 and 1 can not be relaxed from CM to IM.

Example 4. Let $f = e^z-1$, $g = (e^z-1)^2$ and $a = -1$. Then f,g share 0 IM and $1,\infty$ CM. Also $N(r,\infty;f) \equiv 0$ and $N(r,a;f) \equiv 0$ but $(f-a)(g+a-1) \not\equiv a(1-a)$.

Example 5. Let $f = 2-e^z$, $g = e^z(2-e^z)$ and $a = 2$. Then f,g share 1 IM and $0,\infty$ CM. Also $N(r,\infty;f) \equiv 0$ and $N(r,a;f) \equiv 0$ but $(f-a)(g+a-1) \not\equiv a(1-a)$.

Now one may ask the following question: *Is it really impossible to relax in any way the nature of sharing of any one of* 0 *and* 1 *in the theorems stated above*?

In the paper we study this problem. Though we do not know the situation for Theorem 1 we can relax the nature of sharing of 0 and 1 separately in Theorem C and thereby we can improve Theorem A, Theorem B and Theorem C.

To this end we now explain the notion of weighted sharing as introduced in [4, 5].

Definition 2 [4, 5]. Let k be a nonnegative integer or infinity. For $a \in \mathscr{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share a value a with weight k then z_o is a zero of $f-a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\leq k)$ and z_o is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 3 [4]. For $S \subset \mathscr{C} \cup \{\infty\}$, we define $E_f(S, k)$ as $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$, where k is a nonnegative integer or infinity.

Definition 4. For $a \in \mathscr{C} \cup \{\infty\}$, we put

$$\delta_{p)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f \mid \leq p)}{T(r, f)},$$

where p is a positive integer.

Now we state the main results of the paper.

Theorem 2. *Let f and g be two distinct meromorphic functions sharing $(0, 1)$, $(1, \infty)$ and (∞, ∞). If a $(\neq 0, 1)$ is a finite complex number such that $3\delta_{2)}(a; f) + 2\delta_{1)}(\infty; f) > 3$ then a and $1 - a$ are Picard exceptional values of f and g and also ∞ is so and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

Theorem 3. *Let f and g be two distinct meromorphic functions sharing $(0, \infty)$, $(1, 1)$ and (∞, ∞). If a $(\neq 0, 1)$ is a finite complex number such that $3\delta_{2)}(a; f) + 2\delta_{1)}(\infty; f) > 3$ then a and $1 - a$ are Picard exceptional values of f and g and also ∞ is so and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

Example 4 shows that in Theorem 2 sharing $(0, 1)$ can not be relaxed to sharing $(0, 0)$ and Example 5 shows that in Theorem 3 sharing $(1, 1)$ can not be relaxed to sharing $(1, 0)$.

Throughout the paper we denote by f, g two nonconstant meromorphic functions defined in the open complex plane $\mathscr{C}$.

2. Lemmas

In this section we present some lemmas which will be required in the sequel.

Lemma 1. *If f and g share $(0, 0)$, $(1, 0)$ and $(\infty, 0)$ then*

$$\text{(i)} \quad T(r, f) \leq 3T(r, g) + S(r, f)$$

and

$$\text{(ii)} \quad T(r, g) \leq 3T(r, f) + S(r, g).$$

Proof. Since f, g share $(0, 0)$, $(1, 0)$ and $(\infty, 0)$, by the second fundamental theorem we get

$$\begin{aligned}T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &= \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &\le 3T(r,g)+S(r,f),\end{aligned}$$

which is (i).

Similarly we can prove (ii). This proves the lemma. □

Lemma 2. *Let f and g share $(0,1)$, $(1,\infty)$, (∞,∞) and $f\not\equiv g$. Then*

(i) $\overline{N}(r,0;f\,|\ge 2)+N(r,\infty;f\,|\ge 2)+N(r,1;f\,|\ge 2)=S(r,f)$,

(ii) $\overline{N}(r,0;g\,|\ge 2)+N(r,\infty;g\,|\ge 2)+N(r,1;g\,|\ge 2)=S(r,f)$.

Proof. We prove (i) because (ii) follows from (i) since f and g share $(0,1)$, $(1,\infty)$, (∞,∞).

