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THE NUMBER OF FUNCTIONS DEFINING INTERPOLATING VARIETIES

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Abstract

In this paper, we prove that if a disjoint union of a countable number of complex affine subspaces is interpolating for the Hörmander algebra, then it can be written as the common zero set of $\alpha+1$ functions in the Hörmander algebra, where α is the maximum number of codimensions of the complex affine subspaces. Finally, we prove with an example in one complex variable that the number $\alpha+1$ is lowest.

1. Introduction

Let X_ν ($\nu \in \boldsymbol{N}$, the set of positive integers) be k_ν-codimensional complex affine subspaces of $\boldsymbol{C}^n$ $(1 \le k_\nu \le n)$, and put $\alpha = \max_{\nu\in\boldsymbol{N}} k_\nu$. Assume that $X_\nu \cap X_{\nu'} = \emptyset$ for $\nu \neq \nu'$. Let N_ν be the orthogonal linear subspaces of X_ν, where we use the canonical inner product $\langle z, w\rangle = \sum_{l=1}^n z_l \overline{w}_l$ on $\boldsymbol{C}^n$. Set $S_\nu = N_\nu \cap S^{2n-1}$, where $S^{2n-1} = \{u \in \boldsymbol{C}^n : |u| = 1\}$. Then Oh'uchi [O] proved the following result:

Theorem A. *Let $X = \bigcup_{\nu\in\boldsymbol{N}} X_\nu$ be an analytic subset of $\boldsymbol{C}^n$ consisting of disjoint complex affine subspaces X_ν. Let p be a weight function on $\boldsymbol{C}^n$. Then X is interpolating for $A_p(\boldsymbol{C}^n)$ if and only if there exist $f_1, \dots, f_m \in A_p(\boldsymbol{C}^n)$ $(m \ge \alpha)$ and constants $\varepsilon, C > 0$ such that*

$$X \subset Z(f_1, \dots, f_m)(:= \{z \in \boldsymbol{C}^n : f_1(z) = \cdots = f_m(z) = 0\}) \tag{1.1}$$

and

$$\sum_{j=1}^m |D_u f_j(\zeta)| \ge \varepsilon \exp(-Cp(\zeta)) \tag{1.2}$$

for all $u \in S_\nu$, $\zeta \in X_\nu$ and $\nu \in \boldsymbol{N}$.

Here the directional derivative $D_u f$ with a vector $u = (u_1, \dots, u_n) \in S^{2n-1}$ is defined by

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$$D_u f = \sum_{l=1}^{n} \frac{\partial f}{\partial z_l} \cdot u_l.$$

For the terminologies, see §2. It extends the result of Berenstein and Li [BL1, Theorem 2.5], which deals with the case of $k_\nu = n$ for all $\nu \in \boldsymbol{N}$.

Here we would like to discuss how many functions in $A_p(\boldsymbol{C}^n)$ we need to have an equality in (1.1). The main result of this paper is as follows:

Main Theorem. *Let $X = \bigcup_{\nu \in \boldsymbol{N}} X_\nu$ be an analytic subset of $\boldsymbol{C}^n$ consisting of disjoint complex affine subspaces X_ν. Let p be a weight function on $\boldsymbol{C}^n$. Then X is interpolating for $A_p(\boldsymbol{C}^n)$ if and only if there exist $f_1, \dots, f_{\alpha+1} \in A_p(\boldsymbol{C}^n)$ and constants $\varepsilon, C > 0$ such that*

$$X = Z(f_1, \dots, f_{\alpha+1}) \tag{1.3}$$

and

$$\sum_{j=1}^{\alpha+1} |D_u f_j(\zeta)| \geq \varepsilon \exp(-Cp(\zeta)) \tag{1.4}$$

for all $u \in S_\nu$, $\zeta \in X_\nu$ and $\nu \in \boldsymbol{N}$.

In §4, we prove that the number $\alpha + 1$ is lowest by an example in one complex variable.

