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NASH INEQUALITIES FOR COMPACT MANIFOLDS WITH BOUNDARY

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Abstract

In this paper, we shall prove the Nash inequality for a compact manifold with boundary with respect to a weighted measure, using covering arguments of Jerison [10] and Oden-Sung-Wang [13]. We shall also state some results which are easily obtained from that inequality.

1. Introduction

The Nash inequality is equivalent not only to the Sobolev inequality but also to the diagonal upper bound of the heat kernel. Therefore, it is an important factor for the study of solutions of parabolic equations ([7]). In this paper, we shall first prove the Nash inequality for metric balls under the assumptions of the volume doubling property and local Poincaré inequality, by using covering arguments of Jerison [10]. We shall also point out that the Nash inequality, conversely, implies a lower bound of the volume of balls, and hence, a family of Nash inequalities is equivalent to the parabolic Harnack inequality. Using covering arguments of K. Oden, C. J. Sung and J. Wang [13], we also derive the Nash inequality for a compact manifold with boundary with respect to a weighted measure wv_M, where w is a positive function on $\mathrm{Int}\, M := M - \partial M$.

We shall make the above statements mathematically precise. Let M be a compact, connected Riemannian manifold with boundary ∂M. We denote by v_M the Riemannian measure of M. In order to emphasize that M contains the boundary ∂M, we shall often write $\overline{M}$ in place of M. K. Oden, C. J. Sung and J. Wang proved the Poincaré inequality of M with respect to a weighted measure;

Theorem 1.1 (K. Oden, C. J. Sung and J. Wang [13]). *Let w be a given function on M with $w > 0$ on* $\mathrm{Int}\, M := M - \partial M$. *We assume*

1. *M satisfies the volume doubling property, i.e., $\exists c_1 > 0$ s.t. for any ball $B(x, 2r)$ with $x \in \overline{M}$ we have $|B(x, 2r)|/|B(x, r)| \le c_1$, where c_1 is a constant in-*

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dependent of x *and* r, *and for* $A \subset M$, $|A|$ *stands for the volume of* A, *i.e.*, $|A| = \int_A dv_M$.

2. *M satisfies the interior rolling R-ball condition, i.e.*, $\exists R > 0$ *s.t. for all* $x \in \partial M$, $\exists B(p,R) \subset M$ *s.t.* $B(p,R) \cap \partial M = \{x\}$.

3. *M satisfies the weak Poincaré inequality on balls, i.e., for any ball* $B(x,2r) \subset M$, $B(x,2r) \cap \partial M = \emptyset$, *we have*

$$\inf_{k \in \boldsymbol{R}} \int_{B(x,r)} |f-k|^2 \, dv_M \le c_2 r^2 \int_{B(x,2r)} |\nabla f|^2 \, dv_M$$

for all $f \in C^1(\overline{M})$, *where* c_2 *is a constant independent of* x, r *and* f.

4. $\exists c_3 > 1$ *s.t.* $w(x) \le c_3 w(y)$ *for all* $x, y \in M$ *with* $0 < d(x,\partial M) \le 2d(y,\partial M)$.

Then we have the following Poincaré inequality on M:

$$\inf_{f \in C^1(\overline{M})} \frac{\int_M |\nabla f|^2 w \, dv_M}{\inf_{k \in \boldsymbol{R}} \int_M |f-k|^2 w \, dv_M} \ge c(c_1, c_2, c_3, R, \eta, D),$$

where η *is the first nonzero Neumann eigenvalue of* $M_{R/2} = \{x \in M \mid d(x,\partial M) \ge R/2\}$ *and* $D = \operatorname{diam}(M)$, *the diameter of* M.

We shall consider

5. $\exists c_6 > 0$ and $\exists \varphi > 0$ s.t. $c_6 d(x,\partial M)^{2\varphi} \le w(x)$ for all $x \in M - M_R$, where $M_R = \{x \in M \mid d(x,\partial M) \ge R\}$.

In this paper, we shall add this assumption 5 to the ones of Theorem 1.1, and prove the Nash inequality,

$$\left(\int_M |f - f'_M|^2 w \, dv_M\right)^{1+(2/\nu)} \le c \cdot \left(\int_M |\nabla f|^2 w \, dv_M\right)\left(\int_M |f - f'_M| w \, dv_M\right)^{4/\nu},$$

$$\forall \nu \text{ satisfying } \nu > 4\varphi \text{ and } \nu \ge 2\nu_0 := 2(\log c_1)/\log 2, \ \forall f \in C^\infty(M),$$

where

$$f'_M = \int_M f w \, dv_M \Big/ \int_M w \, dv_M,$$

$$c = c(c_1, c_2, c_3, c_6, \nu, \varphi, R, D, |M|) + c(c_3, c_6, \nu, \varphi, R, D) \cdot N,$$

$|M|$ stands for the volume of M, i.e., $|M| = \int_M dv_M$ and N is the "Nash constant" of $M_{R/2}$ (see equation (7) in Theorem 4.1).

We should note that, actually, the assumption 4 implies the assumption 5 with $c_6 = R^{-2\varphi} \inf_{d(x,\partial M)=R/2} w(x)$ and $2\varphi = (\log c_3)/\log 2$, and hence the assumption 5 is not required for a Nash inequality to hold (cf. Remark 4.1). Nevertheless we assume the property 5 because it is important to choose ν as small as possible.

Let u_1 be the normalized first Dirichlet eigenfunction for the Laplacian on M. K. Oden, C. J. Chen and J. Wang set $w = (u_1)^2$ and estimated the constant $c(c_1, c_2, c_3, R, \eta, D)$ in Theorem 1.1 from below by the geometric constants of M.

Similarly, the above constant c can be estimated from above by the geometric constants of M (see Proposition 5.1).

In section 3, we shall also point out that the Nash inequality conversely means the lower bound of the measure in the following way: Let M be the closure of a relatively compact domain in a complete Riemannian manifold N, w be a function such that $w \in C^\infty(M)$ and $0 < w$ on $\mathrm{Int}\, M = M - \partial M$. ($w$ may possibly takes zero-value at a point of the boundary ∂M). We suppose that the following Nash inequality holds:

$$\left(\int_M |f - f'_M|^2 \, d\mu^w\right)^{1+(2/\nu)} \leq \hat{c} \cdot (\mathrm{diam}\, M)^2 \cdot \mu^w(M)^{-2/\nu} \left(\int_M |\nabla f|^2 \, d\mu^w\right) \left(\int_M |f - f'_M| \, d\mu^w\right)^{4/\nu},$$

for all $f \in C^\infty(M)$, where $f'_M = \int_M f \, d\mu^w / \mu^w(M)$. Then, for each $0 < r < \mathrm{diam}\, M$ and $x_0 \in M$, we have the lower bound of the measure

$$\frac{\mu^w(B(x_0, r) \cap M)}{\mu^w(M)} \geq \frac{c(\nu)}{1 + \hat{c}^{\nu/2}} \left(\frac{r}{\mathrm{diam}\, M}\right)^\nu.$$

From this lower bound estimate of the measure, we can easily see that the parabolic Harnack inequality (i.e., Theorem 3.3, 2) is equivalent to a family of Nash inequalities, i.e., there exist constants $\hat{c} > 0$, $r_0 > 0$ and $\nu > 0$ such that

$$\left(\int_{B(x,r)} |f - f'_{B(x,r)}|^2 \, d\mu^w\right)^{1+(2/\nu)} \leq \hat{c} \cdot r^2 \cdot \mu^w(B(x,r))^{-2/\nu} \left(\int_{B(x,r)} |\nabla f|^2 \, d\mu^w\right) \left(\int_{B(x,r)} |f - f'_{B(x,r)}| \, d\mu^w\right)^{4/\nu},$$

$$\forall f \in C^\infty(\overline{B(x,r)}), \quad \forall x \in M, \quad 0 < \forall r < r_0.$$

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2. Abstract results

In this section, we shall state the equivalence between the diagonal heat kernel upper bound, Nash inequality, and Sobolev inequality. We shall also point out that the diagonal heat kernel upper bound implies a lower bound of the eigenvalue of the associated operator.

Let M be a compact, connected Riemannian manifold with boundary ∂M and $w \in C^0(\overline{M})$ be a function which is positive on $\mathrm{Int}\, M := M - \partial M$. We define a weighted measure μ^w, by $\mu^w(E) = \int_E w \, dv_M$ for $E \subset M$. We denote by $H^1(M, \mu^w)$

a Hilbert space obtained by the completion of the space $C^0(M) \cap C^\infty(\mathrm{Int}\, M)$ with respect to the norm

$$\|f\|^2_{H^1(M,\mu^w)} = \int_M (|\nabla f|^2 + f^2)\, d\mu^w.$$

On the Hilbert space $L^2(M,\mu^w)$, we shall consider a closed form

$$\mathscr{E}(f,h) = \int_M \langle \nabla f, \nabla h\rangle\, d\mu^w; \quad f,h \in \mathrm{Dom}(\mathscr{E}) = H^1(M,\mu^w)$$

and denote by A the nonpositive self-adjoint operator associated with this closed form $\mathscr{E}$. Let $p_t(x,y)$ be the heat kernel of the operator A. Slight modifications of the arguments in [2] and [16] enable us to show the following

Theorem 2.1. *Let M, w, A, and p_t be as above. We assume $\int_M w\, dv_M = 1$ for simplicity. Let $v > 0$ be given. Then the following inequalities* (1) *and* (3) *are equivalent, and when $v > 2$, they are also equivalent with* (2) *and* (4):

(1) $p_t(x,x) \le 1 + at^{-v/2} \quad (\forall t > 0, \forall x \in M)$.

(2) $\|f\|^2_{2v/(v-2)} \le b \cdot \mathscr{E}(f,f) \quad \forall f \in C^0(M) \cap C^\infty(\mathrm{Int}(M))$ *with* $\int_M fw\, dv_M = 0$.

(3) $\|f\|_2^{2+(4/v)} \le c \cdot \mathscr{E}(f,f)\|f\|_1^{4/v} \quad \forall f \in C^0(M) \cap C^\infty(\mathrm{Int}(M))$ *with* $\int_M fw\, dv_M = 0$.

(4) $\inf_{\alpha \in \boldsymbol{R}} \|f - \alpha\|^2_{2v/(v-2)} \le d \cdot \mathscr{E}(f,f) \quad \forall f \in C^0(M) \cap C^\infty(\mathrm{Int}(M))$.

Here, we write $\|f\|_p = (\int_M |f|^p w\, dv_M)^{1/p}$. (3) *implies* (1) *with $a = c_1 v^{v/2} c^{v/2}$,* (1) *implies* (3) *with $c = c_2 a^{2/v}$,* (1) *implies* (2) *with $b = c_3 v^2 (v-2)^{-2} a^{2/v}$,* (2) *implies* (3) *with $c = b$,* (2) *implies* (4) *with $d = b$, and* (4) *implies* (2) *with $b = 4d$, where c_1, c_2, c_3 are some numerical constants.*

Corollary 2.1. *The Nash inequality* (3) *in Theorem* 2.1 *implies*

$$\alpha^{-2/v} v^{-1} c^{-1} i^{2/v} \le l_i, \quad |\varphi_i|^2_\infty \le e\alpha(vcl_i)^{v/2} \quad (i = 1,2,\ldots),$$

where $0 = l_0 < l_1 \le l_2 \le \cdots$ are the eigenvalues of $-A$ and $\{\varphi_i\}$ is a complete orthonormal system of $L^2(M,\mu^w)$ consisting of eigenfunctions with φ_i having eigenvalue l_i.

