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A CLASSIFICATION OF POLARIZED SURFACES (X, L) WITH $\kappa(X) \geq 0$, $\dim \mathrm{Bs}|L| \leq 0$, $g(L) = q(X) + m$, AND $h^0(L) = m + 1$

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Abstract

Let (X, L) be a polarized surface and $\dim \mathrm{Bs}|L| \leq 0$. In our previous paper we have studied polarized surfaces with $g(L) = q(X) + m$ and $h^0(L) \geq m + 2$. In this paper, we classify (X, L) with $\kappa(X) \geq 0$, $g(L) = q(X) + m$ and $h^0(L) = m + 1$.

0. Introduction

Let X be a smooth projective variety over the complex number field $\boldsymbol{C}$ with $\dim X = n$, and let L be an ample (resp. a nef and big) line bundle on X. Then we call the pair (X, L) a polarized (resp. quasi-polarized) manifold. The sectional genus $g(L)$ of (X, L) is defined as follows:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical line bundle of X. A classification of (X, L) with small value of sectional genus was obtained by several authors. On the other hand, Fujita proved the following Theorem (see Theorem (II.13.1) in [Fj3]).

Theorem. *Let (X, L) be a polarized manifold. Then for any fixed $g(L)$ and $n = \dim X$, there are only finitely many deformation type of (X, L) unless (X, L) is a scroll over a smooth curve.*

(For a definition of the deformation type of (X, L), see §13 of Chapter II in [Fj3].) By this theorem, Fujita proposed the following Conjecture;

Conjecture (Fujita). *Let (X, L) be a polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = h^1(\mathcal{O}_X)$: the irregularity of X.*

This Conjecture is very difficult and it is unknown even for the case in which X is a surface.

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If $\dim \mathrm{Bs}|L| \le 0$, then we can prove that $g(L) \ge q(X)$ (see Theorem 3.2 in [Fk3]). Furthermore the author proved that if (X, L) is a quasi-polarized manifold with $\dim X = 3$ and $h^0(L) := \dim H^0(L) \ge 2$, then $g(L) \ge q(X)$ (see [Fk5]). Moreover the author obtained the classification of polarized 3-folds (X, L) with the following types;

(1) $g(L) = q(X)$ and $h^0(L) \ge 3$ ([Fk5]),

(2) $g(L) = q(X) + 1$ and $h^0(L) \ge 4$ ([Fk2]),

(3) $g(L) = q(X) + 2$ and $h^0(L) \ge 5$ ([Fk6]).

By considering the result of 3-dimensional case, it is natural to consider the following problem;

Problem. Let (X, L) be a polarized manifold with $\dim X = n$ and $g(L) = q(X) + m$, where m is a nonnegative integer. Assume that $h^0(L) \ge n + m$. Then classify (X, L) with these properties.

In [Fk7], we get a classification of polarized manifolds (X, L) with $n := \dim X \ge 3$, $g(L) = q(X) + m$, $\dim \mathrm{Bs}|L| \le 0$, and $h^0(L) \ge m + n$.

In [Fk9], we studied polarized surfaces (X, L) with $n = 2$, $g(L) = q(X) + m$ and $h^0(L) \ge m + 2$.

Here we remark that if $n \ge 3$, then we can use the adjunction theory for $K_X + (n-2)L$. But if $n = 2$, then we cannot use the theory, so we need to study (X, L) by the value of Kodaira dimension.

In this paper, we consider the case in which $n = 2$, $g(L) = q(X) + m$, $\dim \mathrm{Bs}|L| \le 0$, and $h^0(L) = m + 1$. In particular we study the case where $\kappa(X) \ge 0$. By using this result we get a classification of polarized manifolds (X, L) with $n = \dim X \ge 3$, $g(L) = q(X) + m$, $\mathrm{Bs}|L| = \emptyset$, and $h^0(L) = m + n - 1$. We will study this in a forthcoming paper [Fk10].

We use the customary notation in algebraic geometry.

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1. Preliminaries

Theorem 1.1. *Let (X, L) be a polarized manifold with $n = \dim X \ge 2$. Assume that $|L|$ has a ladder and $g(L) \ge \Delta(L)$, where $\Delta(L)$ is the delta genus of (X, L).*

(1) *If $L^n \ge 2\Delta(L) + 1$, then $g(L) = \Delta(L)$ and $q(X) = 0$.*

(2) *If $L^n \ge 2\Delta(L)$, then $\mathrm{Bs}|L| = \emptyset$.*

(3) *If $L^n \ge 2\Delta(L) - 1$, then $|L|$ has a regular ladder.*

Proof. See (I.3.5) in [Fj3]. □

Theorem 1.2. *Let (X, L) be a polarized manifold with $n = \dim X \ge 2$. If $\dim \mathrm{Bs}|L| \le 0$ and $L^n \ge 2\Delta(L) - 1$, then $|L|$ has a ladder.*

Proof. See (I.4.15) in [Fj3]. □

Definition 1.3 (See Definition 1.1 in [Fj1]). Let (X, L) be a polarized surface. Then (X, L) is called a hyperelliptic polarized surface if $\mathrm{Bs}|L| = \emptyset$, the morphism defined by $|L|$ is of degree two onto its image W, and if $\Delta(W, H) = 0$ for the hyperplane section H on W.

Theorem 1.4. *Let (X, L) be a polarized manifold with* $\dim X = n$ *such that* $\mathrm{Bs}|L| = \emptyset$, $L^n = 2\Delta(L)$, *and* $g(L) > \Delta(L)$. *Then (X, L) is hyperelliptic unless L is simply generated and (X, L) is a Fano-K3 variety.*

Proof. See Theorem 1.4 in [Fj1]. □

Theorem 1.5. *Let (X, L) be a hyperelliptic polarized surface. Then (X, L) is one of the following types;*

Type	L^2	$g(L)$	$q(X)$
(I_a)	2	a	0
(IV_a)	8	$2a+1$	0
$({}^*II_a)$	4	$2a$	0
$(\sum(\delta_1, \delta_2)^+_{a,b})$	$2\lvert\delta\rvert$	$a\lvert\delta\rvert + b - 1$	0
$(\sum(\delta_1, \delta_2)^0_b)$	$2\lvert\delta\rvert$	$b-1$	$b-1$
$(\sum(\mu, \mu)^=_a)$	4μ	$a\mu - 1$	$a-1$
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$4(\mu+\gamma)$	$a\mu + 2a\gamma - \gamma - 1$	0

Furthermore the Kodaira dimension of X is the following

Value of $\kappa(X)$	2	1
(I_a)	$a > 2$	–
(IV_a)	$a > 2$	–
$({}^*II_a)$	$a > 1$	–
$(\sum(\delta_1, \delta_2)^+_{a,b})$	case (5)	case (4)
$(\sum(\delta_1, \delta_2)^0_b)$	–	–
$(\sum(\mu, \mu)^=_a)$	–	–
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$a > 2$	$a = 2$ *and* $\gamma > 2$

Value of $\kappa(X)$	0	$-\infty$
(I_a)	$a = 2$	$a < 2$
(IV_a)	$a = 2$	–
$({}^*II_a)$	–	$a = 1$
$(\sum(\delta_1, \delta_2)^+_{a,b})$	case (3) and (6a)	case (1) and (2)
$(\sum(\delta_1, \delta_2)^0_b)$	–	*any* b
$(\sum(\mu, \mu)^=_a)$	–	*any* a
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$a = \gamma = 2$	$a = 2$ *and* $\gamma = 1$

For the definition of the above types, see [Fj1]. *In particular for the cases of the type* $(\sum(\delta_1, \delta_2)^+_{a,b})$, *see* (5.20) *in* [Fj1].

