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NON COMMUTATIVITY OF SELF HOMOTOPY GROUPS

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Abstract

We study non-commutativity of the self homotopy groups of Lie groups.

1. Introduction

Let G be a connected Lie group and $\mu : G \times G \to G$ the multiplication of G. For any space A with a base point, the based homotopy set $[A, G]$ becomes a group with respect to the binary operation $\mu_* : [A, G] \times [A, G] = [A, G \times G] \to [A, G]$. Even if A is a simple space, it is difficult to calculate the group $[A, G]$. A general result was given by Whitehead (p. 464 of [17]):

$$\text{nil}[A, G] \leq \text{cat } A, \tag{1.1}$$

where nil and cat denote the nilpotency class and the Lusternik-Schnirelmann category with $\text{cat}(*) = 0$, respectively. We study the special case $A = G$. In [9], [12], [13], the group $[G, G]$ has been calculated for $G = SU(3)$, $Sp(2)$, G_2. It shows that $\text{nil}[G, G]$ equals 2 if $G = SU(3)$, $Sp(2)$ and 3 if $G = G_2$. This supports the following conjectures which were proposed in [13] by the first author.

Conjecture 1.1. *If G is simple, then* $\text{nil}[G, G] \geq \text{rank } G$.

Conjecture 1.2. *If G is simple and* $\text{rank } G \geq 2$, *then* $\text{nil}[G, G] \geq 2$, *that is*, $[G, G]$ *is not commutative*.

If 1.1 is affirmative, then so is 1.2. Notice that two conjectures are false in general without the assumption of simpleness of G ([13]).

The purpose of this note is to prove the following which supports the above conjectures.

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THEOREM 1.3. (1) $\mathrm{nil}[SU(4), SU(4)] = 3$.
(2) $\mathrm{nil}[G, G] \geq 2$ *if* $G = SU(5), SU(6), Sp(3)$.
(3) $\mathrm{nil}[G, G] \geq 3$ *if* $G = \mathrm{Spin}(7), \mathrm{Spin}(8), E_6, F_4$.
(4) $\mathrm{nil}[E_8, E_8] \geq 5$.

We study Spin(7), Spin(8), E_6, E_8, F_4 in §2, $SU(4)$ in §3, and $SU(5)$, $SU(6)$, $Sp(3)$ in §4.

We do not distinguish notationally between a map and its homotopy class.

2. mod p non commutativity

For any space Y with a base point, we denote by d_n the diagonal maps $Y \to \underbrace{Y \times \cdots \times Y}_{n}$ and $Y \to \underbrace{Y \wedge \cdots \wedge Y}_{n}$.

Let X be a connected homotopy associative CW Hopf space. The commutator map $c_2 : X \times X \to X$ is the composite of

$$X \times X \xrightarrow{d_2 \times d_2} X \times X \times X \times X \xrightarrow{1 \times tw \times 1} X \times X \times X \times X$$
$$\xrightarrow{1 \times 1 \times \sigma \times \sigma} X \times X \times X \times X \xrightarrow{\mu \times \mu} X \times X \xrightarrow{\mu} X$$

where tw is the twisting map, σ is the inverse and μ is the multiplication of X. Inductively we define $c_n = c_2 \circ (1 \times c_{n-1}) : \underbrace{X \times \cdots \times X}_{n} \to X$ for $n \geq 3$. Of course, when X is a topological group, $c_2(x, y) = xyx^{-1}y^{-1}$ and c_n can be seen as a map $\underbrace{X \wedge \cdots \wedge X}_{n} \to X$ for $n \geq 2$. Given $f, g \in [Y, X]$, its commutator $[f, g] \in [Y, X]$ is represented by the map

$$Y \xrightarrow{d_2} Y \times Y \xrightarrow{f \times g} X \times X \xrightarrow{c_2} X.$$

Let p be an odd prime and $h_*(-)$ the mod p ordinary homology $H_*(-; \boldsymbol{Z}_p)$ or the Morava K-theory $K(n)_*(-)$ with the coefficient $K(n)_* = \boldsymbol{Z}_p[v_n, v_n^{-1}]$, $|v_n| = 2(p^n - 1)$. We assume that $H_i(X; \boldsymbol{Z}_p)$ is finite dimensional for every i. Thus $h_*(X)$ is a Hopf algebra with the multiplication μ_* and the comultiplication d_{2*}. Hence $h_*(X)$ is cocommutative but, in general, not commutative. Given $x \in h_s(X)$ and $y \in h_t(X)$, we define

$$[x, y] = xy - (-1)^{st} yx \in h_{s+t}(X).$$

By direct calculation, we have

LEMMA 2.1 ([14], [19]). *If* $x_1, \ldots, x_n \in h_*(X)$ *are primitive* $(n \geq 2)$, *then*

$$c_{n*}(x_1 \otimes \cdots \otimes x_n) = [x_1, [x_2, \ldots [x_{n-1}, x_n \underbrace{] \ldots]]}_{n-1}$$

and it is primitive.

