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ŁOJASIEWICZ EXPONENT AT INFINITY IN $\boldsymbol{C}[x, y, z]$

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Abstract

We consider the set $\mathscr{F} = \{p(z)x + q(y,z), p \in \boldsymbol{C}[z] \backslash \{0\}, q \in \boldsymbol{C}[y,z]\}$. We connect algebraic properties of a polynomial $f \in \mathscr{F}$, such that f is a variable in $\boldsymbol{C}[x,y,z]$ or f is a tame variable in $\boldsymbol{C}[z][x,y]$ with the Łojasiewicz exponent at infinity of f. We compute this exponent for some polynomials of $\mathscr{F}$.

1. Introduction

Let A be a commutative ring (in this paper A will be $\boldsymbol{C}$ or $\boldsymbol{C}[z]$) and let $A^{[n]} = A[x_1, \dots, x_n]$ be the A-algebra of polynomials in n indeterminates. We say that an automorphism σ of the A-algebra $A^{[n]}$ is *triangular* if, for all i, $\sigma(x_i) = a_i x_i + P_i(x_{i+1}, \dots, x_n)$ where a_i is a unit in A and $P_i \in A[x_{i+1}, \dots, x_n]$. An automorphism is *tame* if it is in the subgroup generated by affine and triangular automorphisms. We denote by $V_n(A)$ the set of polynomials of $A^{[n]}$ which are components of an automorphism of $A^{[n]}$, we call them *variables*. In a same way, we denote by $AV_n(A)$ (resp. $BV_n(A)$, resp. $TV_n(A)$) the set of affine (resp. triangular, resp. tame) variables of $A^{[n]}$ *i.e.* components of an affine (resp. triangular, resp. tame) automorphism.

For a polynomial $f \in \boldsymbol{C}[x_1, \dots, x_n]$, we consider $\operatorname{grad} f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$. We denote by $W_n(\boldsymbol{C})$ the set of polynomials of $\boldsymbol{C}^{[n]}$ without critical value (*i.e.* such that $\operatorname{grad} f$ is nowhere vanishing).

If $f \in W_n(\boldsymbol{C})$, one defines the *Łojasiewicz exponent at infinity*, $L_\infty(f)$, to be the supremum of the set

$$\{v \in \boldsymbol{R} \mid \exists A > 0, \exists B > 0, \forall x \in \boldsymbol{C}^m, \text{ if } \|x\| \geq B, \text{ then } A\|x\|^v \leq \|\operatorname{grad} f(x)\|\}$$

This original analytic definition is equivalent to the following more algebraic one (cf. [PZ] 2.1). We set $\mathscr{A}^n = \{\psi \in (\boldsymbol{C}\{t, t^{-1}\})^n; \operatorname{ord}(\psi) < 0\}$, where $\operatorname{ord}(\psi)$ is the t-adic valuation of ψ. Let $f \in W_n(\boldsymbol{C})$, for $\psi \in \mathscr{A}^n$, we set:

$$L(f, \psi) = \frac{\operatorname{ord}(\operatorname{grad} f)(\psi)}{\operatorname{ord}(\psi)}.$$

We have: $L_\infty(f) = \inf\{L(f, \psi); \psi \in \mathscr{A}^n\}$.

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For an indeterminate $x = x_i$, we define the *x-partial Łojasiewicz exponent* of $f \in W_n(\boldsymbol{C})$ as $L_\infty^x(f) = \inf\{L(f,\psi); \psi \in \mathscr{A}_x^n\}$ where $\mathscr{A}_x^n = \{\psi = (\psi_j)_{1 \le j \le n} \in \mathscr{A}^n; \operatorname{ord}(\psi_i) \ge 0\}$.

When $n = 3$, we set $x = x_1$, $y = x_2$, $z = x_3$, $\mathscr{A} = \mathscr{A}^3$ and $\mathscr{A}_y = \mathscr{A}_y^3$.

For a polynomial $f \in W_2(\boldsymbol{C})$, the number $L_\infty(f)$ has an algebraic significance. Precisely, we have the following two theorems (cf. [N] Theorem 0.4 for the equality in 1) of Theorem 1, cf. [CK1] Theorem 10.2 for Theorem 1 and cf. [H] Proposition 1.5.1 and [CK1] Remark 11.4 for Theorem 2):

Theorem 1. *Let $f \in W_2(\boldsymbol{C})$, the following assumptions are equivalent*:
1) $f \in V_2(\boldsymbol{C}) = TV_2(\boldsymbol{C})$,
2) $L_\infty(f) > -1$.

Theorem 2. *We have*: $L_\infty(W_2(\boldsymbol{C})) = \boldsymbol{Q} \setminus \{-1\}$.

