

## CURVATURE CHARACTERIZATIONS OF TWISTOR SPACES OVER FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS

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### Abstract

In this paper, we study the complex contact structure of a twistor space over a self-dual, Einstein 4-manifold with nonzero scalar curvature. Although the existence of such a structure has been known and well utilized by researchers for several decades now, the Hermitian geometry resulting from the complex contact structure is still in the process of being fully developed. Here we give a characterization of such twistor spaces as those satisfying a curvature (and hence purely geometric) identity. Later, we describe how this result fits in with other areas of research in complex contact geometry.

### 1. Introduction

The study of the twistor spaces over Riemannian four-manifolds began with the seminal results by Atiyah, Hitchin and Singer in [2] regarding the  $SO(4)$ -splitting of the Weyl curvature tensor. Following the heels of this paper, were the results found independently by Salamon ([23]) and Bérard-Bergery ([3]), which characterized the defined almost complex structures on the twistor spaces according to what was happening with the metric structure of the base manifold. In all three lines of research, the object was to encode the information of the downstairs manifold into the complex (or at least almost complex) geometry of the twistor space.

Since the publication of these three important papers, the field of twistor theory has exploded. Although for many researchers, the emphasis continues to be what information about the base manifold can be gleaned from the twistor space, many researchers have studied the twistor spaces as geometric objects of their own. Most notable for our purposes here are the curvature equations derived by Davidov and Muscarov in [8] and the famous Inverse Twistor Construction (see [22]). Although this theorem generalizes to any odd dimension for  $Z$ , this last theorem is stated by Swann in [24] as follows.

**THEOREM 1.1.** *Let  $Z$  be a three-dimensional complex manifold such that*

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1. There is a holomorphic line bundle  $\mathcal{L}$  over  $Z$  and a one-form  $\theta \in \Omega^1(Z, \mathcal{L})$  such that  $\theta \wedge d\theta$  vanishes nowhere, that is,  $Z$  is a complex contact manifold;
2. The fibration  $Z \rightarrow M$  is not holomorphic, but the fibres are complex lines with normal bundle  $2\mathcal{O}(1)$  and these lines are transverse to the contact distribution, that is,  $\theta$  is nowhere zero on the vertical fibres; and
3.  $Z$  admits a real structure  $\sigma$ , that is, an anti-holomorphic involution such that  $\sigma$  preserves the vertical fibres and  $\sigma^*\theta = \bar{\theta}$ .

Then  $Z$  is the twistor space over a four-dimensional self-dual, Einstein Riemannian manifold.

One aspect of twistor geometry that is still being developed is the rich Hermitian geometry that arises from the complex contact structure. There has been much work since the 1950s in this area regarding *real* contact geometry (see, for example, [5]), but the complex contact side of things is still being mapped out. In this paper, we will use this Hermitian geometry associated to the complex contact structure to prove the following theorem.

**THEOREM 6.1.** *Let  $(Z, J)$  be a three-dimensional manifold. Then  $Z$  is biholomorphic to the twistor space of a self-dual, Einstein four-manifold with non-zero scalar curvature, if and only if  $Z$  has a hermitian metric  $h$  and a  $J$ -invariant splitting  $TZ = \mathcal{V} \oplus \mathcal{H}$  such that  $\dim_{\mathbb{R}} \mathcal{H} = 4$ ,  $\mathcal{V}$  is totally geodesic (and hence integrable),  $\mathcal{H}$  is holomorphic with  $h$  bundlelike through  $\mathcal{V}$ , and, for every unit  $U \in \mathcal{V}$ , the Riemannian curvature of  $Z$ ,  $R$ , satisfies*

$$\begin{aligned} R_{XY}U &= \lambda u(Y)\mathcal{H}X + \mu v(Y)J'X \\ &\quad - \lambda u(X)\mathcal{H}Y - \mu v(X)J'Y \\ &\quad + [v_1h(J'X, Y) + v_2h(J''X, Y)]V, \end{aligned}$$

where  $V = -JU$ ,  $u(E) = h(U, E)$ ,  $v(E) = h(V, E)$  and  $J' = J \circ \mathcal{H}$  and  $J'' = J \circ \mathcal{V}$  for some constants,  $\lambda$ ,  $\mu$ ,  $v_1$  and  $v_2$ , such that  $\lambda + \mu \neq 0$  and  $v_2 > 0$ .

The curvature identity in the theorem above is actually a generalization of what is called a “normal complex contact structure.” This notion—derived from an analogue in real contact geometry—was first developed by Ishihara and Konishi in [17] but later expanded more satisfactorily by Korkmaz in [21]. More recently, some work has been done regarding another category of normal complex contact structures, of which the complex Heisenberg group is an element, in [10] and [11].

After a brief tour of complex contact geometry in the first section, we conduct a quick exposition of twistor theory over four-dimensional Riemannian manifolds, including a few well-known results. Next, restricting our attention to the twistor spaces over self-dual, Einstein four-manifolds with non-zero scalar curvature, we explore some aspects of their holomorphic structures, including their complex contact structures. This section ends with a very technical theorem detailing some Hermitian and curvature identities. In the fifth section, we

detail some results regarding holomorphic subbundles of codimension one, so that we can prove the main theorem in the sixth section. Finally, in the last section, we discuss how this results fits within the current research in the Hermitian geometry of complex contact manifolds, giving us the ability to re-state the theorem in more aesthetically pleasing ways.

Unlike the seminal work by Davidov and Muscarov in [8], we will be using the standard wedge coefficients for forms in order to conform with the currently used terminology of complex contact geometry in [18], [21] and [11]. So, for example, if  $\alpha$  and  $\beta$  are vectors, then the 2-vector  $\alpha \wedge \beta$  is defined to be:

$$\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha).$$

A particular consequence of this is that, for any vectors  $E, F$  and 1-form  $\beta$ ,

$$d\beta(E, F) = \frac{1}{2}(E\beta(F) - F\beta(E) - \beta([E, F])).$$

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## 2. A brief introduction to complex contact geometry

Although we will not be using too many of the ideas from this section, it will be useful to give a brief explanation of the current terms regarding complex contact geometry. Recall that a *complex contact manifold* is a complex manifold  $(Z, J)$  of complex dimension  $2n + 1$  such that there is an open atlas  $\mathcal{U} = \{\mathcal{O}\}$  of  $Z$  for which the following statements hold:

1. On each  $\mathcal{O} \in \mathcal{U}$ , there is a holomorphic 1-form  $\eta$  such  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $\mathcal{O}$ .
2. For  $\mathcal{O}, \mathcal{O}' \in \mathcal{U}$  with respective 1-forms  $\eta, \eta'$ , there is holomorphic function  $f : \mathcal{O} \cap \mathcal{O}' \rightarrow \mathbf{C}^*$  such that  $\eta' = f\eta$ .

If we define  $\mathcal{H}^{1,0} = (\bigcup_{\mathcal{O} \in \mathcal{U}} \ker(\eta)) \cap T^{1,0}Z$ , then  $\mathcal{H}^{1,0}$  is a well-defined, holomorphic subbundle of  $T^{1,0}Z$  of maximal rank and complex dimension  $2n$ , called *the holomorphic contact subbundle* of  $Z$ . Set  $\mathcal{H} = \{X \in T^{\mathbf{R}}Z : X - iJX \in \mathcal{H}^{1,0}\}$ . Thus,  $\mathcal{H}$  is a subbundle of  $TZ$  with maximal rank, which is holomorphic in the sense that there are local bases of  $\mathcal{H}$   $\{X_1, \dots, X_{4n}\}$  such that  $\mathcal{L}_{X_j}J = 0$  for  $j = 1, \dots, 4n$ . We will call  $\mathcal{H}$  the *holomorphic contact subbundle* or simply the *contact subbundle*.

Let  $(Z, J)$  be a complex contact manifold with complex contact subbundle  $\mathcal{H}$ . We say that a set of  $\mathbf{C}$ -valued 1-forms  $\{\pi\}$  whose domains cover  $Z$  is a *normalized contact structure*, if any two 1-forms  $\pi, \pi'$  in this set with respective domains  $\mathcal{O}, \mathcal{O}'$  satisfy:

1.  $\mathcal{H} = \ker(\pi)$  on  $\mathcal{O}$ .
2. On  $\mathcal{O} \cap \mathcal{O}'$ ,  $\pi' = f\pi$  for some  $f : \mathcal{O} \cap \mathcal{O}' \rightarrow S^1$ .

Note that for any  $\pi$  in this set,  $\pi = u - iv$  where  $u$  and  $v$  are real 1-forms such that  $v = u \circ J$ . By taking a hermitian metric on the complex line bundle  $L = \{\omega \in \Lambda^{1,0}Z : \omega(\mathcal{H}) = 0\}$ , it is fairly easy to construct a normalized contact structure on  $Z$ . This very construction shows that such structures are far from unique. However, they are handy objects to use because of the next theorem (See [9] for proof of all the following results).

