T. YAMAZAKI
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ON A SURGERY OF *K*-CONTACT MANIFOLDS

Tsutomu Yamazaki

Abstract

In this paper we introduce a surgery of a certain type on *K*-contact manifolds. As an application we classify the diffeomorphism types of all closed simply connected 5-dimensional *K*-contact manifolds of rank 3.

1. Introduction

We consider a contact manifold (M, α) endowed with some geometric structure: a Riemannian metric g such that the contact flow ψ_t is isometric. In this case we call (M, α, g) and ψ_t a *K-contact manifold* and *K-contact flow*, respectively. If M is compact, the closure of the 1-parameter subgroup $\{\psi_t \mid t \in \boldsymbol{R}\}$ in the isometry group of (M, g) generated by a *K*-contact flow is a compact connected abelian Lie group, hence is isomorphic to T^k for some integer k. Thus a compact *K*-contact manifold has an action of the torus T^k containing the contact flow as a dense subgroup. Conversely if a contact manifold (M, α) has a T^k-action such as above, there exists a metric g on M such that (M, α, g) is a *K*-contact manifold ([12]). We call the *K*-contact manifold with this T^k-action a *K*-contact manifold of *rank k*. In this paper, using the property of the T^k-action on a *K*-contact manifold, we show that a surgery along a closed orbit of the *K*-contact flow produces a new *K*-contact manifold.

Let (M, α, g) be a $(2n+1)$-dimensional *K*-contact manifold of rank k. We take a closed orbit O through a point x in M of the *K*-contact flow. (It is known that there exists such an orbit for any *K*-contact flow ([2]).) Then O coincides with the T^k-orbit of x. Thus there exists a T^k-invariant open neighborhood U of O which is T^k-equivariantly diffeomorphic to $S^1 \times D_\varepsilon^{2n}$ with a T^k-action, where $D_\varepsilon^{2n} = \{z \in \boldsymbol{C}^n \mid |z| < \varepsilon\}$. In the category of manifolds with T^k-action, we perform the following surgery. Remove a T^k-invariant closed subset $S^1 \times D_\mu^{2n} = \{(e^{\sqrt{-1}\theta_0}, z) \mid |z| \le \mu\}$ of $S^1 \times D_\varepsilon^{2n}$ from M. We glue this complement and $D_b^2 \times S^{2n-1} = \{(w, z) \in \boldsymbol{C} \times \boldsymbol{C}^n \mid |w| \le b, |z| = 1\}$ along their boundaries $S^1 \times S^{2n-1}$ in such a way that the T^k-action is extended to the whole of $D_b^2 \times S^{2n-1}$. This produces a new manifold $\tilde{M}$ with T^k-action. In section 3 we show that this surgery is carried out in the category of *K*-contact manifolds of rank k. First we describe the normal form of the *K*-contact form on $S^1 \times D_\varepsilon^{2n}$.

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Next we construct a K-contact form and a T^k-action on $D_b^2 \times S^{2n-1}$ such that there exists a T^k-equivariant contact diffeomorphism from $(D_b^2 - \{0\}) \times S^{2n-1}$ to the complement of a closed neighborhood $S^1 \times D_\mu^{2n}$ in $S^1 \times D_\varepsilon^{2n}$. As a result we obtain a contact form $\tilde{\alpha}$ on $\tilde{M}$ such that the 1-parameter subgroup generated by the contact flow of $\tilde{\alpha}$ is dense subgroup in T^k. Therefore there exists a metric $\tilde{g}$ on $\tilde{M}$ such that $(\tilde{M}, \tilde{\alpha}, \tilde{g})$ is a K-contact manifold of rank k. This surgery is called *K-contact surgery*. In the case of the $(2n+1)$-dimensional K-contact manifold of rank $(n+1)$, K-contact surgery increases the number of the closed orbit of the K-contact flow by $(n-1)$.

In section 4, as an application we give the classification of closed simply connected 5-manifolds which admit a structure of K-contact manifold of rank 3.

2. Torus action on K-contact manifold

In this section we briefly describe the definition of K-contact manifold and the T^k-action on it. For detail see [12].

A one-form α on a smooth $(2n+1)$-dimensional manifold M is called a *contact form* if $\alpha \wedge (d\alpha)^n$ is everywhere non-zero. The pair (M, α) is called a *contact manifold*. A contact manifold carries a vector field Z which is uniquely determined by the equations $\alpha(Z) = 1$ and $d\alpha(Z, \cdot) = 0$. We call Z and the flow ψ_t generated by Z the *Reeb vector field* and *the contact flow*, respectively. A $2n$-dimensional distribution $D = \operatorname{Ker} \alpha$ is called a *contact structure*. From the definition of α and D, it is obvious that $d\alpha$ is non-degenerate on D. Namely $d\alpha$ induces a symplectic structure on D. In this case, it is well-known that there exist a positive definite inner product g_T and an almost complex structure J on D satisfying $g_T(X, Y) = d\alpha(X, JY)$ and $g_T(JX, JY) = g_T(X, Y)$ for any $X, Y \in D$. We extend g_T on D to the whole TM as a symmetric bilinear form by requiring $g_T(Z, X) = 0$ for $X \in TM$. Thus we get a Riemannian metric $g = g_T \oplus (\alpha \otimes \alpha)$, which is called a *contact metric*. If there exists a contact metric g such that $L_Z g = 0$, the triple (M, α, g) is called *K-contact manifold* and the flow generated by Z is called a *K-contact flow*.