In the three dimensional case, the authors of [PZ] exhibit a family $\mathscr{P} \subset TV_2(\boldsymbol{C}[z])$ such that $L_\infty(\mathscr{P}) = \boldsymbol{Q}$. This shows that Theorem 1, Theorem 2 can not be extended to this case. Modulo a permutation of coordinates $\mathscr{P} = \{zx + y - 3y^{2n+1}z^{2q} + 2y^{3n+1}z^{3q}; n, q \in \boldsymbol{N}\setminus\{0\}\}$. In spite of this negative observation, we try to find a relation between algebraic properties of a polynomial of $\boldsymbol{C}^{[3]}$ and its Łojasiewicz exponent at infinity. We restrict our study to the family $\mathscr{F} = BV_2(\boldsymbol{C}(z)) \cap \boldsymbol{C}[x,y,z]$ because for $f \in \mathscr{F}$ there exists criteria to check $f \in V_3(\boldsymbol{C})$ and $f \in TV_2(\boldsymbol{C}[z])$. We have: $\mathscr{P} \subset \mathscr{F}$. For $f = p(z)x + q(y,z) \in \mathscr{F}$, we set $\tilde{f} = p(z)x + \tilde{q}(y,z)$ where $\tilde{q}$ is the remainder of the division of q by p in $\boldsymbol{C}[y][z]$. We remark that $\tilde{f}$ is the image of f by $\tau = (x + (\tilde{q}(y,z) - q(y,z))p(z)^{-1}, y, z)$ which is a triangular automorphism of $\boldsymbol{C}[x,y,z]$. We have: $\tilde{\mathscr{P}} = \{zx + y\}$ and $\tilde{\mathscr{F}} = \{p(z)x + q(y,z), p \in \boldsymbol{C}[z], q \in \boldsymbol{C}[y,z]; \deg_z q < \deg p\} \subset \mathscr{F}$.

In section 2, we prove the following result:

Theorem 3. *Let $f \in \mathscr{F} \cap W_3(\boldsymbol{C})$, the following assumptions are equivalent*:
1) $f \in TV_2(\boldsymbol{C}[z])$,
2) $L_\infty(\tilde{f}) = 0$.

Theorem 3 shows that for $f \in \mathscr{F} \cap W_3(\boldsymbol{C})$ the number $L_\infty(f)$ contains an algebraic information. This information is not directly attainable, it appears with the help of the map $f \mapsto \tilde{f}$. In other words, it is not attached to f but to the orbit of f under the action of triangular automorphisms of $\boldsymbol{C}[x,y,z]$.

In section 3, we make some computations to prove the following results:

Theorem 4. *We have*: $L_\infty(\tilde{\mathscr{F}} \cap W_3(\boldsymbol{C}) \setminus V_3(\boldsymbol{C})) = \boldsymbol{Q} \cap]-\infty, -1[$.

Theorem 5. *We have*: $L_\infty(\tilde{\mathscr{F}} \cap V_3(\boldsymbol{C})) \cup \{-1\} = \boldsymbol{Q} \cap (]-\infty, -1/2[\cup \{0\})$.

Theorem 4 and Theorem 5 can be compared with Theorem 2. The following question is still open:

Question 1. Does there exist $\tilde{f} \in \tilde{\mathscr{F}} \cap W_3(\boldsymbol{C})$ such that $L_\infty(\tilde{f}) = -1$?

Using only Łojasiewicz exponent and the map $f \mapsto \tilde{f}$, we can not differentiate variables from non-variables. However, this is possible with help of y-partial Łojasiewicz exponent, in fact we have:

Theorem 6. *Let $f \in \mathscr{F} \cap W_3(\boldsymbol{C})$, the following assumptions are equivalent*:
1) $f \in V_3(\boldsymbol{C})$,
2) $L_\infty^y(\tilde{f}) \geq 0$.

It would be interesting to connect $L_\infty(\tilde{f})$ to the property $f \in TV_3(\boldsymbol{C})$. But we know nothing about this property, for example the following two questions are open:

Question 2. Do we have $TV_3(\boldsymbol{C}) = V_3(\boldsymbol{C})$?

Question 3. Let $Z_3^1(\boldsymbol{C})$ be the set of component of an automorphism σ of $\boldsymbol{C}[x,y,z]$ such that $\sigma(z) = z$. Do we have $TV_3(\boldsymbol{C}) \cap Z_3^1(\boldsymbol{C}) = TV_2(\boldsymbol{C}[z])$?

An affirmative answer to Question 2 would give a negative answer to Question 3.

2. Proofs

Here is our main result:

Theorem 7. *Let $f \in \tilde{\mathscr{F}} \cap W_3(\boldsymbol{C})$, the following assumptions are equivalent*:
1) $f \in AV_2(\boldsymbol{C}[z])$,
2) $L_\infty(f) = 0$,
3) $L_\infty(f) \geq -1/2$.