First we show that $\overline{N}(r,0;f\,|\ge 2)=S(r,f)$. If $\overline{N}(r,0;f)=S(r,f)$ then there is nothing to prove. So we suppose that $\overline{N}(r,0;f)\ne S(r,f)$. Let

$$\phi=\frac{f'}{f-1}-\frac{g'}{g-1}.$$

If $\phi\equiv 0$, we get on integration $f-1=c(g-1)$, where c is a constant. Since $\overline{N}(r,0;f)\ne S(r,f)$, there exists $z_o\in\mathscr{C}$ such that $f(z_o)=g(z_o)=0$. So $c=1$ and hence $f\equiv g$, which is a contradiction. Therefore $\phi\not\equiv 0$.

Since f and g share $(0,1)$, a multiple zero of f is also a multiple zero of g and so it is a zero of ϕ. Therefore, by the first fundamental theorem, the Milloux theorem {p. 55 [1]} and Lemma 1 we get

$$\begin{aligned}\overline{N}(r,0;f\,|\ge 2) &\le N(r,0;\phi)\\ &\le N(r,\phi)+m(r,\phi)+O(1)\\ &= N(r,\phi)+S(r,f).\end{aligned}$$

Now the possible poles of ϕ occur only at the poles of f,g and the zeros of $f-1$, $g-1$. Since f,g share $(1,\infty)$ and (∞,∞), it follows that ϕ has no pole at all. So from above we get

$$\overline{N}(r,0;f\,|\ge 2)=S(r,f).$$

Secondly we show that $N(r,1;f\,|\ge 2)=S(r,f)$. If $N(r,1;f)=S(r,f)$, there is nothing to prove. So we suppose that $N(r,1;f)\ne S(r,f)$. Let

$$\psi=\frac{f'}{f}-\frac{g'}{g}.$$

If $\psi\equiv 0$ then $f\equiv cg$, where c is a constant. Since f,g share $(1,\infty)$ and $N(r,1;f)\ne S(r,f)$, it follows that $c=1$ and so $f\equiv g$. This is impossible and so $\psi\not\equiv 0$.

Since f and g share $(1, \infty)$, it follows that a zero of $f - 1$ with multiplicity m (≥ 2) is also a zero of $g - 1$ with multiplicity m (≥ 2) and so it is a zero of ψ with multiplicity $m - 1$. So by the first fundamental theorem, the Milloux theorem {p. 55 [1]} and Lemma 1 we get

$$\begin{aligned} N(r, 1; f \mid\geq 2) &\leq 2N(r, 0; \psi) \\ &\leq 2N(r, \psi) + 2m(r, \psi) + O(1) \\ &= 2N(r, \psi) + S(r, f). \end{aligned}$$

If f, g share $(b, 0)$, we denote by $\overline{N}_*(r, b; f, g)$ the reduced counting function of those b-points of f whose multiplicities are different from the multiplicities of the corresponding b-points of g.

Since f, g share $(0, 1)$ and (∞, ∞), it follows that poles of ψ occur only at those zeros of f whose multiplicities are different from the multiplicities of the corresponding zeros of g. Since ψ has only simple poles and f, g share $(0, 1)$, it follows from above that

$$\begin{aligned} N(r, 1; f \mid\geq 2) &\leq 2\overline{N}(r, \psi) + S(r, f) \\ &\leq 2\overline{N}_*(r, 0; f, g) + S(r, f) \\ &\leq 2\overline{N}(r, 0; f \mid\geq 2) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Let $F = f/(f - 1)$ and $G = g/(g - 1)$. Then F, G share $(0, 1)$, $(1, \infty)$ and (∞, ∞). So by above we get $N(r, 1; F \mid\geq 2) = S(r, F)$ and hence $N(r, \infty; f \mid\geq 2) = S(r, f)$. This proves the lemma. □

Lemma 3. *If α is a nonconstant entire function then*

$$T(r, \alpha^{(p)}) = S(r, e^{\alpha}),$$

where $\alpha^{(p)}$ is the p^{th} derivative of α.