A K-contact manifold carries the torus action as follows:

Proposition 2.1 ([12]). *Let (M, α) be a $(2n+1)$-dimensional contact manifold and ψ_t be the contact flow. If we assume that M is compact, the following statements are equivalent.*

1) *There exists a contact metric g such that (M, α, g) is a K-contact manifold.*

2) *There exists a smooth effective T^k-action $\{h_u \mid u \in T^k\}$ on M, which contains the contact flow $\{\psi_t = h_{\Psi(t)}\}$ as a dense subgroup. In this situation, $1 \leq k \leq n+1$ follows.*

Remark 2.2. (1) From Proposition 2.1, we see that for some rationally independent real constants $\lambda_0, \ldots, \lambda_{k-1}$ and the standard integral basis $e_0, \ldots, e_{k-1}$ of $\operatorname{Lie}(T^k)$, the Reeb vector field Z on a K-contact manifold takes the form

$$Z(x) = \lambda_0 \frac{d(h_{\exp(te_0)}(x))}{dt}\bigg|_{t=0} + \cdots + \lambda_{k-1} \frac{d(h_{\exp(te_{k-1})}(x))}{dt}\bigg|_{t=0}$$

at $x \in M$.

(2) Given a compact K-contact manifold (M, α, g), the torus T^k in Proposition 2.1 is obtained as the closure of the K-contact flow $\{\psi_t \mid t \in \boldsymbol{R}\}$ in the isometry group of (M, g). In this paper we always choose the basis $e_0, \ldots, e_{k-1}$ of $\mathrm{Lie}(T^k)$ such that $\lambda_0, \ldots, \lambda_{k-1}$ are all positive.

Definition 2.3. (M, α, g) is called a K-contact manifold of rank k if the closure of the K-contact flow $\{\psi_t \mid t \in \boldsymbol{R}\}$ in the isometry group of (M, g) is isomorphic to a k-dimensional torus T^k.

3. K-contact surgery

In this section, first we will give a description of the neighborhood of a closed orbit of the K-contact flow. Using this description, we will introduce a surgery along a closed orbit of the K-contact flow. In [2] the following result is proved:

Theorem 3.1 [2]. *Any K-contact flow has at least two closed orbit.*

It follows that we can carry out a surgery for any K-contact flow.

Let (M, α, g) be a K-contact manifold of rank k. Let Z and ψ_t be the Reeb vector field and the K-contact flow of (M, α, g), respectively. Notice that if the orbit $\psi_t \cdot x$ of the K-contact flow ψ_t through a point x in M is closed, then $\psi_t \cdot x$ coincides with the T^k-orbit $T^k \cdot x$ of x. The T^k-orbit $T^k \cdot x$ of $x \in M$ is T^k-equivariantly diffeomorphic to $T^k/(T^k)_x$, where $(T^k)_x$ denote the isotropy group of x. Hence if $T^k \cdot x$ is T^k-equivariantly diffeomorphic to S^1, $(T^k)_x$ is isomorphic to T^{k-1} or $T^{k-1} \times \boldsymbol{Z}/p\boldsymbol{Z}$, where p is an integer. However we only consider the closed orbit $O := \psi_t \cdot x$ of K-contact flow ψ_t through a point x in M such that $(T^k)_x \cong T^{k-1}$. The case of $(T^k)_x \cong T^{k-1} \times \boldsymbol{Z}/p\boldsymbol{Z}$ directely follows from the case of $(T^k)_x \cong T^{k-1}$, because all of the construction will be carried out to be $\boldsymbol{Z}/p\boldsymbol{Z}$-invariant. We will give a local description of the K-contact form, the Reeb vector field and the T^k-action on the T^k-invariant open neighborhood U of a closed orbit O.

Let D be the contact distribution $\operatorname{Ker} \alpha$ and $\boldsymbol{R}Z$ the trivial line bundle spanned by Z. Then by the definition of the K-contact metric g, we have $T_x(O) = (\boldsymbol{R}Z)_x$ and $T_x(O)^\perp = D_x$, where $T_x(O)^\perp$ is the orthogonal complement of $T_x(O)$ with respect to g. Let B be a ball in the tangent space T_xM at x such that the exponential map $\exp_x$ of g is a diffeomorphism. Put $B_D = B \cap D_x$. Then there exists a T^k-invariant open neighborhood U_1 of O in M which is T^k-equivariantly diffeomorphic to $S^1 \times B_D$ in the following sense.

Consider the map $\tilde{f}$ from $T^k \times B_D$ to a T^k-invariant open neighborhood U_1 of O in M defined by $\tilde{f}(\Theta, y) = \Theta \cdot \exp_x(y)$ for $\Theta = (e^{\sqrt{-1}\theta_0}, e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_{k-1}}) \in$

T^k and $y \in B_D$. Then $\tilde{f}$ is well defined and induces a T^k-equivariant diffeomorphism f from $T^k \times_{(T^k)_x} B_D$ onto U_1. Here $T^k \times_{(T^k)_x} B_D$ is the quotient space by the $(T^k)_x$-action defined by $\tilde{t} \cdot (\Theta, y) = (\tilde{t}\Theta, \tilde{t}^{-1} y)$ for $\tilde{t} \in (T^k)_x \subset T^k$. The T^k-action on it is induced by the one on $T^k \times B_D$ defined by $t \cdot (\Theta, y) = (t\Theta, y)$. Note that the restriction of f to the zero section is the identity map. The map $\tilde{G} : T^k \times B_D \to S^1 \times B_D$ defined by

$$\tilde{G}(\Theta, y) = (e^{\sqrt{-1}\theta_0}, \rho(1, e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_{k-1}}) y),$$

where ρ denote the linear representation $(T^k)_x \to SO(2n; \boldsymbol{R})$ of B_D, induces a T^k-equivariant diffeomorphism $G : T^k \times_{(T^k)_x} B_D \to S^1 \times B_D$. Here the T^k-action on $S^1 \times B_D$ is induced by the one on $T^k \times B_D$. Hence we have the T^k-equivariant diffeomorphism $\tilde{\varphi} = G \circ f^{-1} : U_1 \to S^1 \times B_D$ whose restriction to O is the identity map. In this situation, we have the following:

Proposition 3.2. *There exist a T^k-invariant open neighborhood U of O and a T^k-equivariant diffeomorphism $\varphi : U \to S^1 \times D_\varepsilon^{2n} = \{(e^{\sqrt{-1}\theta_0}, z_1, \dots, z_n) \in S^1 \times \boldsymbol{C}^n \mid \sum_{j=1}^n z_j \bar{z}_j < \varepsilon^2\}$ satisfying the following properties.*

1) *For integers m_{ij}, $i = 1, \dots, n$, $j = 1, \dots, k-1$ and $(t_0, t_1, \dots, t_{k-1}) \in T^k$, the T^k-action on $S^1 \times D_\varepsilon^{2n}$ is given by*

$$(3.1)\qquad \begin{aligned}&(t_0, t_1, \dots, t_{k-1}) \cdot (e^{\sqrt{-1}\theta_0}, z_1, \dots, z_n)\\ &\quad = (t_0 e^{\sqrt{-1}\theta_0}, t_1^{m_{11}} \cdots t_{k-1}^{m_{1k-1}} z_1, \dots, t_1^{m_{n1}} \cdots t_{k-1}^{m_{nk-1}} z_n).\end{aligned}$$

2) *$\varphi \mid O = id$.*

3)

$$(3.2)\qquad \begin{aligned}\alpha_0 := (\varphi^{-1})^* \alpha = 1/\lambda_0 \left\{ 1 - \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_j m_{ij} \right) z_i \bar{z}_i \right\} d\theta_0\\ + \sqrt{-1}/2 \sum_{j=1}^n (z_j\, d\bar{z}_j - \bar{z}_j\, dz_j),\end{aligned}$$

where $\lambda_0, \dots, \lambda_{k-1}$ are rationally independent positive real numbers.

4) *The Reeb vector field Z_0 of α_0 is given by*

$$(3.3)\qquad Z_0 = \lambda_0 \frac{\partial}{\partial \theta_0} + \sum_{j=1}^{k-1} \lambda_j \left(\sum_{i=1}^n m_{ij} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) \right).$$

Proof. The proof is completely analogous to the one in [1]. We consider the contact form $\alpha_1 = (\tilde{\varphi}^{-1})^* \alpha$, where $\tilde{\varphi} : U \to S^1 \times B_D$. Put $q_1^0 = \alpha_1(\partial/\partial\theta_0)$. Then α_1 takes the form $\alpha_1 = q_0^1\, d\theta_0 + \alpha_D$, where α_D is a one-form on $S^1 \times B_D$. Clearly $\alpha_D(\partial/\partial\theta_0) = 0$ and $L_{\partial/\partial\theta_0} \alpha_D = 0$, because α_1, q_0^1, $d\theta_0$ are all T^k-invariant. It also follows from the assumption that α_D is a one-form on B_D which is invariant under the action of $T^{k-1} \cong T^k/S^1$, and hence $d\alpha_D$ is T^{k-1}-invariant two-form on B_D. Since T_xB_D coincides with the contact plane D_x at $x \in M$, the restriction $(d\alpha_1)_x \,|\, T_xB_D$ of $(d\alpha_1)_x$ to T_xB_D is a symplectic two-form on T_xB_D. Moreover from $(d\alpha_1)_x \,|\, T_xB_D = (d\alpha_D)_x \,|\, T_xB_D$, $(d\alpha_D)_x \,|\, T_xB_D$ is a T^{k-1}-invariant symplectic two-form on T_xB_D. Thus there exists a sufficiently small neighborhood $S^1 \times D^{2n}_{\varepsilon_1}$ of S^1 such that $d\alpha_D$ is T^{k-1}-invariant symplectic two-form on $D^{2n}_{\varepsilon_1}$. It follows that, from the equivariant Darboux theorem ([4]), there exists a complex coordinate $(z_1, \dots, z_n)$ on $D^{2n}_{\varepsilon_2} \subset D^{2n}_{\varepsilon_1}$ with $z_j = x_j + \sqrt{-1}\,y_j$ such that

(1) $d\alpha_D = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i = d\{\sqrt{-1}/2\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i)\}$,

(2) the action of T^{k-1} is given by

$$(t_1, \dots, t_{k-1}) \cdot (z_1, \dots, z_n) = (t_1^{m_{11}} \cdots t_{k-1}^{m_{1k-1}} z_1, \dots, t_1^{m_{n1}} \cdots t_{k-1}^{m_{nk-1}} z_n),$$

where m_{ij}, $(i = 1, \dots, n, j = 1, \dots, k-1)$ are integers and $(t_1, \dots, t_{k-1}) \in T^{k-1}$. Extending $d\alpha_D$ on $D^{2n}_{\varepsilon_2}$ to $S^1 \times D^{2n}_{\varepsilon_2}$, we have

$$d\alpha_1 = d\left(q^0\, d\theta_0 + \sqrt{-1}/2\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i)\right)$$

with respect to the coordinate $(e^{\sqrt{-1}\theta_0}, z_1, \dots, z_n)$ on $S^1 \times D^{2n}_{\varepsilon_2}$, where q^0 is a T^k-invariant smooth function on $S^1 \times D^{2n}_{\varepsilon_2}$. Hence we have

$$\alpha_1 = q^0\, d\theta_0 + \sqrt{-1}/2\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i) + dh.$$