Proof. We set $f = p(z)x + q(y,z)$.
We have: grad $f = (p(z), \partial_y q(y,z), p'(z)x + \partial_z q(y,z))$.
1) $\Rightarrow$ 2): We can write $q(y,z) = a(z)y + b(z)$ with $a, b \in \boldsymbol{C}[z]$ and $\gcd(a,p) = 1$. If $p \in \boldsymbol{C}[z] \backslash \boldsymbol{C}$ (resp. $p \in \boldsymbol{C} \backslash \{0\}$), then there exists $z_1 \in \boldsymbol{C}$ such that $p'(z_1) \neq 0$ (resp. $z_1 = 0$). We consider $\psi(t) = (-p'(z_1)^{-1}\partial_z q(t^{-1}, z_1), t^{-1}, z_1) \in \mathscr{A}$ (resp. $\psi(t) = (t^{-1}, 0, 0) \in \mathscr{A}$), then (grad $f)(\psi(t)) = (p(z_1), a(z_1), 0)$. Therefore, ord(grad $f)(\psi(t)) \geq 0$ and $L_\infty(f) \leq L(f, \psi) \leq 0$.
Now, let $\psi(t) = (x(t), y(t), z(t)) \in \mathscr{A}$. Suppose ord(grad $f)(\psi(t)) > 0$, then $\lim_{t \to 0} p(z(t)) = 0$ and $\lim_{t \to 0} a(z(t)) = 0$ which is impossible since $\gcd(a,p) = 1$. Therefore, ord(grad $f)(\psi(t)) \leq 0$ and $L(f, \psi) \geq 0$. Hence $L_\infty(f) \geq 0$ and finally $L_\infty(f) = 0$.
2) $\Rightarrow$ 3): Obvious.
3) $\Rightarrow$ 1): We suppose that $f \notin AV_2(\boldsymbol{C}[z])$ and we prove $L_\infty(f) < -1/2$. There are two cases:

Case 1. There exists a root z_1 of p such that $\partial_y q(y, z_1) \in \boldsymbol{C}[y] \setminus \boldsymbol{C}^*$.

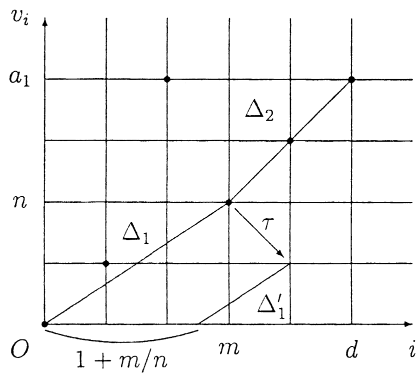
There exists $y_1 \in \boldsymbol{C}$ such that $\partial_y q(y_1, z_1) = 0$. Let $(y(t), z(t)) \in (\boldsymbol{C}\{t\})^2$ be a parametrization of the germ $\partial_y q(y,z) = 0$ in the neighborhood of (y_1, z_1) and let $x(t) = -p'(z(t))^{-1} \partial_z q(y(t), z(t))$. We have: $\operatorname{ord}(x(t)) < 0$ (if $\operatorname{ord}(x(t)) \geq 0$, then $(x(0), y_1, z_1)$ is a critical point of f), thus $\psi(t) = (x(t), y(t), z(t)) \in \mathscr{A}_y$. We have: $\operatorname{ord}(\operatorname{grad} f)(\psi(t)) = \operatorname{ord}(p(t)) = l \operatorname{ord}(z(t) - z_1)$ where l is the multiplicity of z_1 in p. On the other hand, $\operatorname{ord}(\psi(t)) = \operatorname{ord}(x(t)) = \operatorname{ord} \partial_z q(y(t), z(t)) - \operatorname{ord} p'(z(t) - z_1) \geq -(l-1) \operatorname{ord} z(t)$. Hence $L_\infty(f) \leq L(f, \psi) \leq l/(1-l) < -1$.

Case 2. For every root z_1 of p, we have $\partial_y q(y, z_1) \in \boldsymbol{C}^*$ and $d + 1 := \deg_y(q) > 1$.

We write $p(z) = z_0 \prod_{i=1}^{k} (z - z_i)^{l_i}$, with $z_i \in \boldsymbol{C}$ and $l_i \in \boldsymbol{N} \setminus \{0\}$. Let $A_d \in \boldsymbol{C}[z]$ be the term of degree d in $\partial_y q \in \boldsymbol{C}[z][y]$, and let a_i be the vanishing order of z_i in A_d for $1 \leq i \leq k$. If $l_i \leq a_i$ for all $1 \leq i \leq k$, then $\deg(p) = \sum_{i=1}^{k} l_i \leq \sum_{i=1}^{k} a_i \leq \deg_z(A_d) \leq \deg_z(q) < \deg(p)$, which is impossible. Therefore, there exists $i \in \{1, \ldots, k\}$ such that $a_i < l_i$. From now on, we suppose $i = 1$ and $z_i = 0$.