**THEOREM 2.1.** *Given a normalized contact structure  $\{\pi\}$  on a complex contact manifold  $Z$ , there is a unique, two-real-dimensional,  $J$ -invariant subbundle  $\mathcal{V}$  of  $TZ$  such that*

1.  $TZ \cong \mathcal{H} \oplus \mathcal{V}$ .
2. For any element  $\pi = u - iv$  of the normalized contact structure, there is a local basis of  $\mathcal{V}$ ,  $\{U, JU\}$  defined by:
  - (a)  $u(U) = 1, v(U) = 0, u(JU) = 0, v(JU) = -1,$
  - (b)  $du(U, X) = 0, dv(JU, X) = 0$  for any  $X \in \mathcal{H}$ .

Since there is a splitting  $TZ \cong \mathcal{H} \oplus \mathcal{V}$ , we have two natural projections  $TZ \rightarrow \mathcal{H}$  and  $TZ \rightarrow \mathcal{V}$ . We will denote these respective projections by the same name as their corresponding subbundles,  $\mathcal{H}$  and  $\mathcal{V}$ .

For a specific normalized contact structure, we define two new 2-forms,  $\hat{G}$  and  $\hat{H}$ , by

$$\begin{aligned}\hat{G}(X, Y) &= du(\mathcal{H}X, \mathcal{H}Y), \\ \hat{H}(X, Y) &= dv(\mathcal{H}X, \mathcal{H}Y).\end{aligned}$$

Then it is not difficult to show that

$$\begin{aligned}\hat{G} &= du - \sigma \wedge v, \\ \hat{H} &= dv + \sigma \wedge u,\end{aligned}$$

where  $\sigma$  is a connection form of the bundle  $L$ . We call  $\sigma$  the *Ishihara-Konishi connection* of the normalized contact structure and  $d\sigma$  its *Ishihara-Konishi curvature*.

Given a normalized contact structure  $\{\pi\}$ , a hermitian metric  $h$  is called *associated* to  $\eta$ , if it satisfies the following statements:

1. For any unit, vertical vector field  $U \in \Gamma^\infty(\mathcal{V})$ , the real endomorphism  $k_U$  defined by

$$k_U(X) = \text{skew}(X \mapsto \mathcal{H}\nabla_X U)$$

satisfies  $k_U^2 = -id_{\mathcal{H}}$ ,  $k_{JU} = J \circ k_U = -k_U \circ J$ .

2.  $\mathcal{V} \perp \mathcal{H}$ .

Thus, the covariant derivatives  $\nabla U$  and  $\nabla(JU)$  of an associated metric induce a quaternionic structure  $\{k_U, k_{JU}, J' = J \circ \mathcal{H}\}$  on  $\mathcal{H}$ . Associated metrics were shown to exist in [18]. Note also that for any unit vertical vector  $U$ ,

$$d\sigma(U, JU) = \frac{1}{2}h(R_{UJU}JU, U).$$

### 3. The twistor space of a four-dimensional Riemannian manifold

Let  $(M, \langle, \rangle)$  be a 4-dimensional Riemannian manifold. For any open set  $U$  of  $M$ , there is an orthonormal, oriented basis  $\{e_1, e_2, e_3, e_4\}$  of  $TM$  with dual basis  $\{e^1, e^2, e^3, e^4\}$  of  $T^*M$ .

On  $\Lambda^2 = \Lambda^2(TM)$ , the Hodge star operator is an involution, i.e.  $*^2 = \text{id}$ . So, the eigenvalues of  $*$  are  $+1$  and  $-1$ . In fact, the eigenspaces are locally given by:

$$\Lambda_+^2 M = \langle \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4 \rangle, \quad \Lambda_-^2 M = \langle w_1, w_2, w_3, w_4 \rangle,$$

where

$$\begin{aligned} w_1 &= e_1 \wedge e_2 - e_3 \wedge e_4 & \bar{w}_1 &= e_1 \wedge e_2 + e_3 \wedge e_4 \\ w_2 &= e_1 \wedge e_3 - e_4 \wedge e_2 & \bar{w}_2 &= e_1 \wedge e_3 + e_4 \wedge e_2 \\ w_3 &= e_1 \wedge e_4 - e_2 \wedge e_3 & \bar{w}_3 &= e_1 \wedge e_4 + e_2 \wedge e_3. \end{aligned}$$

The inner product  $\langle, \rangle$  on  $TM$  induces a metric on  $\Lambda^2 M$  by setting:

$$\langle \alpha \wedge \beta, \gamma \wedge \delta \rangle = \frac{1}{2} \begin{vmatrix} \langle \alpha, \gamma \rangle & \langle \alpha, \delta \rangle \\ \langle \beta, \gamma \rangle & \langle \beta, \delta \rangle \end{vmatrix} \quad \text{for primitive 2-vectors,}$$

and defining bilinearly on general 2-vectors.

The induced Levi-Civita connection on  $\Lambda^2 M$  from that of the original Riemannian metric  $\langle, \rangle$  on  $TM$  is the bundle connection for this new metric on  $\Lambda^2 M$ . Furthermore, taking  $\{w_1, w_2, w_3\}$  as the oriented, orthonormal basis of  $\Lambda_-^2 M$ , we have a well-defined cross-product on  $\Lambda_-^2 M$  with  $w_1 \times w_2 = w_3$ .

The Riemannian curvature  $\tilde{R}$  of  $(M, \langle, \rangle)$  induces an operator  $\mathcal{R}$  on  $\Lambda^2 M$  by

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = -\langle \tilde{R}_{XY} Z, W \rangle.$$

Also, the bundle connection  $\nabla$  on  $\Lambda^2 M$  induces a bundle curvature given by:

$$R_{X \wedge Y}^\nabla(Z \wedge W) = \tilde{R}_{X \wedge Y} Z \wedge W + Z \wedge \tilde{R}_{X \wedge Y} W.$$

It is well-known that the relationship between these curvatures on  $\Lambda_-^2$  is given by:

$$\langle R_{XY}^\nabla \alpha, \beta \rangle = \langle \mathcal{R}(\alpha \times \beta), X \wedge Y \rangle,$$

for  $\alpha, \beta \in \Lambda_-^2$  (see [8] for details).

With respect to the basis  $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, w_1, w_2, w_3\}$ , we write  $\mathcal{R} : \Lambda_+^2 \oplus \Lambda_-^2 \rightarrow \Lambda_+^2 \oplus \Lambda_-^2$  as a block matrix:

$$\mathcal{R} = \begin{pmatrix} A_1 & 'B \\ B & A_2 \end{pmatrix},$$

where  $A_1$  and  $A_2$  are  $3 \times 3$  anti-symmetric matrices.

Recall the definition of the Weyl curvature tensor:

$$\begin{aligned}\tilde{C}_{XY}Z &= \tilde{R}_{XY}Z \\ &+ \frac{1}{n-2} (\langle \tilde{Q}X, Z \rangle Y - \langle \tilde{Q}Y, Z \rangle X + \langle X, Z \rangle \tilde{Q}Y - \langle Y, Z \rangle \tilde{Q}X) \\ &- \frac{\tau}{(n-1)(n-2)} (\langle X, Z \rangle Y - \langle Y, Z \rangle X).\end{aligned}$$

Let  $\mathcal{W} : \Lambda^2 \rightarrow \Lambda^2$  be defined by

$$\langle \mathcal{W}(X \wedge Y), Z \wedge W \rangle = -\langle \tilde{C}_{XY}Z, W \rangle.$$

It was proven by Atiyah, Hitchin and Singer in [2] that  $\mathcal{W}$  preserves the fibration  $\Lambda^2 = \Lambda_- \oplus \Lambda_+$  so that

$$\mathcal{R} = \begin{pmatrix} \mathcal{W}_+ + \frac{\tau}{6} \text{id} & B \\ B & \mathcal{W}_- + \frac{\tau}{6} \text{id} \end{pmatrix},$$

where  $\mathcal{W}_+$  and  $\mathcal{W}_-$  are the restrictions of  $\mathcal{W}$  on  $\Lambda_+$  and  $\Lambda_-$ , respectively, and the operator  $\mathcal{B}$ , given by

$$\mathcal{B} \equiv \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},$$

is the traceless Ricci operator. We say  $(M, \langle, \rangle)$  is *self-dual*, if  $\mathcal{W}_- = 0$  and *anti-self-dual*, if  $\mathcal{W}_+ = 0$ .

Set  $ZM = \{\sum_{j=1}^3 a_j w_j \in \Lambda_-^2 : (a_1)^2 + \dots + (a_3)^2 = 1\} \cong S^2$ .  $Z$  is called the *twistor space* of  $(M, \langle, \rangle)$ . Let  $\pi : Z \rightarrow M$  be the natural projection. Each element  $z \in Z_x M$  defines an almost complex structure  $S_z$  on  $T_x M$  by:

$$\langle S_z X, Y \rangle = 2\langle z, X \wedge Y \rangle.$$

Since the space of almost complex structures compatible with a given inner product and orientation of  $\mathbf{R}^4$  is  $S^2$ , we see that  $\pi^{-1}(x)$  is the space of all almost complex structures which are compatible with  $(T_x M, \langle, \rangle)$  and which are oppositely oriented to the given orientation of  $\{e_1, e_2, e_3, e_4\}$ .