Here we can choose h to be a T^k-invariant smooth function on $S^1 \times D^{2n}_{\varepsilon_2}$ such that its restriction $h \,|\, S^1 \times \{0\}$ to $S^1 \times \{0\}$ is equal to 0. We put $\alpha_t = \alpha_0 + t dh$ $(0 \le t \le 1)$, where $\alpha_0 = q^0\, d\theta_0 + \sqrt{-1}/2\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i)$. Since the restriction of α_t to S^1 is the contact form α_0 for all t (because $h \,|\, S^1 \times \{0\} = 0$), there exists a smaller neighborhood $S^1 \times D^{2n}_{\varepsilon_3} \subset S^1 \times D^{2n}_{\varepsilon_2}$ on which α_t is the contact form. Let Z_t be the Reeb vector field of α_t and put $X_t = -hZ_t$. Since $X_t(0) = 0$ for all $x \in S^1$, there exists a smaller neighborhood $S^1 \times D^{2n}_{\varepsilon} \subset S^1 \times D^{2n}_{\varepsilon_3}$ of S^1 on which X_t can be integrated to a time dependent embedding $\varphi_t : S^1 \times D^{2n}_{\varepsilon} \to S^1 \times D^{2n}_{\varepsilon_3}$ with $\varphi_t(x) = x$ for all $x \in S^1$. We have

$$\begin{aligned} d/dt(\varphi_t^*\alpha_t) &= \varphi_t^*(L_{X_t}\alpha_t + d\alpha_t/dt) \\ &= \varphi_t^*\{d(\iota_{X_t}\alpha_t) + dh\} \\ &= 0. \end{aligned}$$

This means that $\varphi_1^*\alpha_1 = \varphi_0^*\alpha_0 = \alpha_0$ on $S^1 \times D^{2n}_{\varepsilon}$. Since h and Z_t are T^k-invariant, so is X_t. This implies φ_t is T^k-equivariant. It follows that there exists a T^k-equivariant diffeomorphism $\varphi : U \to S^1 \times D^{2n}_{\varepsilon}$ such that $\varphi \,|\, O = id$ and $\varphi^*\alpha_0 = \alpha$, where the T^k-action on $S^1 \times D^{2n}_{\varepsilon}$ is given by (3.1).

From Remark 2.2, the Reeb vector field Z_0 of α_0 takes the form

$$Z_0 = \lambda_0 \frac{\partial}{\partial \theta_0} + \sum_{j=1}^{k-1} \lambda_j \left\{ \sum_{i=1}^{n} m_{ij} \sqrt{-1} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) \right\}$$

for rationally independent positive real numbers $\lambda_0, \ldots, \lambda_{k-1}$. From the equation $\alpha_0(Z_0) = 1$, we have $q^0 = 1/\lambda_0 \{1 - \sum_{i=1}^{n} (\sum_{j=1}^{k-1} \lambda_j m_{ij}) z_i \bar{z}_i\}$. Therefore we obtain 3). q.e.d.

Remark 3.3. From the effectiveness of the T^k-action, the integers $m_{1j}, \ldots, m_{nj}$ satisfy $g.c.d(m_{1j}, \ldots, m_{nj}) = 1$ for any j and $(m_{11}, \ldots, m_{n1}), \ldots, (m_{1k-1}, \ldots, m_{nk-1})$ are linearly independent in $\boldsymbol{R}^n$.

We have the following modification of the K-contact form and its Reeb vector field on the T^k-invariant open neighborhood U of the closed orbit O of the K-contact flow.

LEMMA 3.4. *For positive integers $N = (\zeta_1, \ldots, \zeta_n)$, take a T^k-equivariant diffeomorphism φ_N from $S^1 \times D_\varepsilon^{2n}$ with the coordinate $(e^{\sqrt{-1}\theta_0}, z_1, \ldots, z_n)$ to $S^1 \times D_\varepsilon^{2n}$ with the coordinate $(e^{\sqrt{-1}\theta_0}, v_1, \ldots, v_n)$ defined by*

$$\varphi_N(e^{\sqrt{-1}\theta_0}, z_1, \ldots, z_n) = (e^{\sqrt{-1}\theta_0}, e^{\sqrt{-1}\zeta_1\theta_0} z_1, \ldots, e^{\sqrt{-1}\zeta_n\theta_0} z_n). \tag{3.4}$$

Then we have the followings.

1) *The T^k-action is given by*

$$\begin{aligned} &(t_0, t_1, \ldots, t_{k-1}) \cdot (e^{\sqrt{-1}\theta_0}, v_1, \ldots, v_n) \\ &\quad = (t_0 e^{\sqrt{-1}\theta_0}, t_0^{\zeta_1} t_1^{m_{11}} \cdots t_{k-1}^{m_{1k-1}} v_1, \ldots, t_0^{\zeta_n} t_1^{m_{n1}} \cdots t_{k-1}^{m_{nk-1}} v_n) \end{aligned} \tag{3.5}$$

on the right hand side (*resp.* (3.1) *on the left hand side*).

(2)

$$\begin{aligned} \alpha_N &:= (\varphi^{-1})^* \alpha_0 \\ &= 1/\lambda_0 \left[1 - \sum_{i=1}^{n} \left\{ \sum_{j=1}^{k-1} (\lambda_j m_{ij}) + \lambda_0 \zeta_i \right\} v_i \bar{v}_i \right] d\theta_0 + \sqrt{-1}/2 \sum_{i=1}^{n} (v_i \, d\bar{v}_i - \bar{v}_i \, dv_i). \end{aligned} \tag{3.6}$$

3) *The Reeb vector field Z_N of α_N is given by*

$$\begin{aligned} Z_N = {} & \lambda_0 \left\{ \frac{\partial}{\partial \theta_0} + \sum_{i=1}^{n} \zeta_i \sqrt{-1} \left(v_i \frac{\partial}{\partial v_i} - \bar{v}_i \frac{\partial}{\partial \bar{v}_i} \right) \right\} \\ & + \sum_{j=1}^{k-1} \lambda_j \left\{ \sum_{i=1}^{n} m_{ij} \sqrt{-1} \left(v_i \frac{\partial}{\partial v_i} - \bar{v}_i \frac{\partial}{\partial \bar{v}_i} \right) \right\}. \end{aligned} \tag{3.7}$$

This lemma is easily obtained by direct calculations.