Proof. Theorem 2.1 implies $p_t(x,x) \le 1 + at^{-v/2}$ for all $t > 0$ and $x \in M$, where $a = \alpha(vc)^{v/2}$ and α is a numerical constant. Therefore for each $\lambda > 0$ and for all $x \in M$,

$$\begin{aligned}\sum_{0<l_i\le\lambda} (\varphi_i)^2(x) &\le e \sum_{0<l_i\le\lambda} e^{-l_i/\lambda}(\varphi_i)^2(x) \\ &\le e\{p_{1/\lambda}(x,x) - 1\} \le ea\lambda^{v/2}.\end{aligned}$$

Integrating the both sides of this inequality, we get

$$\sharp\{l_i \mid 0 < l_i \le \lambda\} \le ea\lambda^{v/2} \quad (\lambda > 0).$$

Thus, Corollary 2.1 follows. $\square$

3. Nash inequality and volume comparison

In this section, we shall prove the following three assertions:

1. The Nash inequality holds under the assumptions of the volume doubling property and weak Poincaré inequality.
2. The Nash inequality implies a volume comparison.
3. The local parabolic Harnack inequality is equivalent to the family of Nash inequalities.

Let M be a complete Riemannian manifold and μ be a positive Borel measure on M. In this section, we shall write $V(x,r) = \mu(B(x,r))$, $|A| = \mu(A)$ and $f_A = \mu(A)^{-1}\int_A f\, d\mu$, for a bounded measurable subset $A \subseteq M$ and a function $f \in C^0(\overline{A})$.

Lemma 3.1. *Let M and μ be as above. We assume the following condition (D) and (P):*

(D) *$V(x,2r) \le c_1 V(x,r)$ for all $x \in M$ and $0 < r \le r_0$, where c_1 is a constant independent of x and r;*

(P) *$\int_{B(x,r)} |f - f_{B(x,r)}|^2\, d\mu \le c_2 r^2 \int_{B(x,2r)} |\nabla f|^2\, d\mu$ for all $x \in M$, $0 < r \le r_0$ and $f \in C^\infty(M)$, where c_2 is a constant independent of x, r and f.*

Then, for each $\nu \ge \nu_0 := (\log c_1)/\log 2$, we have

$$\|f\|_{L^2(B(x,r),\mu)} \le c_4 s \|\nabla f\|_{L^2(B(x,8r),\mu)} + c_5 V(x,r)^{-1/2}(r/s)^{\nu/2}\|f\|_{L^1(B(x,r),\mu)},$$

for all $r \in (0,r_0)$, $s > 0$, and $f \in C^\infty(\overline{B(x,8r)})$ with $\int_{B(x,r)} f\, d\mu = 0$, where c_4 and c_5 are constants depending only on c_1 and c_2.

Proof. From the proof of Theorem 2.1 in Saloff-Coste [14], we see that there exist constants c_6 and c_7 depending only on c_1 and c_2 such that

$$\int_{B(x,r)} f^2\, d\mu \le c_6 s^2 \int_{B(x,8r)} |\nabla f|^2\, d\mu + c_7 V(x,r)^{-1}\left(\frac{r}{s}\right)^{\nu}\left(\int_{B(x,r)} |f|\, d\mu\right)^2 \tag{1}$$

for all $f \in C^\infty(\overline{B(x,8r)})$, $0 < r < r_0$, $0 < s \le r/4$, and $\nu \ge \nu_0$. Since $\int_{B(x,r)} f\, d\mu = 0$, the assumption (P) implies

$$\int_{B(x,r)} f^2\, d\mu \le c_2 r^2 \int_{B(x,2r)} |\nabla f|^2\, d\mu \le c_6 s^2 \int_{B(x,8r)} |\nabla f|^2\, d\mu$$

for $s \ge \sqrt{c_2/c_6}\,r$. Hence, when we replace c_6 with $\max\{c_6, 16c_2\}$ if necessary, the inequality (1) holds for all $s > 0$. This proves Lemma 3.1. $\square$

We shall set $E := B(\xi_1, r_1) \subset M$ and consider the following conditions $(VD)_1$ and $(VD)_2$:

$(VD)_1$ M satisfies the volume doubling condition, i.e.,
$V(x,2r) \le c_1 V(x,r)$ for $x \in M, 0 < r \le r_0$;

$(VD)_2$ $\mu(B(x,2r) \cap E) \le c_1 \mu(B(x,r) \cap E)$ for $x \in \overline{E}, 0 < r \le r_0$.

In order to show the first assertion 1, we shall use the following covering lemma due to Jerison [10]:

LEMMA 3.2 (Whitney decomposition [10]). *We set* $E := B(\xi_1, r_1) \subset M$ *and assume* $(VD)_1$ *or* $(VD)_2$. *Then there exist a pairwise disjoint family* $\mathscr{F} = \{B_i = B(x_i, r_i) \mid i \in I\}$ *of metric balls of* E *and a constant* $c(c_1, r_1/r_0)$ *such that the following assertions hold*:

(1) $E = \bigcup_{i \in I} 2B_i$;

(2) $10^2 \rho(B_i) \leq \mathrm{dist}(B_i, \partial E) \leq 10^3 \rho(B_i) \quad (\forall i \in I)$;

(3) $\sharp\{i \in I \mid \eta \in 32B_i\} \leq c(c_1, r_1/r_0) \quad (\forall \eta \in E)$,

where $c(c_1, r_1/r_0)$ *is a constant depending only on the constants* c_1 *and* $\max\{1, r_1/r_0\}$, *for* $B = B(x, r)$ *and* $a > 0$, *we write* $aB := B(x, ar)$, *and* $\rho(B)$ *stands for the radius of* B.

For $B \in \mathscr{F}$, *let* γ_B *be a geodesic segment from the center of* B *to the center* ξ_1 *of* E. (*This path may not be unique, but will be fixed throughout our arguments*). *Denote* $\mathscr{F}(B) = \{A \in \mathscr{F} \mid 2A \cap \gamma_B \neq \emptyset\}$ *and* $A(\mathscr{F}) = \{B \in \mathscr{F} \mid A \in \mathscr{F}(B)\}$ *for* $A \in \mathscr{F}$. *Then the following holds*:

(4) $\sum_{B \in A(\mathscr{F})} \sharp\mathscr{F}(B) \dfrac{|B|}{|A|} \leq c(c_1) \cdot \log\left(\dfrac{r_1}{\rho(A)}\right)$.

LEMMA 3.3. *Let* $E = B(\xi_1, r_1)$ *and* M *be as in Lemma* 3.2. *We assume one of the two conditions* $(VD)_1$ *and* $(VD)_2$. *Moreover, we suppose that*

$$(N) \quad \int_{B(x,r)} |f - f_{B(x,r)}|^2 \, d\mu$$
$$\leq c_4 s^2 \int_{B(x,8r)} |\nabla f|^2 \, d\mu + c_5 V(x,r)^{-1} \left(\frac{r}{s}\right)^{\nu} \left(\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu\right)^2,$$

for all $B(x, 8r) \subset E$, $s > 0$, $f \in C^\infty(\overline{B(x, 8r)})$ *and* $\nu \geq \nu_0 := (\log c_1)/\log 2$. *Then, if* $2r_1 \leq r_0$, *or if*

$$(2) \qquad |E| = V(\xi_1, r_1) \leq c_3 |A| \left(\frac{2r_1}{\rho(A)}\right)^{\nu_0} \quad \text{for all ball } A \subset E,$$

then we have the Nash inequality:

$\|f - f_E\|_{L^2(E)}^{2+(4/\nu')} \leq c(c_1, c_4, c_5, r_1/r_0, \nu') \cdot (r_1)^2 (c_3)^{2/\nu'} |E|^{-2/\nu'} \|\nabla f\|_{L^2(E)}^2 \|f - f_E\|_{L^1(E)}^{4/\nu'}$,

for all $f \in C^\infty(\bar{E})$, *where* $\nu' = 2\nu$.

Remark 3.1. We note that the inequality (2) follows from the fact that $2r_1 \leq r_0$. Indeed, we first observe that the property $(VD)_1$ or $(VD)_2$ implies that $V(x, r) \leq c_1 V(x, s)(r/s)^{\nu_0}$ for $0 < s < r \leq r_0$. We denote by o_A the center of a ball A. Since $A \subset E$ implies $E = B(\xi_1, r_1) \subset B(o_A, 2r_1)$, we have

$$V(\xi_1, r_1) \leq V(o_A, 2r_1) \leq c_1 V(o_A, \rho(A))(2r_1/\rho(A))^{\nu_0},$$

and hence, the inequality (2) with $c_3 = c_1$ holds.

Proof. Let $\mathscr{F}$ be a Whitney decomposition as in Lemma 3.2 and $f \in C^\infty(\bar{E})$. We choose $B_0 \in \mathscr{F}$ such that $2B_0 \ni \xi_1$. For $B \in \mathscr{F}$, there exist $A_1, \dots, A_l \in \mathscr{F}(B)$ such that

$$A_1 = B, \quad A_l = B_0, \quad 2A_i \cap 2A_{i+1} \neq \emptyset \quad (i = 1, 2, \dots, l-1).$$

By virture of (VD), (N), and Lemma 3.2 (2), we have

$$\int_{4A_k} |f - f_{4A_k}|^2 \, d\mu \le I_{A_k}(s) \quad (\forall s > 0),$$

where we set

$$I_{A_k}(s) := c_4 s^2 \int_{32A_k} |\nabla f|^2 \, d\mu + c_6 |A_k|^{-1} \left(\frac{\rho(A_k)}{s}\right)^{\nu} \left(\int_{4A_k} |f - f_{4A_k}| \, d\mu\right)^2,$$

and $c_6 = c(c_1, c_5)$. Hence

$$\begin{aligned}
&|f_{4A_k} - f_{4A_{k+1}}|^2 |4A_k \cap 4A_{k+1}| \\
&\quad = \int_{4A_k \cap 4A_{k+1}} |f_{4A_k} - f_{4A_{k+1}}|^2 \, d\mu \\
&\quad \le 2 \int_{4A_k \cap 4A_{k+1}} |f - f_{4A_k}|^2 + |f - f_{4A_{k+1}}|^2 \, d\mu \le 2I_k(s_k) + 2I_{k+1}(s_{k+1})
\end{aligned}$$

for all $s_k > 0$ and $s_{k+1} > 0$, where $I_k := I_{A_k}$.