Proof. Since by the Milloux theorem {p. 55 [1]} and by a result of Clunie {p. 54 [1]} we get

$$T(r, \alpha^{(p)}) \leq (p + 1)T(r, \alpha) + S(r, \alpha)$$

and

$$T(r, \alpha) = S(r, e^{\alpha}),$$

the lemma is proved. □

Lemma 4. *If f and g share $(0, 1)$, $(1, \infty)$, (∞, ∞) and $f \not\equiv g$ then*

$$\frac{f - 1}{g - 1} = e^{\alpha} \tag{1}$$

and

$$\frac{g}{f} = h, \tag{2}$$

where α *is an entire function and* h *is a meromorphic function with* $\overline{N}(r, 0; h) = S(r, f)$ *and* $\overline{N}(r, \infty; h) = S(r, f)$.

Proof. Since f and g share $(1, \infty)$, (∞, ∞), it follows that $(f-1)/(g-1)$ has no zero and pole. So there exists an entire function $\alpha = \alpha(z)$ such that

$$\frac{f-1}{g-1} = e^{\alpha}.$$

Now we put

$$h = \frac{g}{f}.$$

Then h is meromorphic and we show that $\overline{N}(r, 0; h) = S(r, f)$ and $\overline{N}(r, \infty; h) = S(r, f)$.

Since f and g share $(0, 1)$, (∞, ∞), it follows that h has a zero at z_o if z_o is a zero of f and g with multiplicities m and n respectively such that $m < n$; and h has a pole at z_o if $n < m$.

Since f and g share $(0, 1)$, it follows by Lemma 2 that

$$\overline{N}(r, 0; h) \leq \overline{N}(r, 0; g \mid\geq 2) = S(r, f)$$

and

$$\overline{N}(r, \infty; h) \leq \overline{N}(r, 0; f \mid\geq 2) = S(r, f).$$

This proves the lemma. □

Lemma 5. *If* f *and* g *share* $(0, 1)$, $(1, \infty)$, (∞, ∞) *and* $f \not\equiv g$ *then for any* a $(\neq 0, 1, \infty)$

$$\overline{N}(r, a; f \mid\geq 3) = S(r, f).$$

Proof. From (1) and (2) we see that

$$f = \frac{1 - e^{\alpha}}{1 - he^{\alpha}}$$

and so

$$f - a = \frac{(1-a) + e^{\alpha}(ah - 1)}{1 - he^{\alpha}}.$$

First we suppose that α is nonconstant. If z_o is a zero of $f - a$ with multiplicity ≥ 3 then z_o is a zero of

$$\frac{d}{dz}[(1-a) + e^{\alpha}(ah - 1)] = \alpha' e^{\alpha}\left[ah - 1 + a\frac{h'}{\alpha'}\right]$$

with multiplicity ≥ 2. So z_o is a zero of α' or z_o is a zero of

$$\frac{d}{dz}\left[ah - 1 + a\frac{h'}{\alpha'}\right] = ah\left[\frac{h'}{h} - \frac{\alpha''}{(\alpha')^2}\cdot\frac{h'}{h} + \frac{1}{\alpha'}\cdot\frac{h''}{h}\right].$$

Therefore

$$\overline{N}(r, a; f \mid\geq 3) \leq N(r, 0; \alpha') + \overline{N}(r, 0; h) + T\left(r, \frac{h'}{h} - \frac{h'}{h}\cdot\frac{\alpha''}{(\alpha')^2} + \frac{1}{\alpha'}\cdot\frac{h''}{h}\right)$$

$$\leq \overline{N}(r, 0; h) + 2T\left(r, \frac{h'}{h}\right) + T(r, \alpha'') + 4T(r, \alpha') + T\left(r, \frac{h''}{h}\right) + O(1).$$

Since by (1), (2), Lemma 1 and Lemma 3 $T(r, \alpha') = S(r, f)$, $T(r, \alpha'') = S(r, f)$, $S(r, h) = S(r, f)$ and by Lemma 4 $\overline{N}(r, 0; h) = S(r, f)$, $\overline{N}(r, \infty; h) = S(r, f)$, it follows by the Milloux theorem {p. 55 [1]} that

$$\begin{aligned}\overline{N}(r, a; f \mid\geq 3) &\leq N\left(r, \frac{h'}{h}\right) + N\left(r, \frac{h''}{h}\right) + S(r, f)\\ &\leq 4\overline{N}(r, 0; h) + 4\overline{N}(r, \infty; h) + S(r, f)\\ &= S(r, f).\end{aligned}$$