We will construct a K-contact fom $\tilde{\alpha}_N$ and a T^k-action on $D_b^2 \times S^{2n-1}$ such that there exists a T^k-equivariant contact diffeomorphism from $(D_b^2 - \{0\}) \times S^{2n-1}$ to the complement of some neighborhood of $S^1 \times \{0\}$ in $S^1 \times D_\varepsilon^{2n}$. Here D_b^2 is the closed ball of radius b in $\boldsymbol{C}$.

For $N = (\zeta_1, \dots, \zeta_n)$ we define a K-contact form $\tilde{\alpha}_N$, a Reeb vector field $\tilde{Z}_N$, and a T^k-action on $\boldsymbol{C} \times S^{2n-1} = \{(re^{\sqrt{-1}\theta_0}, w_1, \dots, w_n) \in \boldsymbol{C} \times \boldsymbol{C}^n \mid \sum_{j=1}^n w_j \overline{w}_j = 1\}$ as follows.

$$\tilde{\alpha}_N = -r^2\, d\theta_0 + (1 + \lambda_0 r^2)\alpha_A, \tag{3.8}$$

where $\alpha_A = \sqrt{-1}/2 \sum_{i=1}^n 1/a_i(w_i\, d\overline{w}_i - \overline{w}_i\, dw_i)$ and $a_i = \lambda_0 \zeta_i + \sum_{j=1}^{k-1} \lambda_j m_{ij}$.

$$\begin{aligned}\tilde{Z}_N = \lambda_0 &\left\{ \frac{\partial}{\partial \theta_0} + \sum_{i=1}^n \zeta_i \sqrt{-1}\left(w_i \frac{\partial}{\partial w_i} - \overline{w}_i \frac{\partial}{\partial \overline{w}_i}\right)\right\} \\ &+ \sum_{j=1}^{k-1} \lambda_j \left\{ \sum_{i=1}^n m_{ij}\left(w_i \frac{\partial}{\partial w_i} - \overline{w}_i \frac{\partial}{\partial \overline{w}_i}\right)\right\}.\end{aligned} \tag{3.9}$$

$$\begin{aligned}&(t_0, t_1, \dots, t_{k-1}) \cdot (re^{\sqrt{-1}\theta_0}, w_1, \dots, w_n) \\ &\quad = (t_0 re^{\sqrt{-1}\theta_0}, t_0^{\zeta_1} t_1^{m_{11}} \cdots t_{k-1}^{m_{1k-1}} w_1, \dots, t_0^{\zeta_n} t_1^{m_{n1}} \cdots t_{k-1}^{m_{nk-1}} w_n)\end{aligned} \tag{3.10}$$

where $(t_0, t_1, \dots, t_{k-1}) \in T^k$.

Take integers $\zeta_1, \dots, \zeta_n$ in Lemma 3.4 such that $a_i \varepsilon^2 > 1 + \delta$ for a positive constant δ and all i, (so that $a_i > 0$ for all i). Put $B(a_1, \dots, a_n)_\mu = \{(z_1, \dots, z_n) \in D_\varepsilon^{2n} \mid \sum_{i=1}^n a_i z_i \bar{z}_i \le \mu\} \subset D_\varepsilon^{2n}$ and $D_\mu^2 = \{re^{\sqrt{-1}\theta} \in \boldsymbol{C} \mid r \le \mu\} \subset \boldsymbol{C}$. Then we have the following:

Proposition 3.5. *For $b = [1/\lambda_0\{(1+\delta)^2 - 1\}]^{1/2}$ there exists a T^k-equivariant diffeomorphism*

$$\Phi : S^1 \times (B(a_1, \dots, a_n)_{1+\delta} \backslash B(a_1, \dots, a_n)_1) \to (D_b^2 - \{0\}) \times S^{2n-1} \tag{3.11}$$

satisfying $\Phi^* \tilde{\alpha}_N = \alpha_0$. Here the T^k-action is given by (3.1) *on the left hand side* (*resp.* (3.10) *on the right hand side*).

Proof. Put $R^2 = \sum_{i=0}^n a_i z_i \bar{z}_i$ and $\alpha_R = 1/r^2\{\sqrt{-1}/2 \sum_{i=1}^n (v_i\, d\bar{v}_i - \bar{v}_i\, dv_i)\}$. Take $a_l = \max\{a_1, \dots, a_n\}$. Consider the diffeomorphism Ψ from $S^1 \times (D_\varepsilon^{2n} - \{0\})$ onto $S^1 \times S^{2n-1} \times (0, a_l \varepsilon^2) = \{(e^{\sqrt{-1}\theta_0}, w_1, \dots, w_n, R) \in S^1 \times \boldsymbol{C}^n \times \boldsymbol{R} \mid \sum_{i=1}^n w_i \overline{w}_i = 1, 0 < R < a_l \varepsilon^2\}$ defined by

$$\Psi(e^{\sqrt{-1}\theta_0}, v_1, \dots, v_n) = (e^{\sqrt{-1}\theta_0}, (\sqrt{a_1}/R)v_1, \dots, (\sqrt{a_n}/R)v_n, R) \tag{3.12}$$

and a one-form $\alpha_A = \sqrt{-1}/2 \sum_{i=1}^n 1/a_i(w_i\, d\overline{w}_i - \overline{w}_i\, dw_i)$ on $S^1 \times S^{2n-1} \times (0, a_l \varepsilon^2)$. Then we have

$$(\Psi^{-1})^* \alpha_N = 1/\lambda_0(1 - R^2)\, d\theta_0 + R^2 (\Psi^{-1})^* \alpha_R \tag{3.13}$$