Since $2A_k \cap 2A_{k+1} \neq \emptyset$, from Lemma 3.2 (2), we see that

$$\frac{99}{10^3 + 3} \rho(A_{k+1}) \le 99\rho(A_k) \le (10^3 + 3)\rho(A_{k+1}),$$

and

$$B(y, 2\min\{\rho(A_k), \rho(A_{k+1})\}) \subset 4A_k \cap 4A_{k+1},$$

where $y \in 2A_k \cap 2A_{k+1}$. (In this proof of Lemma 3.3, integration is always with respect to the measure μ, and hence, in the following, we shall often omit '$d\mu$' for simplicity.) Therefore, it is easy to see that

$$\begin{aligned}
&\int_{4A_k \cup 4A_{k+1}} |f - f_{4A_k}|^2 \\
&\quad \le \int_{4A_k} |f - f_{4A_k}|^2 + 2\int_{4A_{k+1}} (|f - f_{4A_{k+1}}|^2 + |f_{4A_{k+1}} - f_{4A_k}|^2) \\
&\quad \le I_k(s_k) + 2I_{k+1}(s_{k+1}) + 4|4A_{k+1}| \frac{I_k(s_k) + I_{k+1}(s_{k+1})}{|4A_k \cap 4A_{k+1}|} \\
&\quad \le c(c_1)\{I_k(s_k) + I_{k+1}(s_{k+1})\}.
\end{aligned}$$

Therefore, we get

$$\int_{4A_k \cup 4A_{k+1}} |f_{4A_k} - f_{4A_{k+1}}|^2 \le c(c_1)\{I_k(s_k) + I_{k+1}(s_{k+1})\}.$$

Hence, by (VD),

$$|f_{4A_k} - f_{4A_{k+1}}|^2 \le \frac{c(c_1)}{|4A_k \cup 4A_{k+1}|}\{I_k(s_k) + I_{k+1}(s_{k+1})\} \le c(c_1)\left\{\frac{I_k(s_k)}{|A_k|} + \frac{I_k(s_{k+1})}{|A_{k+1}|}\right\}.$$

Therefore,

$$\begin{aligned}
\int_{2B} |f - f_{4B_0}|^2 &= \int_{2B} \left| f - f_{4B} + \sum_{k=1}^{l-1} (f_{4A_k} - f_{4A_{k+1}}) \right|^2 \\
&\le l \int_{2B} \left(|f - f_{4B}|^2 + \sum_{k=1}^{l-1} |f_{4A_k} - f_{4A_{k+1}}|^2 \right) \\
&\le l \left\{ I_1(s_1) + c(c_1) \sum_{k=1}^{l-1} \left(\frac{|B|}{|A_k|} I_k(s_k) + \frac{|B|}{|A_{k+1}|} I_{k+1}(s_{k+1}) \right) \right\} \\
&\le c(c_1) \cdot l \sum_{k=1}^{l-1} \frac{|B|}{|A_k|} I_k(s_k) \\
&\le c(c_1) \cdot \sharp\mathscr{F}(B) \sum_{A \in \mathscr{F}(B)} \frac{|B|}{|A|} I_A(s_A),
\end{aligned}$$

for all $s_A > 0$. Summing up all $B \in \mathscr{F}$, we get, by Lemma 3.2 (1),

$$\begin{aligned}
\int_E |f - f_{4B_0}|^2 &\le \sum_{B \in \mathscr{F}} \int_{2B} |f - f_{4B_0}|^2 \\
&\le c(c_1) \sum_{B \in \mathscr{F}} \sharp\mathscr{F}(B) \sum_{A \in \mathscr{F}(B)} \frac{|B|}{|A|} I_A(s_A) \\
&= c(c_1) \sum_{A \in \mathscr{F}} \left(\sum_{B \in A(\mathscr{F})} \sharp\mathscr{F}(B) \frac{|B|}{|A|} \right) I_A(s_A) \\
&\le c(c_1) \sum_{A \in \mathscr{F}} \log\left(\frac{r_1}{\rho(A)} \right) I_A(s_A)
\end{aligned}$$

for all $s_A > 0$, where we have used Lemma 3.2 (4). In this inequality, we shall set $s_A = \rho(A)^{1-\theta}|A|^{-\delta}s$ for $s > 0$, $\theta \in \boldsymbol{R}$, and $\delta \in \boldsymbol{R}$. Then, we have

$$\int_E |f - f_{4B_0}|^2 \le c(c_1, c_4) \sum_{A \in \mathscr{F}} \log\left(\frac{r_1}{\rho(A)}\right) \frac{\rho(A)^{2-2\theta}}{|A|^{2\delta}} s^2 \int_{32A} |\nabla f|^2$$

$$+ c(c_1, c_5) \sum_{A \in \mathscr{F}} \log\left(\frac{r_1}{\rho(A)}\right) |A|^{-1} (\rho(A)^{\theta} |A|^{\delta} s^{-1})^{\nu} \left(\int_{4A} |f - f_{4A}|\right)^2$$

$$\le c(c_1, c_4) \cdot s^2 \varepsilon_1^{-1} (r_1)^{\varepsilon_1} \sum_{A \in \mathscr{F}} \frac{\rho(A)^{2-2\theta-\varepsilon_1}}{|A|^{2\delta}} \int_{32A} |\nabla f|^2$$

$$+ c(c_1, c_5) \cdot s^{-\nu} (\varepsilon_2)^{-1} (r_1)^{\varepsilon_2} \sum_{A \in \mathscr{F}} \frac{|A|^{\delta\nu - 1}}{\rho(A)^{\varepsilon_2 - \theta\nu}} \left(\int_{4A} |f - f_{4A}|\right)^2$$

for all $\varepsilon_1, \varepsilon_2 > 0$, where we have used the fact that $(\log x)/x^{\varepsilon} \le (e\varepsilon)^{-1}$ for all x, $\varepsilon > 0$. Since

$$\int_{4A} |f_E - f_{4A}| = |4A|\,|f_E - f_{4A}| = |4A| \left| |4A|^{-1} \int_{4A} (f - f_E) \right| \le \int_{4A} |f - f_E|,$$

we have

$$\int_{4A} |f - f_{4A}| \le \int_{4A} (|f - f_E| + |f_E - f_{4A}|) \le 2 \int_{4A} |f - f_E|.$$

Hence, noting Lemma 3.2 (3), we get

$$\sum_{A \in \mathscr{F}} |A|^{-1/2} \left(\int_{4A} |f - f_{4A}|\right)^2 \le \sum_{A \in \mathscr{F}} 4|A|^{-1/2} \left(\int_{4A} |f - f_E|\right)^2$$

$$\le c(c_1) \sum_{A \in \mathscr{F}} \left(\int_{4A} |f - f_E|^2\right)^{1/2} \left(\int_{4A} |f - f_E|\right)$$

$$\le c(c_1) \left(\int_E |f - f_E|^2\right)^{1/2} \sum_{A \in \mathscr{F}} \int_{4A} |f - f_E|$$

$$\le c(c_1, r_1/r_0) \left(\int_E |f - f_E|^2\right)^{1/2} \int_E |f - f_E|$$

$$\le c(c_1, r_1/r_0) \left\{ u \int_E |f - f_E|^2 + u^{-1} \left(\int_E |f - f_E|\right)^2 \right\}$$

for all $u > 0$. Now, we set

(3) $$\delta = 1/(2\nu); \quad \theta = 1/4; \quad \varepsilon_1 = 1/2; \quad \varepsilon_2 = \nu/4.$$

Then,

$$\frac{|A|^{\delta\nu - 1}}{\rho^{\varepsilon_2 - \theta\nu}} = |A|^{-1/2},$$

and from the assumption (2) and $v \geq v_0$,

$$\frac{\rho(A)^{2-2\theta-\varepsilon_1}}{|A|^{2\delta}} = \frac{\rho(A)^{2-2\theta-\varepsilon_1}}{|A|^{1/v}} \leq (c_3)^{1/v}|E|^{-1/v}2r_1\rho(A)^{1-2\theta-\varepsilon_1} = (c_3)^{1/v}|E|^{-1/v}2r_1.$$

Therefore, we obtain, from Lemma 3.2 (3),

$$\begin{aligned}\int_E |f-f_E|^2 &\leq \int_E |f-f_{4B_0}|^2 \\ &\leq c(c_1,c_4,r_1/r_0)\cdot s^2(\varepsilon_1)^{-1}(r_1)^{\varepsilon_1}(c_3)^{1/v}|E|^{-1/v}r_1\int_E|\nabla f|^2 \\ &\quad + c(c_1,c_5,r_1/r_0)\cdot s^{-v}(\varepsilon_2)^{-1}(r_1)^{\varepsilon_2}\left\{u\int_E|f-f_E|^2 + u^{-1}\left(\int_E|f-f_E|\right)^2\right\}\end{aligned}$$

for all $s, u > 0$, and hence

$$\begin{aligned}(1-c_8s^{-v}(\varepsilon_2)^{-1}(r_1)^{\varepsilon_2}u)\int_E|f-f_E|^2 &\leq c_7s^2(\varepsilon_1)^{-1}(r_1)^{\varepsilon_1+1}(c_3)^{1/v}|E|^{-1/v}\int_E|\nabla f|^2 \\ &\quad + c_8s^{-v}(\varepsilon_2)^{-1}(r_1)^{\varepsilon_2}u^{-1}\left(\int_E|f-f_E|\right)^2,\end{aligned}$$

for all $s, u > 0$, where $c_7 = c(c_1,c_4,r_1/r_0)$, $c_8 = c(c_1,c_5,r_1/r_0)$. Hence, setting $u = 2^{-1}c_8^{-1}s^v\varepsilon_2(r_1)^{-\varepsilon_2}$, we have, for all $s > 0$,

$$\begin{aligned}\frac{1}{2}\int_E|f-f_E|^2 &\leq c_7s^2(\varepsilon_1)^{-1}(r_1)^{\varepsilon_1+1}(c_3)^{1/v}|E|^{-1/v}\int_E|\nabla f|^2 \\ &\quad + 2(c_8)^2s^{-2v}(\varepsilon_2)^{-2}(r_1)^{2\varepsilon_2}\left(\int_E|f-f_E|\right)^2.\end{aligned}$$

Here, we note that, in general, when A and B are positive constants, a function

$$As^2 + Bs^{-2v}$$

of $s > 0$ takes the minimum value $A^{v/(v+1)}B^{1/(v+1)}(v^{1/(v+1)} + v^{-v/(v+1)})$ when $s = (Bv/A)^{1/(2v+2)}$. Hence, if we set

$$A = 2c_7(\varepsilon_1)^{-1}(r_1)^{\varepsilon_1+1}(c_3)^{1/v}|E|^{-1/v}\int_E|\nabla f|^2$$

and

$$B = 4(c_8)^2(\varepsilon_2)^{-2}(r_1)^{2\varepsilon_2}\left(\int_E|f-f_E|\right)^2,$$

then we have

$$\int_E|f-f_E|^2 \leq c(v)A^{v/(v+1)}B^{1/(v+1)}.$$

Thus, recalling (3), we obtain

$$\begin{aligned}\|f-f_E\|_{L^2(E)}^{2+(2/\nu)} &\le c(\nu)AB^{1/\nu}\\ &= c(\nu)\cdot c_7(c_8)^{2/\nu}(c_3)^{1/\nu}|E|^{-1/\nu}(r_1)^2\|\nabla f\|_{L^2(E)}^2\|f-f_E\|_{L^1(E)}^{2/\nu}\\ &= c(\nu,c_1,c_4,c_5,r_1/r_0)(c_3)^{1/\nu}(r_1)^2|E|^{-1/\nu}\|\nabla f\|_{L^2(E)}^2\|f-f_E\|_{L^1(E)}^{2/\nu}.\end{aligned}$$