2. Preliminaries

We fix the notation. A plurisubharmonic function $p : \boldsymbol{C}^n \to [0, \infty)$ is called a *weight function* if it satisfies

$$\log(1 + |z|^2) = O(p(z)) \tag{2.1}$$

and there exist constants $C_1, C_2 > 0$ such that for all z, z' with $|z - z'| \leq 1$

$$p(z') \leq C_1 p(z) + C_2. \tag{2.2}$$

Definition 2.1. Let $\mathcal{O}(\boldsymbol{C}^n)$ be the ring of all entire functions on $\boldsymbol{C}^n$ and let p be a weight function on $\boldsymbol{C}^n$. Set

$$A_p(\boldsymbol{C}^n) = \{f \in \mathcal{O}(\boldsymbol{C}^n) \colon \text{There exist constants } A, B > 0 \text{ such that } |f(z)| \leq A \exp(Bp(z)) \text{ for all } z \in \boldsymbol{C}^n\}.$$

Then $A_p(\boldsymbol{C}^n)$ is a subring of $\mathcal{O}(\boldsymbol{C}^n)$. $A_p(\boldsymbol{C}^n)$ is often called *the Hörmander algebra*. The following lemma is easily deduced from (2.1) and (2.2):

Lemma 2.2. *Let p be a weight function on $\boldsymbol{C}^n$. Then the followings hold*:
(1) $\boldsymbol{C}[z_1, \dots, z_n] \subset A_p(\boldsymbol{C}^n)$.

(2) *If* $f \in A_p(\boldsymbol{C}^n)$, *then* $\partial f/\partial z_j \in A_p(\boldsymbol{C}^n)$ *for* $j = 1, \dots, n$.

(3) $f \in \mathcal{O}(\boldsymbol{C}^n)$ *belongs to* $A_p(\boldsymbol{C}^n)$ *if and only if there exists a constant* $K > 0$ *such that*

$$\int_{\boldsymbol{C}^n} |f|^2 \exp(-Kp)\, d\lambda < \infty,$$

where $d\lambda$ *denotes the Lebesgue measure on* $\boldsymbol{C}^n$.

For the proof, see e.g. [H].

Example 2.3. (1) If $p(z) = \log(1 + |z|^2)$, then $A_p(\boldsymbol{C}^n) = \boldsymbol{C}[z_1, \dots, z_n]$.

(2) If $p(z) = |z|^a$ $(a > 0)$, then $A_p(\boldsymbol{C}^n)$ is a space of entire functions which are of order $= a$ and of finite type, or which are of order $< a$.

(3) If $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|^2)$, then $A_p(\boldsymbol{C}^n) = \hat{\mathscr{E}}'(\boldsymbol{R}^n)$, that is, the space of Fourier transforms of distributions with compact support on $\boldsymbol{R}^n$ (see e.g. [E]).

(4) When $p(z) = \exp|z|^a$ $(a > 0)$, p is a weight function if and only if $a \le 1$.

In the rest of this paper, p will always represent a weight function.

Definition 2.4. Let X be an analytic subset of $\boldsymbol{C}^n$, and let $\mathcal{O}(X)$ be the space of analytic functions on X. Then we define

$$A_p(X) = \{f \in \mathcal{O}(X)\text{: There exist constants } A, B > 0 \text{ such that } |f(z)| \le A \exp(Bp(z)) \text{ for all } z \in X\}.$$

Definition 2.5. An analytic subset X in $\boldsymbol{C}^n$ is said to be *interpolating for* $A_p(\boldsymbol{C}^n)$ if the restriction map $R_X : A_p(\boldsymbol{C}^n) \to A_p(X)$ defined by $R_X(f) = f|_X$ is surjective.

The semilocal interpolating theorem by [BT] is useful to show an analytic subset to be interpolating. Let X be given by

$$X = Z(f_1, \dots, f_N) = \{z \in \boldsymbol{C}^n : f_1(z) = \cdots = f_N(z) = 0\}$$

with $f_1, \dots, f_N \in A_p(\boldsymbol{C}^n)$. Then for $\varepsilon, C > 0$, we define

$$S_p(f; \varepsilon, C) = \left\{ z \in \boldsymbol{C}^n : |f(z)| = \left(\sum_{j=1}^{N} |f_j(z)|^2 \right)^{1/2} < \varepsilon \exp(-Cp(z)) \right\},$$

which is an open neighborhood of X. We recall the semilocal interpolation theorem of [BT].

Semilocal Interpolation Theorem. *Let* h *be a holomorphic function in* $S_p(f; \varepsilon, C)$ *such that*

$$|h(z)| \le A_1 \exp(B_1 p(z))$$

for all $z \in S_p(f; \varepsilon, C)$, *where* $\varepsilon, C > 0$. *Then there exist an entire function*

$H \in A_p(\boldsymbol{C}^n)$, *constants* $\varepsilon_0, C_0, A, B > 0$ *and holomorphic functions* $g_1, \dots, g_N$ *in* $S_p(f; \varepsilon_0, C_0)$ *such that*

$$H(z) - h(z) = \sum_{j=1}^{N} g_j(z) f_j(z)$$

and

$$|g_j(z)| \leq A \exp(Bp(z))$$

for all $z \in S_p(f; \varepsilon_0, C_0)$ *and* $j = 1, \dots, N$. *In particular,* $H = h$ *on the variety* $X = Z(f_1, \dots, f_N)$.