For  $z \in Z$ , set  $\mathcal{V}_z = \ker(\pi_*)_z$ . Note that, if  $x = \pi(z)$ , we can identify  $\mathcal{V}_z$  with all elements of  $(\Lambda_- M)_x$  which are orthogonal to  $z$ . So, if  $s$  is a section of  $Z$  and  $X \in T_x M$ , we have  $\langle \nabla_X s, s \rangle = (1/2)X\langle s, s \rangle = 0$ , i.e.  $\nabla_X s \in \mathcal{V}_z$  ( $\nabla$  here is the bundle connection on  $\Lambda^2 M$  induced by the Riemannian connection of  $\langle, \rangle$ ).

For  $X \in T_x M$  and section  $s$  of  $Z$ , set  $X^{\mathcal{H}} = s_* X - \nabla_X s$  and  $\mathcal{H}_z = \{X^{\mathcal{H}} : X \in T_x M, s(x) = z\}$ , or equivalently

$$\mathcal{H}_z = \{s_* X : X \in T_x M, s \text{ a section of } Z \text{ such that } s(x) = z \text{ and } \nabla s = 0 \text{ at } x\}.$$

This definition gives us a splitting  $TM = \mathcal{H} \oplus \mathcal{V}$ . We denote the natural projections from  $TM$  to the respective subbundles by the same letter:

$$\mathcal{H} : TM \rightarrow \mathcal{H}, \quad \mathcal{V} : TM \rightarrow \mathcal{V}.$$

Set  $J' = J \circ \mathcal{H}$  and  $J'' = J \circ \mathcal{V}$ .

**4. Almost complex and hermitian structures on twistor spaces**

**4.1. The almost complex structure**

We define an almost complex structure on  $Z$  as follows. Let  $z \in Z$ . Define  $J$  on  $T_z Z$  by:

$$JX = \begin{cases} X \times z & \text{if } X \in \mathcal{V}_z \\ (\text{hor}_z \circ S_z \circ \pi_*)X & \text{if } X \in \mathcal{H}_z. \end{cases}$$

LEMMA 4.1. *Let  $z_1, z_2 \in \pi^{-1}(p)$ , then*

$$S_{z_1} \circ S_{z_2} = -S_{z_1 \times z_2}.$$

*Proof.* Without loss of generality, assume that  $z_1 = w_1$  and  $z_2 = w_2$ , so that we need only show that  $S_{w_1} \circ S_{w_2} = -S_{w_3}$ . Using the definitions of  $w_1, w_2$  and  $w_3$ , it is easily seen that, with respect to the basis  $\{e_1, e_2, e_3, e_4\}$ ,

$$S_{w_1} = \left( \begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right), \quad S_{w_2} = \left( \begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right),$$

$$S_{w_3} = \left( \begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right).$$

It is then easily verified that  $S_{w_1} \circ S_{w_2} = -S_{w_3}$ .

Then the following result is well-known (see [8]):

**THEOREM 4.2.** *The almost complex structure  $J$  is integrable if and only if  $\langle, \rangle$  is self-dual. Furthermore, suppose that  $\langle, \rangle$  is self-dual, then  $\mathcal{H}$  is holomorphic if and only if  $\langle, \rangle$  is also Einstein.*

**4.2. Hermitian geometry of a twistor space**

From this point on, we will assume that  $\langle, \rangle$  is self-dual and Einstein. For  $t \in \mathbf{R} - (0)$ , define the pseudo-Riemannian metric  $h_t$  on  $Z$  by:

$$h_t = \pi^* \langle, \rangle + t \langle, \rangle^v,$$

where  $\langle, \rangle^v$  is the restriction of the bundle metric on  $\mathcal{V}$ . This metric is called the *Salamon-Bérard-Bergery metric* with vertical coefficient  $t$ . For convenience, we'll refer to these as *SBB metrics*, and frequently the value for  $t$  will be explicitly given or understood.

These metrics have many widely-known properties, the proofs and statements

of which can be found in [8]. We will just list a few here. It should be noted that most of these properties hold true for the SBB metric of the twistor space of any four-dimensional Riemannian manifold, not necessarily only one that is self-dual and Einstein.

**THEOREM 4.3.** *Let  $Z$  be the twistor space over a self-dual, Einstein four-manifold  $(M, \langle, \rangle)$  with SBB metric  $h_t$ . Then the following statements are true.*

1.  $h_t$  is hermitian with respect to the complex structure  $J$ .
2.  $\mathcal{V}$  is a totally geodesic foliation with respect to  $h_t$ .
3. Let  $X$  be a horizontal vector in  $T_z Z$  such that  $\pi_*(X) = \tilde{X}$  and  $W$  be any vertical vector field defined in an open neighborhood of  $z$ . Then, by identifying  $\mathcal{V}_z$  with 2-vectors in  $(\Lambda^2 M)_{|\pi(z)}$  that are perpendicular to  $z$ ,

$$\mathcal{H}(\nabla_X W)_z = -\frac{1}{2} \text{hor}_z((\tilde{R}_z(JW)\tilde{X})),$$

where  $\text{hor}_z : T_x M \rightarrow T_z Z$  is the horizontal lift.

### 4.3. Complex contact geometry of a twistor space

Suppose  $t > 0$  so that  $h_t$  is a Riemannian metric. Let  $U'$  be any local vertical vector field on  $Z$  such that  $\langle U', U' \rangle^v = 1$ . Set

$$U_t = \frac{1}{\sqrt{t}} U', \quad V_t = -J U_t,$$

$$u_t(E) = h_t(U_t, E), \quad v_t(E) = h_t(V_t, E).$$

Then, for  $X, Y \in \mathcal{H}_z$  with  $p = \pi(z)$ ,  $\tilde{X} = \pi_*(X)$ ,  $\tilde{Y} = \pi_*(Y)$ ,

$$\begin{aligned} 2 du_t(X, Y) &= -u_t([X, Y]) \\ &= -h_t(U_t, [X, Y]) \\ &= h_t(U_t, \tilde{R}_p(X, Y)z) \\ &= t \langle \tilde{R}_p(X, Y)z, U_t \rangle \\ &= \sqrt{t} \langle \tilde{R}_p(X, Y)z, U' \rangle \\ &= \sqrt{t} \langle \mathcal{R}(z \times U'), X \wedge Y \rangle \\ &= \sqrt{t} \left( \frac{\tau}{6} \right) \langle z \times U', X \wedge Y \rangle \\ &= \sqrt{t} \left( \frac{\tau}{12} \right) 2 \langle z \times U', X \wedge Y \rangle \\ &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{z \times U'} X, Y \rangle. \end{aligned}$$

If we set  $h_t(k_{U_t} X, Y) = du_t(X, Y)$  and  $k_{U_t}|_{\mathcal{V}} = 0$ , then at  $z \in Z$ ,  $k_{U_t} =$



$\sqrt{t}(\tau/24)S_{z \times U'}$ . In particular,  $k_{U_t|_{\mathcal{H}}}^2 = -\text{id}|_{\mathcal{H}}$  if and only if  $t = (24/\tau)^2$ . We also get the following relationship between  $dv_t$  and  $du_t$ .

LEMMA 4.4. *For  $X, Y \in \mathcal{H}_z$ ,  $dv_t(X, Y) = du_t(JX, Y)$ . Also,  $k_{V_t} = k_{U_t} \circ J = -J \circ k_{U_t}$ . Finally,  $k_{U_t}X = \nabla_X U_t$ .*

*Proof.* Let  $p = \pi(z)$ ,  $\tilde{X} = \pi_*(X)$ , and  $\tilde{Y} = \pi_*(Y)$ . From the above work, we have:

$$\begin{aligned} 2 dv_t(X, Y) &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{z \times V'} \tilde{X}, \tilde{Y} \rangle \\ &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{JU' \times z} \tilde{X}, \tilde{Y} \rangle \\ &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{(U' \times z) \times z} \tilde{X}, \tilde{Y} \rangle \\ &= -\sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{U'} \tilde{X}, \tilde{Y} \rangle. \end{aligned}$$

Also, using the above work,

$$\begin{aligned} 2 du_t(JX, Y) &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{z \times U'} \circ S_z \tilde{X}, \tilde{Y} \rangle \\ &= -\sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{(z \times U') \times z} \tilde{X}, \tilde{Y} \rangle \\ &= -\sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{U'} \tilde{X}, \tilde{Y} \rangle. \end{aligned}$$

This proves the first equation as well as the relation  $k_{V_t} = k_{U_t} \circ J$ .