$$= 1/\lambda_0(1 - R^2)\, d\theta_0 + R^2 \alpha_A. \tag{3.14}$$

Moreover the map H from $S^1 \times S^{2n-1} \times (1, 1+\delta]$ to $(D_b^2 - \{0\}) \times S^{2n-1}$ defined by

$$(3.15)\qquad H(e^{\sqrt{-1}\theta_0}, w_1, \dots, w_n, R) = (\{1/\lambda_0(R^2-1)\}^{1/2} e^{\sqrt{-1}\theta_0}, w_1, \dots, w_n)$$

gives a T^k-equivariant diffeomorphism and satisfies $H^*\tilde{\alpha}_N = (\Psi^{-1}) * \alpha_N$. Take a composition map $\Phi := H \circ \Psi \circ \varphi_N$ of the above Ψ, H and φ_N defined by (3.4), and consider the map restricted to $S^1 \times (B(a_1, \dots, a_n)_{1+\delta} \backslash B(a_1, \dots, a_n)_1)$. Then Φ gives a T^k-equivariant diffeomorphism from $S^1 \times (B(a_1, \dots, a_n)_{1+\delta} \backslash B(a_1, \dots, a_n)_1)$ onto $(D_b^2 - \{0\}) \times S^{2n-1}$ satisfying $\Phi^*\tilde{\alpha}_N = \alpha_0$. q.e.d

Let $\tilde{M}$ be a manifold obtained by gluing $M^* = M \backslash (S^1 \times \varphi^{-1} B(a_1, \dots, a_n)_1)$ and $D_b^2 \times S^{2n-1}$ along $\varphi^{-1}(S^1 \times (B(a_1, \dots, a_n)_{1+\delta} \backslash B(a_1, \dots, a_n)_1))$ and $(D_b^2 - \{0\}) \times S^{2n-1}$ by the map $\Phi \circ \varphi$, that is,

$$\tilde{M} := (M \backslash \varphi^{-1}(S^1 \times B(a_1, \dots, a_n)_1)) \cup_{\Phi\circ\varphi} D_b^2 \times S^{2n-1}.$$

Then we have the following:

Theorem 3.6. *There exists a contact metric $\tilde{g}$ on $\tilde{M}$ such that $(\tilde{M}, \tilde{\alpha}, \tilde{g})$ is a K-contact manifold of rank k.*

Proof. By Proposition 3.5, we see that there exists a contact form $\tilde{\alpha}$ on $\tilde{M}$ which equals to α on M^* and equals to $\tilde{\alpha}_N$ on $D_b^2 \times S^{2n-1}$. Since the T^k-action on M preserves M^* and $\Phi \circ \varphi$ is T^k-equivariant, there exists a well defined T^k-action on $\tilde{M}$. From the construction we see that the closure of the contact flow of every point $x \in \tilde{M}$ coincides with the T^k-orbit of x. Therefore, from Proposition 2.1, there exists a contact metric $\tilde{g}$ on $\tilde{M}$ such that $(\tilde{M}, \tilde{\alpha}, \tilde{g})$ is a K-contact manifold of rank k. q.e.d.

Definition 3.7. Let (M, α, g) be a K-contact manifold of rank k and $\varphi^{-1} \circ \varphi_N : S^1 \times B(a_1, \dots, a_n)_{1+\delta} \to M$ a T^k-equivariant contact embedding. Then the K-contact manifold $(\tilde{M}, \tilde{\alpha}, \tilde{g})$ of rank k is obtained by replacing the image $\varphi^{-1} \circ \varphi_N(S^1 \times B(a_1, \dots, a_n)_{1+\delta})$ with $D_b^2 \times S^{2n-1}$. This procedure is called a K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$ of (M, α, g).

Remark 3.8. (1) Applying K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$ to $(2n+1)$-dimensional K-contact manifold of rank $n+1$, the number of the closed orbits of the K-contact flow increases by $n-1$ at a time. In the case of $(2n+1)$-dimensional K-contact manifold of rank n, a similar phenomenon is also realized, if we take the weight $N = (\zeta_1, \dots, \zeta_n)$ such that $g.c.d(\zeta_1, \dots, \zeta_n) = 1$ and the vectors $(\zeta_1, \dots, \zeta_n)$, $(m_{11}, \dots, m_{n1}), \dots, (m_{1n-1}, \dots, m_{nn-1})$ are an orthonormal basis in $\boldsymbol{R}^n$.

(2) In general we can take only sufficiently small one as the radius ε of the T^k-invariant neighborhood $S^1 \times D_\varepsilon^{2n}$ in Proposition 3.2, so that we have to take sufficiently large one as the weight $N = (\zeta_1, \dots, \zeta_n)$ of K-contact surgery. It seems important to clarify the both relation in concrete examples.

(3) It is possible to perform a reverse operation of the above K-contact surgery in some situation. Let (M, α, g) be a $(2n+1)$-dimensional K-contact manifold of rank k. Consider a unit sphere $S^{2n-1} = \{(w_1, \dots, w_n) \in \boldsymbol{C}^n \mid \sum_{i=1}^n w_i \overline{w}_i = 1\}$ and the K-contact form $\alpha_A = \sqrt{-1}/2 \sum_{i=1}^n 1/a_i(w_i\, d\overline{w}_i - \overline{w}_i\, dw_i)$ in (3.8) and the T^k-action restricted to the factor $(w_1, \dots, w_n)$ of (3.10) on it. Assume that there exists a T^k-equivariant contact embedding $i : S^{2n-1} \hookrightarrow M$. If $i(S^{2n-1})$ has a T^k-invariant open neighborhood which is T^k-equivariantly diffeomorphic to $(D_b^2 \times S^{2n-1}, \tilde{\alpha}_N)$ by the map $\overline{\Phi}$, then we see that there exists a K-contact form $\bar{\alpha}$ on $\overline{M} = M \backslash (\overline{\Phi}^{-1}(\{0\} \times S^{2n-1})) \cup S^1 \times B(a_1, \dots, a_n)_{1+\delta}$ by Proposition 3.5.

