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QUASINORMALITY OF ORDER 1 FOR FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract

Let $\mathscr{F}$ be a family of functions meromorphic on the plane domain D, all of whose zeros are multiple. Suppose that $f^{(k)}(z) \neq 1$ for all $f \in \mathscr{F}$ and $z \in D$. Then if $\mathscr{F}$ is quasinormal on D, it is quasinormal of order 1 there.

1. Introduction

This paper continues our study of the order of quasinormality of families of meromorphic functions on plane domains, all of whose zeros are multiple, initiated in [6].

Recall that a family $\mathscr{F}$ of functions meromorphic on a plane domain $D \subset \boldsymbol{C}$ is said to be quasinormal on D [2] if from each sequence $\{f_n\} \subset \mathscr{F}$ one can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly with respect to the spherical metric on $D \backslash E$, where the set E (which may depend on $\{f_{n_k}\}$) has no accumulation point in D. If E can always be chosen to satisfy $|E| \leq \nu$, $\mathscr{F}$ is said to quasinormal of order ν on D. Thus a family is quasinormal of order 0 on D if and only if it is normal on D. The family $\mathscr{F}$ is said to (quasi)normal at $z_0 \in D$ if it is (quasi)normal on some neighborhood of z_0; thus $\mathscr{F}$ is quasinormal on D if and only if it is quasinormal at each point $z \in D$. On the other hand, $\mathscr{F}$ fails to be quasinormal of order ν on D precisely when there exist points $z_1, z_2, \dots, z_{\nu+1}$ in D and a sequence $\{f_n\} \subset \mathscr{F}$ such that no subsequence of $\{f_n\}$ is normal at z_j, $j = 1, 2, \dots, \nu+1$.

In [6], we proved

Theorem A. *Let $\mathscr{F}$ be a quasinormal family of meromorphic functions on D, all of whose zeros are multiple. If for any $f \in \mathscr{F}$, $f'(z) \neq 1$ for $z \in D$, then $\mathscr{F}$ is quasinormal of order 1 on D.*

Here we extend this result to derivatives of arbitrary order.

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THEOREM. *Let $k \geq 1$ be an integer. Let $\mathscr{F}$ be a quasinormal family of meromorphic functions on D, all of whose zeros have multiplicity at least $k+1$. If for any $f \in \mathscr{F}$, $f^{(k)}(z) \neq 1$ for $z \in D$, then $\mathscr{F}$ is quasinormal of order 1 on D.*

COROLLARY. *Let k and M be positive numbers. Let $\mathscr{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least $k+1$. Suppose that each $f \in \mathscr{F}$ has at most M zeros on D and that $f^{(k)}(z) \neq 1$ on D. Then $\mathscr{F}$ is quasinormal of order 1 on D.*

Indeed, it follows easily from Lemma 2 below that $\mathscr{F}$ is quasinormal of order no greater than M, so the hypotheses of our Theorem are satisfied. That $\mathscr{F}$ need not be normal on D is shown by the following example.

Example 1. Let $D = \{z : |z| < 1\}$ and $\mathscr{F} = \{f_\alpha\}$, where

$$f_\alpha(z) = \frac{(z - \alpha/(k+1))^{k+1}}{k!(z-\alpha)} = \frac{1}{k!}z^k + P_{k-2}(z) + \frac{A}{z-\alpha}, \quad \alpha \in \boldsymbol{C}\backslash\{0\},$$

where P_{k-2} is a polynomial of degree $k-2$ and $A = (1/k!)(k/(k+1))^{k+1}\alpha^{k+1} \neq 0$. Then all zeros of f_α have multiplicity at least $k+1$ and $f_\alpha^{(k)}(z) \neq 1$. However, f_α takes on the values 0 and ∞ in any fixed neighborhood of 0 if α is sufficiently small, so $\mathscr{F}$ fails to be normal at 0.

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2. Notation and preliminary results

Let us set some notation. Throughout, k is a positive integer. We denote by Δ the open unit disc in $\boldsymbol{C}$. For $z_0 \in \boldsymbol{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. We write $f_n \overset{\chi}{\Rightarrow} f$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ on D if the convergence is in the Euclidean metric.

We require the following known results.

LEMMA 1. *Let $\mathscr{F}$ be a family of functions meromorphic on Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathscr{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

a) *points $z_n \in \Delta$, $z_n \to z_0$;*
b) *functions $f_n \in \mathscr{F}$; and*
c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \stackrel{\chi}{\Rightarrow} g(\zeta)$ *on* $\boldsymbol{C}$, *where* g *is a nonconstant meromorphic function on* $\boldsymbol{C}$, *all of whose zeros have multiplicity at least* k, *such that* $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$. *In particular,* g *has order at most* 2.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

This is the local version of [7, Lemma 2] (cf. [4, Lemma 1], [10, pp. 216–217]). The proof consists of a simple change of variable in the result cited from [7]; cf. [5, pp. 299–300].