We write $\partial_y q(y, z) = \sum_{i=0}^{d} A_i(z) y^i$ and let $v_i = v_z(A_i) \in \boldsymbol{N}$. We have: $v_0 = 0$ and $v_i \geq 1$ for $1 \leq i \leq d$. Let Δ_1 be the first side of the Newton polygon of $\partial_y q(y, z)$ in the neighborhood of $(\infty, 0)$ and let n/m be its slope (see the picture below). In particular, $n/m \leq a_1/d$ and since $m \leq d$, we have $n \leq a_1 < l_1$.

Let $I = \{i | (i, v_i) \in \Delta_1\}$, for $i \in I$ we set $c_i = (z^{-v_i} A_i)_{z=0}$. Let $c \in \boldsymbol{C}$ be such that $\sum_{i \in I} c_i c^{v_i} = 0$, since $v_0 = 0$ and $v_i \geq 1$ for all $i \geq 1$, we have $c \neq 0$.



We set $g(X, Y) = \partial_y q(X^{-n}, X^m(c + Y)) = \sum_{i=0}^{d} A_i(X^m(c+Y)) X^{-ni}$. By definition of n and m, we have $g(X, Y) \in \boldsymbol{C}[X, Y]$. By definition of c_i and c we have $g(0, Y) = \sum_{i \in I} c_i (c + Y)^{v_i} \neq 0$ and $g(0, 0) = 0$. Thanks to Puiseux's theorem (cf. [BK] or [C]), there exists $u \in \boldsymbol{N}^*$ and $\beta \in \boldsymbol{C}\{t\}$ such that $g(t^u, \beta(t)) = 0$ in $\boldsymbol{C}\{t\}$. We consider $\psi(t) = (x(t), y(t), z(t))$ where $z(t) = t^{um}(c + \beta(t))$, $y(t) = t^{-un}$ and $x(t) = -p'(z(t))^{-1} \partial_z q(y(t), z(t))$. Since $\operatorname{ord}(y(t)) < 0$, we have: $\psi \in \mathscr{A}$.

Since $\partial_y q(y(t), z(t)) = g(t^u, \beta(t)) = 0$ and $p'(z(t)) x(t) + \partial_z q(y(t), z(t)) = 0$, we have: $\operatorname{ord}(\operatorname{grad} f)(\psi(t)) = \operatorname{ord}(p(z(t))) = l_1 um$.

We write $\partial_z q(y, z) = \sum_{i=0}^{d+1} B_i(z) y^i$ and we set $w_i = v_z(B_i) \in \boldsymbol{N}$. The set

$\{(i,w_i); i \geq 1, B_i \neq 0\}$ is the image of $\{(i,v_i); i \geq 1, A_i \neq 0\}$ by the translation $\tau = (1,-1)$. Let Δ_1' the image by τ of Δ_1, Δ_1' meets the (O,i)-axis at $(1+m/n, 0)$. The order of $\{B_i(z(t))y(t)^i | (i,w_i) \in \Delta_1'\}$ is equal to the order of $y(t)^{1+m/n}$ *i.e.* $-un(1+m/n) = -u(n+m)$, therefore $\operatorname{ord}(\partial_z q(z(t),y(t))) \geq -u(n+m)$ (elements of $\{B_i(z(t))y(t)^i | (i,w_i) \notin \Delta_1'\}$ have a bigger order). On the other hand $\operatorname{ord}(p'(z(t))) = um(l_1 - 1)$. Hence:

$$\begin{aligned}
\operatorname{ord}(x(t)) &= -\operatorname{ord}(p'(z(t))) + \operatorname{ord}(\partial_z q(y(t),z(t))) \\
&\geq -um(l_1-1) - u(n+m) = -u(ml_1+n) \\
\operatorname{ord}(\psi(t)) &= \min\{\operatorname{ord}(x(t)), \operatorname{ord}(y(t)), \operatorname{ord}(z(t))\} \\
&\geq \min\{-u(ml_1+n), -un\} = -u(ml_1+n).
\end{aligned}$$

Finally (since $n < l_1$) we have:

$$L_\infty(f) \leq \frac{\operatorname{ord}(\operatorname{grad} f)(\psi(t))}{\operatorname{ord}(\psi(t))} \leq -\frac{uml_1}{u(ml_1+n)} < -\frac{m}{m+1} \leq -1/2.$$

The proof of Theorem 7 is complete.

Proof of Theorem 3. Using Theorem 7, it is enough to prove equivalence between $f \in TV_2(\boldsymbol{C}[z])$ and $\tilde{f} \in AV_2(\boldsymbol{C}[z])$. Since $f \in BV_2(\boldsymbol{C}(z))$ this can be straight inferred from [EV] Proposition 2 which is a consequence of amalgamated structure of $\operatorname{Aut}_{\boldsymbol{C}(z)} \boldsymbol{C}(z)^{[2]}$ (cf, for example [N] Theorem 3.3).