3. The proof of the main theorem

The sufficiency is included in Theorem A. Then we show the necessity.

Let $X = \bigcup_{\nu \in \boldsymbol{N}} X_\nu$ be an analytic subset of $\boldsymbol{C}^n$ consisting of disjoint complex affine subspaces X_ν of codimension k_ν. Put $\alpha = \sup_{\nu \in \boldsymbol{N}} k_\nu$. Then we define $f_1, \dots, f_\alpha \in A_p(\boldsymbol{C}^n)$ by the following lemma, which follows from the proof of the necessity part of the main theorem in [O, pp. 377–384].

Lemma 3.1. *If* X *is interpolating for* $A_p(\boldsymbol{C}^n)$, *then there exist* α *entire functions* $f_1, \dots, f_\alpha \in A_p(\boldsymbol{C}^n)$ *and constants* $\varepsilon, C > 0$ *such that*

$$X \subset Z(f_1, \dots, f_\alpha) \tag{3.1}$$

and

$$\sum_{j=1}^{\alpha} |D_u f_j(\zeta)| \geq \varepsilon \exp(-Cp(\zeta)) \tag{3.2}$$

for all $u \in S_\nu$, $\zeta \in X_\nu$ *and* $\nu \in \boldsymbol{N}$.

Next we shall give $f_{\alpha+1}$. To do it, we need the following lemma:

Lemma 3.2. *Let* $f_1, \dots, f_\alpha \in A_p(\boldsymbol{C}^n)$ *be in Lemma* 3.1. *Let* $\{X_\nu\} \cup \{Y_\mu\}$ *be the set of all connected components of* $Z(f_1, \dots, f_\alpha)$. *Then there exist constants* $\varepsilon_0, C_0 > 0$ *such that*

$$\sharp\{\nu \in \boldsymbol{N} : X_\nu \cap W \neq \emptyset\} \leq 1$$

for every connected component W *of* $S_p(f; \varepsilon_0, C_0)$. *Moreover, letting* W_ν *be the connected component of* $S_p(f; \varepsilon_0, C_0)$ *including* X_ν, *we have* $W_\nu \cap Y_\mu = \emptyset$ *for every* μ.

Proof. Fix $\nu \in \boldsymbol{N}$ and $\zeta \in X_\nu$. For $j \in \{1, \dots, \alpha\}$ and $u \in S_\nu$, consider the entire function

$$f_{j,u,\zeta}(w) = f_j(wu + \zeta), \quad w \in \boldsymbol{C}.$$

Setting

$$f'_{j,u,\zeta}(0) = \frac{d}{dw}\bigg|_{w=0} f_{j,u,\zeta},$$

we have

$$\sum_{j=1}^{\alpha} |f'_{j,u,\zeta}(0)| \geq \varepsilon \exp(-Cp(\zeta))$$

by (3.2). Hence, for all $u \in S_v$ there exists $j_u \in \{1, \dots, \alpha\}$ such that

$$|f'_{j_u,u,\zeta}(0)| \geq \frac{\varepsilon}{\alpha} \exp(-Cp(\zeta)). \tag{3.3}$$

Put

$$V_{u,\zeta} = \{w \in \boldsymbol{C} : f_{j_u,u,\zeta}(w) = 0\}$$

and

$$d_{u,\zeta} = \begin{cases} \min\{1, \operatorname{dist}(0, V_{u,\zeta} \backslash \{0\})\}, & \text{if } V_{u,\zeta} \backslash \{0\} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Since $f_{j_u} \in A_p(\boldsymbol{C}^n)$, we have

$$|f_{j_u}(zu + \zeta)| \leq A_1 \exp(B_1 p(zu + \zeta))$$

for some constants $A_1, B_1 > 0$ independent of ζ, u and v. Thus (2.2) implies that for $|w| \leq 1$