Lemma 2. *Let* $\mathscr{F}$ *be a family of functions meromorphic on* D *and let* $k \ge 1$ *be an integer. If for each* $f \in \mathscr{F}$ *and* $z \in D$, $f(z) \ne 0$ *and* $f^{(k)}(z) \ne 1$, *then* $\mathscr{F}$ *is normal on* D.

This is a well-known result of Gu [3].

Lemma 3. *Let* $\mathscr{F}$ *be a family of functions meromorphic on* D, *all of whose zeros have multiplicity at least* $k+1$ *and all of whose poles are multiple. If for each* $f \in \mathscr{F}$, $f^{(k)}(z) \ne 1$, $z \in D$, *then* $\mathscr{F}$ *is normal on* D.

This is Theorem 5 in [9].

Lemma 4. *Let* f *be a nonconstant meromorphic function of finite order on* $\boldsymbol{C}$, *all of whose zeros have multiplicity at least* $k+1$. *If* $f^{(k)}(z) \ne 1$ *on* $\boldsymbol{C}$, *then*

$$f(z) = \frac{1}{k!} \frac{(z-a)^{k+1}}{z-b}$$

for some a *and* b $(\ne a)$ *in* $\boldsymbol{C}$.

This follows from Lemmas 6 and 8 of [9].

3. Auxiliary lemmas

The proof of the theorem proceeds by a number of intermediate results.

Lemma 5. *Let* $\{a_j\}$ *be a sequence in* Δ *which has no accumulation points in* Δ. *Let* $\{f_n\}$ *be a sequence of functions meromorphic on* Δ, *all of whose zeros have multiplicity at least* $k+1$, *such that* $f_n^{(k)}(z) \ne 1$ *for all* n *and all* $z \in \Delta$. *Suppose that*

(a) *no subsequence of* $\{f_n\}$ *is normal at* a_1;

(b) *there exists* $\delta > 0$ *such that each* f_n *has a single* (*multiple*) *zero on* $\Delta(a_1, \delta)$; *and*

(c) $f_n \stackrel{\chi}{\Rightarrow} f$ *on* $\Delta \backslash \{a_j\}_{j=1}^{\infty}$.

Then

(d) *there exists* $\eta_0 > 0$ *such that for each* $0 < \eta < \eta_0$, f_n *has a single simple pole on* $\Delta(a_1, \eta)$ *for all sufficiently large* n; *and*

(e) $f(z) = (z - a_1)^k / k!$.

Proof. It suffices to prove that each subsequence of $\{f_n\}$ has a subsequence which satisfies (d) and (e). So suppose we have a subsequence of $\{f_n\}$, which (to avoid complication in notation) we again call $\{f_n\}$.

Since $\{f_n\}$ is not normal at a_1, it follows from Lemma 1 that we can extract a subsequence (which, renumbering, we continue to call $\{f_n\}$), points $z_n \to a_1$, and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on } \boldsymbol{C}, \tag{1}$$

where g is a nonconstant meromorphic function of finite order on $\boldsymbol{C}$, all of whose zeros have multiplicity at least $k+1$. Since $g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) \neq 1$ and $g_n^{(k)} \Rightarrow g^{(k)}$ on the complement of the poles of g, either $g^{(k)} \neq 1$ or $g^{(k)} \equiv 1$, by Hurwitz' Theorem. In the latter case, g is a polynomial of degree k and therefore does not have zeros of multiplicity at least $k+1$. Thus $g^{(k)}(\zeta) \neq 1$ on $\boldsymbol{C}$; so by Lemma 4,

$$g(\zeta) = \frac{1}{k!}\frac{(\zeta - a)^{k+1}}{\zeta - b} \tag{2}$$

for distinct complex numbers a and b. It now follows from the argument principle that there exist sequences $\xi_n \to a$ and $\eta_n \to b$ such that, for sufficiently large n, $g_n(\xi_n) = 0$ and $g_n(\eta_n) = \infty$. Thus, writing $z_{n,0} = z_n + \rho_n\xi_n$, $z_{n,1} = z_n + \rho_n\eta_n$, we have $z_{n,j} \to a_1$ $(j = 0, 1)$, $f_n(z_{n,0}) = 0$ and $f_n(z_{n,1}) = \infty$.

Let us now assume that (d) has been shown to hold. It follows from Lemma 3 that the pole of f_n at $z_{n,1}$ is simple. The limit function f from (c) is either meromorphic on $\Delta\backslash\{a_j\}_{j=1}^{\infty}$ or identically infinite there. Suppose first that it is meromorphic on $\Delta\backslash\{a_j\}_{j=1}^{\infty}$. Then there exists $\delta_0 > 0$ such that f has no poles on $\Gamma = \{z : |z - a_1| = \delta_0\}$ and $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on Γ. We claim that $f^{(k)} \equiv 1$ on $\Delta'(a_1, \delta_0)$. Indeed, otherwise by Hurwitz' Theorem, $f^{(k)} \neq 1$. Now $1/(f_n^{(k)} - 1)$ is analytic on $\Delta(a_1, \delta_0)$ and converges uniformly on Γ to $1/(f^{(k)} - 1)$. By the maximum principle, $1/(f_n^{(k)} - 1)$ converges uniformly on $\Delta(a_1, \delta_0)$, so $\{f_n^{(k)}\}$ is normal at a_1. However, since $f_n^{(k)}(z_{n,0}) = 0$ and $f_n^{(k)}(z_{n,1}) = \infty$ and $z_{n,j} \to a_1$ $(j = 0, 1)$, $\{f_n^{(k)}\}$ is not equicontinuous at a_1, a contradiction.

