A Bertini-type theorem for free arithmetic linear series

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Abstract In this paper, we prove a version of the arithmetic Bertini theorem asserting that there exists a strictly small and generically smooth section of a given arithmetically free graded arithmetic linear series.

0. Introduction

When we generalize results on arithmetic surfaces to those on higher-dimensional arithmetic varieties, it is sometimes very useful to cut the base scheme by a "good" global section s of a given Hermitian line bundle and proceed to induction on dimension. To do this, we have in the context of Arakelov geometry the following result.

FACT ([5, THEOREMS 4.2 AND 5.3])

Let $\overline{A}$ be a C^∞-Hermitian line bundle on a generically smooth projective arithmetic variety X, and let $x_1, \dots, x_q$ be points (not necessarily closed) on X. Suppose that (i) A is ample, (ii) $c_1(\overline{A})$ is positive definite, and (iii) $\mathrm{H}^0(X, mA)$ has a $\mathbb{Z}$-basis consisting of sections with supremum norms less than 1 for every $m \gg 1$. Then there exist a sufficiently large integer $m \geq 1$ and a nonzero section $s \in \mathrm{H}^0(X, mA)$ such that

(1) $\mathrm{div}(s)_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$,
(2) $s(x_i) \neq 0$ for every i, and
(3) $\|s\|_{\sup} < 1$.

For example, this technique plays essential roles in the proofs of the arithmetic Bogomolov–Gieseker inequality on high-dimensional arithmetic varieties (see [5]), of the arithmetic Hodge index theorem in codimension 1 (see [6], [10]), of the arithmetic Siu inequality of Yuan [9], and so on. A purpose of this paper is to give a simple elementary proof of the above fact and to strengthen it to the case of arithmetically free graded arithmetic linear series.

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Let K be a number field. Let X be a projective arithmetic variety that is geometrically irreducible over $\mathrm{Spec}(O_K)$, and let L be an effective line bundle on X. A *graded linear series* belonging to L is a subgraded O_K-algebra

$$R_\bullet := \bigoplus_{m\geq 0} R_m \subseteq \bigoplus_{m\geq 0} \mathrm{H}^0(X, mL).$$

We consider norms $\|\cdot\|_m$ on $R_m \otimes_{\mathbb{Z}} \mathbb{R}$, and assume that the family of norms $\|\cdot\|_\bullet := (\|\cdot\|_m)_{m\geq 0}$ is *multiplicative*, that is,

$$\|s\otimes t\|_{m+n} \leq \|s\|_m \|t\|_n$$

holds for every $s\in R_m$ and $t\in R_n$.

THEOREM A

Let X be a generically smooth projective arithmetic variety, and let A be an effective line bundle on X. We consider a graded linear series

$$R_\bullet := \bigoplus_{m\geq 0} R_m$$

belonging to A and a multiplicative norm $\|\cdot\|_\bullet$ on $R_\bullet \otimes_{\mathbb{Z}} \mathbb{R}$. Suppose the following conditions:

- *R_1 is base point free,*
- *$R_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by R_1 over $\mathbb{Q}$, and*
- *$\bigcap_{m\geq 1}\{x\in X_{\mathbb{Q}} \mid t(x)=0$ for every $t\in R_m$ with $\|t\|_m<1\}=\emptyset$.*

Let $Y^1,\dots,Y^p$ be smooth closed subvarieties of the complex manifold $X(\mathbb{C})$, and let $x_1,\dots,x_q$ be points (not necessarily closed) on X. Then, for every sufficiently large integer $m\gg 1$, there exists a nonzero section $s\in R_m$ such that

(a) $\mathrm{div}(s|_{Y^1}),\dots,\mathrm{div}(s|_{Y^p})$ *are all smooth,*
(b) $s(x_i)\neq 0$ *for every i, and*
(c) $\|s\|_m<1$.

Let $\overline{L}$ be a continuous Hermitian line bundle on X, and let $\|\cdot\|_{\mathrm{sup}}^{(m)}$ be the supremum norm on $\mathrm{H}^0(X,mL)\otimes_{\mathbb{Z}}\mathbb{R}$. We define a $\mathbb{Z}$-submodule of $\mathrm{H}^0(X,mL)$ by

$$\mathrm{F}^{0+}(X,m\overline{L}) := \left\langle s\in \mathrm{H}^0(X,mL) \,\middle|\, \|s\|_{\mathrm{sup}}^{(m)}<1\right\rangle_{\mathbb{Z}}.$$

Then $\bigoplus_{m\geq 0}\mathrm{F}^{0+}(X,m\overline{L})$ is a graded linear series belonging to L. We denote the stable base locus of $\bigoplus_{m\geq 0}\mathrm{F}^{0+}(X,m\overline{L})$ by $\mathrm{SBs}^{0+}(\overline{L})$.