$$|f_{j_u,u,\zeta}(w)| \leq A_2 \exp(B_2 p(\zeta)), \tag{3.4}$$

where $A_2 = A_1 \exp(B_1 C_2)$ and $B_2 = B_1 C_1$. Set

$$g_{u,\zeta}(w) = \frac{f_{j_u,u,\zeta}(w)}{w}.$$

Since $f_{j_u,u,\zeta}$ has a zero at $w = 0$ of order one, $g_{u,\zeta}$ is an entire function on $\boldsymbol{C}$ and

$$g_{u,\zeta}(0) = f'_{j_u,u,\zeta}(0) \neq 0. \tag{3.5}$$

By (3.4), on $|w| = 1$ we have

$$|g_{u,\zeta}(w)| = \frac{|f_{j_u,u,\zeta}(w)|}{|w|} = |f_{j_u,u,\zeta}(w)| \leq A_2 \exp(B_2 p(\zeta)).$$

It follows from the Maximum Modulus Theorem that for $|w| \leq 1$

$$|g_{u,\zeta}(w)| \leq A_2 \exp(B_2 p(\zeta)). \tag{3.6}$$

Then the entire function

$$G_{u,\zeta}(w) = \frac{g_{u,\zeta}(w) - g_{u,\zeta}(0)}{3A_2 \exp(B_2 p(\zeta))}$$

satisfies that $G_{u,\zeta}(0) = 0$ and $|G_{u,\zeta}(w)| < 1$ for $|w| \le 1$. The Schwarz Lemma implies that $|G_{u,\zeta}(w)| \le |w|$ for $|w| \le 1$. In particular, for $w_0 \in V_{u,\zeta} \setminus \{0\}$ with $|w_0| \le 1$, we obtain from (3.3) and (3.5)

$$\begin{aligned} |w_0| \ge |G_{u,\zeta}(w_0)| &= \left| \frac{g_{u,\zeta}(0)}{3A_2 \exp(B_2 p(\zeta))} \right| = \left| \frac{f'_{j_u,u,\zeta}(0)}{3A_2 \exp(B_2 p(\zeta))} \right| \\ &\ge \varepsilon_3 \exp(-C_3 p(\zeta)), \end{aligned}$$

where $\varepsilon_3 = \varepsilon / 3A_2\alpha$ and $C_3 = C + B_2$. Hence,

$$d_{u,\zeta} \ge \varepsilon_3 \exp(-C_3 p(\zeta)). \tag{3.7}$$

Now we need the following Borel-Carathèodory inequality, (cf., e.g., [BG]).

Borel-Carathèodory inequality. *Let h be a function which is holomorphic in a neighborhood of $|w| \le R$ and has no zero in $|w| < R$. If $h(0) = 1$ and $0 \le |w| \le r < R$, then the following estimate follows*:

$$\log|h(w)| \ge -\frac{2r}{R-r} \log \max_{|\omega|=R} |h(\omega)|.$$

Since $g_{u,\zeta}(0) \ne 0$, we apply this inequality to $h(w) = g_{u,\zeta}(w)/g_{u,\zeta}(0)$, $R = d_{u,\zeta}$ and $r = d_{u,\zeta}/2$, to obtain

$$\begin{aligned} \log\left|\frac{g_{u,\zeta}(w)}{g_{u,\zeta}(0)}\right| &\ge -\frac{2 \cdot (d_{u,\zeta}/2)}{d_{u,\zeta} - d_{u,\zeta}/2} \log \max_{|\omega|=d_{u,\zeta}} \left|\frac{g_{u,\zeta}(\omega)}{g_{u,\zeta}(0)}\right| \\ &= -2 \log \max_{|\omega|=d_{u,\zeta}} \left|\frac{g_{u,\zeta}(\omega)}{g_{u,\zeta}(0)}\right| \end{aligned}$$

for $|w| \le d_{u,\zeta}/2$. Then it follows from (3.3), (3.5) and (3.6) that

$$\begin{aligned} |g_{u,\zeta}(w)| &\ge |g_{u,\zeta}(0)| \left(\max_{|\omega|=d_{u,\zeta}} \left|\frac{g_{u,\zeta}(\omega)}{g_{u,\zeta}(0)}\right| \right)^{-2} \\ &= |g_{u,\zeta}(0)|^3 \left(\max_{|\omega|=d_{u,\zeta}} |g_{u,\zeta}(\omega)| \right)^{-2} \\ &\ge \varepsilon_4 \exp(-C_4 p(\zeta)), \end{aligned} \tag{3.8}$$