Thus f has no poles on $\Delta'(a_1, \delta_0)$ and $f_n^{(k)} \Rightarrow 1$ on $\Delta'(a_1, \delta_0)$. We claim now that for every $0 \leq i \leq k$

$$f_n^{(k-i)}(z) \Rightarrow \frac{(z - a_1)^i}{i!} \quad \text{on } \Delta'(a_1, \delta_0). \tag{3}$$

We have already proved this for $i = 0$.

We continue by induction. Suppose that (3) holds for $i = j$ and let $i = j + 1$. For $z, z_0 \in \Delta'(a_1, \delta_0)$, we have

$$f_n^{(k-(j+1))}(z) - f_n^{(k-(j+1))}(z_0) = \int_{z_0}^{z} f_n^{(k-j)}(\zeta)\, d\zeta.$$

By the induction assumption, the last term tends to $(z-a_1)^{j+1}/(j+1)! - (z_0-a_1)^{j+1}/(j+1)!$; thus

$$f_n^{(k-(j+1))}(z) \Rightarrow \frac{(z-a_1)^{j+1}}{(j+1)!} + \beta(z_0),$$

where $\beta(z_0) = \lim_{n\to\infty}[f_n^{(k-(j+1))}(z_0) - (z_0-a_1)^{j+1}/(j+1)!]$.

We now show that $\beta(z_0) = 0$. If not, take r such that $0 < r < \min\{|(j+1)!\beta(z_0)|^{1/(j+1)}, \delta_0\}$. For large enough n, we have

$$(4) \quad \frac{1}{2\pi i}\int_{|\zeta-a_1|=r} \frac{f_n^{(k-j)}(\zeta)}{f_n^{(k-(j+1))}(\zeta)}\,d\zeta = \frac{1}{2\pi i}\int_{|\zeta-a_1|=r} \frac{(\zeta-a_1)^j/j!}{(\zeta-a_1)^{j+1}/(j+1)! + \beta(z_0)}\,d\zeta.$$

Now the right hand term is zero, since the zeros of $(\zeta-a_1)^{j+1}/(j+1)! + \beta(z_0)$ are outside $\Delta(a_1, r)$. By condition (d), the number of poles in $\Delta(a_1, \delta_0)$ of $f_n^{(k-(j+1))}$ in (4) is $k-(j+1)+1 = k-j$, counting multiplicities.

As for the number of zeros, without loss of generality, we may assume $b = 0$ in (2). Then $a \neq 0$, and we have

$$\begin{aligned} g(\zeta) &= \frac{1}{k!}\frac{1}{\zeta}\left[\zeta^{k+1} - (k+1)a\zeta^k + \cdots + (-1)^k\binom{k+1}{k}z^k\zeta + (-1)^{k+1}a^{k+1}\right] \\ &= \frac{1}{k!}\left[\zeta^k - (k+1)a\zeta^{k-1} + \cdots + \frac{(-1)^{k+1}a^{k+1}}{\zeta}\right]. \end{aligned}$$

Hence, for each $0 \le i \le k$, $g^{(i)}(\zeta)$ has exactly $k+1$ zeros in $\boldsymbol{C}$, counting multiplicities. Thus by (1), for large enough n, $f_n^{(i)}(z)$ has at least $k+1$ zeros in $\Delta(a_1, \delta_0)$. We then get by the argument principle that the left hand term in (4) is at least $k+1-(k-j) = j+1$, and we have a contradiction. Thus $\beta(z_0) = 0$, and (3) is proved. Take $i = k$ in (3) to get assertion (e).

Suppose now that $f \equiv \infty$ on $\Delta\backslash\{a_j\}_{j=1}^{\infty}$. Let

$$F_n(z) = f_n(z)\frac{z - z_{n,1}}{(z - z_{n,0})^{k+1}}.$$

By (b), $F_n(z) \neq 0$ on $\Delta(a_1, \delta)$. Applying the maximum principle to the sequence $\{1/F_n\}$ of analytic functions, we see that $F_n \Rightarrow \infty$ on $\Delta(a_1, \delta)$. We have

$$(5) \qquad \begin{aligned} \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} &= \frac{F_n(z_n + \rho_n\zeta)}{\rho_n^k}\,\frac{(\rho_n\zeta + z_n - z_{n,0})^{k+1}}{(\rho_n\zeta + z_n - z_{n,1})} \\ &= F_n(z_n + \rho_n\zeta)\frac{(\zeta - \xi_n)^{k+1}}{\zeta - \eta_n}. \end{aligned}$$

It follows from (1), (2), and (5) that $F_n(z_n + \rho_n\zeta) \to 1$, which contradicts $F_n \Rightarrow \infty$ near a_1. Thus the posssibility $f \equiv \infty$ may be ruled out.