Note that

$$du_t(X, JY) = h_t(k_{U_t}X, JY) = -h_t(J \circ k_{U_t}X, Y)$$

as well as

$$\begin{aligned} 2 du_t(X, JY) &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{z \times U'} \tilde{X}, S_z \tilde{Y} \rangle \\ &= -\sqrt{t} \left( \frac{\tau}{12} \right) \langle S_z \circ S_{z \times U'} \tilde{X}, \tilde{Y} \rangle \\ &= \sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{z \times (z \times U')} \circ S_z \tilde{X}, \tilde{Y} \rangle \\ &= -\sqrt{t} \left( \frac{\tau}{12} \right) \langle S_{U'} \circ S_z \tilde{X}, \tilde{Y} \rangle \\ &= 2h_t(k_{U_t} \circ JX, Y) \end{aligned}$$

Thus,  $k_{U_t} \circ J = -J \circ k_{U_t}$ . Finally, by its definition,  $h_t$  is a bundlelike metric on  $Z$  with respect to the fibration  $\pi : Z \rightarrow M$ , that is,  $(\mathcal{L}_{U_t} h_t)(X, Y) = 0$ . This is equivalent to

$$0 = h_t(\nabla_X U_t, Y) + h_t(\nabla_Y U_t, X).$$

Thus,

$$\begin{aligned} h_t(k_{U_t} X, Y) &= du_t(X, Y) \\ &= \frac{1}{2}(Xu_t(Y) - Yu_t(X) - u_t([X, Y])) \\ &= -\frac{1}{2}h_t(U_t, \nabla_X Y - \nabla_Y X) \\ &= \frac{1}{2}h_t(\nabla_X U_t, Y) - \frac{1}{2}h_t(\nabla_Y U_t, X) \\ &= h_t(\nabla_X U_t, Y) \end{aligned}$$

This proves the lemma.

All of these results allow us to prove the following proposition.

**PROPOSITION 4.5.** *Let  $(M, g)$  be a 4-dimensional Riemannian manifold with twistor space  $Z$ . Then  $(Z, J, \mathcal{H})$  is a complex contact manifold if and only if  $g$  is self-dual and Einstein.*

*Proof.* Clearly, we need only to show that if  $\mathcal{H}$  is holomorphic, then  $\mathcal{H}$  is a complex contact structure.

Fix  $t > 0$ . Let  $U$  be a local unit vertical vector field on  $Z$  with respect to  $h_t$ . Let  $u = u_t$  and  $v = v_t$  be the 1-forms as defined in the previous paragraph. Set

$$\pi = u - iv \quad (\neq 0).$$

Then  $\mathcal{H} = \ker(\pi)$  so that  $\pi \in \mathcal{V}^* \cap \mathcal{H}^\perp$ . Note also that

$$d\pi = du - i dv,$$

which is nondegenerate on  $\mathcal{H}$ .

Also,  $\pi(T^{0,1}M) = (0)$ . Since  $\mathcal{H}$  is holomorphic, there exists a local function  $f$  such that the 1-form  $\omega = f\pi$  is holomorphic. Then

$$\begin{aligned} \omega \wedge d\omega &= (f\pi) \wedge (df \wedge \pi + f d\pi) \\ &= f^2 \pi \wedge d\pi. \end{aligned}$$

Since  $d\pi$  is non-degenerate on  $\mathcal{H}$ , we know that  $\pi \wedge d\pi$  (and hence  $\omega \wedge d\omega$ ) is nowhere zero on its domain. This proves that  $\mathcal{H}$  is a complex contact structure.

By combining the above result with the work from the previous paragraph, we get the following proposition.

PROPOSITION 4.6. *Suppose that  $(M, g)$  is a self-dual, Einstein manifold with scalar curvature  $\tau$ . Then the SBB metric  $h_t$  is associated to the complex contact structure  $\mathcal{H}$  if and only if  $t = (24/\tau)^2$ .*

From this point on, we will drop the subscripts of “ $t$ ” when they are easily understood. For any vertical, unit vector  $U$ , define the local 1-form  $\sigma_U$  by

$$\sigma_U(E) = -h(\nabla_E U, JU).$$

It is easily seen that  $\sigma_{JU} = \sigma_U$  and that  $d\sigma_U$  is a globally-defined 2-form on  $Z$ .

The following technical proposition has several identities that we will need for the main theorem. Note that the first and second are true for *any* associated metric of a complex contact manifold, whereas the rest are particular to the associated metric of twistor spaces. Recall that  $J' = J \circ \mathcal{H}$  and  $J'' = J \circ \mathcal{V}$ .

PROPOSITION 4.7. *Let  $U$  be a vertical vector field. Set  $V = -JU$  and  $\sigma = \sigma_U$ . Let  $u$  and  $v$  be the 1-forms corresponding to  $U$  and  $V$  with respect to  $h$ . Then, for  $E, F \in TZ$ ,*

1.  $h(k_U E, F) = (du - \sigma \wedge v)(E, F)$ .
2.  $\nabla_U k_U = -\sigma(U)k_{JU}$ .
3.  $d\sigma(E, F) = th(J''E, F) + (\tau/24)h(J'E, F)$ .
4.  $d\sigma(E, F) = (1/2)h(R_{EF}U, V) + t(\tau/24)^2 h(J'E, F)$ .
5.  $h(R_{EF}U, V) = 2th(J''E, F) + 2(\tau/24)(1 - t(\tau/24))h(J'E, F)$ .
6.  $\nabla_{JU} k_U = (\tau/24)(1 - 2t(\tau/24))J' - \sigma(JU)k_{JU}$ .
7.  $\nabla_U J = (\tau/24)(1 - 2t(\tau/24))k_{JU}$ .

*Proof.* Proof of 1. By definition,  $h(k_U X, Y) = du(X, Y)$  for two horizontal vectors  $X$  and  $Y$ . Now suppose that  $W = U$  or  $V$  and  $F = U, V$  or in  $\mathcal{H}$ . Then

$$\begin{aligned} 2 du(W, F) &= -u([W, F]) \\ &= -h(\nabla_W F, U) + h(\nabla_F W, U) \\ &= h(\nabla_W U, F) - h(\nabla_F U, W) \\ &= -\sigma(F)v(W) + \sigma(W)v(F) \\ &= 2(\sigma \wedge v)(W, F). \end{aligned}$$

This proves the first statement.

Proof of 2. Let  $\tilde{k}_U$  be the 2-form given by  $\tilde{k}_U(E, F) = h(k_U E, F)$ . Then, by 1.,

$$d\tilde{k}_U = -d\sigma \wedge v + \sigma \wedge dv$$

Now,

$$3 d\tilde{k}_U(E, F, G) = h((\nabla_E k_U)F, G) + h((\nabla_G k_U)E, F) + h((\nabla_F k_U)G, E).$$

So, for  $Y, Z \in \mathcal{H}$ ,

$$3(\sigma \wedge dv)(U, Y, Z) = h((\nabla_U k_U)Y, Z) + h((\nabla_Z k_U)U, Y) + h((\nabla_Y k_U)Z, U).$$

Hence,

$$\begin{aligned} h((\nabla_U k_U)Y, Z) &= (\sigma \wedge dv)(U, Y, Z) - h((\nabla_Z k_U)U, Y) - h((\nabla_Y k_U)Z, U) \\ &= \sigma(U) dv(Y, Z) - h(\nabla_Z U, k_U Y) - h(\nabla_Y (k_U Z), U) \\ &= \sigma(U) dv(Y, Z) - h(\nabla_Z U, k_U Y) + h(k_U Z, \nabla_Y U) \\ &= \sigma(U) dv(Y, Z) - h(k_U Z, k_U Y) + h(k_U Z, k_U Y) \\ &= \sigma(U)h(k_U Y, Z) \\ &= -\sigma(U)h(k_{JU} Y, Z). \end{aligned}$$

This proves the second statement.

Proof of 3. Let  $U$  be any unit vertical field with respect to  $h_t$  and  $X$  be a horizontal vector field. Set  $V = -JU$ . Since  $\mathcal{V}$  is totally geodesic with respect to  $h_t$ ,

$$\begin{aligned} 0 &= h_t(R_{XU}U, V) \\ &= h_t(\nabla_X \nabla_U U, V) - h_t(\nabla_U \nabla_X U, V) - h_t(\nabla_{[X, U]}U, V) \\ &= h_t(\nabla_X(\sigma(U)U), V) - h_t(\nabla_U(\sigma(X)V + 2k_U X), V) - \sigma([X, U]) \\ &= X\sigma(U) - U\sigma(X) - \sigma([X, U]) \\ &= 2d\sigma(X, U). \end{aligned}$$

So, we see that  $d\sigma(\mathcal{V}, \mathcal{H}) = 0$ .