By the Borel theorem, the mod p cohomology $H^*(X; \boldsymbol{Z}_p)$ is a tensor product of truncated polynomial algebras and exterior algebras generated by even and odd dimensional elements respectively. In particular, mod p-cohomology of exceptional Lie groups have form

$$H^*(X; \boldsymbol{Z}_p) = \bigotimes_{i,j} \boldsymbol{Z}_p[y_i]/(y_i^p) \otimes \Lambda(x_j),$$

where $|y_i|$ is even and $|x_j|$ is odd. The mod p homology is the dual of the cohomology and is additively isomorphic to the cohomology. Let us denote by z_j (resp. y_i) the dual of x_j (resp. y_i). We have

Theorem 2.2 ([6], [7]). *Let G be an exceptional Lie group having p-torsion in homology. Then for some n with $2 \leq n \leq 3$, we have*

(1) $K(n)_*(G) \cong K(n)_* \otimes H_*(G; Z_p)$.

(2) *For each $z_j \neq z_3$, there is y_i such that $c_{2*}(y_i \otimes z_j) \neq 0$ in $K(n)_*(G)$.*

Notice that G of the above theorem is one of F_4, E_6, E_7, E_8 for $p = 3$ and E_8 for $p = 5$. In these cases, all y_i and z_j are primitive. Except the case $G = E_8$ for $p = 3$, Theorem 2.2 holds for $n = 2$.

By definition, we easily have

Lemma 2.3. *$c_{2*}(1 \otimes 1) = 1$ and $c_{2*}(\alpha \otimes 1) = c_{2*}(1 \otimes \alpha) = 0$ for $\alpha \in \tilde{h}_*(X)$.*

Localization technique works for our purpose. For any prime number p (including the case $p = 2$), let $X_{(p)}$ be the p-localization of X. Then $[X_{(p)}, X_{(p)}] \cong [X, X]_{(p)}$ and

$$\mathrm{nil}[X, X] = \max_p \{\mathrm{nil}[X_{(p)}, X_{(p)}]\}. \tag{2.1}$$

Now we consider the concrete cases. Harper [2], Harris [3] and Wilkerson [18] showed that there are decompositions of mod p spaces (not as H-spaces):

$$F_4 \simeq_3 F_4' \times F_4'', \quad E_6 \simeq_3 F_4 \times (E_6/F_4), \quad E_8 \simeq_5 E_8' \times E_8''$$

where

$$H^*(F_4'; Z_3) = Z_3[y_8]/(y_8^3) \otimes \Lambda(x_3, x_7), \quad H^*(F_4''; Z_3) = \Lambda(x_{11}, x_{15}),$$
$$H^*(E_8'; Z_5) = Z_5[y_{12}]/(y_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{27}, x_{35}),$$
$$H^*(E_8''; Z_5) = \Lambda(x_{15}, x_{23}, x_{39}, x_{47}).$$

The action $[y_8, -]$ (resp. $[y_{12}, -]$) in $K(2)_*(G)$ for $(G, p) = (F_4, 3)$ (resp. $(E_8, 5)$) is given as follows [6]:

$$z_3 \to z_{11} \to -v_2 z_3, \quad z_7 \to z_{15} \to -v_2 z_7$$

$$\text{(resp. } z_3 \to z_{15} \to z_{27} \to z_{39} \to -v_2 z_3, \quad z_{11} \to z_{23} \to z_{35} \to z_{47} \to -v_2 z_{11}).$$

COROLLARY 2.4. *If* (G,p) *is* $(F_4,3)$, $(E_6,3)$ *or* $(E_8,5)$, *then* $\mathrm{nil}[G,G]\geq p$.

Proof. Under the condition, we have $y^p=-v_2y$ in $K(2)_*(G)$ by (1.4) of [20]. As is well-known,

$$\mathrm{ad}^k(y)(z):=[y,[y,\dots[y,z\underbrace{]\dots]]}_{k}=\sum_{l=0}^{k}\binom{k}{l}(-1)^l y^{k-l}zy^l.$$

In particular $\mathrm{ad}^p(y)(z)=y^pz-zy^p=-v_2[y,z]\neq 0$, whence $\mathrm{ad}^{p-1}(y)(z)\neq 0$.

Let f be the composite of

$$F_{4(3)}\xrightarrow{\text{proj}}F'_{4(3)}\xrightarrow{\subset}F_{4(3)}.$$

Then

$$f_*(y_8)=y_8\quad\text{and}\quad f_*(z_{15})=0. \tag{2.2}$$

By direct computation of the diagonal map, we have

$$d_{3*}(y_8^2z_{15})=2y_8\otimes y_8\otimes z_{15}+a,$$

where $a=\sum a_1\otimes a_2\otimes a_3$ such that $a_i=z_{15}$ for some $i\leq 2$ or $a_i=1$ for one or two i's and $a_j\in\widetilde{K(2)}_*(F_{4(3)})(=\widetilde{K(2)}_*(F_4))$ for all j with $a_j\neq 1$. Since $c_{3*}(f\times f\times id)_*(a)=0$ by 2.3 and (2.2), it follows from 2.1 that we have