4. 5-dimensional K-contact manifolds of rank 3

In this section we will study the T^k-equivariant diffeomorphism type and the diffeomorphism type of the result of a K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$.

First we will consider the T^k-equivariant diffeomorphism type. Let M_1 and M_2 be two n-manifolds with an effective T^k-actions (we call them T^k-manifold). Suppose that there exist orbits O_1 and O_2 which are diffeomorphic to S^1 on M_1 and M_2. Then there is an operation to obtain a new T^k-manifold as follows ([6], [10]). Using the T^k-invariant Riemannian metric, for $i = 1, 2$, we take a framing of the normal bundle v_i of O_i in M_i. Then v_i is T^k-equivariantly diffeomorphic to $S^1 \times D^{n-1}$. Each orbit O_i has a T^k-invariant tubular neighborhood U_i which is diffeomorphic to the above $S^1 \times D^{n-1}$. Removing T^k-invariant tubular neighborhoods $S^1 \times D^{n-1}$ of O_1 and O_2 from M_1 and M_2, attach their complements along boundaries $S^1 \times S^{n-2}$ by a T^k-equivariant diffeomorphism. The resulting manifold is also a T^k-manifold and is denoted by $M_1 \sharp_{S^1} M_2$.

The operation of K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$ can also be formulated as above. We take a K-contact manifold (M, α, g) of rank k and $S^{2n+1} = \{(w_0, \dots, w_n) \in \boldsymbol{C}^{n+1} \mid \sum_{j=0}^n w_j \overline{w}_j = 1\}$ with the T^k-action defined by

$$(t_0, t_1, \dots, t_{k-1}) \cdot (w_0, \dots, w_n) = (t_0 w_0, t_0^{\zeta_1} t_1^{m_{11}} \cdots t_{k-1}^{m_{1k-1}} w_1, \dots, t_0^{\zeta_n} t_1^{m_{n1}} \cdots t_{k-1}^{m_{nk-1}} w_n)$$

for $(t_0, \dots, t_n) \in T^k$ and $\zeta_1, \dots, \zeta_n$, $m_{11}, \dots, m_{nk-1} \in \boldsymbol{Z}$. Moreover take the closed orbit of the K-contact flow on M and $S^1 = \{(w_0, \dots, w_n) \in S^{2n+1} \mid w_1 = \cdots = w_n = 0\} \subset S^{2n+1}$ as T^k-orbits to be diffeomorphic to S^1. Then we obtain a T^k-manifold $M \sharp_{S^1} S^{2n+1}$, which is T^k-equivariantly diffeomorphic to the manifold $\tilde{M}$ obtained by the K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$. (Note that the complement of the T^k-invariant tubular neighborhood of S^1 in S^{2n+1} is T^k-equivariantly diffeomorphic to $D_b^2 \times S^{2n-1}$.) Thus the following proposition holds.

PROPOSITION 4.1. *Let (M, α, g) be a $(2n+1)$-dimensional K-contact manifold of rank k. Then $M \sharp_{S^1} S^{2n+1} \sharp_{S^1} \cdots \sharp_{S^1} S^{2n+1}$ carries a structure of K-contact manifold of rank k.*

To study the diffeomorphism type of the result of a K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$, we need the following notation.

DEFINITION 4.2 ([6]). Let S^1 be a smooth embedded circle in a smooth n-manifold M. A framing $f_1, \dots, f_{n-1}$ of the normal bundle of S^1 is called canonical if and only if there exists a 2-disk D^2 smooth embedded in M with boundary S^1 and the frames $f_1, \dots, f_{n-2}$ can be extended to a framing of the normal bundle of D^2. A framing of the normal bundle of S^1 is called twisted if and only if it is not canonical.

Under the above notation, we have the following:

LEMMA 4.3 ([6]). (1) *Let M_1 and M_2 be two n-dimensional T^k-manifolds with T^k-orbits O_1 and O_2 which are diffeomorphic to S^1. Suppose an embedded S^1 bounds a disk in each of M_1 and M_2. Then* (a) *If framings of O_1 and O_2 are cononical, then $M_1 \sharp_{S^1} M_2$ is diffeomorphic to $M_1 \sharp M_2 \sharp (S^2 \times S^{n-2})$.* (b) *If a framing of O_1 is twisted and that of O_2 is canonical, then $M_1 \sharp_{S^1} M_2$ is diffeomorphic to $M_1 \sharp M_2 \sharp (S^2 \tilde{\times} S^{n-2})$. Here $S^2 \tilde{\times} S^{n-2}$ is the non-trivial oriented S^{n-2}-bundle over S^2.*

(2) *If the second Stiefel-Whitney class $w_2(M)$ of n-manifold M is non-zero and $n \geq 5$, then $M \sharp (S^2 \tilde{\times} S^{n-2})$ is diffeomorphic to $M \sharp (S^2 \times S^{n-2})$.*

Let (M, α, g) be a $(2n+1)$-dimensional K-contact manifold of rank k. Let $\varphi^{-1} \circ \varphi_N : S^1 \times B(a_1, \dots, a_n)_{1+\delta} \to M$ be the T^k-equivariant contact embedding in Definition 3.7 which is determined by the framing of the closed orbit O of the K-contact flow. Suppose dim $M \geq 5$ and the embedded S^1 by the map restricted to $S^1 \times \{0\}$ of $\varphi^{-1} \circ \varphi_N$ bounds a 2-disk. From Lemma 4.3 together with Theorem 3.6 we have the following:

COROLLARY 4.4. (1) *The case of $w_2(M) = 0$. If the framing of the closed orbit O of the K-contact flow is canonical, then a manifold $\tilde{M}$ obtained by a K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$ is diffeomorphic to $M \sharp (S^2 \times S^{2n-1})$. If the framing of O is twisted, then $\tilde{M}$ is diffeomorphic to $M \sharp (S^2 \tilde{\times} S^{2n-1})$.*

(2) *The case of $w_2(M) \neq 0$. $\tilde{M}$ is always diffeomorphic to $M \sharp (S^2 \times S^{2n-1})$.*

Remark 4.5. (1) By suitably choosing the weight $N = (\zeta_1, \dots, \zeta_n)$, both type of (1) in Corollary 4.4 is realized.