Next we suppose that α is a constant. Let $e^{\alpha} = c$, a constant. Since f is nonconstant, it follows that h is nonconstant and we get

$$f - a = \frac{(1 - a) + c(ah - 1)}{1 - ch}.$$

If z_o is a zero of $f - a$ with multiplicity ≥ 3 then z_o is a zero of

$$\frac{d}{dz}[(1 - a) + c(ah - 1)] = ach' = ach\left(\frac{h'}{h}\right)$$

with multiplicity ≥ 2. Therefore by Lemma 4 we get

$$\begin{aligned}\overline{N}(r, a; f \mid\geq 3) &\leq \overline{N}(r, 0; h) + T\left(r, \frac{h'}{h}\right)\\ &= \overline{N}(r, 0; h) + N\left(r, \frac{h'}{h}\right) + S(r, f)\\ &= 2\overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r, f)\\ &= S(r, f).\end{aligned}$$

This proves the lemma. □

LEMMA 6 [3]. *Let f_1, f_2, f_3 be meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then*

$$T(r, f_1) \leq \sum_{i=1}^{3} N_2(r, 0; f_i) + \max_{1\leq i,j\ (i\neq j)\leq 3}\{N_2(r, \infty; f_i) + \overline{N}(r, \infty; f_j)\} + S(r),$$

where $N_2(r,b;f_i) = \overline{N}(r,b;f_i) + \overline{N}(r,b;f_i \mid \geq 2)$ *for some* $b \in \mathscr{C} \cup \{\infty\}$ *and* $S(r) = \sum_{i=1}^{3} S(r,f_i)$.

Lemma 7 [9]. *Let* f_1, f_2, f_3 *be three nonconstant meromrophic functions such that* $f_1 + f_2 + f_3 \equiv 1$ *and let* $g_1 = -f_1/f_3$, $g_2 = 1/f_3$ *and* $g_3 = -f_2/f_3$. *If* f_1, f_2, f_3 *are linearly independent then* g_1, g_2, g_3 *are also linearly independent.*

Lemma 8. *Let* f *and* g *be distinct and share* $(0,1)$, $(1,\infty)$ *and* (∞,∞). *Let*

$$f_1 = \frac{(f-a)(1-he^{\alpha})}{1-a}, \quad f_2 = \frac{-ahe^{\alpha}}{1-a} \quad \text{and} \quad f_3 = \frac{e^{\alpha}}{1-a},$$

where a $(\neq 0,1,\infty)$ *be a complex number and* h *and* α *be defined as in Lemma* 4. *If* f_1, f_2, f_3 *are linearly independent then*

$$\text{(i)} \quad N(r,0;f \mid \leq 1) \leq N(r,a;f \mid \leq 2) + S(r,f)$$

and

$$\text{(ii)} \quad N(r,1;f \mid \leq 1) \leq N(r,a;f \mid \leq 2) + S(r,f).$$

Proof. Since $(1-a)f_1 \equiv 1 - e^{\alpha} - a(1-he^{\alpha})$, it follows by Lemma 4 that $\overline{N}(r,\infty;f_1) = S(r,f)$. Also $\overline{N}(r,\infty;f_2) = S(r,f)$ and $\overline{N}(r,\infty;f_3) \equiv 0$. First we suppose that e^{α} is nonconstant.

Now by Lemma 4 and Lemma 6 we get

$$\begin{aligned} T(r,e^{\alpha}) &\leq N_2(r,0;f_1) + 2\overline{N}(r,0;f_2) + N_2(r,0;f_3) + S(r,f) \\ &= N_2(r,0;f_1) + 2\overline{N}(r,0;h) + S(r,f) \\ &= N_2(r,0;f_1) + S(r,f). \end{aligned} \tag{3}$$