Now, suppose that  $X, Y \in \mathcal{H}_z$  with  $p = \pi(z)$ ,  $\tilde{X} = \pi_*(X)$ ,  $\tilde{Y} = \pi_*(Y)$ . Then, from the identities of Theorem 4.3, we get

$$\begin{aligned} 2d\sigma(X, Y)|_z &= X\sigma(Y)|_z - Y\sigma(X)|_z - \sigma([X, Y])|_z \\ &= X\langle D_{\pi_* Y}U, z \times U \rangle - Y\langle D_{\pi_* X}U, z \times U \rangle - \langle D_{[\pi_* X, \pi_* Y]}U, z \times U \rangle \\ &= \langle R^\nabla(\tilde{X}, \tilde{Y})U, z \times U \rangle \\ &= \frac{\tau}{6} \langle (\tilde{X} \wedge \tilde{Y})^-, z \rangle \\ &= \frac{\tau}{12} h_t(J'X, Y). \end{aligned}$$

Finally,  $t\langle \cdot, \cdot \rangle^v$  is the standard metric on  $S^2$  with sectional curvature  $\sqrt{t}$ . So, for any vertical vectors  $W_1, W_2$ ,

$$d\sigma(W_1, W_2) = t\langle JW_1, W_2 \rangle^v.$$

Proof of 4. Let  $E, F \in TZ$ . Then

$$\begin{aligned}
 2 d\sigma(E, F) &= E\sigma(F) - F\sigma(E) - \sigma([E, F]) \\
 &= Eh(\nabla_F U, V) - Fh(\nabla_E U, V) - h(\nabla_{[E, F]} U, V) \\
 &= h(\nabla_E \nabla_F U, V) + h(\nabla_F U, \nabla_E V) - h(\nabla_F \nabla_E U, V) \\
 &\quad - h(\nabla_E U, \nabla_F V) - h(\nabla_{[E, F]} U, V) \\
 &= h(R_{EF} U, V) + h(k_U F, k_V E) - h(k_U E, k_V F) \\
 &= h(R_{EF} U, V) - h(k_U F, k_{JU} E) + h(k_U E, k_{JU} F) \\
 &= h(R_{EF} U, V) + h((k_U k_{JU} - k_{JU} k_U) E, F) \\
 &= h(R_{EF} U, V) + 2t \left( \frac{\tau}{24} \right)^2 h(J'E, F)
 \end{aligned}$$

Proof of 5. This comes from combining statements 3. and 4.

Proof of 6. By statement 1., for  $Y, Z \in \mathcal{H}$ ,

$$\begin{aligned}
 &h((\nabla_V k_U) Y, Z) + h((\nabla_Y k_U) Z, V) + h((\nabla_Z k_U) V, Y) \\
 &= (-d\sigma \wedge v + \sigma \wedge dv)(V, Y, Z).
 \end{aligned}$$

Now,

$$\begin{aligned}
 \text{LHS} &= -h((\nabla_{JU} k_U) Y, Z) + h(\nabla_Y (k_U Z), V) - h(k_U \nabla_Z V, Y) \\
 &= -h((\nabla_{JU} k_U) Y, Z) - h(k_U Z, \nabla_Y V) - h(k_U k_V Z, Y) \\
 &= -h((\nabla_{JU} k_U) Y, Z) + h(k_U Z, k_{JU} Y) + h(k_U k_{JU} Z, Y) \\
 &= -h((\nabla_{JU} k_U) Y, Z) - t \left( \frac{\tau}{24} \right)^2 h(Z, JY) + t \left( \frac{\tau}{24} \right)^2 h(JZ, Y) \\
 &= -h((\nabla_{JU} k_U) Y, Z) - 2t \left( \frac{\tau}{24} \right)^2 h(Z, JY),
 \end{aligned}$$

whereas

$$\begin{aligned}
 \text{RHS} &= -d\sigma \wedge v(Y, Z, V) + \sigma(V) dv(Y, Z) \\
 &= -d\sigma(Y, Z) + \sigma(JU)h(k_{JU} Y, Z) \\
 &= -\frac{\tau}{24} h(JY, Z) + \sigma(JU)h(k_{JU} Y, Z).
 \end{aligned}$$

Thus,

$$-h((\nabla_{JU} k_U) Y, Z) - 2t \left( \frac{\tau}{24} \right)^2 h(Z, JY) = -\frac{\tau}{24} h(JY, Z) + \sigma(JU)h(k_{JU} Y, Z).$$

This proves the statement.

Proof of 7. Note that by substituting “ $JU$ ” for “ $U$ ” in statement 6, we get:

$$\nabla_U k_{JU} = -\frac{\tau}{24} \left(1 - t \frac{\tau}{24}\right) J' + \sigma(U) k_U.$$

It is easily seen that  $\nabla_U J = \nabla_U J'$ . Then

$$\begin{aligned} \nabla_U J &= -\nabla_U (k_{JU} \circ k_U) \\ &= -(\nabla_U k_{JU}) k_U - k_{JU} \nabla_U k_U \\ &= -\left(-\frac{\tau}{24} \left(1 - 2t \frac{\tau}{24}\right) J' + \sigma(U) k_U\right) k_U - k_{JU} (-\sigma(U) k_{JU}) \\ &= \frac{\tau}{24} \left(1 - 2t \frac{\tau}{24}\right) k_{JU}. \end{aligned}$$

### 5. Holomorphic subbundles of complex codimension one

In this section, we will study some of the properties of a holomorphic subbundle with codimension one for a complex manifold. Throughout this section, we will assume that  $(Z, J)$  is a complex manifold with  $TZ = \mathcal{H} \oplus \mathcal{V}$  where both  $\mathcal{H}$  and  $\mathcal{V}$  are  $J$ -invariant and  $\dim_{\mathbb{C}} \mathcal{V} = 1$ .

Recall that  $\mathcal{H}$  is holomorphic if and only if it is locally spanned by a set of infinitesimal automorphisms of  $J$ , i.e. there is a local basis  $\{X_1, \dots, X_{2N}\}$  such that  $\mathcal{L}_{X_j} J = 0$  for each  $j$ . This is equivalent to the dual subbundle  $\mathcal{V}^*$  being holomorphic, that is, there is a local holomorphic 1-form,  $\omega$ , such that  $\mathcal{H} = \ker(\omega)$ . Thus,  $\mathcal{H}$  is holomorphic if and only if there is a complex-valued 1-form  $\omega$  such that

$$\mathcal{H} = \ker(\omega), \quad \bar{\partial}\omega = 0.$$

We will use this fact to show the following proposition.

**PROPOSITION 5.1.** *Let  $(Z, J)$  be a complex manifold with  $J$ -invariant subbundle fibration  $TZ = \mathcal{H} \oplus \mathcal{V}$ . Let  $u$  and  $v = u \circ J$  be a pair of local 1-forms such that  $\mathcal{H} \subset \ker(u) \cap \ker(v)$ . If  $\mathcal{H}$  is holomorphic, then, for  $X, Y \in \mathcal{H}$ ,*

$$du(X, Y) = -du(JX, JY)$$

$$dv(X, Y) = du(JX, Y).$$

*Proof.* Let  $X$  and  $Y$  be local holomorphic sections of  $\mathcal{H}$ , i.e. as explained above,  $X, Y \in \mathcal{H}$  and  $\mathcal{L}_X J = \mathcal{L}_Y J = 0$ . Then

$$\begin{aligned} du(JX, JY) &= JXu(JY) - JYu(JX) - u([JX, JY]) \\ &= -u([JX, JY]) \\ &= -u(J^2[X, Y]) \\ &= u([X, Y]) \\ &= -du(X, Y). \end{aligned}$$

Since  $\mathcal{H}$  is spanned by a basis of holomorphic sections, we know that the first part of the proposition holds in general.

Similarly, using the same  $X$  and  $Y$  as described above, we have

$$\begin{aligned} dv(X, Y) &= Xv(Y) - Yv(X) - v([X, Y]) \\ &= -v([X, Y]) \\ &= -u \circ J([X, Y]) \\ &= -u([JX, Y]) \\ &= du(JX, Y). \end{aligned}$$

This proves the proposition.

### 6. Main theorem

We are now in a suitable position to state and prove the main theorem.

**THEOREM 6.1.** *Let  $(Z, J)$  be a three-dimensional manifold. Then  $Z$  is biholomorphic to the twistor space of a self-dual, Einstein four-manifold with non-zero scalar curvature, if and only if  $Z$  has a hermitian metric  $h$  and a  $J$ -invariant splitting  $TZ = \mathcal{V} \oplus \mathcal{H}$  such that  $\dim_{\mathbb{R}} \mathcal{H} = 4$ ,  $\mathcal{V}$  is totally geodesic (and hence integrable),  $\mathcal{H}$  is holomorphic with  $h$  bundlelike through  $\mathcal{V}$ , and, for every unit  $U \in \mathcal{V}$ , the Riemannian curvature of  $Z$ ,  $R$ , satisfies*

$$\begin{aligned} R_{XY}U &= \lambda u(Y)\mathcal{H}X + \mu v(Y)J'X \\ &\quad - \lambda u(X)\mathcal{H}Y - \mu v(X)J'Y \\ &\quad + [v_1 h(J'X, Y) + v_2 h(J''X, Y)]V, \end{aligned}$$

where  $V = -JU$ ,  $u(E) = h(U, E)$ ,  $v(E) = h(V, E)$  and  $J' = J \circ \mathcal{H}$  and  $J'' = J \circ \mathcal{V}$  for some constants,  $\lambda$ ,  $\mu$ ,  $v_1$  and  $v_2$ , such that  $\lambda + \mu \neq 0$  and  $v_2 > 0$ .