We have shown that when (d) obtains, (e) does as well. Now let us show that (d) must hold. Suppose not. Then, taking a subsequence and renumbering, we may assume that on any neighborhood of a_1, f_n has at least two poles for sufficiently large n. Keeping the notation established above, let $z_{n,2} \neq z_{n,1}$ be such that $f_n(z_{n,2}) = \infty$ and f_n has no poles in $\Delta'(z_{n,1}, |z_{n,1} - z_{n,2}|)$. Write $z_{n,2} = z_n + \rho_n \eta_n^*$. Then $z_{n,2} \to a_1$ but $\eta_n^* \to \infty$ since the right hand side of (2) has but a single simple pole. Set

$$G_n(\zeta) = \frac{f_n(z_{n,1} + (z_{n,2} - z_{n,1})\zeta)}{(z_{n,2} - z_{n,1})^k}.$$

Since $z_{n,2} - z_{n,1} \to 0$, $G_n(\zeta)$ is defined for any $\zeta \in \boldsymbol{C}$ if n is sufficiently large; and $G_n^{(k)}(\zeta) \neq 1$. Note that $G_n(1) = \infty$. Also,

$$G_n(0) = \infty \quad G_n\left(\frac{z_{n,0} - z_{n,1}}{z_{n,2} - z_{n,1}}\right) = 0$$

and

$$\frac{z_{n,0} - z_{n,1}}{z_{n,2} - z_{n,1}} = \frac{\xi_n - \eta_n}{\eta_n^* - \eta_n} \to 0,$$

so $\{G_n\}$ is not normal at 0. On the other hand, for n sufficiently large, G_n has only a single zero (which tends to 0 as $n \to \infty$) on any compact subset of $\boldsymbol{C}$. Since $G_n'(\zeta) \neq 1$, it follows from Lemma 2 that $\{G_n\}$ is normal on $\boldsymbol{C} \backslash \{0\}$. Taking a subsequence and renumbering, we may assume that $G_n \overset{\chi}{\Rightarrow} G$ on $\boldsymbol{C} \backslash \{0\}$. Since G_n has only a single pole on Δ, conditions (a), (b), (c), and (d) hold for the sequence $\{G_n\}$ (defined, say, on $\Delta(0,2)$) with $a_1 = 0$ and $\delta = 1$. Thus, by the first part of the proof, $G(\zeta) = \zeta^k/k!$. But this contradicts $G(1) = \infty$. This completes the proof of Lemma 5.

Definition. Let $z_1, z_2 \in \boldsymbol{C}$ and put $\tilde{z} = (z_1 + z_2)/2$. We say that (z_1, z_2) is a k-nontrivial pair of zeros of f if

(i) $f(z_1) = f(z_2) = 0$ and

(ii) there exists z_3 such that $|z_3 - \tilde{z}| < |z_1 - z_2|$ and $|f'(z_3)|/|z_1 - z_2|^{k-1} > 1$.

Note that (ii) is equivalent to

(ii′) there exists z^* such that $|z^*| < 1$ and $|h'(z^*)| > 1$, where

$$h(z) = \frac{f(\tilde{z} + (z_1 - z_2)z)}{(z_1 - z_2)^k}.$$

Since $|h'(z)| \geq h^{\#}(z)$, it suffices to have $h^{\#}(z^*) > 1$ in (ii′).

Our next result deals with the situation in which the functions f_n have more than a single zero in each neighborhood of a point of non-normality.

Lemma 6. *Let $\{f_n\}$ be a sequence of functions meromorphic on Δ, all of whose zeros have multiplicity at least $k+1$, such that $f_n^{(k)}(z) \neq 1$ for all n and all $z \in \Delta$. Suppose that*

(a) *no subsequence of* $\{f_n\}$ *is normal at* z_0, *and*

(b) *for each* $\delta > 0$, f_n *has at least two distinct zeros on* $\Delta(z_0, \delta)$ *for sufficiently large* n.

Then for each $\delta > 0$, f_n *has a* k*-nontrivial pair* (a_n, c_n) *of zeros on* $\Delta(z_0, \delta)$ *for sufficiently large* n, *and*

$$\left\{\frac{f_n(d_n + (a_n - c_n)\zeta)}{(a_n - c_n)^k}\right\}$$

is not normal on Δ. *Here* $d_n = (a_n + c_n)/2$.