$$\begin{aligned}[f,[f,id]]_*(y_8^2z_{15})&=c_{3*}(f\times f\times id)_*d_{3*}(y_8^2z_{15})=2\,\mathrm{ad}^2(y_8)(z_{15})\\&=-\mathrm{ad}^2(y_8)(z_{15})\neq 0\end{aligned}$$

in $K(2)_*(F_4)$. Hence $[f,[f,id]]\neq 0$ and $\mathrm{nil}[F_{4(3)},F_{4(3)}]\geq 3$ so that $\mathrm{nil}[F_4,F_4]\geq 3$ by (2.1).

Let $\tilde f$ be the composite of

$$E_{6(3)}\xrightarrow{\text{proj}}F_{4(3)}\xrightarrow{f}F_{4(3)}\xrightarrow[\subset]{i}E_{6(3)}.$$

We have $i^*[\tilde f,[\tilde f,id]]=i_*[f,[f,id]]\neq 0$ in $[F_{4(3)},E_{6(3)}]$, because $i_*:[F_{4(3)},F_{4(3)}]\to[F_{4(3)},E_{6(3)}]$ is injective. Hence $[\tilde f,[\tilde f,id]]\neq 0$ in $K(2)_*(E_{6(3)})$ and $\mathrm{nil}[E_{6(3)},E_{6(3)}]\geq 3$ so that $\mathrm{nil}[E_6,E_6]\geq 3$ by (2.1).

Let g be the composite of

$$E_{8(5)}\xrightarrow{\text{proj}}E'_{8(5)}\xrightarrow{\subset}E_{8(5)}.$$

Then

$$g_*(y_{12})=y_{12}\quad\text{and}\quad g_*(z_{15})=0. \tag{2.3}$$

We have

$$d_{5*}(y_{12}^4z_{15})=(5-1)!\,y_{12}\otimes y_{12}\otimes y_{12}\otimes y_{12}\otimes z_{15}+a$$

where $a = \sum a_1 \otimes \cdots \otimes a_5$ such that $a_i = z_{15}$ for some $i \leq 4$ or $a_i = 1$ for at least one and at most four i's and $a_j \in \widetilde{K(2)}_*(E_{8(5)})(= \widetilde{K(2)}_*(E_8))$ for all j with $a_j \neq 1$. Since $c_{5*}(g \times g \times g \times g \times id)_*(a) = 0$ by 2.3 and (2.3), it follows from 2.1 that we have

$$[g,[g,[g,[g,id]]]]_*(y_{12}^4 z_{15}) = 24\,\mathrm{ad}^4(y_{12})(z_{15}) = -\mathrm{ad}^4(y_{12})(z_{15}) \neq 0$$

in $K(2)_*(E_8)$ and $\mathrm{nil}[E_{8(5)}, E_{8(5)}] \geq 5$ so that $\mathrm{nil}[E_8, E_8] \geq 5$ by (2.1). $\square$

Proposition 2.5. $\mathrm{nil}[\mathrm{Spin}(8), \mathrm{Spin}(8)] \geq \mathrm{nil}[\mathrm{Spin}(7), \mathrm{Spin}(7)] \geq 3$.

Proof. Since the bundle $\mathrm{Spin}(7) \to \mathrm{Spin}(7)/G_2 = S^7$ has a 3 section, there is a mod 2 equivalence $\mathrm{Spin}(7) \simeq_2 S^7 \times G_2$. In particular the inclusion $i_{(2)} : G_{2(2)} \to \mathrm{Spin}(7)_{(2)}$ has a homotopy left inverse. Thus the following homomorphism $i_{(2)*}$ is injective and $i_{(2)}^*$ is surjective:

$$[\mathrm{Spin}(7)_{(2)}, \mathrm{Spin}(7)_{(2)}] \xleftarrow{i_{(2)*}} [\mathrm{Spin}(7)_{(2)}, G_{2(2)}] \xrightarrow{i_{(2)}^*} [G_{2(2)}, G_{2(2)}].$$

Hence

$$\mathrm{nil}[\mathrm{Spin}(7)_{(2)}, \mathrm{Spin}(7)_{(2)}] \geq \mathrm{nil}[\mathrm{Spin}(7)_{(2)}, G_{2(2)}] \geq \mathrm{nil}[G_{2(2)}, G_{2(2)}].$$

Since the localization is an exact functor, it follows from Theorem 2.3 of [13] that the last number is three. We then have $\mathrm{nil}[\mathrm{Spin}(7), \mathrm{Spin}(7)] \geq 3$ by (2.1).