(2) If the embedded S^1 to any closed orbit of the K-contact flow bounds a disk, then $\tilde{M}$ obtained by a K-contact surgery is always diffeomorphic either to $M \sharp (S^2 \times S^{2n-1})$ or to $M \sharp (S^2 \tilde{\times} S^{2n-1})$.

As an application, we classify all manifolds which are diffeomorphic to closed simply connected 5-dimensional K-contact manifold of rank 3. They are closed simply connected 5-manifolds with an effective T^3-action, which are classified by Oh [10].

Theorem 4.6 (Oh). *Let M be a closed simply connected 5-manifold with an effective T^3-action. Suppose that the number of the T^3-orbits which is diffeomorphic to S^1 equals to r. Then M is diffeomorphic to*

$$S^5 \quad \text{if } r = 3,$$

$$\underbrace{(S^2 \times S^3)\sharp \cdots \sharp (S^2 \times S^3)}_{r-3} \quad \text{if } w_2(M) = 0,$$

$$(S^2 \tilde{\times} S^3)\sharp \underbrace{(S^2 \times S^3)\sharp \cdots \sharp (S^2 \times S^3)}_{r-4} \quad \text{if } w_2(M) \neq 0.$$

Here $w_2(M)$ is the second Stiefel-Whitney class of M and $S^2 \tilde{\times} S^3$ denotes the nontrivial oriented S^3-bundle over S^2.

Remark 4.7. Let M be a closed simply connected 5-manifold which admits a structure of K-contact manifold of rank 3 with r distinct closed orbits of the K-contact flow. Then $r \geq 3$. Moreover if $w_2(M) \neq 0$, then $r \geq 4$.

From Theorem 4.6 together with Corollary 4.4, we have the following theorem:

Theorem 4.8. *Let M be a closed simply connected 5-manifold. Then M admits a structure of K-contact manifold of rank 3 with r distinct closed orbits of the K-contact flow if and only if M is diffeomorphic to*

$$S^5 \quad \text{if } r = 3,$$

$$\underbrace{(S^2 \times S^3)\sharp \cdots \sharp (S^2 \times S^3)}_{r-3} \quad \text{if } w_2(M^5) = 0,$$

$$(S^2 \tilde{\times} S^3)\sharp \underbrace{(S^2 \times S^3)\sharp \cdots \sharp (S^2 \times S^3)}_{r-4} \quad \text{if } w_2(M^5) \neq 0.$$

Proof. The effective T^3-action on 5-manifold has no $T^2 \times \boldsymbol{Z}/p\boldsymbol{Z}$ as an isotropy group ([10]). Therefore if the T^3-orbit of $x \in M$ is T^3-equivariantly diffeomorphic to S^1, the isotropy group $(T^3)_x$ is isomorphic to T^2. It follows that the number of the closed orbits of the K-contact flow coincides with the number of the T^3-orbit which are diffeomorphic to S^1. For rationally independent positive constants λ_0, λ_1, λ_2, consider the contact form $\alpha_\lambda = \sqrt{-1}/2 \cdot \sum_{j=0}^2 \lambda_j (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j)$ on $S^5 = \{(z_0, z_1, z_2) \in \boldsymbol{C}^3 \mid \sum_{j=0}^2 z_j \bar{z}_j = 1\}$ and the T^3-action on it given by

$$(t_0, t_1, t_2) \cdot (z_0, z_1, z_2) = (t_0 z_0, t_1 z_1, t_2 z_2),$$

where $(t_0, t_1, t_2) \in T^3$. Then the closure of the orbit of contact flow of arbitrary point $z \in S^5$ coincides with the T^3-orbit of z. Thus there exists a metric g_λ such that $(S^5, \alpha_\lambda, g_\lambda)$ is the K-contact manifold of rank 3 (See [11], [12]). The number of T^3-orbits which is diffeomorphic to S^1 equals to 3. It follows that if we apply K-contact surgery of weight $N = (\zeta_1, \dots, \zeta_n)$ to $(S^5, \alpha_\lambda, g_\lambda)$, then we obtain the K-contact manifold of rank 3, by Corollary 4.4 and Remark 4.5, which is diffeomorphic to $S^2 \times S^3$ or $S^2 \tilde{\times} S^3$. By Remark 3.9 the number of the closed orbits of the K-contact flow increase by 1, and hence it equals 4. Repeat K-contact surgery. Consequently we obtain the K-contact manifold of rank 3 which is diffeomorphic to $\underbrace{(S^2 \times S^3)\sharp \cdots \sharp(S^2 \times S^3)}_{r-3}$ or $(S^2 \tilde{\times} S^3)\sharp \underbrace{(S^2 \times S^3)\sharp \cdots \sharp(S^2 \times S^3)}_{r-4}$ and has closed orbits of the K-contact flow to be equal to r. Hence by Theorem 4.6 we obtain Theorem 4.8. q.e.d.

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Department of Mathematics
Faculty of Science
Saitama University
Shimo-Okubo 255
Saitama, 338-8570
Japan
E-mail: tyamazak@rimath.saitama-u.ac.jp