Proof. As in the proof of the previous lemma, it follows from (a) and Lemmas 1 and 4 that for each subsequence of $\{f_n\}$ there exists a (sub)subsequence (which, renumbering, we continue to denote by $\{f_n\}$), points $z_n \to z_0$, numbers $\rho_n \to 0^+$, and distinct $a, b \in \boldsymbol{C}$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} g(\zeta) = \frac{1}{k!}\frac{(\zeta - a)^{k+1}}{\zeta - b} \quad \text{on } \boldsymbol{C}. \tag{6}$$

Thus there exist $\xi_n \to a$, $\eta_n \to b$ so that $a_n = z_n + \rho_n\xi_n \to z_0$, $b_n = z_n + \rho_n\eta_n \to z_0$ and $g_n(\xi_n) = f_n(a_n) = 0$, $g_n(\eta_n) = f_n(b_n) = \infty$ for n sufficiently large.

By assumption, there also exists $c_n \neq a_n$, $c_n \to z_0$, such that $f_n(c_n) = 0$. Thus $c_n = z_n + \rho_n\xi_n^*$ and $\xi_n^* \to \infty$ by (6). Setting $d_n = (a_n + c_n)/2$, we see that the function

$$h_n(\zeta) = \frac{f_n(d_n + (a_n - c_n)\zeta)}{(a_n - c_n)^k}$$

is defined for any $\zeta \in \boldsymbol{C}$ if n is sufficiently large. We claim that $\{h_n\}$ is not normal at $\zeta = 1/2$. Indeed, we have

$$\frac{a_n - d_n}{a_n - c_n} \to \frac{1}{2}, \qquad \frac{b_n - d_n}{a_n - c_n} \to \frac{1}{2},$$

$$h_n\left(\frac{a_n - d_n}{a_n - c_n}\right) = f_n(a_n) = 0, \qquad h_n\left(\frac{b_n - d_n}{a_n - c_n}\right) = f_n(b_n) = \infty,$$

so $\{h_n\}$ fails to be equicontinuous in a neighborhood of $1/2$. It follows from Marty's Theorem that

$$\lim_{n\to\infty} \sup_{|\zeta - 1/2| \le 1/4} h_n^\#(\zeta) = \infty.$$

Thus (a_n, c_n) is a k-nontrivial pair of zeros of f_n for n sufficiently large.

LEMMA 7. *Let* $\{f_n\}$ *be a sequence of functions meromorphic on* Δ, *all of whose zeros have multiplicity at least* $k + 1$, *such that* $f_n^{(k)}(z) \neq 1$ *for all* n *and all* $z \in \Delta$. *Suppose that*

(a) *there exist* $d \in \Delta$, $a_n \to d$, $c_n \to d$, *and* $z_0 \in \boldsymbol{C}$ *such that for every* $\delta > 0$,

$$h_n(z) = \frac{f_n(d_n + (a_n - c_n)z)}{(a_n - c_n)^k}$$

has at least two distinct zeros on $\Delta(z_0, \delta)$ *for sufficiently large* n, *where* $d_n = (a_n + c_n)/2$; *and*

(b) *no subsequence of* $\{h_n\}$ *is normal at* z_0.

Then for n *sufficiently large*, f_n *has a* k*-nontrivial pair of zeros* $(z_{n,1}^*, z_{n,2}^*)$ *such that* $z_{n,j}^* \to d$ $(j = 1, 2)$ *and* $|z_{n,1}^* - z_{n,2}^*| < |a_n - c_n|$.

Proof. As before, it follows from Lemmas 1 and 4 that to each subsequence of $\{h_n\}$ there corresponds a subsequence (which we continue to write as $\{h_n\}$), $z_n \to z_0$, and $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{h_n(z_n + \rho_n \zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} \frac{1}{k!} \frac{(\zeta - a)^{k+1}}{\zeta - b} \quad \text{on } \boldsymbol{C}.$$

Thus there exist $\xi_{n,0} \to b$, $\xi_{n,1} \to a$ so that $z_{n,j} = z_n + \rho_n \xi_{n,j} \to z_0$ $(j = 0, 1)$ and $g_n(\xi_{n,0}) = h_n(z_{n,0}) = \infty$, $g_n(\xi_{n,1}) = h_n(z_{n,1}) = 0$. By (a), there exist $z_{n,2} \to z_0$, $z_{n,2} \neq z_{n,1}$, such that $h_n(z_{n,2}) = 0$. Setting $z_{n,2} = z_n + \rho_n \xi_{n,2}$, we have $\xi_{n,2} \to \infty$. Now put

$$z_{n,j}^* = d_n + (a_n - c_n) z_n + \rho_n (a_n - c_n) \xi_{n,j} \quad j = 0, 1, 2.$$

Clearly $z_{n,j}^* \to d$, $j = 0, 1, 2$. Define

$$G_n(\zeta) = \frac{f_n((z_{n,1}^* + z_{n,2}^*)/2 + (z_{n,1}^* - z_{n,2}^*)\zeta)}{(z_{n,1}^* - z_{n,2}^*)^k}.$$

Then $\{G_n\}$ is not normal at $\zeta = 1/2$. Indeed,

$$G_n\left(\frac{2\xi_{n,0} - \xi_{n,1} - \xi_{n,2}}{2(\xi_{n,1} - \xi_{n,2})}\right) = \infty, \quad G_n(1/2) = 0.$$

Since $(2\xi_{n,0} - \xi_{n,1} - \xi_{n,2})/2(\xi_{n,1} - \xi_{n,2}) \to 1/2$, $\{G_n\}$ is not equicontinuous at $\zeta = 1/2$. As before, it follows from Marty's Theorem that $(z_{n,1}^*, z_{n,2}^*)$ is a k-nontrivial pair of zeros of f_n. Now $|z_{n,1}^* - z_{n,2}^*| = |a_n - c_n|\,|z_{n,1} - z_{n,2}|$; therefore, since $z_{n,j} \to z_0$ $(j = 1, 2)$, we have $|z_{n,1}^* - z_{n,2}^*| < |a_n - c_n|$ for large enough n, as required.