Theorem 8. *Let* $f = p(z)x + q(y,z) \in \mathscr{F} \cap W_3(\boldsymbol{C})$, *the following assumptions are equivalent*:

1) *Every root* z_1 *of* p *is such that* $\partial_y q(y,z_1) \in \boldsymbol{C}^*$,
2) $f \in V_2(\boldsymbol{C}[z])$,
3) $f \in V_3(\boldsymbol{C})$.

Proof. 1) $\Rightarrow$ 2): Assumption 1) is equivalent to say that $q(y,z) = \sum q_i(z)y^i$ with q_1 (resp. q_i $(i \geq 2)$) unit (resp. nilpotent) modulo $p\boldsymbol{C}[z][y]$ and the Russell-Sathaye's theorem (cf. [R] Proposition 2.2) implies $f \in V_2(\boldsymbol{C}[z])$.

2) $\Rightarrow$ 3): Obvious.

3) $\Rightarrow$ 1): Let $Z = \{z_1, \ldots, z_n\}$ be the set of roots of p. For all $t \in \boldsymbol{C}$, the polynomial $f - t$ is a variable thus the surface $S_t = \{(x,y,z) \in \boldsymbol{C}^3; f(x,y,z) = t\}$ is isomorphic to $\boldsymbol{C}^2$ and $\chi(S_t) = \chi(\boldsymbol{C}^2) = 1$ (Euler's characteristics).

The map $(y,z) \mapsto (p(z)^{-1}(t - q(y,z)), y, z)$ is a homeomorphism between $\boldsymbol{C}^2 \backslash (\boldsymbol{C} \times Z)$ and $S_t \backslash (\boldsymbol{C}^2 \times Z)$, thus $\chi(S_t \backslash (\boldsymbol{C}^2 \times Z)) = \chi(\boldsymbol{C}^2 \backslash (\boldsymbol{C} \times Z))$. Since $\chi(S_t) = \chi(S_t \backslash (\boldsymbol{C}^2 \times Z)) + \chi(S_t \cap (\boldsymbol{C}^2 \times Z))$ and $\chi(\boldsymbol{C}^2) = \chi(\boldsymbol{C}^2 \backslash (\boldsymbol{C} \times Z)) + \chi(\boldsymbol{C} \times Z)$, we have: $\chi(S_t \cap (\boldsymbol{C}^2 \times Z)) = \chi(\boldsymbol{C} \times Z) = \chi(Z) = n$.

On the other hand $\chi(S_t \cap (\boldsymbol{C}^2 \times Z)) = \sum_{i=1}^n \chi(q(y,z_i) = t)$ and for a generic t we have $\chi(S_t \cap (\boldsymbol{C}^2 \times Z)) = \sum_{i=1}^n \deg(q(y,z_i))$. Finally $\sum_{i=1}^n \deg(q(y,z_i)) = n$. For $1 \leq i \leq n$, we have $\deg(q(y,z_i)) \geq 1$ (if there exists i such that $q(y,z_i) = t \in \boldsymbol{C}$

then $z - z_i$ divises $f - t$ which is impossible) hence $\deg(q(y, z_i)) = 1$ for all $1 \leq i \leq n$ which proves 1).

Proof of Theorem 6. Suppose $f \in V_3(\boldsymbol{C})$ and let $\psi(t) = (x(t), y(t), z(t)) \in \mathscr{A}_y$.

Suppose $\operatorname{ord}(\operatorname{grad} f)(\psi(t)) > 0$, then $\lim_{t\to 0} p(z(t)) = 0$ *i.e.* $\lim_{t\to 0} z(t) = z_1$ where z_1 is a root of p. Since $\operatorname{ord}(y) \geq 0$, we have $0 = \lim_{t\to 0} \partial_y q(y(t), z(t)) = \partial_y q(y(0), z_1)$ which contradicts Theorem 8. Therefore, $\operatorname{ord}(\operatorname{grad} f)(\psi(t)) \leq 0$ and $L(f, \psi) \geq 0$. Hence $L_\infty^y(f) \geq 0$.

Now, suppose $f \notin V_3(\boldsymbol{C})$, by Theorem 8, p has a root z_1 such that $\partial_y q(y, z_1) \in \boldsymbol{C}[y] \backslash \boldsymbol{C}^*$. The Case 1 of Theorem 7 implies $L_\infty^y(f) < -1$.

Examples. $L_\infty^y(z^2x + z + y^2) = -2$, $L_\infty^y(z^2x + y) = 0$ and $L_\infty^y(zx + y) = 1$.

3. Computations

In this section, we explain how to compute $L_\infty(f)$ for $f \in \tilde{\mathscr{F}}$.