Since the bundle $\mathrm{Spin}(8) \to \mathrm{Spin}(8)/\mathrm{Spin}(7) = S^7$ has a section, there is a homeomorphism $\mathrm{Spin}(8) \approx S^7 \times \mathrm{Spin}(7)$. In particular the inclusion $i : \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ has a left inverse. By the same method as above, we have $\mathrm{nil}[\mathrm{Spin}(8), \mathrm{Spin}(8)] \geq \mathrm{nil}[\mathrm{Spin}(7), \mathrm{Spin}(7)]$. This completes the proof. $\square$

Remark 2.6. By using mod 2 versions of 2.2 and 2.3, we can prove non-commutativity of $[\mathrm{Spin}(7), \mathrm{Spin}(7)]$.

3. $SU(4)$

The purpose of this section is to prove

Proposition 3.1. $\mathrm{nil}[SU(4), SU(4)] = 3$.

We have $\mathrm{nil}[SU(4), SU(4)] \leq \mathrm{cat}\, SU(4) = 3$ by (1.1) and [15]. It then suffices to show the existence of three maps $a_1, a_2, a_3 : SU(4) \to SU(4)$ such that $[a_1, [a_2, a_3]] \neq 0$.

Let $p : SU(4) \to S^7$, $p : SU(3) \to S^5$, $p' : SU(4) \to SU(4)/Sp(2) = S^5$ be the canonical projections, $\theta : S^3 = SU(2) \to SU(n)$ $(n \geq 3)$, $i : SU(3) \to SU(4)$ the inclusions, and ι_n the identity map of S^n.

Recall from [1], [10] the following:

$$\pi_5(SU(3)) = \mathrm{Z}\{[2]\} \overset{i_*}{\cong} \pi_5(SU(4)), \quad p_*[2] = 2\iota_5,$$

$$\pi_7(SU(4)) = \mathrm{Z}\{[6]\}, \quad p_*[6] = 6\iota_7, \quad p'_* : \pi_8(SU(4)) \cong \pi_8(S^5) = \mathrm{Z}_{24},$$

$$\pi_{12}(SU(3)) \overset{i_*}{\cong} \pi_{12}(SU(4)) = \mathrm{Z}_{60}\{\langle i_*[2], [6]\rangle\}, \tag{3.1}$$

$$\pi_{15}(SU(3)) = \mathrm{Z}_{36} \overset{i_*}{\subset} \pi_{15}(SU(4)) = \mathrm{Z}_{72} \oplus \mathrm{Z}_2. \tag{3.2}$$

There exists a map g which makes the following diagram commutative up to homotopy:

$$\begin{tikzcd}
SU(4) \arrow[r, "d_3"] \arrow[d, equal] & SU(4) \wedge SU(4) \wedge SU(4) \arrow[d, "1 \wedge p' \wedge p"] & & \\
SU(4) \arrow[d, "q"] & SU(4) \wedge S^5 \wedge S^7 \arrow[rr, "1 \wedge i_*[2] \wedge [6]"] & & SU(4) \wedge SU(4) \wedge SU(4) \arrow[d, "c_3"] \\
S^{15} \arrow[r, "g"] & S^3 \wedge S^5 \wedge S^7 \arrow[u, "\theta \wedge 1 \wedge 1"] \arrow[rr, "\langle \theta, \langle i_*[2], [6]\rangle\rangle"] & & SU(4)
\end{tikzcd}$$

By using integral cohomology, we see that g is a homotopy equivalence. Hence

$$[1, [i \circ [2] \circ p', [6] \circ p]] = \pm q^* \langle \theta, \langle i_*[2], [6]\rangle\rangle.$$

We shall prove non-triviality of these elements. Let $\eta_2 : S^3 \to S^2$ be the Hopf map and write $\eta_n = \Sigma^{n-2}\eta_2$. Then $\pi_{n+1}(S^n) = \mathrm{Z}_2\{\eta_n\}$ for $n \geq 3$ by [16]. There is a cell-decomposition:

$$SU(4) = S^3 \cup_{\eta_3} e^5 \cup e^8 \cup e^7 \cup e^{10} \cup e^{12} \cup_\xi e^{15}.$$

We have an exact sequence:

$$[\Sigma SU(4)^{(14)}, SU(4)] \xrightarrow{\Sigma\xi^*} \pi_{15}(SU(4)) \xrightarrow{q^*} [SU(4), SU(4)]$$

where $X^{(k)}$ denotes the k-skeleton of a CW-complex X. The following implies that the order of $[1, [i \circ [2] \circ p', [6] \circ p]]$ is a multiple of three so that 3.1 follows.

LEMMA 3.2. (1) *The order of* $\langle \theta, \langle i_*[2], [6]\rangle\rangle$ *is a multiple of three.*
(2) $2^7[\Sigma SU(4)^{(14)}, SU(4)] = 0$.