We see that $(1-a)f_1 \equiv (f-a)(1-he^{\alpha}) \equiv 1 - e^{\alpha} - a(1-he^{\alpha})$ and $f = (1-e^{\alpha})/(1-he^{\alpha})$. So z_o will be a possible zero of f_1 if either z_o is a zero of $f-a$ or z_o is a common zero of $1-e^{\alpha}$ and $1-he^{\alpha}$. Therefore

$$N_2(r,0;f_1) \leq N_2(r,a;f) + N(r,0;1-he^{\alpha}) - N(r,\infty;f).$$

So from (3) we get

$$T(r,e^{\alpha}) \leq N_2(r,a;f) + N(r,0;1-he^{\alpha}) - N(r,\infty;f) + S(r,f). \tag{4}$$

Since $f = (1-e^{\alpha})/(1-he^{\alpha})$, it follows from Lemma 4, the first fundamental theorem and (4) that

$$\begin{aligned} \overline{N}(r,0;f) &\leq N(r,0;1-e^{\alpha}) - N(r,0;1-he^{\alpha}) + N(r,\infty;f) + \overline{N}(r,\infty;h) \\ &= N(r,1;e^{\alpha}) - N(r,0;1-he^{\alpha}) + N(r,\infty;f) + S(r,f) \\ &\leq T(r,e^{\alpha}) - N(r,0;1-he^{\alpha}) + N(r,\infty;f) + S(r,f) \\ &\leq N_2(r,a;f) + S(r,f). \end{aligned} \tag{5}$$

Since by Lemma 2 $\overline{N}(r,0;f \mid \geq 2) = S(r,f)$, by Lemma 5 $\overline{N}(r,a;f \mid \geq 3) = S(r,f)$ and $N_2(r,a;f) = N(r,a;f \mid \leq 2) + 2\overline{N}(r,a;f \mid \geq 3)$, it follows from (5) that

$$N(r,0;f\,|\leq 1) \leq N(r,a;f\,|\leq 2) + S(r,f).$$

If e^{α} is a constant, it follows that $\overline{N}(r,0;f) = S(r,f)$ because $f-1 \equiv e^{\alpha}(g-1)$, $f \not\equiv g$ and f,g share $(0,1)$. So (i) is trivially true.

If h is constant then $h \neq 1$ because $f \not\equiv g$. So from

$$f-1 = \frac{(1-h)e^{\alpha}}{1-he^{\alpha}},$$

it follows that $\overline{N}(r,1;f) = S(r,f)$. Hence (ii) is obvious. Therefore we suppose that h is nonconstant.

Let $g_1 = -f_1/f_3 = -e^{-\alpha}(f-a)(1-he^{\alpha})$, $g_2 = 1/f_3 = (1-a)e^{-\alpha}$ and $g_3 = -f_2/f_3 = ah$. Then $g_1+g_2+g_3 \equiv 1$ and by Lemma 7 g_1, g_2, g_3 are linearly independent. Applying Lemma 6 to g_1, g_2, g_3 we get

$$T(r,h) \leq N_2(r,a;f) + N(r,0;1-he^{\alpha}) - N(r,\infty;f) + S(r,f). \tag{6}$$

Since

$$f-1 \equiv \frac{(1-h)e^{\alpha}}{1-he^{\alpha}},$$

it follows from Lemma 4, the first fundamental theorem, Lemma 5, Lemma 2 and (6) that

$$\begin{aligned} N(r,1;f\,|\leq 1) &= \overline{N}(r,1;f) + S(r,f) \\ &\leq \overline{N}(r,1;h) - N(r,0;1-he^{\alpha}) + N(r,\infty;f) + S(r,f) \\ &\leq N_2(r,a;f) + S(r,f) \\ &= N(r,a;f\,|\leq 2) + S(r,f). \end{aligned}$$

This proves the lemma. □

Lemma 9. *Let f and g be nonconstant meromorphic functions such that $af+bg \equiv c$, where a, b, c are nonzero constants. Then*

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f).$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,c/a;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