Let $f = p(z)x + q(y, z) \in \tilde{\mathscr{F}}$.

If $f \in AV_2(\boldsymbol{C}[z])$, then $L_\infty(f) = 0$.

Now suppose $f \notin AV_2(\boldsymbol{C}[z])$, we have $L_\infty(f) < 0$ thus:

$$L_\infty(f) = \inf\left\{L(f, \psi); \psi \in \mathscr{A}; \lim_{t\to 0} \|(\operatorname{grad} f)(\psi(t))\| = 0\right\} \quad (*).$$

We write $p(z) = z_0 \prod_{i=1}^k (z - z_i)^{l_i}$, with $z_i \in \boldsymbol{C}$ and $l_i \in \boldsymbol{N} \backslash \{0\}$.

Let $L_i = \inf\{L(f, \psi); \psi = (x, y, z) \in \mathscr{A}, z \to z_i\}$ for $1 \leq i \leq k$, since $(*)$ we have: $L_\infty(f) = \min\{L_i; 1 \leq i \leq k\}$.

To compute L_1 (for example) one can suppose $z_1 = 0$. Let $l = l_1 = v_z(p(z))$.

Lemma (The way to choose $x(t)$). *Let* $\mathscr{E}_{a,b} = \{(\operatorname{ord} \partial_y q(y, z), \operatorname{ord} \partial_z q(y, z));$ $\operatorname{ord}(y) = a, \operatorname{ord}(z) = b\}$,

$\mathscr{D}_{a,b} = \{(\lambda, \mu) \in \mathscr{E}_{a,b}; \mu - (l-1)b < -1\}$,

$\mathscr{B}_{a,b}^- = \{(\lambda, \mu) \in \mathscr{E}_{a,b}; \mu - (l-1)b \leq -1\}$,

$\mathscr{B}_{a,b}^+ = \{(\lambda, \mu) \in \mathscr{E}_{a,b}; \mu - (l-1)b > -1\}$.

1. *If* $a < 0$, *we set*:

$M_{a,b} = \min\{\min\{lb, \lambda, \mu\}/a; (\lambda, \mu) \in \mathscr{E}_{a,b}\}$ *and*

$$N_{a,b} = \min\left\{\frac{\min\{lb, \lambda\}}{\min\{\mu - (l-1)b, a\}}; (\lambda, \mu) \in \mathscr{E}_{a,b}\right\}.$$

2. *If* $a \geq 0$, *we set*:

$M_{a,b} = \min\{-\min\{lb, \lambda, \mu\}; (\lambda, \mu) \in \mathscr{D}_{a,b}\}$ *and*

$N_{a,b} = \min\{N_{a,b}^-, N_{a,b}^+\}$ *with*

$N_{a,b}^+ = -\min\{\lambda, (l-1)b - 1\}$ *and*

$$N^-_{a,b} = \min\left\{\frac{\min\{lb,\lambda\}}{\mu-(l-1)b}; (\lambda,\mu)\in\mathscr{B}^-_{a,b}\right\}.$$

Then $L_1 = \min\{M_{a,b}, N_{a,b}; a\in\boldsymbol{Z}, b\in\boldsymbol{N}\backslash\{0\}\}$.

Proof. Let $\psi = (x,y,z)\in\mathscr{A}$ be such that $\mathrm{ord}(y) = a\in\boldsymbol{Z}$ and $\mathrm{ord}(z) = b\in\boldsymbol{N}\backslash\{0\}$. Let $(\lambda,\mu) = (\mathrm{ord}\,\partial_y q(y,z), \mathrm{ord}\,\partial_z q(y,z))$ We have:

$$L(f,\psi) = \frac{\min\{lb,\lambda,\mathrm{ord}(p'(z)x+\partial_z q(y,z))\}}{\min\{\mathrm{ord}(x),a\}}.$$

1. If $a<0$, there are two cases:
   1.1. If $\mathrm{ord}(x) > \mu-(l-1)b$, then
   $$L(f,\psi) = \frac{\min\{lb,\lambda,\mu\}}{\min\{\mathrm{ord}(x),a\}} \geq \frac{\min\{lb,\lambda,\mu\}}{a}$$
   with equality, for example, when $x=0$.
   1.2. If $\mathrm{ord}(x)\leq\mu-(l-1)b$, then
   $$L(f,\psi) = \frac{\min\{lb,\lambda,\mathrm{ord}(p'(z)x+\partial_z q(y,z))\}}{\min\{\mathrm{ord}(x),a\}} \geq \frac{\min\{lb,\lambda\}}{\min\{\mu-(l-1)b,a\}}$$
   with equality, for example, when $x = -\partial_z q(y,z)/p'(z)$.
2. If $a\geq 0$, then $\mathrm{ord}(x)<0$.
   2.1. The case $\mathrm{ord}(x) > \mu-(l-1)b$ can occur if and only if $(\lambda,\mu)\in\mathscr{D}_{a,b}$ and then
   $$L(f,\psi) = \frac{\min\{lb,\lambda,\mu\}}{\mathrm{ord}(x)} \geq -\min\{lb,\lambda,\mu\}$$
   with equality, for example, when $x = t^{-1}$.
   2.2. The case $\mathrm{ord}(x)\leq\mu-(l-1)b$ can be dealt with in the same way as when $a<0$ if $(\lambda,\mu)\in\mathscr{B}^-_{a,b}$, but if $(\lambda,\mu)\in\mathscr{B}^+_{a,b}$, then
   $$L(f,\psi) = \frac{\min\{\lambda,(l-1)b+\mathrm{ord}(x)\}}{\mathrm{ord}(x)} \geq -\min\{\lambda,(l-1)b-1\}$$
   with equality, for example, when $x=t^{-1}$.