where $\varepsilon_4 = \varepsilon^3/\alpha^3 A_2^2$ and $C_4 = 3C + 2B_2$. Let

$$\hat{d}_\zeta = \varepsilon_3 \exp(-C_3 p(\zeta)),$$

where ε_3 and C_3 are the same as in (3.7). Since $\hat{d}_\zeta \le d_{u,\zeta}$ by (3.7), it follows from (3.8) that for $|w| = \hat{d}_\zeta/2$

$$|f_{j_u,u,\zeta}(w)| = |wg_{u,\zeta}(w)| \geq \varepsilon_5 \exp(-C_5 p(\zeta)),$$

where $\varepsilon_5 = \varepsilon_3\varepsilon_4/2$ and $C_5 = C_3 + C_4$. Thus we have proved that for every $u \in S_\nu$, there exists $j_u \in \{1, \dots, \alpha\}$ such that

$$|f_{j_u}(wu+\zeta)| \geq \varepsilon_5 \exp(-C_5 p(\zeta))$$

for $|w| = \hat{d}_\zeta/2$. Hence we have

$$\begin{aligned} (3.9) \qquad |f(wu+\zeta)| &= \left(\sum_{j=1}^{\alpha} |f_j(wu+\zeta)|^2\right)^{1/2} \\ &\geq |f_{j_u}(wu+\zeta)| \geq \varepsilon_5 \exp(-C_5 p(\zeta)). \end{aligned}$$

Note that the constants ε_5 and C_5 are independent of u, ζ and ν.

For arbitrary $\nu \in \boldsymbol{N}$ and $\zeta \in X_\nu$, we consider a neighborhood

$$U_{\nu,\zeta} = \left\{ z = wu + \zeta \in \{\zeta\} + N_\nu : |w| \leq \frac{\hat{d}_\zeta}{2}, u \in S_\nu \right\}$$

of ζ in $\{\zeta\} + N_\nu$. For all $z \in \partial U_{\nu,\zeta}$, there exists $u_{z-\zeta} \in S_\nu$ such that

$$z = \frac{\hat{d}_\zeta}{2} u_{z-\zeta} + \zeta.$$

Then it follows from (3.9) that

$$(3.10) \qquad |f(z)| = \left| f\left(\frac{\hat{d}_\zeta}{2} u_{z-\zeta} + \zeta\right) \right| \geq \varepsilon_5 \exp(-C_5 p(\zeta)).$$

Let $\hat{V}_\nu$ be the component of $S_p(f; \varepsilon_5, C_5)$ containing X_ν. Then it is clear that

$$\hat{V}_\nu \subset \bigcup_{\zeta \in X_\nu} U_{\nu,\zeta}$$

by (3.10). Now we claim that for $\zeta' \in X_{\nu'}$ and $\nu' \neq \nu$,

$$\zeta' \notin \bigcup_{\zeta \in X_\nu} U_{\nu,\zeta}.$$

In fact, by ζ we denote the orthogonal projection of ζ' to X_ν, so that $\zeta' \in \{\zeta\} + N_\nu$. Then there exists $u_{\zeta'-\zeta} \in S_\nu$ such that

$$\zeta' = |\zeta' - \zeta| u_{\zeta'-\zeta} + \zeta.$$

(3.7) implies that

$$|\zeta' - \zeta| \geq d_{u_{\zeta'-\zeta}} \geq \varepsilon_3 \exp(-C_3 p(\zeta)) = \hat{d}_\zeta.$$

Hence $\zeta' \notin U_{\nu,\zeta}$. For $\tilde{\zeta} \in X_\nu \setminus \{\zeta\}$, it is clear that $\zeta' \notin U_{\nu,\tilde{\zeta}}$. This proves the lamma for $\varepsilon_0 = \varepsilon_5$ and $C_0 = C_5$. $\square$

Proof of the main theorem. Define a holomorphic function h in $S_p(f;\varepsilon_0,C_0)$ by

$$h(z)=\begin{cases}0, & \text{if } z\in W_\nu,\\ 1, & \text{if } z\in S_p(f;\varepsilon_0,C_0)\backslash\bigcup_{\nu\in\boldsymbol{N}}W_\nu.\end{cases}$$

Then we have

$$|h(z)|\le 1\cdot\exp(1\cdot p(z))$$

for every $z\in S_p(f;\varepsilon_0,C_0)$. Hence it follows from the semilocal interpolation theorem that there exists an entire function $H\in A_p(\boldsymbol{C}^n)$ such that $H|_{Z(f_1,\dots,f_\alpha)}\equiv h|_{Z(f_1,\dots,f_\alpha)}$, that is,

$$H(z)=\begin{cases}0, & \text{if } z\in X_\nu,\\ 1, & \text{if } z\in Z(f_1,\dots,f_\alpha)\backslash\bigcup_{\nu\in\boldsymbol{N}}X_\nu.\end{cases}$$