This proves the lemma. □

3. Proof of Theorem 2 and Theorem 3

Proof of Theorem 2. Let f_1, f_2, f_3 be defined as in Lemma 8. Suppose, if possible, f_1, f_2, f_3 are linearly independent. Then by the second fundamental theorem, Lemma 2, Lemma 5 and Lemma 8 we get

$$\begin{aligned}2T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &= N(r,0;f\,|\le 1)+N(r,1;f\,|\le 1)+N(r,a;f\,|\le 2)\\ &\quad + N(r,\infty;f\,|\le 1)+S(r,f)\\ &\le 3N(r,a;f\,|\le 2)+N(r,\infty;f\,|\le 1)+S(r,f),\end{aligned}$$

which implies

$$3\delta_{2)}(a;f)+\delta_{1)}(\infty;f)\le 2.$$

This contradicts the given condition. So there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1f_1+c_2f_2+c_3f_3\equiv 0. \tag{7}$$

If possible, let $c_1=0$. Then from (7) and the definitions of f_2, f_3 it follows that h is a constant. Since $f\not\equiv g$, we see that $h\ne 1$ and so 1 becomes a Picard's exceptional value of f because f,g share $(1,\infty)$ and $g\equiv hf$.

Again since

$$f\equiv\frac{1}{h}+\frac{h-1}{h(1-he^{\alpha})},$$

it follows that $1/h$ is also a Picard's exceptional value of f. So by the second fundamental theorem and Lemma 2 we get

$$T(r,f)\le N(r,\infty;f\,|\le 1)+S(r,f),$$

which implies $\delta_{1)}(\infty;f)=0$. This contradicts the given condition. So $c_1\ne 0$.

Also we see that

$$f_1+f_2+f_3\equiv 1. \tag{8}$$

Eliminating f_1 from (7) and (8) we get

$$cf_2+df_3\equiv 1, \tag{9}$$

where c, d are constants and $|c|+|d|\ne 0$.

Now we consider the following cases.

Case I. Let $c\ne 0$ and $d\ne 0$. Then from (9) we get

$$\frac{-ache^{\alpha}}{1-a}+\frac{de^{\alpha}}{1-a}\equiv 1. \tag{10}$$

If one of he^{α} and e^{α} is constant then from (10) it follows that the other is also constant and from (1) and (2) we see that f becomes a constant, which is impossible. So he^{α} and e^{α} are nonconstant.

From (10) we get by Lemma 9 and Lemma 4 that

$$T(r,e^{\alpha}) \le \overline{N}(r,0;e^{\alpha}) + \overline{N}(r,0;h) + \overline{N}(r,\infty;e^{\alpha}) + S(r,e^{\alpha}) = S(r,f) + S(r,e^{\alpha}). \tag{11}$$

Again from (10) we get

$$d - ach \equiv \frac{1-a}{e^{\alpha}}.$$

This implies that $\overline{N}(r,d/ac;h) \equiv 0$ and $\overline{N}(r,\infty;h) \equiv 0$. So by the second fundamental theorem we get in view of Lemma 4

$$T(r,h) \le \overline{N}(r,0;h) + \overline{N}(r,d/ac;h) + \overline{N}(r,\infty;h) + S(r,h) = S(r,f) + S(r,h). \tag{12}$$

Since

$$f \equiv \frac{1-e^{\alpha}}{1-he^{\alpha}},$$

it follows that

$$T(r,f) = O(T(r,e^{\alpha})) + O(T(r,h)). \tag{13}$$

From (11), (12) and (13) we see that there exists a sequence of values of r tending to infinity for which $T(r,f) = o\{T(r,f)\}$. This is a contradiction.

Case II. Let $c = 0$ but $d \ne 0$. From (9) we see that e^{α} is a constant. Since $f \not\equiv g$, it follows from (1) that $e^{\alpha} \ne 1$. So it again follows from (1) that $\overline{N}(r,0;f) \equiv 0$ because f,g share $(0,1)$. Also from (1) and (2) we get

$$f \equiv \frac{1-e^{\alpha}}{1-he^{\alpha}}.$$

By the second fundamental theorem, Lemma 2 and Lemma 4 we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,1-e^{\alpha};f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le \overline{N}(r,0;h) + N(r,\infty;f \mid \le 1) + S(r,f) \\ &= N(r,\infty;f \mid \le 1) + S(r,f), \end{aligned}$$

which implies that $\delta_{1)}(\infty;f) = 0$. This contradicts the given condition.