Hence, when we set $\nu' = 2\nu$, we get

$$\|f-f_E\|_{L^2(E)}^{2+(4/\nu')} \le c(\nu',c_1,c_4,c_5,r_1/r_0)(c_3)^{2/\nu'}(r_1)^2|E|^{-2/\nu'}\|\nabla f\|_{L^2(E)}^2\|f-f_E\|_{L^1(E)}^{4/\nu'}.$$

This completes the proof of Lemma 3.3. □

Putting Lemma 3.1 and 3.2 together, we have the following

THEOREM 3.1. *Let M be a complete Riemannian manifold, μ be a positive measure on M, and $E = B(\xi_1, r_1) \subset M$. We assume the following three conditions*:

(1) (*volume doubling condition*)

$$(\mathrm{VD})_1 \quad V(x,2r) \le c_1 V(x,r) \quad \textit{for } \forall x \in M,\ 0 < \forall r \le r_0$$

or

$$(\mathrm{VD})_2 \quad \mu(B(x,2r)\cap E) \le c_1\mu(B(x,r)\cap E) \quad \textit{for } \forall x \in \bar{E},\ 0 < \forall r \le r_0;$$

(2) (*weak Poincaré inequality*)

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2\, d\mu \le c_2 r^2 \int_{B(x,2r)} |\nabla f|^2\, d\mu$$

for $\forall B(x,2r) \subset E$, $0 < \forall r \le r_0$, $\forall f \in C^\infty(\bar{E})$;

(3) $2r_1 \le r_0$ *or*

$$|E| = V(\xi_1, r_1) \le c_3|A|\left(\frac{2r_1}{\rho(A)}\right)^{\nu_0} \quad \textit{for all ball } A \subset E.$$

Then we have the Nash inequality

$$\|f-f_E\|_{L^2(E)}^{2+(4/\nu)} \le c(c_1,c_2,\nu,r_0/r_1)(r_1)^2(c_3)^{2/\nu}|E|^{-2/\nu}\|\nabla f\|_{L^2(E)}^2\|f-f_E\|_{L^1(E)}^{4/\nu}$$

for all $f \in C^\infty(\bar{E})$ *and all* $\nu \ge 2\nu_0 = 2(\log c_1)/\log 2$. *In this inequality, when* $2r_1 \le r_0$, $c(c_1,c_2,\nu,r_0/r_1)$ *does not depend on* r_0/r_1 *and we can take* $c_3 = c_1$.

Remark 3.2. Theorem 2.1 implies that under the assumptions of Theorem 3.1, we also have the Sobolev inequality

$$\inf_{\alpha\in\boldsymbol{R}}\|f-\alpha\|_{L^{2\nu/(\nu-2)}(E)}^2 \le c(c_1,c_2,\nu,r_0/r_1)(r_1)^2(c_3)^{2/\nu}|E|^{-2/\nu}\|\nabla f\|_{L^2(E)}^2 \quad (\forall f \in C^\infty(\bar{E})).$$

A Neumann eigenvalue estimate is also obtained by means of Corollary 2.1.

Remark 3.3. After writing this paper, the author found that P. Hajłasz and P. Koskela [9] proved the Sobolev inequality in a more general situation than Theorem 3.1 by a different method. For this fact and other related results, see their paper.

The Nash inequality on M implies a lower bound of the measure of the intersection of a ball and M. Thus, it restricts the boundary behavior of the measure from below;

THEOREM 3.2. *Let M be the closure of a relatively compact domain in a complete Riemannian manifold N, w be a function such that $w \in C^\infty(M)$ and $0 < w$ on* $\operatorname{Int} M = M - \partial M$ *(w may possibly takes zero at a point of the boundary ∂M). We suppose that there exists a constant $\hat{c} > 0$ such that the following Nash inequality holds*:

$$\left(\int_M |f - f_M|^2 \, d\mu^w\right)^{1+(2/\nu')} \leq \hat{c} \cdot (\operatorname{diam} M)^2 \cdot \mu^w(M)^{-2/\nu'} \left(\int_M |\nabla f|^2 \, d\mu^w\right) \left(\int_M |f - f_M| \, d\mu^w\right)^{4/\nu'},$$

for all $f \in C^\infty(M)$, where $f_M = \int_M f \, d\mu^w / \mu^w(M)$. Then, for each $0 < r <$ diam *M and $x_0 \in M$, we have*

$$\frac{\mu^w(B_M(x_0, r)_0)}{\mu^w(M)} \geq \frac{c(\nu')}{1 + \hat{c}^{\nu'/2}} \left(\frac{r}{\operatorname{diam} M}\right)^{\nu'},$$

where $c(\nu')$ is a constant depending only on ν' and $B_M(x_0, r)_0$ stands for the connected component (containing x_0) of the intersection $B(x_0, r) \cap \operatorname{Int} M$ of the metric ball $B(x_0, r)$ of N and $\operatorname{Int} M$.

Proof. Let $p_t(x, y)$ be the heat kernel of the operator on $L^2(M, \mu^w/\mu^w(M))$ associated with the closed form

$$\mathscr{E}(f, h) = \int_M \langle \nabla f, \nabla h \rangle \, d\mu^w \quad (f, h \in \operatorname{Dom}(\mathscr{E}) = H^1(M, \mu^w)).$$

Then, from Theorem 2.1, we have

$$p_t(x, x) \leq 1 + a t^{-\nu'/2} \quad (\forall t > 0, \forall x \in M),$$

where $a = \alpha(\nu' \hat{c})^{\nu'/2} (\operatorname{diam} M)^{\nu'}$ and α is some numerical constant. Take a point $x_0 \in M$ and $r > 0$, and consider the form

$$\mathscr{E}_D(f, h) = \int_{B_M(x_0, r)_0} \langle \nabla f, \nabla h \rangle \, d\mu^w$$

for $f, h \in \mathrm{Dom}(\mathscr{E}_D) := \{f \in C^\infty(\overline{B_M(x_0, r)_0}); f|_{\mathrm{Int}\, M \cap \partial B_M(x_0,r)_0} = 0\}$. Let $h_t(x, y)$ be the heat kernel on $L^2(B_M(x_0, r)_0, \mu^w/\mu^w(B_M(x_0, r)_0))$ associated with the closure of $\mathscr{E}_D$. Then maximum principle implies that

$$\begin{aligned} h_t(x, y) &\le \frac{\mu^w(B_M(x_0, r)_0)}{\mu^w(M)} p_t(x, y) \\ &\le \frac{\mu^w(B_M(x_0, r)_0)}{\mu^w(M)} \{1 + at^{-\nu'/2}\} \end{aligned}$$

for $t > 0$ and $x, y \in B_M(x_0, r)_0$. Hence, when we set $\tau_0 = (\mathrm{diam}\, M)^2$ and $c_4 = (\mu^w(B_M(x_0, r)_0)/(\mu^w(M)))(a + \tau_0^{\nu'/2})$, we have, for $0 < t \le \tau_0$ and $x, y \in B_M(x_0, r)_0$,

$$h_t(x, y) \le c_4 t^{-\nu'/2}.$$

Therefore, Theorem 2.2 in [14] implies that

$$\begin{aligned} &\left(\int_{B_M(x_0,r)_0} |f|^2 \frac{d\mu^w}{\mu^w(B_M(x_0, r)_0)}\right)^{1+(2/\nu')} \\ &\quad \le c_6 \left(\int_{B_M(x_0,r)_0} |\nabla f|^2 \frac{d\mu^w}{\mu^w(B_M(x_0, r)_0)} + \tau_0^{-1} \int_{B_M(x_0,r)_0} |f|^2 \frac{d\mu^w}{\mu^w(B_M(x_0, r)_0)}\right) \\ &\qquad \times \left(\int_{B_M(x_0,r)_0} |f| \frac{d\mu^w}{\mu^w(B_M(x_0, r)_0)}\right)^{4/\nu'} \end{aligned}$$

for all $f \in C^\infty(\overline{B_M(x_0, r)_0})$ with $f|_{\mathrm{Int}\, M \cap \partial B_M(x_0,r)_0} = 0$, where $c_6 = \alpha c_4^{2/\nu'}$ and α is some numerical constant. Hence, by the same arguments as in the proof of Lemma 2.5 in [11] (see also Akutagawa [1]), we can see that, for $0 < s \le \min\{\sqrt{\tau_0}, r\}$,

$$\begin{aligned} \frac{\mu^w(B_M(x_0, s)_0)}{\mu^w(B_M(x_0, r)_0)} &\ge c(\nu') c_6^{-\nu'/2} s^{\nu'} = c(\nu') c_4^{-1} s^{\nu'} \\ &= c(\nu') \frac{s^{\nu'}}{a + \tau_0^{\nu'/2}} \frac{\mu^w(M)}{\mu^w(B_M(x_0, r)_0)}. \end{aligned}$$

Thus, setting $s = r \le \sqrt{\tau_0} = \mathrm{diam}\, M$, we get

$$\begin{aligned} \frac{\mu^w(B_M(x_0, r)_0)}{\mu^w(M)} &\ge c(\nu') \frac{r^{\nu'}}{c(\nu')\hat{c}^{\nu'/2}(\mathrm{diam}\, M)^{\nu'} + c(\nu')(\mathrm{diam}\, M)^{\nu'}} \\ &= \frac{c(\nu')}{1 + \hat{c}^{\nu'/2}} \left(\frac{r}{\mathrm{diam}\, M}\right)^{\nu'} \end{aligned}$$

for $0 < r < \mathrm{diam}\, M$ and $x_0 \in M$. □

Now, let M be a complete Riemannian manifold and w be a positive smooth function on M. Let positive constants $\hat{c}$ and r_0 be given, and let us assume that M and w satisfy a family of local Nash inequalities

$$\|f - f_{B(\xi_1,r_1)}\|_{L^2(B(\xi_1,r_1),\mu^w)}^{2+(4/\nu)}$$
$$\leq \hat{c}\cdot(r_1)^2\cdot\mu^w(B(\xi_1,r_1))^{-2/\nu}\|\nabla f\|_{L^2(B(\xi_1,r_1),\mu^w)}^2\|f - f_{B(\xi_1,r_1)}\|_{L^1(B(\xi_1,r_1),\mu^w)}^{4/\nu},$$
$$\forall f\in C^\infty(\overline{B(\xi_1,r_1)}),\ \ \forall\xi_1\in M,\ \ 0<\forall r_1<r_0.$$