*Proof.* Let  $Z$  be the twistor space of a four-dimensional self-dual, Einstein manifold  $(M, \langle, \rangle)$  with scalar curvature  $\tau \neq 0$ . Let  $h = h_t$  be the SBB metric on  $Z$  for some  $t > 0$ . We will first prove a number of claims, which, when taken together, will give us the desired curvature condition.

Before we go on to the claims, recall that the O'Neill tensors  $T$  and  $A$  are defined by:

$$\begin{aligned} A(E, F) &= \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \\ T(E, F) &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \end{aligned}$$

Since  $\mathcal{V}$  is totally geodesic, we know that  $T = 0$ .

**CLAIM 1.** *Suppose  $X, Y \in \mathcal{H}$ . Then  $R_{XY}U = 2(\tau/24)(1 - t(\tau/24))h(J'X, Y)V$ .*

By one of O'Neill's equations,  $h(R_{XY}Z, U) = -h((\nabla_Z A)(X, Y), U)$  for any  $Z \in \mathcal{H}$ . By 4.3, at  $z \in Z$ ,  $h((\nabla_Z A)(X, Y), U) = t\langle \nabla_Z \mathcal{R}(X, Y), z \times U \rangle$ . Furthermore, since  $\mathcal{R} = \text{constant}(\text{id})$  on  $\Lambda_-$ ,  $\nabla \mathcal{R} = 0$ . Thus,  $h(R_{XY}Z, U) = 0$ , or  $\mathcal{H}R_{XY}U = 0$ . The fifth equation of Proposition 4.7 gives the rest of the claim.

CLAIM 2.  $R_{UV}U = 2tV$ .

Since  $\mathcal{V}$  is totally geodesic, we know that  $\mathcal{H}R_{UV} = 0$ . Proposition 4.7, part 5 tells us then that  $R_{UV}U = 2tV$ .

CLAIM 3.  $\mathcal{H}R_{UY}U = -t(\tau/24)^2 Y$  for  $Y \in \mathcal{H}$ .

$$\begin{aligned} \mathcal{H}R_{UY}U &= \mathcal{H}\nabla_U \nabla_Y U - \mathcal{H}\nabla_Y \nabla_U U - \mathcal{H}\nabla_{[U, Y]}U \\ &= \mathcal{H}\nabla_U(k_U Y + \sigma(Y)U) + \mathcal{H}\nabla_Y(\sigma(U)JU) - \mathcal{H}k_U[U, Y] \\ &= \mathcal{H}\nabla_U(k_U Y) + \sigma(U)\mathcal{H}\nabla_Y(JU) - \mathcal{H}k_U(\nabla_U Y) + \mathcal{H}k_U \nabla_Y U \\ &= \mathcal{H}(\nabla_U k_U)Y + \sigma(U)k_{JU}Y + k_U k_U Y \\ &= -\sigma(U)k_{JU}Y + \sigma(U)k_{JU}Y - t\left(\frac{\tau}{24}\right)^2 Y \\ &= -t\left(\frac{\tau}{24}\right)^2 Y \end{aligned}$$

CLAIM 4.  $\mathcal{H}R_{XV}U = (\tau/24)(1 - t(\tau/24))JX$  for  $X \in \mathcal{H}$ .

For  $X, Z \in \mathcal{H}$ ,

$$\begin{aligned} h(R_{XV}U, Z) &= h(\nabla_X \nabla_V U - \nabla_V \nabla_X U - \nabla_{[X, V]}U, Z) \\ &= h(\nabla_X(\sigma(V)V), Z) - h(\nabla_V(\sigma(X)V + k_U X), Z) \\ &\quad - h(\nabla_{\nabla_X V}U, Z) + h(\nabla_{\nabla_V X}U, Z) \\ &= \sigma(V)h(\nabla_X V, Z) - h(\nabla_V(k_U X), Z) \\ &\quad - h(\nabla_{k_V X}U, Z) + h(k_U \nabla_V X, Z) \\ &= \sigma(V)h(k_V X, Z) - h((\nabla_V k_U)X, Z) - h(k_U k_V X, Z) \\ &= \sigma(JU)h(k_{JU}X, Z) + h((\nabla_{JU}k_U)X, Z) + h(k_U k_{JU}X, Z) \\ &= \sigma(JU)h(k_{JU}X, Z) + h\left(\frac{\tau}{24}\left(1 - 2t\frac{\tau}{24}\right)JX, Z\right) - \sigma(JU)h(k_{JU}X, Z) \\ &\quad + t\left(\frac{\tau}{24}\right)^2 h(J'X, Z) \\ &= h\left(\frac{\tau}{24}\left(1 - t\frac{\tau}{24}\right)J'X, Z\right) \end{aligned}$$



Combining Claims 1 through 4, we get

$$\begin{aligned} R_{XY}U &= t\left(\frac{\tau}{24}\right)^2 u(Y)\mathcal{H}X + \frac{\tau}{24}\left(1 - t\frac{\tau}{24}\right)v(Y)J'X \\ &\quad - t\left(\frac{\tau}{24}\right)^2 u(X)\mathcal{H}Y - \frac{\tau}{24}\left(1 - t\frac{\tau}{24}\right)v(X)J'Y \\ &\quad + \left[2\frac{\tau}{24}\left(1 - t\left(\frac{\tau}{24}\right)\right)h(J'X, Y) + 2th(J''X, Y)\right]V. \end{aligned}$$

In this context, the coefficients  $\mu$  and  $\lambda$  as indicated in the main theorem are given by:

$$\begin{aligned} \lambda &= t\left(\frac{\tau}{24}\right)^2, \\ \mu &= \frac{\tau}{24}\left(1 - t\frac{\tau}{24}\right), \end{aligned}$$

so that  $\lambda + \mu = \tau/24 \neq 0$ .

We now suppose that  $Z$  satisfies the given curvature identity. Since  $h(R_{JU}U, JU) = 1$ , the leaves of the foliation  $\mathcal{V}$  foliate 2-spheres in  $Z$ . By Theorem 4.4 in [13] and the fact that  $S^2$  is simply-connected,  $Z$  fibres over a 4-manifold  $M = Z/\mathcal{V}$ . Let  $\pi$  be the projection from  $Z$  to  $M$ . The metric  $h$  is bundle-like with respect to  $\mathcal{V}$  on  $Z$ , so that there is a Riemannian metric  $g$  on  $M$  such that the projection  $\pi : (Z, h) \rightarrow (M, g)$  is a Riemannian submersion.

Let  $Z_g$  be the twistor space of  $(M, g)$  with projection  $\pi_g : Z_g \rightarrow M$ . Viewing  $Z_g$  as the space of almost complex structures on  $M$ , which are hermitian with respect to  $g$ , we define a map  $\phi : Z \rightarrow Z_g$  as follows.

For  $z \in Z$  and  $x = \pi(z)$ , let  $\phi(z) \in (\pi_g)^{-1}(x)$  be the almost complex structure on  $T_xM$  given by:

$$\phi(z)X = (\pi_* \circ J \circ \text{hor}_z)(X) \quad \text{for each } X \in T_xM,$$

where  $\text{hor}_z : T_xM \rightarrow \mathcal{H}_z$  is the horizontal lift.

Since all of the maps involved with the definition of  $\phi$  are clearly continuous,  $\phi$  is a continuous map. Also,  $\phi$  maps the vertical leaves of  $Z$  to the vertical leaves of  $Z_g$ , that is, restricted to a vertical leaf of  $Z$ ,  $\phi$  is simply a continuous map from  $S^2$  to  $S^2$ . Thus, if we can show that  $\phi$  is locally one-to-one, i.e. an immersion, we will have proven the theorem.

Let  $x \in M, z \in \pi^{-1}(x)$ . Let  $U$  be a unit vector in  $\mathcal{V}_z$  and extend it as a geodesic vector field about a neighborhood of  $z$ . Let  $t \mapsto \gamma(t) \in \pi^{-1}(x)$  be the unique geodesic such that  $\gamma(0) = z$  and  $\gamma_*(0) = U_z$ . Since  $\phi(\gamma(t)) = \pi_* \circ J \circ \text{hor}_{\gamma(t)} \in Z_g$ , we know that  $\phi_*(U) \neq 0$  if and only if

$$\frac{d}{dt}((\pi_* \circ J \circ \text{hor}_{\gamma(t)})X)_{t=0} \neq 0 \quad \text{for any nonzero } X \in T_xM.$$

Fix a nonzero  $X \in T_x M$ . Let  $\{\psi_s\}$  be the local family of isometries which generate  $U$  (so that  $\gamma(t) = \psi_t(z)$ ). In order to continue, we will need some very technical results derived from the curvature identity.