Proof. (1) Let $\beta \in \pi_{12}(SU(3))$ be a generator. Then, from (3.1) and (3.2), it suffices to show that the order of $\langle \theta, \beta\rangle \in \pi_{15}(SU(3))$ is a multiple of three. By (15.14) of [5], we have

$$p_*\langle \theta, \beta\rangle = \langle \iota_3, p_*\beta\rangle_r \tag{3.3}$$

where $\langle\ ,\ \rangle_r : \pi_s(S^3) \times \pi_t(SU(3)/S^3) \to \pi_{s+t}(SU(3)/S^3)$ is the relative Samelson product. It follows from [10], [16] that $p_* : \pi_{12}(SU(3)) \to \pi_{12}(S^5) = \mathrm{Z}_3\{\alpha_2(5)\}$

$\oplus Z_2 \oplus Z_5$ is surjective, $\pi_{15}(S^8) = Z_3\{\alpha_2(8)\} \oplus Z \oplus Z_8 \oplus Z_5$, and $\pi_{15}(S^5) = Z_9\{\beta_1(5)\} \oplus Z_8 \oplus Z_2$, $3\beta_1(5) = -\alpha_1(5) \circ \alpha_2(8)$. Also by (16.2) of [5], we have $\langle \iota_3, \iota_5 \rangle_r = J_C(\iota_3)$, where $J_C : \pi_3(SU(2)) \to \pi_3(SO(4)) \xrightarrow{J} \pi_7(S^4) \xrightarrow{\Sigma} \pi_8(S^5)$ is the complex J-homomorphism. Since J_C is surjective in this case, we have

$$\text{(3.4)} \qquad \langle \iota_3, \iota_5 \rangle_r \in \pi_8(S^5) = Z_8 \oplus Z_3\{\alpha_1(5)\} \quad \text{is a generator.}$$

It follows from (16.5) of [5] that $\langle \iota_3, \Sigma x \rangle_r = \langle \iota_3, \iota_5 \rangle_r \circ \Sigma^4 x$ for any $x \in \pi_m(S^4)$. Hence $\langle \iota_3, \alpha_2(5) \rangle_r = \langle \iota_3, \iota_5 \rangle_r \circ \alpha_2(8) = \pm\alpha_1(5) \circ \alpha_2(8) = \pm 3\beta_1(5) \neq 0$ by (3.4). Therefore the 3-component of $\langle \iota_3, p_*\beta \rangle_r$ is $\pm 3\beta_1(5)$. Since $p_* : \pi_{15}(SU(3)) \to \pi_{15}(S^5)$ is injective by [10], it follows from (3.3) that the order of $\langle \theta, \beta \rangle$ is a multiple of three.

(2) Let $\gamma : S^5 \to CP^2$ be the canonical map. Then $SU(4)^{(7)} = \Sigma CP^3 = S^3 \cup_{\eta_3} e^6 \cup_{\Sigma\gamma} e^7$. By Proposition 1.15 of [11], $\Sigma^3\gamma$ is homotopic to the composite of $S^8 \xrightarrow{2g} S^5 \overset{j}{\subset} \Sigma^3 CP^2$, where $g \in \pi_8(S^5)$ is a generator. Write $A_4 = [\Sigma SU(4)^{(7)}, SU(4)]$. There is a commutative diagram:

$$\begin{tikzcd}
{[S^5 \cup_{\eta_5} e^7, SU(4)]} \arrow[r, "\Sigma^3\gamma^*"] \arrow[d, "j^*"'] & \pi_8(SU(4)) \arrow[r] & A_4 \arrow[r] & {[S^4 \cup_{\eta_4} e^6, SU(4)] = 0} \\
\pi_5(SU(4)) \arrow[ur, "2g^*"'] & & &
\end{tikzcd}$$

Since $\pi_6(SU(4)) = 0$, j^* is surjective and $p'_*(2g)^* i_*[2] = 2\iota_5 \circ 2g = 4g$. Hence $\mathrm{Im}(\Sigma^3\gamma^*) = 4\pi_8(SU(4))$ and $A_4 = Z_4$. Write $A_1 = [\Sigma SU(4)^{(14)}, SU(4)]$, $A_2 = [\Sigma SU(4)^{(10)}, SU(4)]$, and $A_3 = [\Sigma SU(4)^{(8)}, SU(4)]$. The following diagram implies that $2^7 A_1 = 0$:

$$\begin{tikzcd}
\pi_{13}(SU(4)) = Z_4 \arrow[d, "q^*"] & \pi_{11}(SU(4)) = Z_4 \arrow[d, "q^*"] & \pi_9(SU(4)) = Z_2 \arrow[d, "q^*"] & \\
A_1 \arrow[r, "j^*"] & A_2 \arrow[r, "j^*"] & A_3 \arrow[r, "j^*"] & A_4
\end{tikzcd} \qquad \square$$

4. $SU(5), SU(6), Sp(3)$

Let (G, d) be $(SU, 2)$ or $(Sp, 4)$. Let $\theta : S^3 \subset G(n)$ be the inclusion map and $\alpha \in \pi_{dn-1}(G(n)) = Z$ a generator. We refer to [8] for homotopy groups of Lie groups.