Then, by Theorem 3.2, we have the volume comparison

$$\text{(4)}\qquad \frac{\mu^w(B(\xi_1,s))}{\mu^w(B(\xi_1,r_1))}\geq\frac{c(\nu)}{1+\hat{c}^{\nu/2}}\left(\frac{s}{r_1}\right)^\nu\quad\text{for } 0<\forall s<\forall r_1\leq r_0,\ \ \forall\xi_1\in M.$$

Moreover, we have local Poincaré inequalities

$$\text{(5)}\qquad \|f - f_{B(\xi_1,r_1)}\|_{L^2(B(\xi_1,r_1),\mu^w)}^2\leq\hat{c}\cdot(r_1)^2\|\nabla f\|_{L^2(B(\xi_1,r_1),\mu^w)}^2,$$
$$\forall f\in C^\infty(\overline{B(\xi_1,r_1)}),\ \ 0<\forall r_1\leq r_0,\ \ \forall\xi_1\in M,$$

because the Schwarz inequality implies that

$$\mu^w(B(\xi_1,r_1))^{-2/\nu}\|f - f_{B(\xi_1,r_1)}\|_{L^1(B(\xi_1,r_1),\mu^w)}^{4/\nu}\leq\|f - f_{B(\xi_1,r_1)}\|_{L^2(B(\xi_1,r_1),\mu^w)}^{4/\nu}.$$

Now, we recall the following theorem due to Salloff-Coste (see also A. A. Grigor'yan [8]):

THEOREM 3.3 (Salloff-Coste [14]). *The following two properties* 1 *and* 2 *are equivalent.*

1. *The following properties* (a) *and* (b) *hold for some constants* $r_0>0$, $c_1>0$ *and* $c_2>0$:

(a) $\mu^w(B(x,2r))\leq c_1\mu^w(B(x,r))$, $0<\forall r<r_0$, $\forall x\in M$

(b) $\int_{B(x,r)}|f - f_{B(x,r)}|^2\,d\mu^w\leq c_2r^2\int_{B(x,2r)}|\nabla f|^2\,d\mu^w$, $\quad 0<\forall r<r_0$, $\quad\forall x\in M$, $\forall f\in C^\infty(M)$.

2. *There exists* $r_1>0$, *and there exists a constant* c *depending only on the parameters* $0<\varepsilon<\eta<\delta<1$, *such that, for any* $x\in M$, *any real* s, *and any* $0<r<r_1$, *any nonnegative solution* u *of* $(\partial_t+\mathscr{L})u=0$ *in* $Q=(s-r^2,s)\times B(x,r)$ *satisfies*

$$\sup_{Q_-}\{u\}\leq c\inf_{Q_+}\{u\},$$

where $Q_-=[s-(\delta r)^2,s-(\eta r)^2]\times B(x,\delta r)$, $Q_+=[s-(\varepsilon r)^2,s)\times B(x,\delta r)$ *and* $\mathscr{L}u=-\Delta u-\langle\nabla\log w,\nabla u\rangle$.

As is shown above, a family of local Nash inequalities implies the properties (a) and (b) in Theorem 3.3, and hence, from Theorem 3.1 and 3.3 we see that a family of local Nash inequalities is also equivalent to local parabolic Harnack inequalities;

COROLLARY 3.1. *The properties* 1 *and* 2 *in Theorem* 3.3 *are also equivalent to the following property* 3:

3. *There exist constants* $\hat{c} > 0$, $r_0 > 0$ *and* $v > 0$ *such that*

$$\|f - f_{B(x,r)}\|_{L^2(B(x,r),\mu^w)}^{2+(4/v)}$$
$$\leq \hat{c} \cdot r^2 \cdot \mu^w(B(x,r))^{-2/v} \|\nabla f\|_{L^2(B(x,r),\mu^w)}^2 \|f - f_{B(x,r)}\|_{L^1(B(x,r),\mu^w)}^{4/v},$$
$$\forall f \in C^\infty(\overline{B(x,r)}), \ \forall x \in M, \ 0 < \forall r < r_0.$$

4. Nash inequality with respect to weighted measure

This section is devoted to proving the following theorem:

THEOREM 4.1. *Let* M *be a compact Riemannian manifold with* C^∞ *boundary* ∂M *and* w *be a positive* C^∞ *function on* Int $M := M - \partial M$. w *possibly takes zero value at a point of* ∂M. *We assume the following properties* (D), (R), (P), (w_1) *and* (w_2):

(D) (*volume doubling property*) $|B(x,2r)| \leq c_1 |B(x,r)|$ *for all* $x \in \overline{M}$ *and* $r > 0$, *where* $|B(x,r)| = \int_{B(x,r)} dv_M$.

(R) M *satisfies the "interior rolling* R*-ball condition", that is,* $\exists R > 0$ $\forall x \in \partial M$ $\exists B(p,R) \subset M$ *s.t.* $\overline{B(p,R)} \cap \partial M = \{x\}$.

(P) (*weak Poincaré inequality*)

For all $B(x,2r) \subset M$ *with* $B(x,2r) \cap \partial M = \emptyset$, *and all* $f \in C^\infty(M)$,

$$\inf_{\alpha \in \boldsymbol{R}} \int_{B(x,r)} |f - \alpha|^2 \, dv_M \leq c_2 r^2 \int_{B(x,2r)} |\nabla f|^2 \, dv_M,$$

where c_2 *is a positive constant independent of* x, r *and* f.

(w_1) $\exists c_3 > 1$ *s.t.* $w(x) \leq c_3 w(y)$ *for all* $x, y \in M$ *with* $0 < d(x,\partial M) \leq 2d(y,\partial M)$.

(w_2) $\exists c_6 > 0$ $\exists \varphi > 0$ *s.t.* $c_6 d(x,\partial M)^{2\varphi} \leq w(x)$ *for all* $x \in M - M_R$, *where* $M_R = \{x \in M \mid d(x,\partial M) \geq R\}$.

Then we have the Nash inequality on M:

$$\left(\int_M |f - f'_M|^2 w \, dv_M\right)^{1+(2/v)} \tag{6}$$
$$\leq c \cdot \left(\int_M |\nabla f|^2 w \, dv_M\right)\left(\int_M |f - f'_M| w \, dv_M\right)^{4/v}$$

for all v *satisfying* $v > 4\varphi$ *and* $v \geq 2v_0$, *and all* $f \in C^\infty(\overline{M})$, *where* $f'_M = \int_M f \, d\mu^w / \mu^w(M)$,

$$c = c(c_1, c_2, c_3, c_6, v, \varphi) R^{(v-4\varphi)/v} |M|^{-2/v} D$$
$$+ c(c_3, D/R, v) c_6^{-2/v} R^{-4\varphi/v} c_N(M_{R/2}) |M_{R/2}|^{-2/v} \operatorname{diam}(M_{R/2})^2,$$

$c_N(M_{R/2})$ *is a constant satisfying*

$$(7)\quad \left(\int_{M_{R/2}} |f - f_{M_{R/2}}|^2 \, dv_M\right)^{1+(2/\nu)} \le N \int_{M_{R/2}} |\nabla f|^2 \, dv_M \cdot \left(\int_{M_{R/2}} |f - f_{M_{R/2}}| \, dv_M\right)^{4/\nu},$$

$$N := c_N(M_{R/2}) \cdot |M_{R/2}|^{-2/\nu} \cdot \operatorname{diam}(M_{R/2})^2, \quad \forall f \in C^\infty(\overline{M_{R/2}}),$$

$f_{M_{R/2}} = \int_{M_{R/2}} f \, dv_M / |M_{R/2}|$, *and* D *stands for the diameter of* M.

Remark 4.1. We should note that actually, the condition (w_2) is not required for a Nash inequality to hold.

In fact, the condition (w_1) implies, for example, the following inequality:

$$w(x) \ge \frac{\inf\{w(x) \mid d(x, \partial M) = R/2\}}{R^{2\varphi_1}} \, d(x, \partial M)^{2\varphi_1},$$

where $2\varphi_1 = (\log c_3)/\log 2$. This can be seen as follows: Let us set $f(t) = \inf\{w(x) \mid t \le d(x, \partial M) \le R\}$ for $0 < t \le R$. Then the function f is nondecreasing and the condition (w_1) implies that $f(2t)/f(t) \le c_3$ for $0 < t \le R/2$. For $0 < s < R$, if we take the integer k such that $2^{k-1} < R/s \le 2^k$, we see that

$$f(R/2) \le f(2^{k-1}s) \le (c_3)^{k-1} f(s) < (R/s)^{(\log c_3)/\log 2} f(s).$$

Hence, if we set $2\varphi_1 = (\log c_3)/\log 2$, then $\varphi_1 > 0$ and $f(s) \ge f(R/2) R^{-2\varphi_1} s^{2\varphi_1}$. Therefore, if $d(x, \partial M) = s$, then $w(x) \ge f(s) \ge f(R/2) R^{-2\varphi_1} d(x, \partial M)^{2\varphi_1}$.

Thus, when we set $c_6 := \inf\{w(x) \mid d(x, \partial M) = R/2\}/R^{2\varphi_1}$ and $\varphi := \varphi_1$, the property (w_2) above holds. In spite of this fact, we assume the condition (w_2) because it is important to choose $\nu > 0$ as small as possible. Indeed, in Proposition 5.1, we shall consider the weight function $w = (u_1)^2$, where u_1 is the normalized first Dirichlet eigenfunction of M, and set $\varphi = 1$.

Remark 4.2. As is seen from the following proof of Theorem 4.1, in order to get the Nash inequality (6), it suffice to assume that the doubling property (D) and weak Poincaré inequality (P) hold only on the neighborhood $M - M_R$ of the boundary ∂M.