Let  $V = -JU$ . Set  $k_U E = \mathcal{H} \nabla_E U$ , so that  $du(X, Y) = h(k_U X, Y)$  for any  $X, Y \in \mathcal{H}$ , where the 1-form  $u$  is defined by  $u(E) = h(U, E)$ . Also, set  $\sigma(E) = -h(\nabla_E U, JU)$ .

Since  $h$  is a bundlelike metric,  $k_U$  is skew-symmetric with respect to  $h$ . Also, the first equation of Proposition 5.1 tells us that  $k_U \circ J = -J \circ k_U$ , whereas the second tells us that  $k_{JU} = J \circ k_U$ .

CLAIM.  $\mathcal{L}_U J = ((\lambda + \mu)/\lambda)k_{JU}$  on  $\mathcal{H}$ .

*Proof of Claim.* Using the fact that  $\mathcal{V}$  is totally geodesic, for  $Y, Z \in \mathcal{H}$ ,

$$\begin{aligned} -\lambda h(Y, Z) &= h(R_{UY}U, Z) \\ &= h(\nabla_U \nabla_Y U - \nabla_Y \nabla_U U - \nabla_{[U, Y]}U, Z) \\ &= h(\nabla_U(k_U Y) - \nabla_Y(\sigma(U)V) - \nabla_{\nabla_U Y}U + \nabla_{\nabla_Y U}U, Z) \\ &= h(\nabla_U(k_U Y) - \sigma(U)\nabla_Y V - k_U(\nabla_U Y) + k_U^2 Y, Z) \\ &= h((\nabla_U k_U)Y + \sigma(U)k_{JU}Y, Z) + h(k_U^2 Y, Z). \end{aligned}$$

Since the transformation  $\nabla_U k_U + \sigma(U)k_{JU}$  is skew-symmetric and  $k_U^2 + \text{id}$  is symmetric (with respect to  $h$ ),

$$\nabla_U k_U + \sigma(U)k_{JU} = 0, \quad k_U^2 = -\lambda \text{id on } \mathcal{H}.$$

Note that this also implies that  $k_U \circ k_{JU} = -k_{JU} \circ k_U = \lambda J'$ .

Similarly,

$$\begin{aligned} -\mu h(JX, Z) &= h(R_{XJU}U, Z) \\ &= h(\nabla_X \nabla_{JU}U - \nabla_{JU} \nabla_X U - \nabla_{[X, JU]}U, Z) \\ &= h(\nabla_X(-\sigma(JU)JU) - \nabla_{JU}(k_U X) - \nabla_{\nabla_X(JU)}U + \nabla_{\nabla_{JU}X}U, Z) \\ &= -\sigma(JU)h(\nabla_X(JU), Z) - h((\nabla_{JU}k_U)X, Z) - h(k_U \circ k_{JU}X, Z) \\ &= -\sigma(JU)h(k_{JU}X, Z) - h((\nabla_{JU}k_U)X, Z) - \lambda h(JX, Z). \end{aligned}$$

Thus,  $\nabla_{JU}k_U = -\sigma(JU)k_{JU} - (\mu - \lambda)J'$ , or

$$\nabla_U k_{JU} = \sigma(U)k_U + (\mu - \lambda)J'.$$

Then,

$$\begin{aligned} \lambda \nabla_U J &= -\nabla_U(k_{JU} \circ k_U) \\ &= -(\nabla_U k_{JU}) \circ k_U - k_{JU} \circ (\nabla_U k_U) \\ &= -(\sigma(U)k_U + 2(\mu - \lambda)J') \circ k_U - k_{JU} \circ (-\sigma(U)k_{JU}) \\ &= (\mu - \lambda)k_{JU}. \end{aligned}$$

Now, suppose that  $X$  is a horizontal, projectable vector field. Then

$$\begin{aligned}
 \mathcal{H}(\mathcal{L}_U J)X &= \mathcal{H}([U, JX] - J[U, X]) \\
 &= \mathcal{H}\nabla_U(JX) - \mathcal{H}\nabla_{JX}U - J'\nabla_U X + J'\nabla_X U \\
 &= \mathcal{H}(\nabla_U J)X - \mathcal{H}\nabla_{JX}U + J'\nabla_X U \\
 &= \left(\frac{\mu - \lambda}{\lambda}\right)k_{JU} - k_U JX + Jk_U X \\
 &= \left(\frac{\mu - \lambda}{\lambda} + 2\right)k_{JU}X \\
 &= \left(\frac{\lambda + \mu}{\lambda}\right)k_{JU}X.
 \end{aligned}$$

This proves the claim.

Now, continuing with the proof of the main theorem,

$$\begin{aligned}
 \frac{d}{dt}((\pi_* \circ J \circ \text{hor}_{\gamma(t)})X)_{t=0} &= \frac{d}{dt}((\pi_* \circ J \circ (\psi_{-t})_* \circ \text{hor}_{\gamma(0)})X)_{t=0} \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} [(\pi_* \circ J \circ (\psi_{-t})_* \circ \text{hor}_{\gamma(0)})X - (\pi_* \circ J \circ \text{hor}_{\gamma(0)})X] \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} [\pi_* \circ ((\psi_t)_* J \circ (\psi_{-t})_* - J) \circ \text{hor}_{\gamma(0)}X] \\
 &= (\pi_* \circ \mathcal{L}_U J \circ \text{hor}_{\gamma(0)})X \\
 &= \left(\frac{\lambda + \mu}{\lambda}\right)(\pi_* \circ k_{JU} \circ \text{hor}_{\gamma(0)})X \\
 &\neq 0.
 \end{aligned}$$

Thus,  $\phi_*(U) \neq 0$ , and that proves the theorem.

### 7. Remarks

The main theorem had its origin in a desire to understand the low-dimensional twistor spaces from a complex contact perspective. This had already been achieved to some extent in the higher dimensions by Theorem 6.1 in [11], which we restate here.

**THEOREM 7.1.** *Let  $M$  be a complex contact manifold with contact subbundle  $\mathcal{H}$  and normalized contact structure  $\{\pi = u - iv\}$ . Let  $TM = \mathcal{H} \oplus \mathcal{V}$  be the corresponding splitting with local vertical vector fields  $\{U, V\}$  corresponding to the real forms  $u$  and  $v$ . Suppose  $g$  is an associated metric on  $M$ . Then  $(M, g)$  is isometric to a twistor space for a quaternionic-Kähler manifold with positive scalar curvature if and only if*

$$\begin{aligned} R_{XY}U &= u(Y)X - u(X)Y + v(Y)JX - v(X)JY + 2g(JX, Y)V, \\ R_{XY}V &= -v(Y)X + v(X)Y - u(Y)JX + u(X)JY - 2g(JX, Y)U. \end{aligned}$$

This theorem and the main theorem of this paper both spring from two current topics in complex contact geometry: normal complex contact manifolds and fibrated complex contact manifolds.

### 7.1. Normal complex contact manifolds

The first subject area that motivates Theorem 6.1 is that of normal complex contact manifolds. Modelled after the original definition in the real contact category, a *normal complex contact manifold* is a complex contact manifold  $Z$  with associated metric  $h$  such that, for every unit  $U \in \mathcal{V}$ , and any  $X, Y, Z \in TZ$ ,

$$\begin{aligned} h(R_{XY}U, Z) &= u(Y)h(X, Z) - v(Y)(h(JX, Z) - d\sigma(X, Z)) \\ &\quad - u(X)h(Y, Z) + v(X)(h(JY, Z) - d\sigma(Y, Z)) \\ &\quad + v(Z)[-2h(J'X, Y) + d\sigma(X, Y) - d\sigma(k_U X, k_U Y)]V. \end{aligned}$$

By setting  $d\sigma^\sharp$  to be the  $(1, 1)$ -tensor defined by  $h(d\sigma^\sharp(E), F) = d\sigma(E, F)$ , we can rewrite this definition as follows:

$$\begin{aligned} R_{XY}U &= u(Y)X - v(Y)JX - u(X)Y + v(X)JY \\ &\quad + v(Y)d\sigma^\sharp(X) - v(X)d\sigma^\sharp(Y) \\ &\quad + [-2h(J'X, Y) + d\sigma(X, Y) - d\sigma(k_U X, k_U Y)]V. \end{aligned}$$

This concept was originally due to Ishihara and Konishi in [17] but was recently redefined by Korkmaz in [21]. Using this definition, we can rewrite Theorem 6.1 as follows.

**THEOREM 7.2.** *A three-dimensional complex manifold  $(Z, J)$  is biholomorphic to the twistor space of a self-dual, Einstein four-manifold with non-zero scalar curvature if and only if  $Z$  is a normal complex contact manifold such that the Ishihara-Konishi curvature  $d\sigma$  is non-degenerate on  $TZ$  and positive on  $\mathcal{V}$ .*

*Proof.* First, we show that the SBB metrics such that  $t = (\tau/24)^{-2}$ , i.e. those which are associated to the complex contact structure, are normal. Note that Proposition 4.7 already tells us that  $d\sigma$  is non-degenerate on  $TZ$  and positive on  $\mathcal{V}$ .