This prove that $L_1 = \min\{M_{a,b}, N_{a,b}; a\in\boldsymbol{Z}, b\in\boldsymbol{N}\backslash\{0\}\}$.

Remark. In the lemma, we can change $\mathscr{E}_{a,b}$ to the set of his maximal elements.

Proposition 1. *For* $l\geq 2$, *we have* $L_\infty(z^l x+y^2+z) = -l/(l-1)$.

Proof. Let $f = z^l x+y^2+z$ and $q = y^2+z$ then $\partial_y q = 2y$ and $\partial_z q = 1$. We have: $\mathscr{E}_{a,b} = \{(a,0)\}$ then $M_{a,b}=0$ and

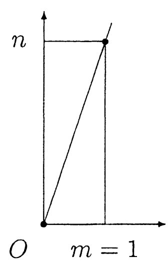
$$N_{a,b} = \frac{\min\{lb, a\}}{\min\{(1-l)b, a\}}.$$

Hence $L_\infty(f) = -l/(l-1)$.

Remark. If $\partial_y q(y, 0) \in \boldsymbol{C}[y] \setminus \boldsymbol{C}^*$, we write $\partial_y q(y, z) = \sum_{i=0}^{d} A_i(z) y^i$ and $v_i = v_z(A_i) \in \boldsymbol{N}$. We consider the sides Δ_i $(1 \le i \le k)$ of the Newton polygon of $\partial_y q(y, z)$ in the neighborhood of $(\infty, 0)$ and let n_i/m_i their slopes.

Let $\mathscr{E}'_{i,u} = \mathscr{E}_{-un_i, un_i}$, $M'_{i,u} = M_{-un_i, un_i}$ and $N'_{i,u} = N_{-un_i, un_i}$ then: $L_1 = \min\{M'_{i,u}, N'_{i,u}; 1 \le i \le k, u \in \boldsymbol{N} \setminus \{0\}\}$.

Proposition 2. *For $1 \le n < l$, we have $L_\infty(z^l x + y + z^n y^2) = -l/(l+n)$.*



Proof. Let $f = y + y^2 z^n + x z^l$ and $q = y + y^2 z^n$ then $\partial_y q = 1 + 2yz^n$ and $\partial_z q = n y^2 z^{n-1}$.

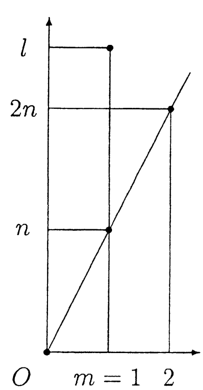
The Newton polygon of $\partial_y q(y, z)$ in the neighborhood of $(\infty, 0)$ has only one side Δ_1, its slope is $n/1$.

The unique maximal element of $\mathscr{E}'_{1,u}$ is $(\infty, -u(n+1))$, thus $M'_{1,u} \ge 0$ and $N'_{1,u} = -l/(n+l)$.

Hence $L_\infty(f) = -l/(n+l)$.

Remark. Theorem 8 shows that for $1 \le n < l$, $z^l x + y + z^n y^2 \in V_3(\boldsymbol{C})$.

Proposition 3. *For $1 \le n < l/2$, we have $L_\infty(z^{l+1} x + y + z^n y^2(-2 + z^n y + 2n z^{l-n})) = -(l-n)/n$.*



Proof. Let $f = z^{l+1}x + y + z^n y^2(-2 + z^n y + 2nz^{l-n})$ and $q = y + z^n y^2 \cdot (-2 + z^n y + 2nz^{l-n})$ then $\partial_y q = 1 - 4yz^n + 3y^2 z^{2n} + 4nyz^l$ and $\partial_z q = 2ny^2 z^{n-1} \cdot (-1 + yz^n + lz^{l-n})$.

The Newton polygon of $\partial_y q(y, z)$ in the neighborhood of $(\infty, 0)$ has only one side Δ_1, his slope is $n/1$.