LEMMA 8. *Let* $\{f_n\}$ *be a sequence of functions meromorphic on* Δ, *all of whose zeros have multiplicity at least* $k + 1$, *such that* $f_n^{(k)}(z) \neq 1$ *for all* n *and all* $z \in \Delta$. *Suppose that*

(a) $\{f_n\}$ *is normal on* $\Delta'(0, 1)$, *but no subsequence of* $\{f_n\}$ *is normal at* 0; *and*

(b) *there exists* $\delta > 0$ *such that* f_n *has a single* (*multiple*) *zero on* $\Delta(0, \delta)$ *for all sufficiently large* n.

Then there exists a subsequence of $\{f_n\}$ (*which we continue to call* $\{f_n\}$) *such that for any* $a \in \boldsymbol{C}$, $f_n - a$ *has at most* $k + 1$ *zeros* (*counting multiplicity*) *on* $\Delta(0, 1/2)$.

Proof. Taking a subsequence and renumbering, we may assume that $f_n \overset{\chi}{\Rightarrow} f$ on $\Delta'(0,1)$. By Lemma 5, $f(z) = z^k/k!$. Suppose that $|a| < (2/3)^k/k!$. Taking Γ to be the circle $\{|z| = 3/4\}$ traversed once in the positive direction, we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{f_n'(z)}{f_n(z)-a}\,dz \to \frac{1}{2\pi i}\int_\Gamma \frac{z^{k-1}/(k-1)!}{z^k/k!-a}\,dz = k.$$

However, the left hand side is the number of a-points of f_n minus the number of poles of f_n inside Γ, counting multiplicities. By Lemma 5, there exists $0 < \delta < 3/4$ such that f_n has a single simple pole on $\Delta(0,\delta)$ for n sufficiently large. Since f_n converges uniformly to $z^k/k!$ on $\{z : \delta \le |z| \le 3/4\}$, there exists N_1 such that if $n \ge N_1$ f_n has a single simple pole in $\Delta(0, 3/4)$. Hence for $n \ge N_1$, f_n takes on the value a (counting multiplicities) exactly $k+1$ times on $\Delta(0, 3/4)$.

Suppose now that $|a| > (2/3)^k/k!$. Let Γ' be the circle $\{|z| = 5/9\}$ traversed in the positive direction. Then

$$\frac{1}{2\pi i}\int_{\Gamma'} \frac{f_n'(z)}{f_n(z)-a}\,dz \to \frac{1}{2\pi i}\int_{\Gamma'} \frac{z^{k-1}/(k-1)!}{z^k/k!-a}\,dz = 0,$$

so the number of a-points minus the number of poles of f_n (counting multiplicity) inside Γ' is 0 for large n. It follows as before that there exists N_2 such that f_n takes on the value a exactly once (counting multiplicities) on $\Delta(0, 5/9)$ if $n \ge N_2$. Dropping the elements f_n with $n < \max(N_1, N_2)$ and renumbering, we obtain the desired sequence.

LEMMA 9. *Let f be a meromorphic function on $\boldsymbol{C}$, all of whose zeros have multiplicity at least $k+1$, such that $f^{(k)}(z) \ne 1$, $z \in \boldsymbol{C}$. Then either*

(i) *f is rational*; *or*

(ii) *there exist k-nontrivial pairs (a_n, c_n) of zeros of f such that $|a_n - c_n| \to 0$ and a sequence of functions*

$$h_n(\zeta) = \frac{f(d_n + (a_n - c_n)\zeta)}{(a_n - c_n)^k}$$

which is not normal on Δ; here $d_n = (a_n + c_n)/2$.

Proof. Suppose f is not rational. Then by Lemma 4, f has infinite order, so there exist $z_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$S(\Delta(z_n, \varepsilon_n), f) = \frac{1}{\pi}\iint_{|z-z_n|\le\varepsilon_n} [f^\#(z)]^2\,dxdy \to \infty. \tag{7}$$

Indeed, otherwise there would exist $\varepsilon > 0$ and $M > 0$ such that $S(\Delta(\zeta, \varepsilon), f) \le M$ for all $\zeta \in \boldsymbol{C}$. From this follows

$$S(r) = \frac{1}{\pi}\iint_{|z|<r} [f^\#(z)]^2\,dxdy = O(r^2),$$

so that (cf. [10, p. 217]) f would have order at most 2, a contradiction. In particular, there exist $z_n^* \in \Delta(z_n, \varepsilon_n)$ such that $f^{\#}(z_n^*) \to \infty$. Let $f_n(z) = f(z + z_n^*)$. Then no subsequence of $\{f_n\}$ is normal at 0.