Proof. See [Fj1]. (Here we remark that the case (6b) of type $(\sum(\delta_1, \delta_2)^+_{a,b})$ is impossible because $\dim X = 2$.) □

DEFINITION 1.6 (See Definition 1.9 in [Fk1]). (1) Let (X, L) be a quasi-polarized surface. Then (X, L) is called L-minimal if $LE > 0$ for any (-1)-curve E on X.

(2) Let (X, L) and (Y, A) be quasi-polarized surfaces. Then (Y, A) is called an L-minimalization of (X, L) if there exists a birational morphism $\mu : X \to Y$ such that $L = \mu^*(A)$ and (Y, A) is A-minimal. (We remark that an L-minimalization of (X, L) always exists.)

(3) Let (X, L) and (X', L') be polarized surfaces. Then (X, L) is called a simple blowing up of (X', L') if X is a blowing up of X' at $x \in X'$ and $(E, L_E) \cong (\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(1))$ for the exceptional divisor E.

Remark 1.6.1. Let X be a smooth projective surface and let L be an ample line bundle on X. Then (X, L) is L-minimal.

THEOREM 1.7. *Let (X, L) be a quasi-polarized surface with $h^0(L) \geq 2$ and $\kappa(X) = 2$. Assume that $g(L) = q(X) + m$ for $m \geq 0$. Then $L^2 \leq 2m$. Moreover if $L^2 = 2m$ and (X, L) is L-minimal, then $X \cong C_1 \times C_2$ and $L \equiv C_1 + 2C_2$, where C_1 and C_2 are smooth curves with $g(C_1) \geq 2$ and $g(C_2) = 2$. (Here $\equiv$ denotes the numerical equivalence of divisors.)*

Proof. See Theorem 3.1 in [Fk4]. □

Remark 1.7.1. Let (X, L) be as in Theorem 1.7. Then $L^2 \leq 2m$ is equivalent to $K_X L \geq 2q(X) - 2$.

THEOREM 1.8. *Let (X, L) be a quasi-polarized surface with $\kappa(X) = 0$ or 1. Assume that $g(L) = q(X) + m$.*

(1) *$L^2 \leq 2m + 2$ holds.*

(2) *If $L^2 = 2m + 2$ and (X, L) is L-minimal, then (X, L) is one of the following;*

(2-1) *$\kappa(X) = 0$ case.*
X is an Abelian surface and L is any nef and big divisor.

(2-2) *$\kappa(X) = 1$ case.*
$X \cong F \times C$ and $L \equiv C + (m+1)F$, where F and C are smooth curves with $g(C) \geq 2$ and $g(F) = 1$. If $h^0(L) > 0$, then $L = C + \sum_{x \in I} m_x F_x$, where F_x is a fiber of the second projection over $x \in C$, I is a set of a finite point of C, and m_x is a positive integer with $\sum_{x \in I} m_x = m + 1$. (Here $D_1 = D_2$ denotes $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ for two divisors D_1 and D_2.)

(3) *If (X, L) is a polarized surface with $\kappa(X) = 1$ and $L^2 \leq 2m + 1$, then $L^2 \leq 2m$.*

Proof. For the proof of (1), (2-1), and (2-2), see Theorem 2.1 in [Fk4]. Next we consider the case (3). Let $\pi : X \to C$ be an elliptic fibration over a smooth curve C. Assume that $L^2 = 2m + 1$.

If $g(C) = 0$, then $q(X) \leq 1$ and $g(L) \leq m + 1$. But since L is ample and $\kappa(X) = 1$, we get that $K_X L \geq 1$ and $g(L) \geq m + 2$. This is impossible. So we may assume that $g(C) \geq 1$.

Let $\mu : X \to S$ be a relative minimalization of $f : X \to C$ and let $A := \mu_*(L)$. Then A is ample. Let $h : S \to C$ be an elliptic fibration such that $f = h \circ \mu$.

(A) The case in which $g(C) = 1$.

If $q(X) = g(C) = 1$, then this is impossible by the same argument as above.

If $q(X) = g(C) + 1 = 2$, then, by the canonical bundle formula, h has at least two multiple fibers since $\kappa(X) = 1$. So we get that $K_X L \geq K_S A \geq 2$. Hence $g(L) > m + 2$ and this is also impossible.

(B) The case in which $g(C) \geq 2$.

If $q(X) = g(C)$, then $K_X L \geq K_S A \geq 4g(C) - 4 = 4q(X) - 4$. Hence

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(4q(X) - 4 + L^2) \\ &= 1 + \frac{1}{2}(4q(X) - 4 + 2m + 1) \\ &= 1 + 2q(X) + m - \frac{3}{2} \\ &= q(X) + m - \frac{1}{2} + q(X) \\ &\geq q(X) + m + \frac{3}{2} \end{aligned}$$

and this is also impossible.

So we assume that $q(X) = g(C) + 1$.

If $LF \geq 2$, then we get that

$$\begin{aligned} K_X L \geq K_S A &\geq (2g(C) - 2)LF \\ &\geq 4g(C) - 4 \\ &= 2g(C) + 2 + 2g(C) - 6 \\ &= 2q(X) + 2g(C) - 6. \end{aligned}$$

Hence

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(2q(X) + 2g(C) - 6 + 2m + 1) \\ &= 1 + q(X) + g(C) - 3 + m + \frac{1}{2} \\ &= q(X) + g(C) - 2 + m + \frac{1}{2} \\ &> q(X) + m \end{aligned}$$

and this is impossible. Hence we may assume that $LF = 1$. In particular $\mu = \mathrm{id}$, and f has no multiple fiber because L is ample. Hence $K_X L =$

$2g(C)-2$. But this is impossible because L^2 is odd. This completes the proof of Theorem 1.8. □

Remark 1.8.1. Let (X, L) be as in Theorem 1.8. Then $L^2 \leq 2m+2$ is equivalent to $K_X L \geq 2q(X)-4$.