Theorem 4.1 ([1]). *The order of the Samelson product* $\langle \theta, \alpha \rangle \in \pi_{dn+2}(G(n))$ *is*

$$\begin{cases} n(n+1) & (G = SU \text{ and } n \geq 3) \\ n(2n+1)\varepsilon_n & (G = Sp \text{ and } n \geq 2) \end{cases}$$

where ε_n *is* 1 *or* 4 *according as* n *is even or odd.*

Proof. The case of $Sp(n)$ follows from Theorem 2 of [1]. By Theorem 1 of [1], the order of $j_*\langle\theta,\alpha\rangle \in \pi_{2n+2}(SU(n+1))$ is $n(n+1)$ for $n \geq 2$, where $j : SU(n) \subset SU(n+1)$. It then follows from the structure of $j_* : \pi_{2n+2}(SU(n)) \to \pi_{2n+2}(SU(n+1))$ (see [8]) that the order of $\langle\theta,\alpha\rangle$ is $n(n+1)$ for $n \geq 3$. $\square$

Recall that $G(n)$ has a cell-decomposition:

$$G(n) = G(n-1) \cup e^{dn-1} \cup_{\rho_n} e^{dn+2} \cup \{\text{cells of dimension} \geq d(n+1)+2\}.$$

We use always this decomposition. Write $Y(n) = G(n-1)^{(dn+1)} \cup e^{dn-1}$ and $Z(n) = Y(n) \cup_{\rho_n} e^{dn+2}$. Let $p : G(n) \to S^{dn-1}$ be the canonical projection. For simplicity we denote by 1 the identity maps, by j the inclusion maps, and by q the quotient maps. There exists a map g which makes the following diagram commutative up to homotopy:

$$\begin{tikzcd}
G(n) \arrow[r, "d_2"] & G(n) \wedge G(n) \arrow[d, "1 \wedge p"] & \\
Z(n) \arrow[u, "j"] \arrow[d, "q"] & G(n) \wedge S^{dn-1} \arrow[r, "1 \wedge \alpha"] & G(n) \wedge G(n) \arrow[d, "c_2"] \\
S^{dn+2} \arrow[r, "g"'] & S^3 \wedge S^{dn-1} \arrow[u, "\theta \wedge 1"] \arrow[r, "\langle\theta,\alpha\rangle"'] & G(n)
\end{tikzcd}$$

By using the integral cohomology, we have that g is a homotopy equivalence so that

$$j^*[1, \alpha \circ p] = \pm q^*\langle\theta,\alpha\rangle.$$

If these elements are non-zero, then $[1, \alpha \circ p] \neq 0$ and $[G(n), G(n)]$ is non-commutative. To study non-triviality of $q^*\langle\theta,\alpha\rangle$, we compare the orders of $\langle\theta,\alpha\rangle$ and the image of $\Sigma\rho_n^*$:

$$[\Sigma Y(n), G(n)] \xrightarrow{\Sigma\rho_n^*} \pi_{dn+2}(G(n)) \xrightarrow{q^*} [Z(n), G(n)]. \tag{4.1}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_{dn+2}(\Sigma G(n-1)^{(dn+1)}) \arrow[r, "j_*"] & \pi_{dn+2}(\Sigma Y(n)) \arrow[r] \arrow[d, "j_*"] & \pi_{dn+2}(\Sigma Y(n), \Sigma G(n-1)^{(dn+1)}) \arrow[d, "q_*", "\cong"'] \\
& \pi_{dn+2}(\Sigma G(n)) \arrow[r, "\Sigma p_*"'] & \pi_{dn+2}(S^{dn})
\end{tikzcd}$$

Since $j_*(\Sigma\rho_n) = 0$, there exists $\tilde{\rho}_n \in \pi_{dn+2}(\Sigma G(n-1)^{(dn+1)})$ such that $j_*(\tilde{\rho}_n) = \Sigma\rho_n$. Hence $\Sigma\rho_n^*$ in (4.1) decomposes as

$$[\Sigma Y(n), G(n)] \xrightarrow{j^*} [\Sigma G(n-1)^{(dn+1)}, G(n)] \xrightarrow{\tilde{\rho}_n^*} \pi_{dn+2}(G(n)) \tag{4.2}$$

and so we have

$$\mathrm{Im}(\Sigma\rho_n^*) \subset \mathrm{Im}(\tilde{\rho}_n^*).$$

We can show that j^* of (4.2) is surjective so that $\mathrm{Im}(\Sigma\rho_n^*) = \mathrm{Im}(\tilde{\rho}_n^*)$. But we do not use this.

Problem 4.2. Is there a prime p satisfying $v_p(\sharp\mathrm{Im}(\tilde{\rho}_n^*)) < v_p(\sharp\langle\theta,\alpha\rangle)$?

Here $\sharp$ denotes the order and $v_p(m)$ is the exponent of p in the prime decomposition of an integer m. Notice that $q^*\langle\theta,\alpha\rangle$ is non-zero if Problem 4.2 is affirmative.