Suppose there exists $\delta > 0$ such that f_n has only a single (multiple) zero ξ_n on $\Delta(0, \delta)$. Since no subsequence of $\{f_n\}$ is normal at 0, $\xi_n \to 0$ by Lemma 2. Thus, again by Lemma 2, $\{f_n\}$ is normal on $\Delta'(0, \delta)$. It follows from Lemma 8 that there exist $n_1 < n_2 < \cdots$ such that for any $a \in \boldsymbol{C}$, $f_{n_j} - a$ has at most $k+1$ zeros (counting multiplicity) on $\Delta(0, \delta/2)$. Thus, for large enough j,

$$S(\Delta(z_{n_j}, \varepsilon_{n_j}), f) \le S(\Delta(0, \delta/2), f_{n_j}) \le k+1$$

which contradicts (7).

Thus, for each $\delta > 0$, f_n has at least two distinct zeros on $\Delta(0, \delta)$ for sufficiently large n. The result now follows immediately from Lemma 6.

4. Proof of the Theorem

Suppose the Theorem is false. Then there exists a sequence $\{a_j^*\} \subset D$ with no accumulation point in D and such that $a_1^* \ne a_2^*$ and a sequence $\{f_n\} \subset \mathscr{F}$ such that $f_n \overset{\chi}{\Rightarrow} f$ on $D \backslash \{a_j^*\}$ but no subsequence of $\{f_n\}$ is normal at a_1^* or a_2^*. We may assume that $a_1^* = 0$ and $D = \Delta$. The argument given in the proof of Lemma 5 shows that $f_n^{(k)} \Rightarrow 1$ on $\Delta \backslash \{a_j^*\}$ or $f = \infty$, so $f \not\equiv 0$.

If there exists $\delta > 0$ such that f_n has only a single (multiple) zero on each $\Delta(a_j^*, \delta)$ $(j = 1, 2)$ for large enough n, it follows from Lemma 5 that $f(z) = (z - a_j^*)^k / k!$ $(j = 1, 2)$ on $\Delta \backslash \{a_j^*\}$. Thus $a_1^* = a_2^*$, a contradiction.

Therefore, one may suppose that for any $\delta > 0$, f_n has at least two distinct zeros on $\Delta(0, \delta)$ for sufficiently large n. By Lemma 6, f_n has a k-nontrivial pair of zeros in $\Delta(0, \delta)$ for n large enough. Therefore, some subsequence of $\{f_n\}$ (which, as usual, we continue to call $\{f_n\}$) has a k-nontrivial pair of zeros (z_n, w_n) such that $|z_n| < 1/n$, $|w_n| < 1/n$. There exist $\delta_0 > 0$ and $1 < s < 2$ such that $f_n \overset{\chi}{\Rightarrow} f$ on $\Delta'(0, 2\delta_0)$ and f does not vanish for $\delta_0 \le |z| \le s\delta_0$. For $1/n < \delta_0$, let (a_n, c_n) be a k-nontrivial pair of zeros of f_n in $\Delta(0, \delta_0)$ whose distance is minimal. Clearly, $a_n - c_n \to 0$. Set $d_n = (a_n + c_n)/2$. Then $d_n \in \Delta(0, \delta_0)$; and, passing to a subsequence, we may assume that $d_n \to a$, so $|a| \le \delta_0$. Since f and f_n have no zeros on $\{z : \delta_0 \le |z| \le s\delta_0\}$ if n is large enough, (a_n, c_n) is a k-nontrivial pair of zeros of f_n on $\Delta(0, s\delta_0)$ whose distance is minimal.

Set

$$h_n(\zeta) = \frac{f_n(d_n + (a_n - c_n)\zeta)}{(a_n - c_n)^k}.$$

Then for each $\zeta \in \boldsymbol{C}$, $h_n(\zeta)$ is defined if n is sufficiently large. Clearly, all zeros of h_n have multiplicity at least $k+1$ and $h_n^{(k)}(\zeta) \ne 1$. We claim that no subsequence of $\{h_n\}$ is normal on $\boldsymbol{C}$. Otherwise, taking a subsequence and renumbering, we would have $h_n \overset{\chi}{\Rightarrow} h$ on $\boldsymbol{C}$. Since (a_n, c_n) is a k-nontrivial pair of zeros of f_n, $h_n(\pm 1/2) = h_n'(\pm 1/2) = \cdots = h_n^{(k)}(\pm 1/2) = 0$, and $\sup_\Delta |h_n^{(k)}(z)| > 1$.