Case III. Let $c \ne 0$ and $d = 0$. Then from (9) we see that $he^{\alpha} = p$, a constant, say. Then $p \ne 1$ because $f \not\equiv g$. So we get

$$f - a \equiv \frac{(1-a+ap) - e^{\alpha}}{1-p}. \tag{14}$$

From (14) we see that $T(r,f) = T(r,e^{\alpha}) + O(1)$. If $1 - a + ap \neq 0$, it follows from (14) and Lemma 3 that

$$N(r,a;f\,|\geq 2) \leq 2N(r,0;\alpha') \leq 2T(r,\alpha') = S(r,e^{\alpha}).$$

Hence

$$\begin{aligned} N(r,a;f\,|\leq 2) &= N(r,a;f) + S(r,f) \\ &= N(r,1-a+ap;e^{\alpha}) + S(r,f) \\ &= T(r,e^{\alpha}) + S(r,f) \\ &= T(r,f) + S(r,f). \end{aligned}$$

This implies that $\delta_{2)}(a;f) = 0$, which contradicts the given condition.

Therefore $1 - a + ap = 0$ i.e. $p = (a-1)/a$. Hence from (14) we get

$$f - a \equiv -ae^{\alpha}. \tag{15}$$

Also from (2) and (15) we get

$$g + a - 1 \equiv \frac{a-1}{e^{\alpha}}. \tag{16}$$

From (15) and (16) we obtain

$$(f-a)(g+a-1) \equiv a(1-a).$$

This proves the theorem. □

Proof of Theorem 3. Let $F = 1 - f$ and $G = 1 - g$. Then F, G are distinct and share $(0,1)$, $(1,\infty)$, (∞,∞). Also $\delta_{2)}(1-a;F) = \delta_{2)}(a;f)$ and $\delta_{1)}(\infty;F) = \delta_{1)}(\infty;f)$. So by Theorem 2 we get

$$(F-1+a)(G-a) \equiv a(1-a)$$

i.e.

$$(f-a)(g+a-1) \equiv a(1-a).$$

This proves the theorem. □

4. Application

As an application of Theorem 2 and Theorem 3 we prove the following result.

THEOREM 4. *Let a and b $(\neq 0,1)$ be two finite complex numbers and $S_1 = \{a+\alpha : \alpha^n + b = 0\}$, $S_2 = \{a+\beta : \beta^n + b = 1\}$, $S_3 = \{\infty\}$ where n (≥ 3) be a positive integer. If either*

$$E_f(S_1,1) = E_g(S_1,1), \quad E_f(S_2,\infty) = E_g(S_2,\infty), \quad E_f(S_3,\infty) = E_g(S_3,\infty)$$

or

$$E_f(S_1,\infty)=E_g(S_1,\infty),\quad E_f(S_2,1)=E_g(S_2,1),\quad E_f(S_3,\infty)=E_g(S_3,\infty)$$

then one of the following holds:

$$\text{(i)}\quad f-a\equiv t(g-a)\quad \textit{where}\ t^n=1$$

and

$$\text{(ii)}\quad (f-a)(g-a)\equiv s\quad \textit{where}\ 4s^n=1.$$

Proof. We suppose that $E_f(S_1,1)=E_g(S_1,1)$, $E_f(S_2,\infty)=E_g(S_2,\infty)$, $E_f(S_3,\infty)=E_g(S_3,\infty)$ because for the other case the theorem can be proved similarly using Theorem 3.

Let $F=(f-a)^n+b$ and $G=(g-a)^n+b$. If $F\equiv G$ then case (i) holds. Let $F\not\equiv G$. Clearly $\delta_{2)}(b;F)=1$ and $\delta_{1)}(\infty;F)=1$. Since F,G share $(0,1)$, $(1,\infty)$, (∞,∞), it follows from Theorem 2 that

$$(F-b)(G+b-1)\equiv b(1-b)$$

i.e.

$$(f-a)^n\{(g-a)^n+2b-1\}\equiv b(1-b). \tag{17}$$

From (17) we see that ∞ and $a+\sqrt[n]{(1-2b)}$ are Picard's exceptional values of g where $n\geq 3$, but this is impossible unless $1-2b=0$. So from (17) we get $(f-a)(g-a)\equiv s$. This proves the theorem. □

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