Corollary 4.1. *Let* M *be a metric ball of radius* R *in a complete Riemannian manifold and assume* (D), (P), (w_1), *and* (w_2). *Then the Nash inequality* (6) *with* $R = D$ *and* $c_N(M_{R/2}) = 0$, *holds.*

Remark 4.3. As is seen from Theorem 2.1, the Nash inequality (6) is equivalent to the following Sobolev inequality:

$$\inf_{\alpha \in \boldsymbol{R}} \left(\int_M |f - \alpha|^{2\nu/(\nu-2)} \, d\mu^w\right)^{(\nu-2)/\nu} \le c\mu^w(M)^{-2/\nu} \int_M |\nabla f|^2 \, d\mu^w,$$

$$\forall \nu \text{ satisfying } \nu > 4\varphi \text{ and } \nu \ge 2\nu_0, \quad \forall f \in C^\infty(\overline{M}).$$

The inequality (7) is also equivalent to the following:

$$\inf_{\alpha\in\boldsymbol{R}}\left(\int_{M_{R/2}}|f-\alpha|^{2\nu/(\nu-2)}\,dv_M\right)^{(\nu-2)/\nu}$$
$$\leq c_N(M_{R/2})\cdot|M_{R/2}|^{-2/\nu}\cdot\operatorname{diam}(M_{R/2})^2\int_{M_{R/2}}|\nabla f|^2\,dv_M,\quad \forall f\in C^\infty(\overline{M_{R/2}}).$$

In order to prove Theorem 4.1, we shall use the following covering lemma due to Oden, Sung and Wang:

LEMMA 4.1 (Whitney decomposition [13]). *Let M be a compact Riemannian manifold with boundary which satisfies the doubling property (D). Then, there exists a pairwise disjoint family $\mathscr{F}=\{B_i=B(x_i,r_i)\,|\,i\in I\}$ of geodesic balls in* Int M *satisfying the following*:

(1) $\bigcup_{i\in I}2B_i=\operatorname{Int}M$;

(2) $\operatorname{dist}(B(x_i,r_i),\partial M)=10^3r_i$;

(3) *There exists a constant $c(c_1)$ depending only on c_1 such that, for all $\eta\in\operatorname{Int}M$, $\sharp\{B_i\in\mathscr{F}\,|\,\eta\in 32B_i\}\leq c(c_1)$.*

Denote $M_R:=\{x\in M\,|\,d(x,\partial M)\geq R\}$ and $\mathscr{L}:=\{B_i\in\mathscr{F}\,|\,x_i\notin M_R\}$. For $B_i=B(x_i,r_i)\in\mathscr{L}$, the interior rolling R-ball condition implies that there exist $y_i\in\partial M$ and $B(q_i,R)\subset\operatorname{Int}(M)$ such that $d(x_i,y_i)=d(x_i,\partial M)$ and $\overline{B(q_i,R)}\cap\partial M=\{y_i\}$. Let $\overline{q_iy_i}$ be the minimal geodesic segment from $q_i\in\partial M_R$ to y_i. Then $x_i\in\overline{q_iy_i}$. Denote by l_i the segment $\overline{q_ix_i}$ of $\overline{q_iy_i}$. For $B_i\in\mathscr{L}$, we then define $\mathscr{F}(B_i)=\{A\in\mathscr{F}\,|\,2A\cap l_i\neq\emptyset\}$. Let $\mathscr{H}=\{A\in\mathscr{F}\,|\,A\in\mathscr{F}(B_i)$ for some $B_i\in\mathscr{L}\}$.

(4) *$A\in\mathscr{F}(B)$ implies $\rho(A)\geq(10^3/(10^3+3))\rho(B)$.*

(5) *For $A\in\mathscr{H}$, let $A(\mathscr{L})=\{B\in\mathscr{L}\,|\,A\in\mathscr{F}(B)\}$. Then*

$$|A|^{-1}\sum_{B\in A(\mathscr{L})}\sharp\mathscr{F}(B)|B|\leq c(c_1)\log\left(\frac{R}{\rho(A)}\right).$$

In the following, for $A\subset M$, we denote $f_A=|A|^{-1}\int_A f\,dv_M$, $f'_A=(\int_A w\,dv_M)^{-1}\int_A fw\,dv_M$, and $|A|=\int_A dv_M$.

LEMMA 4.2. *Let M be a compact manifold with C^∞ boundary ∂M. We assume that M satisfies the following family of inequalities*:

$$\text{(8)}\qquad \int_{B(x,r)}|f-f_{B(x,r)}|^2\,dv_M\leq c_5t^2r^2\int_{B(x,8r)}|\nabla f|^2\,dv_M$$
$$+c_5t^{-\nu}|B(x,r)|^{-1}\left(\int_{B(x,r)}|f-f_{B(x,r)}|\,dv_M\right)^2,$$
$$\forall B(x,8r)\subset\operatorname{Int}M,\ \ \forall f\in C^\infty(\overline{B(x,8r)}),\ \ \forall t>0,\ \ \forall\nu\geq\nu_0.$$

Moreover, we suppose that the pair (M,w) satisfies properties (D), (w_1), and (w_2). Then, the Nash inequality (6) holds.

Proof. Let $\mathscr{F}$ be a Whitney decomposition as in Lemma 4.1 and let $f \in C^\infty(\overline{M})$. By Lemma 4.1 (1), for $B \in \mathscr{L}$, we can take $A_1, \dots, A_l \in \mathscr{F}(B)$ satisfying

$$A_1 = B, \quad q \in 2A_l, \quad 2A_i \cap 2A_{i+1} \neq \emptyset \quad (i = 1, 2, \dots, l-1).$$

Let l' be the integer such that

$$1 \le l' \le l, \quad 4A_j \not\subset M_{R/2} \quad (j = 1, 2, \dots, l'-1), \quad 4A_{l'} \subset M_{R/2}.$$

Denote $f_0' = f'_{M_{R/2}} = \int_{M_{R/2}} fw \, dv_M / \int_{M_{R/2}} w \, dv_M$.
Since $\int_{4A_i} |f - f_{4A_i}| \le 2 \int_{4A_i} |f - f'_{4A_i}|$, we have

$$\int_{4A_i} |f - f_{4A_i}| w \le c(c_3) \int_{4A_i} |f - f'_{4A_i}| w. \tag{9}$$

We note that for each A_i,

$$x, y \in 32A_i \Rightarrow w(x) \le c_3 w(y). \tag{10}$$

By (8), (9) and (10), we can see that

$$\begin{aligned} \int_{4A_i} |f - f'_{4A_i}|^2 w &\le c(c_3, c_5) t^2 \rho(A_i)^2 \int_{32A_i} |\nabla f|^2 w \\ &\quad + c(c_3, c_5) t^{-\nu} \left(\int_{4A_i} w \right)^{-1} \left(\int_{4A_i} |f - f'_{4A_i}| w \right)^2, \quad \forall t > 0, \ \forall \nu \ge \nu_0. \end{aligned}$$

For simplicity, we denote by $J_{A_i}(t) = J_i(t)$ the right hand side of the above inequality;

$$\begin{aligned} J_{A_i}(t) &= J_i(t) \\ &= c(c_3, c_5) t^2 \rho(A_i)^2 \int_{32A_i} |\nabla f|^2 w \\ &\quad + c(c_3, c_5) t^{-\nu} \left(\int_{4A_i} w \right)^{-1} \left(\int_{4A_i} |f - f'_{4A_i}| w \right)^2. \end{aligned}$$

Then,

$$\begin{aligned} |f'_{4A_i} - f'_{4A_{i+1}}|^2 \left(\int_{4A_i \cap 4A_{i+1}} w \right) &\le 2 \int_{4A_i} |f - f'_{4A_i}|^2 w + 2 \int_{4A_{i+1}} |f - f'_{4A_{i+1}}|^2 w \\ &\le 2J_i(t_i) + 2J_{i+1}(t_{i+1}), \quad (t_i > 0, i = 1, 2, \dots, l-1). \end{aligned}$$

Since

$$d(A_i, \partial M) = 10^3 \rho(A_i) \ge 10^3 \frac{10^3}{10^3 + 3} \rho(B) = \frac{10^3}{10^3 + 3} d(B, \partial M), \tag{11}$$

we see, by the assumption (w_1),

$$\frac{\int_{4B} w}{\int_{4A_i} w} \le c(c_1, c_3) \frac{|B|}{|A_i|}.$$

Therefore, since $4A_{l'} \subset M_{R/2}$,

$$\left(\int_{4B} w\right)|f'_{4A_{l'}} - f'_0|^2 = \frac{\int_{4B} w}{\int_{4A_{l'}} w}\left(\int_{4A_{l'}} |f'_{4A_{l'}} - f'_0|^2 w\right)$$
$$\le 2\frac{\int_{4B} w}{\int_{4A_{l'}} w}\left(\int_{4A_{l'}} |f - f'_{4A_{l'}}|^2 w + \int_{M_{R/2}} |f - f'_0|^2 w\right)$$
$$\le c(c_1, c_3)\frac{|B|}{|A_{l'}|}\left(J_{l'}(t_{l'}) + \int_{M_{R/2}} |f - f'_0|^2 w\right) \quad (\forall t_{l'} > 0).$$

Hence,

$$\text{(12)} \quad \int_{4B} |f - f'_0|^2 w \le 2\int_{4B} |f - f'_{4B} + \sum_{i=1}^{l'-1}(f'_{4A_i} - f'_{4A_{i+1}})|^2 w + 2\left(\int_{4B} w\right)|f'_{4A_{l'}} - f'_0|^2$$
$$\le 2l\int_{4B} |f - f'_{4B}|^2 w + 2l\left(\int_{4B} w\right)\sum_{i=1}^{l'-1}|f'_{4A_i} - f'_{4A_{i+1}}|^2$$
$$+ 2\left(\int_{4B} w\right)|f'_{4A_{l'}} - f'_0|^2$$
$$\le 2lJ_1(t_1) + 4l\sum_{i=1}^{l'-1}\frac{\int_{4B} w}{\int_{4A_i \cap 4A_{i+1}} w}\{J_i(t_i) + J_{i+1}(t_{i+1})\}$$
$$+ c(c_1, c_3)\frac{|B|}{|A_{l'}|}\left\{J_{l'}(t_{l'}) + \int_{M_{R/2}} |f - f'_0|^2 w\right\}.$$

We note here that

$$\text{(13)} \qquad \frac{\int_{4B} w}{\int_{4A_i \cap 4A_{i+1}} w} \le c(c_3)\frac{|4B|}{|4A_i \cap 4A_{i+1}|} \le c(c_1, c_3)\min\left\{\frac{|B|}{|A_i|}, \frac{|B|}{|A_{i+1}|}\right\}$$

and

$$\text{(14)} \qquad |A_{l'}| \ge c(c_1, R/D)|M|.$$

Indeed, the first inequality of (13) is due to (11).
When we take a point $y \in 2A_i \cap 2A_{i+1}$, we have

$$\text{(15)} \qquad B(y, 2\min\{\rho(A_i), \rho(A_{i+1})\}) \subset 4A_i \cap 4A_{i+1}.$$

Since $10^3\rho(A_i) - 10^3\rho(A_{i+1}) = d(A_i, \partial M) - d(A_{i+1}, \partial M) \le \rho(A_i) + \rho(A_{i+1})$, we have also

$$\text{(16)} \qquad (10^3 - 1)\rho(A_i) \le (10^3 + 1)\rho(A_{i+1}).$$

The second inequality of (13) follows from (15), (16) and the doubling property (D).

The inequality (14) is due to (D) and the fact that $4A_{l'} \subset M_{R/2}$. Indeed, by (D), we have

$$|A_i| \geq c_1^{-1}\left(\frac{\rho(A_i)}{D}\right)^{v_0}|M|, \quad v_0 = (\log c_1)/\log 2. \tag{17}$$

And, since the fact that $4A_{l'} \subset M_{R/2}$ implies $\rho(A_{l'}) \geq R/2(10^3 - 3)$, we have

$$|A_{l'}| \geq c_1^{-1}\left(\frac{R}{2(10^3-3)D}\right)^{v_0}|M| = c(c_1)\left(\frac{R}{D}\right)^{v_0}|M|,$$

and (14) follows.