Proposition 1.9. *Let X be a smooth projective surface of general type. Then $p_g(X) \geq 2q(X)-4$. If this equality holds and X is minimal, then $X \cong C_1 \times C_2$ for smooth projective curves C_1 and C_2, where $p_g(X) = h^0(K_X)$ and $q(X) = h^1(\mathcal{O}_X)$.*

Proof. See Théorème in [Bea]. □

Proposition 1.10. *Let X be a smooth projective surface of general type such that X is minimal. Assume that $q(X) \geq 1$. Then $K_X^2 \geq 2p_g(X)$.*

Proof. See Théorème 6.1 and Addendum in [De]. □

Theorem 1.11. *Let (X, L) be a quasi-polarized surface with $\kappa(X) \geq 0$. Assume that* $\dim \mathrm{Bs}|L| \leq 0$. *Then $g(L) \geq 2q(X)-1$.*

Proof. See Corollary 3.2 in [Fk0]. □

2. Main Theorem

Theorem 2.1. *Let (X, L) be a polarized surface such that* $\dim \mathrm{Bs}|L| \leq 0$, *$h^0(L) = m+1$, and $\kappa(X) \geq 0$, where $m = g(L) - q(X)$. Assume that $m \geq 1$. Then (X, L) is one of the following types;*

(M-1) *(X, L) is a minimal surface of general type with $L^2 = 1$, $g(L) = 3$, and $q(X) = 2$.*

(M-2) *$\pi : X \to C$ is a minimal elliptic fibration over a smooth curve C and (X, L) is one of the following;*

(M-2-1) *$3 = q(X) = g(C) + 1$, $\chi(\mathcal{O}_X) = 0$, $LF = 2$, and π has no multiple fiber. In this case X is a double covering of $\boldsymbol{P}^1$-bundle on C.*

(M-2-2) *π has just 2 multiple fibers $2F_1$ and $2F_2$, $\chi(\mathcal{O}_X) = 0$, $2 = q(X) = g(C) + 1$, $K_X \equiv F_1 + F_2$, $LF = 2$ for a general fiber F.*

(M-2-3) *$\chi(\mathcal{O}_X) = 0$, $q(X) = g(C)$, and π has just one multiple fiber with $m_i = 2$ and $LF_i = 1$.*

(M-2-4) *$\chi(\mathcal{O}_X) = 0$, $q(X) = g(C) + 1 = 1$, $K_X L = 1$ and π has four multiple fibers m_1F_1, m_2F_2, m_3F_3, and m_4F_4 with one of the following (here we assume that $LF_4 \geq LF_3 \geq LF_2 \geq LF_1$);*

m_1	m_2	m_3	m_4	LF_1	LF_2	LF_3	LF_4
3	2	2	2	2	3	3	3
4	2	2	2	1	2	2	2

(M-2-5) $\chi(\mathcal{O}_X) = 0$, $q(X) = g(C) + 1 = 1$, $K_X L = 1$ *and* π *has three multiple fibers and one of the following lists* (*here we assume that* $LF_3 \geq LF_2 \geq LF_1$);

m_1	m_2	m_3	LF_1	LF_2	LF_3
4	4	4	1	1	1
4	3	3	3	4	4
6	3	3	1	2	2
7	3	2	6	14	21
8	3	2	3	8	12
9	3	2	2	6	9
12	3	2	1	4	6
5	4	2	4	5	10
6	4	2	2	3	6
8	4	2	1	2	4
6	6	2	1	1	3
5	5	2	2	2	5

(M-2-6) $\chi(\mathcal{O}_X) = 0$, $q(X) = g(C) + 1$, $g(C) = 1$ (*resp.* 0), $LF = 2$ *and the number of its multiple fiber is three* (*resp. five*).

(M-3-1) (X, L) *is the type* (I_a) *in Theorem* 1.5 *with* $a = m = 2$ *and* $\kappa(X) = 0$.

(M-3-2) (X, L) *is the type* (IV_a) *in Theorem* 1.5 *with* $a = 2$, $m = 5$, *and* $\kappa(X) = 0$.

(M-3-3) (X, L) *is the type* $(\sum(\delta_1, \delta_2)^+_{a,b})$ *in Theorem* 1.5, *and case* (3) *or case* (6a) *in* (5.20) *of* [Fj1]. *In this case* $\kappa(X) = 0$.

(M-3-4) (X, L) *is the type* $(\sum(\mu + 2\gamma, \mu)^-_a)$ *in Theorem* 1.5 *with* $a = \gamma = 2$, $m = 2\mu + 5$, *and* $\kappa(X) = 0$.

(M-3-5) X *is a K*3*-surface with* $q(X) = 0$ *and* $L^2 = 2m - 2$.

(M-3-6) (X, L) *is a polarized abelian surface such that* (X, L) *is not isomorphic to the following type*: $X \cong E_1 \times E_2$ *and* $L = p_1^* L_1 + p_2^* L_2$, *where* E_i *is an elliptic curve and* L_i *is a line bundle on* E_i *with* $\deg L_1 = 1$ *and* $\deg L_2 \geq 1$.

(N) *Let* $X = X_0 \to X_1 \to \cdots \to X_l = X'$ *be the minimal model of* X. *We put* $L_0 := L$, $\mu_i : X_{i-1} \to X_i$, *and* $L_i := (\mu_i)_*(L_{i-1})$. *Then* $L_{i-1} = \mu_i^* L_i - \alpha_i E_i$ *and* $\alpha_i > 0$ *for any* i, *where* E_i *is a* (-1)*-curve of* μ_i. *We put* $L' := L_l$.

(N-1) (X, L) *is a simple blowing up of* (X', L') *and* X' *has a minimal elliptic fibration* $\pi' : X' \to C$ *over a smooth curve* C *such that* (X', L') *is one of the following*;

(N-1-1) $g(C) = 2$, $q(X') = 3$, $\chi(\mathcal{O}_{X'}) = 0$, $L'F' = 2$ *and* π' *has no multiple fibers, where* F' *is a general fiber of* π',

(N-1-2) π' *has just two multiple fibers*, $2F_1$ *and* $2F_2$, $\chi(\mathcal{O}_{X'}) = 0$, $g(C) = 1$, $q(X') = 2$, *and* $L'F' = 2$.

(N-2) (X', L') *is a polarized abelian surface and* $\sum_i \alpha_i \leq 3$.

Proof. Assume that $L^2 \leq 2m - 2$. Here we put $t = 2m - 2 - L^2$. In this case, we calculate the delta genus $\Delta(L)$;

$$\begin{aligned}\Delta(L) &= 2 + L^2 - h^0(L)\\ &= 1 + L^2 - m\\ &= \frac{1}{2}L^2 - \frac{1}{2}t\\ &\leq \frac{1}{2}L^2.\end{aligned}$$

Hence $L^2 \geq 2\Delta(L)$. So we can use the result of Fujita. Since $\dim \mathrm{Bs}|L| \leq 0$ and

$$\begin{aligned}g(L) &= 1 + \frac{1}{2}(K_X + L)L\\ &> \frac{1}{2}L^2\\ &= \Delta(L),\end{aligned}$$

we get that $|L|$ has a ladder and $\mathrm{Bs}|L| = \emptyset$ by Theorem 1.1 and 1.2.

If $L^2 \geq 2\Delta(L) + 1$, then $q(X) = 0$ and $g(L) = \Delta(L) = m$ by Theorem 1.1. Therefore $L^2 \geq 2\Delta(L) + 1 = 2g(L) + 1 = 3 + (K_X + L)L \geq 3 + L^2$ and this is impossible. So we get that $L^2 = 2\Delta(L)$, and if X is not K3-surface, then (X, L) is a hyperelliptic polarized surface by Theorem 1.4. Since $h^0(L) = m + 1$, we obtain that

$$\begin{aligned}L^2 &= 2\Delta(L)\\ &= 4 + 2L^2 - 2(m + 1).\end{aligned}$$

That is, $L^2 = 2m - 2$. Here we use Fujita's classification of hyperelliptic polarized surfaces. Since $\kappa(X) \geq 0$, by Theorem 1.5 we find that $q(X) = 0$ and since $L^2 = 2m - 2$ and $g(L) = m$, we get that $K_X L = 0$. Since L is ample, we have $\kappa(X) = 0$. Hence (X, L) is one of the following:

If (X, L) is the type (I_a), then $a = m = 2$ and $\kappa(X) = 0$. (This is the type (M-3-1) in Theorem 2.1.)