It is clear that $\alpha+1$ functions $f_1,\dots,f_\alpha$, $H\in A_p(\boldsymbol{C}^n)$ satisfy (1.3) and (1.4). □

4. The sharpness of $\alpha+1$

Finally, we remark that the number '$\alpha+1$' in the main theorem is lowest. We prove this remark by giving an example of an interpolating variety for $A_p(\boldsymbol{C})$ which can not write as the zero set of a function in $A_p(\boldsymbol{C})$, where $p(z)=|z|$. Let $X=\{\zeta_\nu\}_{\nu\in\boldsymbol{N}}$ be a discrete variety in $\boldsymbol{C}$. Then Nevanlinna's counting function is defined as follows: $n(r,X)=\sharp\{\nu\in\boldsymbol{N}:|\zeta_\nu|\le r\}$ and

$$N(r,X)=\int_0^r\frac{n(t,X)-n(0,X)}{t}\,dt+n(0,X)\log r.$$

For $k\in\boldsymbol{N}$ and $r>0$, we define

$$B(r;k:X)=\frac{1}{k}\sum_{0<|\zeta_\nu|\le r}\left(\frac{1}{\zeta_\nu}\right)^k$$

and for $r_1,r_2>0$

$$B(r_1,r_2;k:X)=B(r_1;k:X)-B(r_2;k:X).$$

Then the following proposition which gives the relationship between an entire function in $A_p(\boldsymbol{C})$ and its zero set is deduced by the Fourier series method by Rubel and Taylor [RT]:

Proposition 4.1 (cf. [RT, Theorem 5.2]). *Let p be a radical weight of finite order on $\boldsymbol{C}$, that is, it satisfies $p(z)=p(|z|)$ and*

$$\varlimsup_{r\to\infty}\frac{\log p(r)}{\log r}<\infty.$$

Then there exists $f \in A_p(\boldsymbol{C})$ *such that* $X = Z(f)$ *if and only if*

(1) *X has finite p-density, that is, we have a constant* $A > 0$ *satisfying*

$$N(r, X) \leq Ap(r)$$

for every $r > 0$.

(2) *X is p-balanced, that is, there exists a constant* $A > 0$ *such that*

$$|B(r_1, r_2; k : X)| \leq \frac{Ap(r_1)}{r_1^k} + \frac{Ap(r_2)}{r_2^k}$$

for all $r_1, r_2 > 0$ *and* $k \in \boldsymbol{N}$.

Example 4.2. Put $X = \{\nu\}_{\nu \in \boldsymbol{N}} \subset \boldsymbol{C}$. Applying Theorem A (or [BL2, Corollary 3.5]) to $f(z) = \sin \pi z \in A_{|\cdot|}(\boldsymbol{C})$, we know that X is interpolating for $A_{|\cdot|}(\boldsymbol{C})$. Hence it follows from the main theorem that X can be written as the common zero set of two entire functions in $A_{|\cdot|}(\boldsymbol{C})$. We shall prove that X can be written as the zero set of no entire function in $A_{|\cdot|}(\boldsymbol{C})$.

Put $k = 1$ and $r_2 = 1/2$. Then calculating $|B(r, 1/2; 1 : X)|$, we have

$$\left|B\left(r, \frac{1}{2}; 1 : X\right)\right| = \sum_{\nu=1}^{[r]} \frac{1}{\nu},$$

where $[r]$ is the greatest integer not greater than r. Hence there does not exist a constant $A > 0$ such that

$$\left|B\left(r, \frac{1}{2}; 1 : X\right)\right| \leq \frac{A|r|}{r} + \frac{A/2}{1/2} = 2A$$

for any $r > 0$. Thus X does not satisfy the condition (2) in Proposition 4.1, so that X can be written as a zero set of no entire function in $A_{|\cdot|}(\boldsymbol{C})$.

Finally, the main theorem and this example lead to the following conjecture:

Conjecture. *Let X be an interpolating variety for* $A_p(\boldsymbol{C}^n)$ *and let* α *be the maximum number of codimentions of all irreducible components of X. Then X can be written as the common zero set of* $\alpha + 1$ *entire functions in* $A_p(\boldsymbol{C}^n)$.

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