Using the proof of Theorem 6.1, we see that, for an SBB metric such that  $t = (\tau/24)^{-2}$ ,

$$\begin{aligned} R_{XY}U &= u(Y)\mathcal{H}X + \left(\frac{\tau}{24} - 1\right)v(Y)J'X - u(X)\mathcal{H}Y - \left(\frac{\tau}{24} - 1\right)v(X)J'Y \\ &\quad + 2\left[\left(\frac{\tau}{24} - 1\right)h(J'X, Y) + \left(\frac{\tau}{24}\right)^{-2}h(J''X, Y)\right]V. \end{aligned}$$

Now, we use Proposition 4.7 to show that the right-hand side of the definition of normality gives us the same expression as above. The following computation uses two crucial facts. First,  $v = u \circ J$ , so that  $v \circ J = -u$ . Also, we can rewrite the vertical bundle metric  $h(J''^*, *)$  by  $h(J''E, F) = -2(u \wedge v)(E, F)$ .

$$\begin{aligned}
 \text{RHS} &= u(Y)X - v(Y)JX - u(X)Y + v(X)JY \\
 &\quad + v(Y) \left( \left( \frac{\tau}{24} \right)^{-2} J''X + \frac{\tau}{24} J'X \right) - v(X) \left( \left( \frac{\tau}{24} \right)^{-2} J''Y + \frac{\tau}{24} J'Y \right) \\
 &\quad + \left[ -2h(J'X, Y) + \left( \frac{\tau}{24} \right)^{-2} h(J''X, Y) + \frac{\tau}{24} h(J'X, Y) \right. \\
 &\quad \quad \left. - \frac{\tau}{24} h(J'k_U X, k_U Y) \right] V \\
 &= u(Y)X + v(Y) \left( -JX + \left( \frac{\tau}{24} \right)^{-2} J''X + \frac{\tau}{24} J'X \right) \\
 &\quad - u(X)Y - v(X) \left( -JY + \left( \frac{\tau}{24} \right)^{-2} J''Y + \frac{\tau}{24} J'Y \right) \\
 &\quad + \left[ -2h(J'X, Y) + \left( \frac{\tau}{24} \right)^{-2} h(J''X, Y) + 2 \cdot \frac{\tau}{24} h(J'X, Y) \right] V \\
 &= u(Y)\mathcal{H}X + u(Y)v(X)V + v(Y) \left( -J'X + \frac{\tau}{24} J'X \right) \\
 &\quad + v(Y)v \left( -JX + \left( \frac{\tau}{24} \right)^{-2} J''X \right) V \\
 &\quad - u(X)\mathcal{H}Y - u(X)v(Y)V - v(X) \left( -J'Y + \frac{\tau}{24} J'Y \right) \\
 &\quad - v(X)v \left( -JY + \left( \frac{\tau}{24} \right)^{-2} J''Y \right) V \\
 &\quad \left[ 2 \left( \frac{\tau}{24} - 1 \right) h(J'X, Y) + \left( \frac{\tau}{24} \right)^{-2} h(J''X, Y) \right] V \\
 &= u(Y)\mathcal{H}X + u(Y)v(X)V + \left( \frac{\tau}{24} - 1 \right) v(Y)J'X - \left( \left( \frac{\tau}{24} \right)^{-2} - 1 \right) v(Y)u(X)V
 \end{aligned}$$

$$\begin{aligned}
& -u(X)\mathcal{H}Y - u(X)v(Y)V - \left(\frac{\tau}{24} - 1\right)v(X)J'Y \\
& + \left(\left(\frac{\tau}{24}\right)^{-2} - 1\right)v(X)u(Y)V \\
& \left[2\left(\frac{\tau}{24} - 1\right)h(J'X, Y) + \left(\frac{\tau}{24}\right)^{-2}h(J''X, Y)\right]V \\
& = u(Y)\mathcal{H}X + \left(\frac{\tau}{24} - 1\right)v(Y)J'X - u(X)\mathcal{H}Y - \left(\frac{\tau}{24} - 1\right)v(X)J'Y \\
& 2\left[\left(\frac{\tau}{24} - 1\right)h(J'X, Y) + \left(\frac{\tau}{24}\right)^{-2}h(J''X, Y)\right]V.
\end{aligned}$$

Thus, every associated SBB metric is a normal complex contact metric.

We now assume that  $(Z, h)$  is a normal complex contact manifold. From page 1357 of [21], we know that

$$\begin{aligned}
h((\nabla_E J)F, G) &= -u(E)(d\sigma(G, k_U F) - 2h(k_{JU} F, G)) \\
&+ v(E)(d\sigma(G, k_{JU} F) - 2h(k_U F, G)).
\end{aligned}$$

In particular,  $h((\nabla_U J)Y, Z) = d\sigma(Z, k_U Y) - 2h(k_{JU} Y, Z)$ . Given that  $k_U \circ k_{JU} = -k_{JU} \circ k_U$ , it is easily seen that  $\mathcal{H}(\nabla_U J)k_U = k_U(\nabla_U J)\mathcal{H}$ . Also,  $(\nabla J)J = -J(\nabla J)$ . Combining both of these facts with the above identity, we find that, for any  $X, Y \in TZ$ ,

$$\begin{aligned}
d\sigma(k_U X, k_U Y) &= -d\sigma(\mathcal{H}X, \mathcal{H}Y), \\
d\sigma(JX, JY) &= -d\sigma(X, Y).
\end{aligned}$$

Now, since  $\dim_{\mathbb{R}} \mathcal{H} = 4$ ,  $\dim_{\mathbb{R}} \mathcal{V} = 2$ ,  $d\sigma$  is nondegenerate on  $\mathcal{H}$  and positive on  $\mathcal{V}$ , we know that, for any  $X, Y \in TZ$ ,

$$d\sigma(X, Y) = \alpha_1 h(J'X, Y) + \alpha_2 h(J''X, Y),$$

for some  $\alpha_1 \neq 0$  and  $\alpha_2 > 0$ . Substituting this into the definition for normality and simplifying, we get that, for any  $X, Y \in TZ$ ,

$$\begin{aligned}
R_{XY}U &= u(Y)\mathcal{H}X + (\alpha_1 - 1)v(Y)J'X \\
&- u(X)\mathcal{H}Y - (\alpha_1 - 1)v(X)J'Y \\
&+ [2(\alpha_1 - 1)h(J'X, Y) + 2\alpha_2 h(J''X, Y)]V.
\end{aligned}$$

With  $\lambda = 1$ ,  $\mu = \alpha_1 - 1$ ,  $v_1 = 2(\alpha_1 - 1)$  and  $v_2 = 2\alpha_2$ , Theorem 6.1 then gives the result.

## 7.2. Fibrated complex contact manifolds

The other subcategory of complex contact manifolds that motivates these results is that of fibrated complex contact manifold. A *fibrated complex contact*

*manifold* is a complex contact manifold  $Z$  foliated by leaves of  $\mathcal{V}$  in such a way that a fibration is formed,

$$R \rightarrow Z \rightarrow Z/\mathcal{V} = M.$$

In this case,  $R$  is necessarily a one-dimensional complex manifold with constant curvature, so that  $R$  can only be  $\mathbb{C}$ , the hyperbolic plane,  $S^2$  up to a subgroup of isometries.

A variety of fibrated complex contact manifolds are well-known. Twistorial complex contact manifolds are clearly examples of  $S^2$ -fibrated complex contact manifolds. The complex Heisenberg group is an example of a  $\mathbb{C}$ -fibration; in fact, in [10], the author has shown that one can construct a complex torus fibration for every possible dimension. More recently, Blair, Davidov and Muscarov have recently constructed a so-called hyperbolic twistor space, which gives an example with vertical leaves forming the hyperbolic plane (see [6]).

Characterizing fibrated complex contact manifolds by their vertical leaves remains an unsolved problem. In the case at hand, it is unknown whether twistor spaces over quaternionic Kähler manifolds with nonzero scalar curvature are the only  $S^2$ -fibrated complex contact manifolds. Scrutiny of the proof of Theorem 6.1 reveals that we can rewrite it as follows.

**THEOREM 7.3.** *A three-dimensional complex manifold  $(Z, J)$  is biholomorphic to the twistor space of a self-dual, Einstein four-manifold with non-zero scalar curvature if and only if  $Z$  is an  $S^2$ -fibrated complex contact manifold with a bundlelike associated metric such that  $\text{rank}(\mathcal{H}\mathcal{L}_U J) = 4$  for every non-zero  $U \in \mathcal{V}$ .*

The condition regarding  $\mathcal{L}_U J$  corresponds roughly with conditions 2 and 3 of Theorem 1.1. This makes it seem likely that there are other  $S^2$ -fibrated complex contact manifolds which are not twistorial, although the author is not aware of any such examples.

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