Let c be a root of $C(T) = 1 - 4T^n + 3T^{2n}$ we consider: $g_c(X, Y) = \partial_y q(X^{-n}, X(c + X))$ and $h_c(X, Y) = X^{n+1}\partial_z q(X^{-n}, X(c + X))$ then $g_c(X, Y) = 1 - 4(c + Y)^n + 3(c + Y)^{2n} + 4nX^{l-n}(c + Y)^l$ $h_c(X, Y) = 2n(c + Y)^{n-1} \cdot (-1 + (c + Y)^n + lX^{l-n}(c + Y)^{l-n})$.

If $c^n \neq 1$, then the germ $\{h_c = 0\}$ is empty and this case give no maximal element of $\mathscr{E}'_{1,u}$.

If $c^n = 1$ we consider: $g_{c,c_1}(X, Y) = X^{n-l}g_c(X, X^{l-n}(c_1 + Y))$ and $h_{c,c_1}(X, Y) = X^{n-l}h_c(X, X^{l-n}(c_1 + Y))$.

We have $g_{c,c_1}(0, 0) = 2nc^{n-1}c_1 + 4nc^l$ and $h_{c,c_1}(0, 0) = 2n(nc^{n-1}c_1 + lc^l)$.

For $g_{c,c_1}(0, 0) = h_{c,c_1}(0, 0) = 0$ we must have $2nc = lc^l$ which is impossible since $n < l/2$. Therefore, one of the two germs $\{g_{c,c_1} = 0\}$ or $\{g_{c,c_1} = 0\}$ is empty. Thus $(\infty, u(l - 2n - 1))$ and $(n - l, \infty)$ are the maximal elements of $\mathscr{E}'_{1,u}$.

Hence $M_{1,u} = \min\{-(l - 2n - 1)/n, -(l - n)/n\} = -(l - n)/n$ and $N_{1,u} = \min\{-(l + 1)/(2n + 1), -(l - n)/n\} = -(l - n)/n$, hence $L_\infty(f) = \min\{-l/(n + l), -(l - n)/n\} = -(l - n)/n$.

Remark. Theorem 8 shows that for $1 \leq n < l/2$, $z^{l+1}x + y + z^n y^2(-2 + z^n y + 2nz^{l-n}) \in V_3(\boldsymbol{C})$.

Proposition 4. *For $1 \leq n < l/2$ and $k \geq 2$, we have $L_\infty(z^{l+1}(z - 1)^k x + y + z^n y^2(-2 + z^n y + 2nz^{l-n})) = \min\{-(l - n)/n, k/(1 - k)\}$.*

Proof. Let $f = z^{l+1}(z - 1)^k x + y + z^n y^2(-2 + z^n y + 2nz^{l-n})$.

Let $L_0 = \inf\{L(f, \psi); \psi = (x, y, z) \in \mathscr{A}, z \to 0\}$ and $L_1 = \inf\{L(f, \psi); \psi = (x, y, z) \in \mathscr{A}, z \to 1\}$.

As in Proposition 1 we can compute $L_1 = k/(1 - k)$, as in Proposition 3 we can compute $L_0 = -(l - n)/n$ and $L_\infty(f) = \min\{L_0, L_1\}$.

Remark. Theorem 8 shows that for $1 \leq n < l/2$ and $k \geq 2$, $z^{l+1}(z - 1)^k x + y + z^n y^2(-2 + z^n y + 2nz^{l-n}) \notin V_3(\boldsymbol{C})$.

Remark. For $f \in W_n(\boldsymbol{C})$, we have: $L_\infty(f) \in \boldsymbol{Q}$ cf. [CK2].

Proof of Theorem 4. Let $f \in \tilde{\mathscr{F}} \cap W_3(\boldsymbol{C}) \backslash V_3(\boldsymbol{C})$, by Theorem 8, p has a root z_1 such that $\partial_y q(y, z_1) \in \boldsymbol{C}[y] \backslash \boldsymbol{C}^*$. The Case 1 of Theorem 7 implies $L_\infty(f) < -1$.

Conversely, Proposition 4 shows that for every rational number $r < -1$ there exists $f \in \tilde{\mathscr{F}} \cap W_3(\boldsymbol{C}) \backslash V_3(\boldsymbol{C})$ such that $L_\infty(f) = r$.

Proof of Theorem 5. Let $f \in \tilde{\mathscr{F}} \cap V_3(\boldsymbol{C})$, Theorem 7 implies that $L_\infty(f) \in]-\infty, -1/2[\cup \{0\}$. Conversely, Proposition 1 and Proposition 3 show that for every rational number $r < -1/2$, $r \neq -1$ there exists $f \in \tilde{\mathscr{F}} \cap V_3(\boldsymbol{C})$ such that $L_\infty(f) = r$.

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