It follows easily that $h^{(k)}(\zeta) \neq 1$ on $\boldsymbol{C}$ and that h is nonconstant. Since all zeros of h have multiplicity at least $k+1$, Lemma 4 shows that h must be transcendental. It then follows from Lemma 9 that there exist infinitely many k-nontrivial pairs (ξ_j, η_j) of zeros of h such that $\xi_j \to \infty$ and $\xi_j - \eta_j \to 0$, and z_j^* with $|z_j^* - (\xi_j + \eta_j)/2| < |\xi_j - \eta_j|$ and $h^{\#}(z_j^*) \to \infty$.

Fix j such that $h^{\#}(z_j^*) \geq 2$ and $|\xi_j - \eta_j| < 1$. Then there exist $\xi_{n,j} \to \xi_j$ and $\eta_{n,j} \to \eta_j$ such that for n sufficiently large, $h_n(\xi_{n,j}) = h_n(\eta_{n,j}) = 0$ and $|z_j^* - (\xi_{n,j} + \eta_{n,j})/2| < |\xi_{n,j} - \eta_{n,j}|$. Put

$$\xi_{n,j}^* = d_n + (a_n - c_n)\xi_{n,j} \quad \eta_{n,j}^* = d_n + (a_n - c_n)\eta_{n,j} \quad z_{n,j}^* = d_n + (a_n - c_n)z_j^*.$$

Then

$$\left| z_{n,j}^* - \frac{\xi_{n,j}^* + \eta_{n,j}^*}{2} \right| = |a_n - c_n| \left| z_j^* - \frac{\xi_{n,j} + \eta_{n,j}}{2} \right| < |a_n - c_n| \, |\xi_{n,j} - \eta_{n,j}| = |\xi_{n,j}^* - \eta_{n,j}^*|,$$

where $\xi_{n,j}^* \to a$, $\eta_{n,j}^* \to a$ and $|a| < s\delta_0$; also, for n sufficiently large, $|f_n'(z_{n,j}^*)/(a_n - c_n)^{k-1}| = |h_n'(z_j^*)| \geq h_n^{\#}(z_j^*) > 1$. We conclude that $(\xi_{n,j}^*, \eta_{n,j}^*)$ is a k-nontrivial pair of zeros of f_n on $\Delta(0, s\delta_0)$. However,

$$|\xi_{n,j}^* - \eta_{n,j}^*| = |a_n - c_n| \, |\xi_{n,j} - \eta_{n,j}| < |a_n - c_n|$$

if n is sufficiently large. This contradicts the fact that (a_n, c_n) is a k-nontrivial pair of zeros of f_n in $\Delta(0, s\delta_0)$ whose distance is minimal.

Thus no subsequence of $\{h_n\}$ is normal on $\boldsymbol{C}$. Let E be the set on which $\{h_n\}$ is not normal. Suppose that for each $\zeta \in E$, there is a neighborhood on which h_n has only a single (multiple) zero for sufficiently large n. Then by Lemma 2, $\{h_n\}$ is quasinormal at each point of E and hence on all of $\boldsymbol{C}$. Let $\zeta_0 \in E$. Taking a subsequence, we may assume that no subsequence of $\{h_n\}$ is normal at ζ_0 and that $\{h_n\}$ converges locally spherically uniformly on $\boldsymbol{C} \backslash E_0$, where $E_0 \subset E$ is a discrete set containing ζ_0. By Lemma 5, $h_n \overset{\chi}{\Rightarrow} (\zeta - \zeta_0)^k/k!$ on $\boldsymbol{C} \backslash E_0$. Taking additional subsequences and diagonalizing, we may assume that no subsequence of $\{h_n\}$ is normal at any point of E_0. We claim that $E_0 = \{\zeta_0\}$. Indeed, otherwise there exists $\zeta_1 \in E_0$, $\zeta_1 \neq \zeta_0$; then, as before, it follows from Lemma 5 that $h_n(\zeta) \overset{\chi}{\Rightarrow} (\zeta - \zeta_1)^k/k!$ on $\boldsymbol{C} \backslash E_0$, so that $\zeta_1 = \zeta_0$, $E_0 = \{\zeta_0\}$, and $h_n(\zeta) \overset{\chi}{\Rightarrow} (\zeta - \zeta_0)^k/k!$ on $\boldsymbol{C} \backslash \{\zeta_0\}$. But this contradicts $h_n(\pm 1/2) = 0$. Hence there exists $\zeta_0 \in E$ such that for each $\delta > 0$, there is a subsequence of $\{h_n\}$ (which we continue to call $\{h_n\}$) such that each h_n has at least two distinct zeros in $\Delta(\zeta_0, \delta)$ for sufficiently large n. Then by Lemma 7, for n sufficiently large, f_n has a nontrivial pair of zeros $(w_{n,1}^*, w_{n,2}^*)$ such that $w_{n,j}^* \to a$ $(j = 1, 2)$ and $|w_{n,1}^* - w_{n,2}^*| < |a_n - c_n|$. This contradicts the fact that (a_n, c_n) is a nontrivial pair of zeros of f_n in $\Delta(0, s\delta_0)$ whose distance is minimal.

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