Proposition 4.3. *Problem 4.2 is affirmative when $G(n)$ is one of the following*:

$$Sp(2), Sp(3), SU(3), SU(4), SU(5), SU(6).$$

Proof. It is easy to show the following: $\tilde{\rho}^* = 0$ for $Sp(2)$, $SU(3)$ and $2\cdot\tilde{\rho}^* = 0$ for $SU(4)$. Hence the result follows from 4.1 for these cases. We omit the details.

$Sp(3)$. Consider the following exact sequence:

$$\pi_{11}(Sp(3)) \xrightarrow{q^*} [\Sigma Sp(2)^{(13)}, Sp(3)] \xrightarrow{j^*} [S^4\cup_\eta e^8, Sp(3)]$$

Here $Sp(2)^{(13)} = Sp(2) = S^3\cup e^7\cup e^{10}$. We have $j^* : [S^4\cup e^8, Sp(3)] \cong \pi_4(Sp(3)) = Z_2$ by [8]. Hence $2[\Sigma Sp(2)^{(13)}, Sp(3)] \subset \mathrm{Im}(q^*)$ and so $2\cdot\mathrm{Im}(\tilde{\rho}^*) \subset \mathrm{Im}(q\circ\tilde{\rho})^*$. Hence $48\cdot\mathrm{Im}(\tilde{\rho}^*) \subset 24\cdot\mathrm{Im}(q\circ\tilde{\rho})^* = 0$, since $q\circ\tilde{\rho}\in\pi_{14}(S^{11}) = Z_{24}$. Therefore $v_7(\sharp\mathrm{Im}(\tilde{\rho}^*)) = 0 < v_7(\sharp\langle\theta,\alpha\rangle) = 1$.

Let n be 5 or 6. Then, we have $SU(n-1)^{(2n+1)} = SU(n-1)^{(2n)}$ and $SU(n-1)^{(2n-1)} = SU(n-1)^{(2n-2)} = SU(n-2)^{(2n-2)}\cup e^{2n-3}$. Since $2\pi_{2n+2}(\Sigma SU(n-1)^{(2n)}, \Sigma SU(n-1)^{(2n-2)}) = 0$, there exists $\hat{\rho}\in\pi_{2n+2}(\Sigma SU(n-1)^{(2n-2)})$ such that $j_*\hat{\rho} = 2\tilde{\rho}\in\pi_{2n+2}(\Sigma SU(n-1)^{(2n)})$. By Theorem (2.1) of [4], we have $\pi_{2n+2}(\Sigma SU(n-1)^{(2n-2)}, \Sigma SU(n-2)^{(2n-2)}) = 0$. Hence there exists $\bar{\rho}\in\pi_{2n+2}(\Sigma SU(n-2)^{(2n-2)})$ such that $j_*\bar{\rho} = \hat{\rho}\in\pi_{2n+2}(\Sigma SU(n-1)^{(2n-2)})$. Thus

$$(4.3)\qquad 2\cdot\mathrm{Im}(\tilde{\rho}^*) \subset \mathrm{Im}\{\bar{\rho}^* : [\Sigma SU(n-2)^{(2n-2)}, SU(n)] \to \pi_{2n+2}(SU(n))\}.$$

Let $n=5$ in (4.3). Since $[S^4\cup e^6, SU(5)] = 0$, $q^* : \pi_9(SU(5)) \to [\Sigma SU(3), SU(5)]$ is surjective. Hence $\mathrm{Im}(\bar{\rho}^*) = \mathrm{Im}(q\circ\bar{\rho})^*$. Since $q\circ\bar{\rho}\in\pi_{12}(S^9) = Z_{24}$, $24\cdot\mathrm{Im}(\bar{\rho}^*) = 0$ and $48\cdot\mathrm{Im}(\tilde{\rho}^*) = 0$ by (4.3). Thus $v_5(\sharp\mathrm{Im}(\tilde{\rho}^*)) = 0 < v_5(\sharp\langle\theta,\alpha\rangle) = 1$.

Let $n=6$ in (4.3). First we prove

$$(4.4)\qquad [\Sigma SU(4)^{(8)}, SU(6)] \underset{\cong}{\xrightarrow{j^*}} [\Sigma SU(3), SU(6)] \underset{\cong}{\xleftarrow{q^*}} \pi_9(SU(6)) = Z.$$

Since $\pi_4(SU(6)) = \pi_6(SU(6)) = [S^4\cup_{\eta_4} e^6, SU(6)] = 0$, there are exact sequences:

$$(4.5)\quad [S^5\cup_{\eta_5} e^7, SU(6)] \xrightarrow{\Sigma^2\rho_3^*} \pi_9(SU(6)) = \mathbf{Z} \xrightarrow{q^*} [\Sigma SU(3), SU(6)] \longrightarrow 0,$$

$$\pi_7(SU(6)) \xrightarrow{q^*} [S^5\cup_{\eta_5} e^7, SU(6)] \xrightarrow{j^*} \pi_5(SU(6)) \longrightarrow 0.$$