If (X, L) is the type (IV_a), then $a = 2$, $m = 5$, and $\kappa(X) = 0$. (This is the type (M-3-2) in Theorem 2.1.)

If (X, L) is the type $(\sum^n(\delta_1, \delta_2)^+_{a,b})$, then the case (3) or the case (6a) in (5.20) in [Fj1] occur. (This is the type (M-3-3) in Theorem 2.1.)

If (X, L) is the type $(\sum(\mu + 2\gamma, \mu)^-_a)$, then $a = \gamma = 2$ and $m = 2\mu + 5$. (This is the type (M-3-4) in Theorem 2.1.)

If X is a K3-surface, then $q(X) = 0$ and $L^2 = 2m - 2$. By Riemann-Roch Theorem and Vanishing Theorem, we get that $h^0(L) = m + 1$. (This is the type (M-3-5) in Theorem 2.1.)

From now on we assume that $L^2 \geq 2m - 1$.

(A) The case in which X is minimal.

Here we divide the case (A) into the following:

(A.1) The case in which $\kappa(X) = 2$.
(A.2) The case in which $\kappa(X) = 1$.
(A.3) The case in which $\kappa(X) = 0$.

(A.1) The case in which $\kappa(X) = 2$.

Then $L^2 \leq 2m$ by Theorem 1.7. If $L^2 = 2m$, then $X \cong C_1 \times C_2$ and $L \equiv C_1 + 2C_2$, where C_1 (resp. C_2) is a smooth projective curve with $g(C_1) \geq 2$ (resp. $g(C_2) = 2$). But this is impossible. Actually since $\dim \mathrm{Bs}|L| \leq 0$, we get that for a general fiber C_2 of the projection $C_1 \times C_2 \to C_1$ $\mathrm{Bs}|L_{C_2}| = \emptyset$. But since $LC_2 = 1$ we get that $g(C_2) = 0$ and this is impossible. Hence we may assume that $L^2 \leq 2m - 1$. By the above hypothesis we may assume that $L^2 = 2m - 1$. Here we use a Beauville's result. Since X is minimal with $\kappa(X) = 2$, we get that $p_g(X) \geq 2q(X) - 3$ unless $X \cong C_1 \times C_2$. But if $X \cong C_1 \times C_2$, then $K_X L$ is even and here since we assume that $L^2 = 2m - 1$, we obtain that $K_X L = 2q(X) - 1$ is odd. So this is impossible.

If $q(X) = 0$, then $K_X L = 2q(X) - 1 = -1$ and this is impossible. Hence $q(X) \geq 1$. If $q(X) = 1$, then $K_X L = 1$ and $L^2 = 2m - 1$. Here we remark that $p_g(X) \geq q(X)$ because X is of general type. By Proposition 1.10, we get that $(K_X^2) \geq 2p_g(X) \geq 2q(X)$ and

$$1 = (K_X L)^2 \geq (K_X^2)(L^2) \geq 2L^2$$

and this is impossible.

So we may assume that $q(X) \geq 2$. By Proposition 1.10, we get that

$$K_X^2 \geq 2p_g(X) \geq 2(2q(X) - 3) = 4q(X) - 6.$$

By Hodge index Theorem, we obtain that

$$(*) \qquad \begin{aligned} (K_X L)^2 &\geq (K_X^2)(L^2) \\ &\geq (4q(X) - 6)(2m - 1) \\ &\geq 2(2q(X) - 3)(2q(X) - 3) \end{aligned}$$

because by Theorem 1.11

$$q(X) + m = g(L) \geq 2q(X) - 1.$$

Hence $K_X L \geq \sqrt{2}(2q(X) - 3)$. On the other hand $K_X L = 2q(X) - 1$. Therefore $2q(X) - 1 = K_X L \geq \sqrt{2}(2q(X) - 3)$ and we infer that $(2\sqrt{2} - 2)q(X) \leq 3\sqrt{2} - 1$. So we obtain that

$$q(X) \leq \frac{3\sqrt{2} - 1}{2\sqrt{2} - 2} = 3.914\cdots$$

Thus we have $q(X) \leq 3$.

If $q(X) = 3$ (resp. $q(X) = 2$), then $K_X L = 5$ (resp. 3) and by using $(*)$, we get the following list:

(A.1.α) $q(X) = 3$, $K_X L = 5$, $m \leq 2$, and $g(L) \leq 5$,

(A.1.β) $q(X) = 2$, $K_X L = 3$, $m \leq 2$, and $g(L) \leq 4$.

Here we remark that if $m = 2$, then $L^2 = 2m - 1 = 3$, $h^0(L) = m + 1 = 3$. Hence $\Delta(L) = 2$, that is, $L^2 = 2\Delta(L) - 1$.

(A.1.α.1) Assume that (X, L) is the case (A.1.α) with $q(X) = 3$, $K_X L = 5$ and $m = 1$. Then $4 = g(L) \geq 2q(X) - 1 = 5$ and this is impossible.

(A.1.α.2) Assume that (X, L) is the case (A.1.α) with $q(X) = 3$, $K_X L = 5$ and $m = 2$. Then $L^2 = 3$ and $g(L) = 5$. Since $h^0(L) = 3$, we have $L^2 = 2\Delta(L) - 1$. If $\dim \mathrm{Bs}|L| = 0$, then $q(X) = 0$ by Fujita's classification of (X, L) with $\Delta(L) = 2$. (See [Fj2].) So we may assume that $\mathrm{Bs}|L| = \emptyset$. Then there exists a triple covering $\pi : S \to \boldsymbol{P}^2$. Then by Lemma 3.2 in [Bes], we get that

$$\chi(\mathcal{O}_X) = \frac{g(g+1)}{2} + 2 - c_2$$

and

$$K_X^2 = 2g^2 - 4g + 11 - 3c_2,$$

where c_2 is the second Chern class of the Tschirnhausen bundle of π (see [Bes]). Since $g(L) = 5$, we get that

$$1 - 3 + p_g(X) = \frac{5(5+1)}{2} + 2 - c_2 = 17 - c_2$$

and

$$K_X^2 = 50 - 20 + 11 - 3c_2 = 41 - 3c_2.$$

Therefore $c_2 = 19 - p_g(X)$ and

$$\begin{aligned} K_X^2 &= 41 - 3(19 - p_g(X)) \\ &= 3p_g(X) - 16. \end{aligned}$$

On the other hand since $K_X^2 \geq 2p_g(X) \geq 2q(X) = 6$, we get that $6 \leq K_X^2 = 3p_g(X) - 16$. Hence $p_g(X) \geq 8$. In particular $K_X^2 \geq 2p_g(X) \geq 16$. Since $L^2 = 3$, we get that

$$\begin{aligned} (K_X L)^2 &\geq (K_X^2)(L^2) \\ &\geq 48. \end{aligned}$$

But this is a contradiction because $K_X L = 5$. So this case cannot occur.