By (12), (13) and (14), we obtain

$$\int_{4B}|f - f_0'|^2 w \leq c(c_1, c_3)\sharp\mathscr{F}(B)\sum_{i=1}^{l'}\frac{|B|}{|A_i|}J_i(t_i) + c(c_1, c_3)\left(\frac{D}{R}\right)^{v_0}\frac{|B|}{|M|}\int_{M_{R/2}}|f - f_0'|^2 w.$$

It is not hard to see that Lemma 4.1 (2) implies $A_1, A_2, \ldots, A_{l'} \in \mathscr{L}$. Summing over all $B \in \mathscr{L}$, we get, by Lemma 4.1 (1), (3) and (5),

$$\begin{aligned}
&\int_{\bigcup_{B\in\mathscr{L}}4B}|f - f_0'|^2 w \\
&\quad\leq c(c_1, c_3)\sum_{B\in\mathscr{L}}\sharp\mathscr{F}(B)\sum_{A\in\mathscr{F}(B)\cap\mathscr{L}}\frac{|B|}{|A|}J_A(t_A) + c(c_1, c_3)\left(\frac{D}{R}\right)^{v_0}\int_{M_{R/2}}|f - f_0'|^2 w \\
&\quad\leq c(c_1, c_3)\sum_{A\in\mathscr{H}\cap\mathscr{L}}\left(|A|^{-1}\sum_{B\in A(\mathscr{L})}\sharp\mathscr{F}(B)|B|\right)J_A(t_A) \\
&\qquad + c(c_1, c_3)\left(\frac{D}{R}\right)^{v_0}\int_{M_{R/2}}|f - f_0'|^2 w \\
&\quad\leq c(c_1, c_3)\sum_{A\in\mathscr{H}\cap\mathscr{L}}\left(\log\left(\frac{R}{\rho(A)}\right)\right)J_A(t_A) + c(c_1, c_3)\left(\frac{D}{R}\right)^{v_0}\int_{M_{R/2}}|f - f_0'|^2 w \\
&\quad\leq c(c_1, c_3, c_5)\sum_{A\in\mathscr{H}\cap\mathscr{L}}\left\{\left(\frac{R}{\rho(A)}\right)^{\varepsilon_1}\varepsilon_1^{-1}\rho(A)^2(t_A)^2\int_{32A}|\nabla f|^2 w\right. \\
&\qquad\qquad\left. + \left(\frac{R}{\rho(A)}\right)^{\varepsilon_2}\varepsilon_2^{-1}(t_A)^{-v}\left(\int_{4A}w\right)^{-1}\left(\int_{4A}|f - f_{4A}'|w\right)^2\right\} \\
&\qquad + c(c_1, c_3)\left(\frac{D}{R}\right)^{v_0}\int_{M_{R/2}}|f - f_0'|^2 w \qquad (\forall t_A > 0,\ \forall\varepsilon_i > 0\ (i = 1, 2)).
\end{aligned}$$

For simplicity, we shall set

$$I_A = \left(\frac{R}{\rho(A)}\right)^{\varepsilon_1}\varepsilon_1^{-1}\rho(A)^2(t_A)^2; \quad II_A = \left(\frac{R}{\rho(A)}\right)^{\varepsilon_2}\varepsilon_2^{-1}(t_A)^{-v}\left(\int_{4A}w\right)^{-1/2}$$

and put $t_A = \rho(A)^{-\theta}|A|^{-\delta}s$, where s, θ and $\delta > 0$ are positive constants. Then, we have

$$\begin{aligned} I_A &= \varepsilon_1^{-1}R^{\varepsilon_1}\rho(A)^{2-\varepsilon_1-2\theta}|A|^{-2\delta}s^2 \\ &\leq \varepsilon_1^{-1}R^{\varepsilon_1}\rho(A)^{2-\varepsilon_1-2\theta-2\delta\nu}c_1^{2\delta}|M|^{-2\delta}D^{2\delta\nu}s^2, \end{aligned}$$

where we have used the inequality (17) and the assumption $\nu \geq \nu_0$. We also have

$$\begin{aligned} II_A &\leq c(c_1, c_3)\varepsilon_2^{-1}R^{\varepsilon_2}\rho(A)^{\theta\nu-\varepsilon_2}|A|^{\delta\nu-(1/2)}w(o_A)^{-1/2}s^{-\nu} \\ &\leq c(c_1, c_3, c_6)\varepsilon_2^{-1}R^{\varepsilon_2}\rho(A)^{\theta\nu-\varepsilon_2-\varphi}|A|^{\delta\nu-(1/2)}s^{-\nu}, \end{aligned}$$

where we have used Lemma 4.1 (2) and the assumption (w_2). For $\nu \geq \nu_0$ with $\nu > 2\varphi$, we set

$$\delta = \frac{1}{2\nu}; \quad \theta = \frac{1-\varepsilon_1}{2}; \quad \varepsilon_1 = \frac{\nu - 2\varphi}{2\nu}; \quad \varepsilon_2 = \frac{\nu - 2\varphi}{4}.$$

Then we obtain

$$\begin{aligned} &\int_{\bigcup_{B\in\mathscr{L}} 4B} |f - f_0'|^2 w \\ &\leq c(c_1, c_3, c_5, \nu)(\nu - 2\varphi)^{-1}R^{(\nu-2\varphi)/(2\nu)}|M|^{-1/\nu}Ds^2 \int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 32A} |\nabla f|^2 w \\ &\quad + c(c_1, c_3, c_5, c_6, \nu)(\nu - 2\varphi)^{-1}R^{(\nu-2\varphi)/4}s^{-\nu} \sum_{A\in\mathscr{H}\cap\mathscr{L}} \left(\int_{4A} w\right)^{-1/2}\left(\int_{4A} |f - f_{4A}'|w\right)^2 \\ &\quad + c(c_1, c_3)\left(\frac{D}{R}\right)^{\nu_0}\int_{M_{R/2}} |f - f_0'|^2 w. \end{aligned}$$

Since $\int_{4A} |f - f_{4A}'|w \leq 2\int_{4A} |f - \alpha|w$ for all $\alpha \in \boldsymbol{R}$, and since $\int_{4A} |f - f_0'|w \leq (\int_{4A} |f - f_0'|^2 w)^{1/2}(\int_{4A} w)^{1/2}$, we have

$$\begin{aligned} &\sum_{A\in\mathscr{H}\cap\mathscr{L}} \left(\int_{4A} w\right)^{-1/2}\left(\int_{4A} |f - f_{4A}'|w\right)^2 \\ &\qquad \leq 4\sum_{A\in\mathscr{H}\cap\mathscr{L}} \left(\int_{4A} w\right)^{-1/2}\left(\int_{4A} |f - f_0'|w\right)\left(\int_{4A} |f - f_M'|w\right) \\ &\qquad \leq 4\sum_{A\in\mathscr{H}\cap\mathscr{L}} \left(\int_{4A} |f - f_0'|^2 w\right)^{1/2}\left(\int_{4A} |f - f_M'|w\right) \\ &\qquad \leq c(c_1)\left(\int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 4A} |f - f_0'|^2 w\right)^{1/2}\int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 4A} |f - f_M'|w \\ &\qquad \leq c(c_1)\left\{\varepsilon\int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 4A} |f - f_0'|^2 w + \varepsilon^{-1}\left(\int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 4A} |f - f_M'|w\right)^2\right\} \end{aligned}$$

for all $\varepsilon > 0$. Hence

$$\begin{aligned}
&\int_{\bigcup_{B\in\mathscr{L}} 4B} |f - f_0'|^2 w \\
&\quad \le c(c_1, c_3, c_5, v)(v - 2\varphi)^{-1} R^{(v-2\varphi)/(2v)} |M|^{-1/v} Ds^2 \int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 32A} |\nabla f|^2 w \\
&\qquad + c(c_1, c_3, c_5, c_6, v)(v - 2\varphi)^{-1} R^{(v-2\varphi)/4} s^{-v} \\
&\qquad\quad \times \left\{ \varepsilon \int_{\bigcup_{A\in\mathscr{L}} 4A} |f - f_0'|^2 w + \varepsilon^{-1} \left(\int_{\bigcup_{A\in\mathscr{L}} 4A} |f - f_M'| w \right)^2 \right\} \\
&\qquad + c(c_1, c_3) \left(\frac{D}{R} \right)^{v_0} \int_{M_{R/2}} |f - f_0'|^2 w.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\{1 - c(c_1, c_3, c_5, c_6, v, \varphi) R^{(v-2\varphi)/4} s^{-v} \varepsilon\} \int_{\bigcup_{A\in\mathscr{L}} 4A} |f - f_0'|^2 w \\
&\quad \le c(c_1, c_3, c_5, v, \varphi) R^{(v-2\varphi)/(2v)} |M|^{-1/v} Ds^2 \int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 32A} |\nabla f|^2 w \\
&\qquad + c(c_1, c_3, c_5, c_6, v, \varphi) R^{(v-2\varphi)/4} s^{-v} \varepsilon^{-1} \left(\int_{\bigcup_{A\in\mathscr{L}} 4A} |f - f_M'| w \right)^2 \\
&\qquad + c(c_1, c_3) \left(\frac{D}{R} \right)^{v_0} \int_{M_{R/2}} |f - f_0'|^2 w.
\end{aligned}$$

In the above inequality, we shall choose $\varepsilon > 0$ so that

$$1/2 = c(c_1, c_3, c_5, c_6, v, \varphi) R^{(v-2\varphi)/4} s^{-v} \varepsilon.$$

Then, we have

$$\begin{aligned}
\text{(18)} \quad \int_M |f - f_0'|^2 w &\le \int_{\bigcup_{B\in\mathscr{L}} 4B} |f - f_0'|^2 w + \int_{M_{R/2}} |f - f_0'|^2 w \\
&\le c(c_1, c_3, c_5, v, \varphi) R^{(v-2\varphi)/(2v)} |M|^{-1/v} Ds^2 \int_{\bigcup_{A\in\mathscr{H}\cap\mathscr{L}} 32A} |\nabla f|^2 w \\
&\quad + c(c_1, c_3, c_5, c_6, v, \varphi) R^{(v-2\varphi)/2} s^{-2v} \left(\int_{\bigcup_{A\in\mathscr{L}} 4A} |f - f_M'| w \right)^2 \\
&\quad + \left(c(c_1, c_3) \left(\frac{D}{R} \right)^{v_0} + 1 \right) \int_{M_{R/2}} |f - f_0'|^2 w,
\end{aligned}$$

for all v satisfying $v \ge v_0$ and $v > 2\varphi$, and all $f \in C^\infty(\overline{M})$.