Since the attaching map of the top cell of any Lie group is stably trivial, the composite $S^7 \xrightarrow{\rho_3} S^3\cup_{\eta_3} e^5 \xrightarrow{q} S^5$ is null-homotopic. Hence $\Sigma^2\rho_3^*(\mathrm{Im}(q^*)) = 0$. Since $\pi_6(SU(4)) = 0$, the map $S^5 \xrightarrow{|2|} SU(3) \subset SU(4)$ can be extended to a map $f : S^5\cup_{\eta_5} e^7 \to SU(4)$. Write $\tilde{f} = j\circ f : S^5\cup_{\eta_5} e^7 \to SU(6)$. Since $\Sigma^2\rho_3^*(f) \in \pi_9(SU(4)) = \mathbf{Z}_2$ and $\pi_9(SU(6)) = \mathbf{Z}$, it follows that $\Sigma^2\rho_3^*(\tilde{f}) = 0$. Thus $\Sigma^2\rho_3^* = 0$ in (4.5), since $\tilde{f}$ and $\mathrm{Im}(q^*)$ generate $[S^5\cup_{\eta_5} e^7, SU(6)]$. Therefore

$$(4.6)\qquad q^* : \pi_9(SU(6)) \cong [\Sigma SU(3), SU(6)].$$

Since $\pi_8(SU(6)) = 0$, we have an exact sequence:

$$0 \longrightarrow [\Sigma SU(4)^{(8)}, SU(6)] \xrightarrow{j^*} [\Sigma SU(3), SU(6)] \xrightarrow{\Sigma\tau^*} \pi_7(SU(6)) = \mathbf{Z}$$

where $\tau : S^6 \to SU(3)$ is the attaching map of the 7-dimensional cell of $SU(4)$. Since $q\circ\Sigma\tau = 0$, (4.6) implies that $\Sigma\tau^* = 0$ so that j^* is an isomorphism. This ends the proof of (4.4).

Since $\pi_{10}(SU(6)) = 0$, we have an exact sequence:

$$\pi_{11}(SU(6)) \xrightarrow{q^*} [\Sigma SU(4)^{(10)}, SU(6)] \xrightarrow{\Sigma j^*} [\Sigma SU(4)^{(8)}, SU(6)] \longrightarrow 0$$

Since $q\circ\bar{\rho} \in \pi_{14}(S^{11}) = \mathbf{Z}_{24}$, we have

$$(4.7)\qquad 24\cdot\bar{\rho}^*(\mathrm{Im}(q^*)) = 0.$$

Let $h \in [\Sigma SU(4)^{(10)}, SU(6)]$ be such that $\bar{h} := \Sigma j^* h$ is the composite of

$$\Sigma SU(4)^{(8)} \xrightarrow{q} S^9 \xrightarrow{[4!]} SU(5) \xrightarrow{\subset} SU(6)$$

where $[4!]$ is a generator of $\pi_9(SU(5)) = \mathbf{Z}$. By (4.4), $\bar{h}$ is a generator of $[\Sigma SU(4)^{(8)}, SU(6)] = \mathbf{Z}$. Let $\omega : S^9 \to SU(4)^{(8)}$ be the attaching map of the unique 10-dimensional cell of $SU(4)$. Since $q\circ\Sigma\omega \in \pi_{10}(S^9) = \mathbf{Z}_2$, there exists $k : \Sigma SU(4)^{(10)} \to S^9$ such that the following diagram is commutative up to homotopy:

$$\begin{array}{ccccccccc}
S^{10} & \xrightarrow{\Sigma\omega} & \Sigma SU(4)^{(8)} & \xrightarrow{q} & S^9 & \xrightarrow{[4!]} & SU(5) & \xrightarrow{j} & SU(6) \\
 & & \downarrow{\scriptstyle \Sigma j} & & \downarrow{\scriptstyle 2\iota_9} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\
 & & \Sigma SU(4)^{(10)} & \xrightarrow{k} & S^9 & \xrightarrow{[4!]} & SU(5) & \xrightarrow{j} & SU(6)
\end{array}$$

where the latter two 2's are power maps $x \mapsto x^2$. We then have $(\Sigma j)^*(j\circ[4!]\circ k) = 2\bar{h} = (\Sigma j)^*(2h)$ and

$$2h - j\circ[4!]\circ k \in \mathrm{Im}\{q^* : \pi_{11}(SU(6)) \to [\Sigma SU(4)^{(10)}, SU(6)]\}.$$

Since $k \circ \bar{\rho} \in \pi_{14}(S^9) = 0$, we have $2 \cdot \bar{\rho}^*(h) \in \bar{\rho}^*(\mathrm{Im}(q^*))$. Thus $48 \cdot \bar{\rho}^*(h) = 0$ by (4.7), and hence $48 \cdot \mathrm{Im}(\bar{\rho}^*) = 0$. Therefore $96 \cdot \mathrm{Im}(\tilde{\rho}^*) = 0$ by (4.3). Hence $v_7(\sharp \mathrm{Im}(\tilde{\rho})) = 0 < v_7(\sharp\langle\theta, \alpha\rangle) = 1$. This completes the proof. □

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