(A.1.β.1) Assume that (X, L) is the case (A.1.β) with $q(X) = 2$, $K_X L = 3$ and $m = 1$. Then $g(L) = 3$, $h^0(L) = m + 1 = 2$ and $L^2 = 2m - 1 = 1$. (This is the type (M-1) in Theorem 2.1.)

(A.1.β.2) Assume that (X, L) is the case (A.1.β) with $q(X) = 2$, $K_X L = 3$ and $m = 2$. Then $q(X) = 2$ and $K_X^2 \geq 2p_g(X) \geq 2q(X) = 4$. Since $L^2 = 3$, we get that $(K_X L)^2 \geq (K_X^2)(L^2) \geq 12$. But since $K_X L = 3$, this is a contradiction.

(A.2) The case in which $\kappa(X) = 1$.

Then there exists an elliptic fibration over a smooth curve C; $\pi : X \to C$. The canonical bundle formula of π is the following:

$$K_X \equiv (2g(C) - 2 + \chi(\mathcal{O}_X))F + \sum_i (m_i - 1)F_i,$$

where F is a general fiber of π and $m_i F_i$ is a multiple fiber of π.

If $L^2 \geq 2m + 1$, then we can prove that $\dim \mathrm{Bs}|L| = 1$ by Theorem 1.8 (2) and (3). So we may assume that $L^2 \leq 2m$. We have only to check the case where $L^2 = 2m$ or $L^2 = 2m - 1$.

(A.2.1) The case in which $L^2 = 2m$.

Then $K_X L = 2q(X) - 2$ and $q(X) \geq 2$ because $K_X L > 0$.

If $q(X) = g(C)$, then $K_X L \geq (2g(C) - 2 + \chi(\mathcal{O}_X))LF = (2q(X) - 2 + \chi(\mathcal{O}_X))LF$. Hence $LF = 1$ and $\chi(\mathcal{O}_X) = 0$. But since $h^0(L_F) \geq 2$ for a general fiber F, we get that $\Delta(L_F) = 0$ and $g(F) = 0$. But this is impossible.

If $q(X) = g(C) + 1$, then $\chi(\mathcal{O}_X) = 0$ and

$$K_X L = (2g(C) - 2)LF + \sum_i (m_i - 1)LF_i.$$

Here we remark that $q(X) \geq 2$ since $2q(X) - 2 = K_X L > 0$. In particular $g(C) \geq 1$.

If $LF \geq 2$, then

$$\begin{aligned} K_X L &\geq 4(g(C) - 1) + \sum_i (m_i - 1)LF_i \\ &= 2(g(C) + 1) + 2g(C) - 6 + \sum_i (m_i - 1)LF_i \\ &= 2q(X) + 2g(C) - 6 + \sum_i (m_i - 1)LF_i. \end{aligned}$$

If $g(C) \geq 2$, then $g(C) = 2$ and $K_X L = 2q(X) - 2 = 4$ and π has no multiple fiber.

If $g(C) = 1$, then $q(X) = g(C) + 1 = 2$ and $K_X L = 2$. By the canonical bundle formula, we get that π has just 2 multiple fibers and $\sum_i (m_i - 1)LF_i = 2$, that is, $m_i = 2$ and $LF_i = 1$ for $i = 1, 2$ and $K_X \equiv F_1 + F_2$. In particular $LF = 2$ for a general fiber F of π. Therefore the type of (X, L) is one of the following;

(A.2.1.1) $3 = q(X) = g(C) + 1$, $\chi(\mathcal{O}_X) = 0$, $LF = 2$, and π has no multiple fiber. (This is the type (M-2-1) in Theorem 2.1.)

(A.2.1.2) π has just 2 multiple fibers $2F_1$ and $2F_2$, $\chi(\mathcal{O}_X) = 0$, $2 = q(X) = g(C) + 1$, $K_X \equiv F_1 + F_2$, $LF = 2$ for a general fiber F. (This is the type (M-2-2) in Theorem 2.1.)

We study the case (A.2.1.1). By the condition of (A.2.1.1), we get that π is a smooth fibration. We put $\pi_*(L) = \mathcal{E}$. Then $\mathcal{E}$ is a locally free sheaf of rank 2. Furthermore

$$\pi^* \circ \pi_*(L) \to L$$

is surjective because F is an elliptic curve with $h^0(L_F) = 2$ and $\mathrm{Bs}|L_F| = \emptyset$. So we get that there exists a finite double covering $\rho : X \to \boldsymbol{P}_C(\mathcal{E})$ with $L = \rho^* \mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$

(1). Let $B \subset \boldsymbol{P}_C(\mathscr{E})$ be the branch locus of ρ. Then $B \in |2D|$ for some line bundle D on $\boldsymbol{P}_C(\mathscr{E})$ and B is smooth. By the canonical line bundle formula for ρ, we get that $K_X = \rho^*(K_{\boldsymbol{P}_C(\mathscr{E})} + D)$. Since

$$\begin{aligned} K_{\boldsymbol{P}_C(\mathscr{E})} &= -2C_0 + (2g(C) - 2 - e)F \\ &= -2C_0 + (2 - e)F, \end{aligned}$$

where C_0 is the minimal section of $\boldsymbol{P}_C(\mathscr{E}) \to C$ and $e = -C_0^2$, we have $D \equiv 2C_0 + eF$ because $K_X \equiv 2F_\pi$.

(A.2.2) The case in which $L^2 = 2m - 1$.

Then $K_X L = 2q(X) - 1$ and $q(X) \geq 1$ because $K_X L > 0$. By the canonical bundle formula we get that

$$K_X L = (2g(C) - 2 + \chi(\mathcal{O}_X))LF + \sum_i (m_i - 1)LF_i.$$

Since $h^0(L) = m + 1$ and $\dim \operatorname{Bs}|L| \leq 0$, we find that $LF \geq 2$ for a general fiber F of $\pi : X \to C$.

Here we divide the case (A.2.2) into the following cases:

(a.1) The case in which $q(X) = g(C)$.

(a.2) The case in which $q(X) = g(C) + 1$.

(a.1) The case in which $q(X) = g(C)$.

Then

$$\begin{aligned} K_X L &\geq 2(2q(X) - 2) \\ &= 2q(X) - 1 + 2q(X) - 3. \end{aligned}$$

If $q(X) \geq 2$, then this is impossible. Hence $q(X) = 1$ and then $K_X L = 2q(X) - 1 = 1$. If $\chi(\mathcal{O}_X) > 0$, then $K_X L \geq 2$. So we get that $\chi(\mathcal{O}_X) = 0$ and $\sum_i (m_i - 1)LF_i = 1$. Therefore π has just one multiple fiber with $m_i = 2$ and $LF_i = 1$. (This is the type (M-2-3) in Theorem 2.1.)

(a.2) The case in which $q(X) = g(C) + 1$.

Here we remark that $LF \geq 2$ and $\chi(\mathcal{O}_X) = 0$. We divide two cases by the value of LF.

(a.2.1) The case where $LF \geq 3$.

(a.2.2) The case where $LF = 2$.