In the following, we shall set $\sigma = 2\nu$ in the inequality (18), and prove the Nash inequality (6) with ν replaced by σ. Now, we shall consider the following Nash inequality on $M_{R/2}$:

$$\left(\int_{M_{R/2}} |f - f_{M_{R/2}}|^2\right)^{1+(2/\sigma)} \leq N \int_{M_{R/2}} |\nabla f|^2 \cdot \left(\int_{M_{R/2}} |f - f_{M_{R/2}}|\right)^{4/\sigma},$$

$$N := c_N(M_{R/2}) \cdot |M_{R/2}|^{-2/\sigma} \cdot \operatorname{diam}(M_{R/2})^2, \quad \forall f \in C^\infty(\overline{M_{R/2}}).$$

Since the assumption (w_1) implies

$$w(x) \leq c(c_3, D/R) w(y) \quad \forall x, \forall y \in M_{R/2},$$

it is not hard to see that

$$\text{(19)} \qquad \int_{M_{R/2}} |f - f'_{M_{R/2}}|^2 w \leq c(c_3, D/R, \sigma) N^{\sigma/(\sigma+2)} \left(\int_{M_{R/2}} |\nabla f|^2 w\right)^{\sigma/(\sigma+2)} \times \left(\int_{M_{R/2}} |f - f'_{M_{R/2}}| w\right)^{4/(\sigma+2)} \left(\sup_{M_{R/2}} w\right)^{-2/(\sigma+2)}.$$

For $x_0 \in \partial M_{R/2}$, (w_1) and (w_2) imply

$$c_6 (R/2)^{2\varphi} \leq w(x_0) \leq c(c_3, D/R) w(y) \quad (\forall y \in M_{R/2}).$$

Hence, we have

$$\text{(20)} \qquad \left(\sup_{M_{R/2}} w\right)^{-2/(\sigma+2)} \leq c_6^{-2/(\sigma+2)} c(c_3, D/R, \sigma) R^{-4\varphi/(\sigma+2)}.$$

We also have

$$\text{(21)} \qquad \int_{M_{R/2}} |f - f'_{M_{R/2}}| w \leq 2 \int_{M_{R/2}} |f - f'_M| w.$$

From (19), (20) and (21), we obtain

$$\text{(22)} \qquad \int_{M_{R/2}} |f - f'_{M_{R/2}}|^2 w \leq c' \left(\int_{M_{R/2}} |\nabla f|^2 w\right)^{\sigma/(\sigma+2)} \left(\int_{M_{R/2}} |f - f'_M| w\right)^{4/(\sigma+2)},$$

$$c' = c(c_3, D/R, \sigma) c_6^{-2/(\sigma+2)} R^{-4\varphi/(\sigma+2)} N^{\sigma/(\sigma+2)}.$$

Optimizing over $s > 0$ in the inequality (18) with $v = \sigma/2$, and using (22), we obtain

$$\int_M |f - f'_{M_{R/2}}|^2 w \le c(c_1, c_3, c_5, c_6, \sigma, \varphi) R^{(\sigma-4\varphi)/(\sigma+2)} |M|^{-2/(\sigma+2)} D^{\sigma/(\sigma+2)}$$

$$\times \left(\int_{\bigcup_{A \in \mathscr{H} \cap \mathscr{L}} 32A} |\nabla f|^2 w \right)^{\sigma/(\sigma+2)} \left(\int_{\bigcup_{A \in \mathscr{L}} 4A} |f - f'_M| w \right)^{4/(\sigma+2)}$$

$$+ c(c_3, D/R, \sigma) c_6^{-2/(\sigma+2)} R^{-4\varphi/(\sigma+2)} N^{\sigma/(\sigma+2)} \left(\int_{M_{R/2}} |\nabla f|^2 w \right)^{\sigma/(\sigma+2)}$$

$$\times \left(\int_{M_{R/2}} |f - f'_M| w \right)^{4/(\sigma+2)}$$

$$\le \mathscr{C} \left(\int_M |\nabla f|^2 w \right)^{\sigma/(\sigma+2)} \left(\int_M |f - f'_M| w \right)^{4/(\sigma+2)},$$

where we set

$$\mathscr{C} = c(c_1, c_3, c_5, c_6, \sigma, \varphi) R^{(\sigma-4\varphi)/(\sigma+2)} |M|^{-2/(\sigma+2)} D^{\sigma/(\sigma+2)}$$
$$+ c(c_3, D/R, \sigma) c_6^{-2/(\sigma+2)} R^{-4\varphi/(\sigma+2)} N^{\sigma/(\sigma+2)}.$$

Therefore, for all $\sigma > 4\varphi$ with $\sigma \ge 2v_0$ and all $f \in C^\infty(\overline{M})$, we have

$$\left(\int_M |f - f'_M|^2 w \right)^{1+(2/\sigma)} \le \left(\int_M |f - f'_{M_{R/2}}|^2 w \right)^{1+(2/\sigma)}$$
$$\le \mathscr{C}^{1+(2/\sigma)} \left(\int_M |\nabla f|^2 w \right) \left(\int_M |f - f'_M| w \right)^{4/\sigma}.$$

Here, in general, since $(A+B)^x \le 2^x(A^x + B^x)$ for positive constants x, A and B, we have

$$\mathscr{C}^{1+(2/\sigma)} \le c(c_1, c_3, c_5, c_6, \sigma, \varphi) R^{(\sigma-4\varphi)/\sigma} |M|^{-2/\sigma} D$$
$$+ c(c_3, D/R, \sigma) c_6^{-2/\sigma} R^{-4\varphi/\sigma} c_N(M_{R/2}) |M_{R/2}|^{-2/\sigma} \operatorname{diam}(M_{R/2})^2$$

Thus, we have the Nash inequality (6) with v replaced by σ. □

Theorem 4.1 follows from Lemma 3.1 and 4.2.

Remark 4.4. We note that the dumbbell-like example in R. Chen [4] shows that the interior rolling R-ball condition is necessary for the 'Nash constant' to be bounded from above.

5. An application

In this section, as an application of Theorem 4.1, we shall derive the Sobolev inequality with respect to the measure $(u_1)^2\, dv_M$, where u_1 is the normalized first Dirichlet eigenfunction.

Let M be a compact, connected Riemannian manifold with C^∞ boundary ∂M. Denote by v_M the Riemannian measure of M. We shall consider the following Dirichlet eigenvalue problem:

$$\begin{cases} \Delta u = -\lambda u \\ u|_{\partial M} \equiv 0. \end{cases}$$

Let $\{u_i\}_{i=1}^{\infty}$ be a complete orthonormal system of $L^2(M, v_M)$ consisting of Dirichlet eigenfunctions with u_i having eigenvalue $-\lambda_i$. We shall take the sign of u_1 to be positive: $u_1 > 0$ on Int M. When we set $\phi_i = u_i/u_1$, a direct computation shows that ϕ_i satisfies the following equation:

$$\Delta\phi_i + 2\nabla \log u_1 \cdot \nabla\phi_i + (\lambda_i - \lambda_1)\phi_i = 0.$$

Moreover, it is known that

$$\phi_i \in C^\infty(\overline{M}), \quad \frac{\partial\phi_i}{\partial\vec{n}} = 0,$$

where $\vec{n}$ stands for the outward unit normal vector field on ∂M (for this result, see [I. Singer-B. Wong-S. T. Yau-S. S. T. Yau [15]]). We shall define the closed form $\mathscr{E}_{u_1}$ on the Hilbert space $L^2(M, (u_1)^2 v_M)$ in the following way:

$$\mathscr{E}_{u_1}(f, g) = \int_M \langle\nabla f, \nabla g\rangle (u_1)^2\, dv_M,$$
$$f, g \in \mathrm{Dom}(\mathscr{E}_{u_1}) = H^1(M, (u_1)^2 v_M),$$

where $H^1(M, (u_1)^2 v_M)$ is the Hilbert space constructed by the completion of the space $C^\infty(\overline{M})$ with respect to the norm $\|f\|^2 := \mathscr{E}_{u_1}(f,f) + \int_M f^2 (u_1)^2\, dv_M$. Let A_{u_1} be the nonpositive self-adjoint operator on $L^2(M, (u_1)^2 v_M)$ associated with $\mathscr{E}_{u_1}$. Then, we see that $\{\phi_i\}_{i=1}^{\infty}$ is a complete orthonormal system of $L^2(M, (u_1)^2 v_M)$ consisting of eigenfunctions of A_{u_1} with ϕ_i having eigenvalue $-(\lambda_i - \lambda_1)$. Setting $w = (u_1)^2$ and applying Theorem 1.3 in [13] and Theorem 3.7 in [J. Wang [17]], we have the following

Proposition 5.1. *We assume that* Int $M = M - \partial M$ *is a relatively compact domain in a complete Riemannian manifold L with its Ricci curvature* Ric_L *bounded from below by $-K$ for some constant $K \geq 0$. Moreover, we suppose that M satisfies the interior rolling R-ball condition (R) and that the second fundamental form $\Pi_{\partial M}$ of ∂M with respect to the outward unit normal $\vec{n}$ is bounded from above by H for some constant $H \geq a' > 0$, that is,* $\Pi_{\partial M}(X, X) := -\langle\nabla_X(\partial/\partial\vec{n}), X\rangle \leq$

$H|X|^2$, where $-(a')^2$ is a lower bound of the sectional curvature on $M - M_{R/2}$. Then, we have, for any v satisfying $v \geq 2v_0$ and $v > 4$, there exists a constant $c(v, K, H, R, D, \dim M, \mathrm{Vol}(M))$ such that the following Sobolev inequality holds:

$$\inf_{\alpha \in \boldsymbol{R}} \left(\int_M |f - \alpha|^{2v/(v-2)} (u_1)^2 \, dv_M \right)^{(v-2)/v} \leq c' \int_M |\nabla f|^2 (u_1)^2 \, dv_M, \quad f \in C^\infty(\overline{M}), \tag{23}$$

where D stands for the intrinsic diameter of M, $v_0 = v_0(K, R) = (\log c_1)/\log 2$ and c_1 is a volume doubling constant on $M - M_R$ as before.

Proof. The property (P) is satisfied by the fact $\mathrm{Ric}_M \geq -K$ (see example 1 in [14]). In the similar way to the one in [13], we can prove that for $w = (u_1)^2$, M satisfies the conditions (w_1) and (w_2) with $\varphi = 1$, $c_3 = c(K, H, R, D, \dim M, \mathrm{Vol}\,(M))$ and $c_6 = c(K, H, R, D, \dim M, \mathrm{Vol}\,(M))$. We note that, by the comparison theorem and by the argument about the interval of the existance of solution, we can see that the second fundamental form of $M_{R/2}$ satisfies $-c(R, H, \dim M) \leq \Pi_{\partial M_{R/2}} \leq \max\{H, a'\} = H$. Therefore, we also have

$$c_N(M_{R/2})|M_{R/2}|^{-2/v} \operatorname{diam}(M_{R/2}) \leq c(v, \dim M, H, R, K, |M_{R/2}|, \operatorname{diam}(M_{R/2})),$$

by Theorem 3.7 in [17] and by [4], where $\operatorname{diam}(M_{R/2})$ is the intrinsic diameter of $M_{R/2}$. Hence, Proposition 5.1 easily follows from Theorem 4.1. □

Corollary 2.1 immediately implies the following

COROLLARY 5.1. *The inequality* (23) *implies the following estimate*:

$$\alpha^{2/v}(cv)^{-1} \leq \lambda_2 - \lambda_1.$$

Moreover, (23) *implies the decay of eigenfunctions*:

$$u_i(x) \leq \alpha(vc(\lambda_i - \lambda_1))^{v/4} u_1(x) \quad (i = 2, 3, \ldots),$$

where α is some numerical constant.

Remark 5.1. Actually, the optimal value of the constant v in Proposition 5.1 is $\dim M + 2$. This fact and other results will be proved in H. Kumura [12].

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