(a.2.1) The case where $LF \geq 3$.

Then

$$\begin{aligned} K_X L &\geq 3(2g(C) - 2) + \sum_i (m_i - 1)LF_i \\ &= 2(g(C) + 1) + 4g(C) - 8 + \sum (m_i - 1)LF_i \\ &= 2q(X) + 4g(C) - 8 + \sum_i (m_i - 1)LF_i. \end{aligned}$$

If $g(C) \geq 2$, then this is impossible because $K_X L = 2q(X) - 1$. So we get that $g(C) \leq 1$ and $q(X) \leq 2$. Furthermore we divide the case (a.2.1) into two cases:

(a.2.1.1) The case where $g(C) = 1$.
(a.2.1.2) The case where $g(C) = 0$.
(a.2.1.1) The case where $g(C) = 1$.
Then $q(X) = 2$ and $K_X L = 2q(X) - 1 = 3$. By the canonical bundle formula we get $K_X L = \sum_i (m_i - 1) LF_i$. Since $g(C) = 1$ and $\chi(\mathcal{O}_X) = 0$, π has a multiple fiber because $\kappa(X) = 1$. Since π has at least two multiple fibers (see [Se2]), π has two or three multiple fibers.

If π has just three multiple fibers $m_1 F_1$, $m_2 F_2$, and $m_3 F_3$, then we get that $m_1 = m_2 = m_3 = 2$ and $LF_1 = LF_2 = LF_3 = 1$. But since $LF \geq 3$, this is impossible.

If π has just two multiple fibers $m_1 F_1$ and $m_2 F_2$, we get that $(m_1, m_2) = (2, 3)$ or $(2, 2)$, where we assume $m_1 \leq m_2$.

If $(m_1, m_2) = (2, 3)$, then $LF_1 = 1$ and $2LF_2 = 2$, that is, $LF_i = 1$ for any i. But then $LF = L(m_1 F_1) = 2$ and $LF = L(m_2 F_2) = 3$ and this is impossible.

If $(m_1, m_2) = (2, 2)$, then $LF_1 = 2$ and $LF_2 = 1$ or $LF_1 = 1$ and $LF_2 = 2$. But then $L(m_1 F_1) \neq L(m_2 F_2)$. This is also impossible.

(a.2.1.2) The case where $g(C) = 0$.
Then $q(X) = 1$ and $K_X L = 1$.

Claim. *The number s of multiple fibers of π is at most four.*

Proof. Assume that $s \geq 6$. Let $\{m_i F_i\}_i$ be a multiple fiber of π. Here we assume that $LF_i \leq LF_{i+1}$ for any i. Then

$$\begin{aligned} 1 = K_X L &= -2LF + \sum_i (m_i - 1) LF_i \\ &\geq (m_1 LF_1 + m_2 LF_2) - 2LF + (m_3 - 1) LF_3 - LF_2 \\ &\quad + (m_4 - 1) LF_4 - LF_1 + (m_5 - 1) LF_5 + (m_6 - 1) LF_6 \\ &\geq 2. \end{aligned}$$

Therefore $s \leq 5$.

If $s = 5$, then by the same argument as above we get that $m_5 = 2$ and $LF_5 = 1$. By assumption, we get that $LF_1 = \cdots = LF_5 = 1$ and $LF = L(m_5 F_5) = 2$ for a general fiber F of π. But since $LF \geq 3$ in this case, this is impossible. Therefore $s \leq 4$. □

Here we remark that $s \geq 3$ in this case because $\kappa(X) = 1$. We assume that $LF_i \leq LF_{i+1}$ for any i. We divide the case (a.2.1.2) into the following two cases:
(b.1) The case in which $s = 4$.
(b.2) The case in which $s = 3$.
(b.1) The case in which $s = 4$.
Then by hypothesis we get that $(m_3 - 1) LF_3 - LF_2 = 0$ and $(m_4 - 1) LF_4 - LF_1 = 1$. The first equality implies that $m_3 = 2$ and $LF_2 = LF_3$. By the second equality there are two possible cases.

(α) $m_4 = 2$ and $LF_4 = LF_1 + 1$,

(β) $m_4 = 3$ and $LF_1 = LF_4 = 1$.

If the case (β) occurs, then by hypothesis $LF_1 = LF_2 = LF_3 = LF_4$ and $m_1 = m_2 = m_3 = m_4$. But since $m_3 = 2$ and $m_4 = 3$, this is impossible.

If the case (α) occurs, then $LF_3 = LF_2 = LF_1$ or $LF_4 = LF_3 = LF_2$. Since $m_4 = 2$ and $m_3 = 2$, we get that $LF_4 = LF_3 = LF_2$ and $LF_4 = LF_1 + 1$. Since $m_1 LF_1 = 2LF_4 = 2(LF_1 + 1)$, we get that

$$LF_1 = \frac{2}{m_1 - 2}.$$

Hence $m_1 = 3$ or 4 because LF_1 is integer. If $m_1 = 3$, then $LF_1 = 2$ and if $m_1 = 4$, then $LF_1 = 1$. Hence we get the following list;

m_1	m_2	m_3	m_4	LF_1	LF_2	LF_3	LF_4
3	2	2	2	2	3	3	3
4	2	2	2	1	2	2	2

(This is the type (M-2-4) in Theorem 2.1.)

(b.2) The case in which $s = 3$. (This is the type (M-2-5) in Theorem 2.5.) Then we get that $(m_3 - 1)LF_3 - LF_1 - LF_2 = 1$.

Claim. $m_3 \le 4$.

Proof. If $m_3 \ge 5$, then

$$\begin{aligned} 1 &= (m_3 - 1)LF_3 - LF_1 - LF_2 \\ &= (LF_3 - LF_1) + (LF_3 - LF_2) + (m_3 - 3)LF_3 \\ &\ge 2LF_3 \ge 2 \end{aligned}$$

and this is a contradiction. □

By the value of m_3, we divide the case (b.2) into the following:

(b.2.1) The case in which $m_3 = 4$.

(b.2.2) The case in which $m_3 = 3$.

(b.2.3) The case in which $m_3 = 2$.

(b.2.1) The case in which $m_3 = 4$.

Then $(LF_3 - LF_1) + (LF_3 - LF_2) + LF_3 = 1$. Therefore $LF_3 = 1$ and $LF_3 = LF_2 = LF_1$, so we get that $m_1 = m_2 = 4$.

(b.2.2) The case in which $m_3 = 3$.

Then $(LF_3 - LF_1) + (LF_3 - LF_2) = 1$. So $LF_3 = LF_2$ and $LF_3 = LF_1 + 1$. Therefore $m_2 = 3$. Since $m_1 LF_1 = 3LF_3 = 3(LF_1 + 1)$, we get that $(m_1 - 3)LF_1 = 3$. Since LF_1 is an integer, we obtain that $3/(m_1 - 3)$ is integer. Therefore we have $m_1 = 4, 6$.

If $m_1 = 4$ (resp. $m_1 = 6$), then $LF_1 = 3$ (resp. $LF_1 = 1$). Hence we get that

(1) $(m_1, m_2, m_3) = (4, 3, 3)$, $LF_1 = 3$, $LF_2 = LF_3 = 4$

(2) $(m_1, m_2, m_3) = (6, 3, 3)$, $LF_1 = 1$, $LF_2 = LF_3 = 2$.

(b.2.3) The case in which $m_3 = 2$.

Then $LF_3 = LF_2 + LF_1 + 1$. Hence we find that
(1) $m_1 LF_1 = 2LF_3 = 2LF_2 + 2LF_1 + 2$,
(2) $m_2 LF_2 = 2LF_3 = 2LF_2 + 2LF_1 + 2$.
On the other hand, since $LF_1 = (2/m_1)LF_3$ and $LF_2 = (2/m_2)LF_3$, we get that $LF_3 = (2/m_1)LF_3 + (2/m_2)LF_3 + 1$. Therefore

$$\left(1 - \frac{2}{m_1} - \frac{2}{m_2}\right) LF_3 = 1,$$

that is,

$$LF_3 = \frac{m_1 m_2}{(m_1 - 2)(m_2 - 2) - 4}.$$

Here we remark that $m_2 \geq 3$ because $LF_3 > LF_2$.

Furthermore we divide the case (b.2.3) into the following three cases:
(b.2.3.1) The case in which $m_2 = 3$.
(b.2.3.2) The case in which $m_2 = 4$.
(b.2.3.3) The case in which $m_2 \geq 5$.
(b.2.3.1) The case in which $m_2 = 3$.
Then

$$LF_3 = \frac{3m_1}{m_1 - 6} = 3 + \frac{18}{m_1 - 6}.$$

Since $LF_3 > 0$, we get that $m_1 \geq 7$. Since $18/(m_1 - 6)$ is integer and $LF_1 = 6/(m_1 - 6)$, the candidate of m_1 is the following;

m_1	LF_1	LF_2	LF_3
7	6	14	21
8	3	8	12
9	2	6	9
12	1	4	6

(b.2.3.2) The case in which $m_2 = 4$.
Here we remark that $m_1 \geq 4$. In this case we get that

$$\begin{aligned} LF_3 &= \frac{4m_1}{2(m_1 - 2) - 4} \\ &= \frac{2m_1}{m_1 - 4} \\ &= 2 + \frac{8}{m_1 - 4}. \end{aligned}$$

Since $LF_2 > 0$ and $LF_1 = 4/(m_1 - 4)$, we find that $m_1 \geq 5$ and

m_1	m_2	m_3	LF_1	LF_2	LF_3
5	4	2	4	5	10
6	4	2	2	3	6
8	4	2	1	2	4

(b.2.3.3) The case in which $m_2 \geq 5$.

Then $m_1 \geq 5$ and since $K_X L = 1$ and $m_3 = 2$ we get that

$$LF_1 + LF_2 \leq (m_1 - 4)LF_1 + (m_2 - 4)LF_2 = 4.$$

Therefore $(LF_1, LF_2) = (1,1)$, $(1,2)$, $(1,3)$, $(2,2)$. Since $LF_3 = LF_1 + LF_2 + 1$, $(m_1 - 4)LF_1 + (m_2 - 4)LF_2 = 4$, and $m_3 = 2$, we get the following;

m_1	m_2	m_3	LF_1	LF_2	LF_3
6	6	2	1	1	3
5	5	2	2	2	5

(a.2.2) The case where $LF = 2$.

Then

$$\begin{aligned} K_X L &= 2(2g(C) - 2) + \sum_i (m_i - 1)LF_i \\ &= 4g(C) - 4 + \sum_i (m_i - 1)LF_i \\ &= 2(g(C) + 1) - 6 + 2g(C) + \sum_i (m_i - 1)LF_i \\ &= 2q(X) + 2g(C) - 6 + \sum_i (m_i - 1)LF_i. \end{aligned}$$

Hence $g(C) \leq 2$. Here we remark that

$$\sum_i (m_i - 1)LF_i = \text{number of multiple fibers}$$

because $LF = 2$. In particular $m_i = 2$ and $LF_i = 1$ for any i. If $g(C) = 2$ (resp. 1, 0), then $\sum_i (m_i - 1)LF_i = 1$ (resp. 3, 5). On the other hand, π has at least two multiple fibers. Therefore $g(C) \leq 1$ and $\sum_i (m_i - 1)LF_i = 3$ or 5. (This is the type (M-2-6) in Theorem 2.1.)

(A.3) The case in which $\kappa(X) = 0$.

Then $g(L) = 1 + (1/2)L^2 = q(X) + m$. Then by Riemann-Roch Theorem and the classification of projective surfaces, we get that X is an abelian surface or K3 surface because $h^0(L) = m + 1$. But here we assume $L^2 \geq 2m - 1$. So we get that X is an abelian surface. In particular $L^2 = 2m + 2$.

Here we remark the following: Let (Y, A) be a polarized abelian surface. If $\dim \mathrm{Bs}|A| = 1$, then $Y \cong E_1 \times E_2$ and $A = p_1^* L_1 + p_2^* L_2$, where E_i is an elliptic curve and L_i is a line bundle on E_i with $\deg L_1 = 1$ and $\deg L_2 \geq 1$. (See [LB].) Therefore if (X, L) is not the above type, then $\dim \mathrm{Bs}|L| \leq 0$. (This is the type (M-3-6) in Theorem 2.1.)

(B) The case in which X is not minimal.

Let $X = X_0 \to X_1 \to \cdots \to X_l = X'$ be the minimal model of X. We put $L_0 := L$, $\mu_i : X_{i-1} \to X_i$, and $L_i := (\mu_i)_*(L_{i-1})$. Then $L_{i-1} = \mu_i^* L_i - \alpha_i E_i$ and

$\alpha_i > 0$ for any i, where E_i is a (-1)-curve of μ_i. We put $L' := L_l$. Here we remark that $\dim \mathrm{Bs}|L_l| \le 0$. Then

$$g(L') = g(L) + \sum_{i=1}^{l} \frac{\alpha_i^2 - \alpha_i}{2}$$

and

$$(L')^2 = L^2 + \sum_{i=1}^{l} \alpha_i^2.$$

So we get that

$$g(L') = q(X) + m + \sum_{i=1}^{l} \frac{\alpha_i^2 - \alpha_i}{2}$$

and

$$(L')^2 \ge 2m - 1 + \sum_{i=1}^{l} \alpha_i^2$$

because $L^2 \ge 2m - 1$ by assumption. Here we put $m' = m + \sum_{i=1}^{l} (\alpha_i^2 - \alpha_i)/2$. Then we get that

$$\begin{aligned}
(L')^2 &\ge 2m - 1 + \sum_{i=1}^{l} \alpha_i^2 \\
&= 2m - 1 + \sum_{i=1}^{l} (\alpha_i^2 - \alpha_i) + \sum_{i=1}^{l} \alpha_i \\
&= 2m' - 1 + \sum_{i=1}^{l} \alpha_i \\
&\ge 2m'.
\end{aligned}$$

(B.1) The case in which X is of general type.

Then since $\dim \mathrm{Bs}|L'| \le 0$, we get that $(L')^2 \le 2m'$ by Theorem 1.7. Hence we have $(L')^2 = 2m'$. But then $X' \cong C \times F$ and $L \equiv C + 2F$, where C and F are smooth projective curves with $g(C) \ge 2$ and $g(F) = 2$. This is impossible by the same argument as in the case (A-1) above.

(B.2) The case in which the Kodaira dimension of X is 1.

Then X' has an elliptic fibration over a smooth projective curve C; $\pi : X' \to C$. Then by Theorem 1.8 (2) and (3) we get that $(L')^2 \le 2m'$ since $\dim \mathrm{Bs}|L'| \le 0$. So we get that $(L')^2 = 2m'$. In particular $\sum_i \alpha_i = 1$ and (X, L) is a simple blowing up of (X', L'). Furthermore $m = m'$. So we get that $h^0(L') \ge h^0(L) = m + 1 = m' + 1$.

If $h^0(L') \ge m' + 2$, then $(L')^2 \ge 2\Delta(L')$ and we can check this case by using Fujita Theory. First we remark that $g(L') > m' \ge \Delta(L')$ since $(L')^2 = 2m'$. By Theorem 1.4 and Theorem 1.5, in this case $q(X) = q(X') = 0$ because $\kappa(X) = 1$. But $K_{X'} L' = 2q(X) - 2 = -2$ and this is impossible. So we assume that $h^0(L) =$

$m'+1$. Then by the same argument as in the case (A.2.1) above we get the type of (X', L'), that is,

(B.2.1) $g(C)=2$, $q(X)=3$, $\chi(\mathcal{O}_X)=0$, $L'F'=2$ and π has no multiple fibers, where F' is a general fiber of π (this is the type (N-1-1) in Theorem 2.1),

(B.2.2) π has just two multiple fibers, $2F_1$ and $2F_2$, $\chi(\mathcal{O}_X)=0$, $g(C)=1$, $q(X)=2$, and $L'F'=2$ (this is the type (N-1-2) in Theorem 2.1).

(B.3) The case in which $\kappa(X)=0$.

In this case X' is an abelian surface or bielliptic surface because $K_{X'}L' \leq 2q(X')-2$. But if $(L')^2=2m'$, then $\sum_i \alpha_i=1$ and $g(L)=g(L')$, that is, $m=m'$. Since $h^0(L') \geq h^0(L) \geq m+1=m'+1$, we get that $h^0(L') \geq m'+1$. But this is impossible because $h^0(L')=(L')^2/2$. Hence $(L')^2=2m'+2$. Then $g(L')=2+m'$ and X' is an abelian surface because $q(X')=2$ in this case. Furthermore we have $\sum_i \alpha_i \leq 3$. (This is the type (N-2) in Theorem 2.1.) These complete the proof of Theorem 2.1. □

Remark 2.2. Here we consider the type (M-2-1) in Theorem 2.1. Let $\rho : X \to \boldsymbol{P}_C(\mathscr{E})$ be the double covering. Let $B \subset \boldsymbol{P}_C(\mathscr{E})$ be the branch locus of ρ. Then $B \in |2D|$ for some divisor on $\boldsymbol{P}_C(\mathscr{E})$. Since X and $\boldsymbol{P}_C(\mathscr{E})$ is smooth, we need that B is smooth. So we check the condition that $|2D|$ has a smooth member. Here we assume that $\mathscr{E}$ is normalized. Let C_0 be the minimal section of $\boldsymbol{P}_C(\mathscr{E}) \to C$ and let F be a fiber of $\boldsymbol{P}_C(\mathscr{E}) \to C$. We put $e=-C_0^2$. Then $D \equiv 2C_0+eF$ by the proof of Theorem 2.1.

Assume that $e \geq 0$. Then an irreducible curve on $\boldsymbol{P}_C(\mathscr{E})$ is one of the following types (see [Ha]);

(1) C_0,

(2) F,

(3) aC_0+bF, $a>0$, and $b \geq ae$.

Assume that $B \in |2D|$ is not irreducible. Then we remark that F is not an irreducible component of B because $F(B-F)>0$. If C_0 is an irreducible component of B, then $0=C_0(3C_0+2eF)=-3e+2e=-e$. Hence $e=0$. If C_0 is not an irreducible component of B, then any irreducible component of B is the type xC_0+yF with $x>0$ and $y \geq ex$. If $y>xe$, then xC_0+yF is ample and this is a contradiction because B is smooth. So we have $y=xe$ and

$$\begin{aligned} 0 &= (xC_0+yF)((4-x)C_0+(2e-y)F) \\ &= -x(4-x)e+x(2e-y)+y(4-x) \\ &= (ex-2y)(x-2) \\ &= -y(x-2). \end{aligned}$$

Hence $y=0$ or $x=2$.

If $y=0$, then $e=0$ because $x>0$.

If $x=2$, then $y=2e$ and $B-(2C_0+2eF)=2C_0$. Since C_0 is not an irreducible component of B, we get that $2C_0$ is numerically equivalent to an irreducible curve. Hence $e=0$.

In any case we have $e = 0$ and $B = 4C_0$ if B is not irreducible. Since C_0 is not an irreducible component of B, we get that $B = C_1 + C_2$ where C_i is an irreducible curve with $C_i \equiv 2C_0$ for $i = 1, 2$.

Assume that B is irreducible and $e > 0$. Then by the above condition, we have $2e \geq 4e > 0$ and this is impossible. Hence $e = 0$. Therefore $B \equiv 4C_0$ and $e = 0$ in this case.

Assume that $e < 0$. Then an irreducible curve on $\boldsymbol{P}_C(\mathscr{E})$ is one of the following types;

(1′) C_0,

(2′) F,

(3′) $aC_0 + bF$, where $a = 1$ and $b \geq 0$ or $a \geq 2$ and $b \geq (1/2)ae$.

Since $B \in |2D| = |4C_0 + 2eF|$, F is not an irreducible component of B because B is smooth.

If C_0 is an irreducible component of B, then $C_0(3C_0 + 2eF) = -3e + 2e = -e > 0$ and this is impossible because B is smooth. Therefore C_0 is not an irreducible component of B.

Since

$$2D \equiv 4C_0 + 2eF = \sum_i (a_i C_0 + b_i F)$$

and $2e = (1/2) \times 4 \times e$, we get that $a_i \geq 2$ and $b_i = (1/2)a_i e$ for any i. So if B is not irreducible, then since $\sum_i a_i = 4$, we get that $a_i = 2$ and $b_i = e$. In this case $(2C_0 + eF)^2 = -4e + 4e = 0$. Therefore we have the following two types:

(1″) If B is not irreducible, then $B = C_1 + C_2$, where $C_i \equiv 2C_0 + eF$ for each i.

(2″) If B is irreducible, then $B \equiv 4C_0 + 2eF$.

Therefore we get the following types:

(M-2-1-1) If $e \geq 0$ and B is not irreducible, then $e = 0$ and $B = C_1 + C_2$, where $C_i = 2C_0$ for $i = 1$ or 2.

(M-2-1-2) If $e \geq 0$ and B is irreducible, then $e = 0$ and $B = 4C_0$.

(M-2-1-3) If $e < 0$ and B is not irreducible, then $B = C_1 + C_2$, where $C_i \equiv 2C_0 + eF$ for each i.

(M-2-1-4) If $e < 0$ and B is irreducible, then $B \equiv 4C_0 + 2eF$.

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