COROLLARY B

Let X be a generically smooth projective arithmetic variety, and let $\overline{A}$ be a continuous Hermitian line bundle on X. Suppose that $\mathrm{SBs}(A)=\emptyset$ and $\mathrm{SBs}^{0+}(\overline{A})\cap X_{\mathbb{Q}}=\emptyset$. Let $Y^1,\dots,Y^p$ be smooth closed subvarieties of the complex manifold $X(\mathbb{C})$, and let $x_1,\dots,x_q$ be points (not necessarily closed) on X. Then there exist a sufficiently large integer $m\geq 1$ and a nonzero section $s\in \mathrm{H}^0(X,mA)$ such that

(a) $\operatorname{div}(s|_{Y^1}), \dots, \operatorname{div}(s|_{Y^p})$ *are all smooth,*
(b) $s(x_i) \neq 0$ *for every* i*, and*
(c) $\|s\|_{\sup}^{(m)} < 1$.

COROLLARY C
Let X *be a generically smooth normal projective arithmetic variety, let* $\overline{L} := (L, |\cdot|_{\overline{L}})$ *be a continuous Hermitian line bundle on* X*, and let* $x_1, \dots, x_q$ *be points (not necessarily closed) on* $X \setminus \mathrm{SBs}^{0+}(\overline{L})$*. If* $\mathrm{SBs}^{0+}(\overline{L}) \subsetneq X$*, then there exist a sufficiently large integer* $m \geq 1$ *and a nonzero section* $s \in \mathrm{H}^0(X, mL)$ *such that*

(a) $\operatorname{div}(s)_{\mathbb{Q}}$ *is smooth off* $\mathrm{SBs}^{0+}(\overline{L})$,
(b) $s(x_i) \neq 0$ *for every* i*, and*
(c) $\|s\|_{\sup}^{(m)} < 1$.

Notation and conventions. Let k denote a field, and let $\mathbb{P}^n := \mathbb{P}(k^{n+1})$ denote the projective space of one-dimensional quotients of k^{n+1}. Let $\mathrm{pr}_2 : \mathbb{P}^n \times_k \mathbb{P}^m \to \mathbb{P}^m$ denote the second projection. We denote the natural coordinate variables of $\mathbb{P}^n$ (resp., of $\mathbb{P}^m$) by $X_0, \dots, X_n$ (resp., by $Y_0, \dots, Y_m$) or simply by $X_\bullet$ (resp., by $Y_\bullet$).

Let Y be a smooth variety over k. The *singular locus* of a morphism $\varphi : X \to Y$ over k is a Zariski-closed subset of X defined as

$$\operatorname{Sing}(\varphi) := \{x \in X \mid \varphi \text{ is not smooth at } x\}.$$

A *projective arithmetic variety* X is a reduced irreducible scheme that is projective and flat over $\operatorname{Spec}(\mathbb{Z})$. We say that X is *generically smooth* if $X_{\mathbb{Q}} := X \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Q})$ is smooth.

1. Bertini's theorem with degree estimate

In this section, we consider a geometric case. Let $X \subseteq \mathbb{P}^n$ be a projective variety over an algebraically closed field k that is defined by a homogeneous prime ideal $I_X \subseteq k[X_0, \dots, X_n]$, let $\mathcal{O}_X(1)$ be the hyperplane line bundle on X, and let

$$\deg X := \deg\bigl(c_1\bigl(\mathcal{O}_X(1)\bigr)^{\cdot \dim X}\bigr)$$

be the degree of X in $\mathbb{P}^n$. Let $k[X] := k[X_0, \dots, X_n]/I_X$ be the homogeneous coordinate ring of X, and let $k[X]_l$ be the homogeneous part of $k[X]$ of degree l. There exists a polynomial function $\varphi_X(l)$ such that $\deg \varphi_X = \dim X$, all coefficients are nonnegative, and

$$\dim_k k[X]_l \leq \varphi_X(l) \tag{1.1}$$

for all $l \geq 0$. Let $Z \subseteq X \times_k \mathbb{P}^m$ be a Zariski-closed subset defined by a system of polynomial equations:

$$u_1(X_\bullet; Y_\bullet) = 0 \pmod{I_X}, \qquad \dots, \qquad u_h(X_\bullet; Y_\bullet) = 0 \pmod{I_X},$$

where $u_i \in k[X_0, \dots, X_n; Y_0, \dots, Y_m]$ has homogeneous degree $\deg_{X_\bullet} u_i$ (resp., $\deg_{Y_\bullet} u_i$) in the set of variables $X_\bullet$ (resp., $Y_\bullet$). We recall the following fact from the elimination theory.

LEMMA 1.1

Let $p := \max_i\{\deg_{X_\bullet} u_i\}$*, and let* $q := \max_i\{\deg_{Y_\bullet} u_i\}$*. If the set-theoretic image* $\mathrm{pr}_2(Z)$ *does not coincide with* $\mathbb{P}^m$*, then* $\mathrm{pr}_2(Z)$ *is contained in a hypersurface of* $\mathbb{P}^m$ *defined by a single homogeneous polynomial of degree less than or equal to*

$$\varphi_X(\deg X \cdot p^{\dim X+1}) \cdot q.$$

Proof

First, we can take a geometric point $y_{0,\bullet} = (y_{0,0} : \cdots : y_{0,m}) \in \mathbb{P}^m \setminus \mathrm{pr}_2(Z)$. By an effective Nullstellensatz (see [3, Corollary 1.4]), there exists a positive integer $\ell \leq \deg X \cdot p^{\dim X+1}$ such that

$$(X_0, \ldots, X_n)^\ell \subseteq \big(u_1(X_\bullet; y_{0,\bullet}), \ldots, u_h(X_\bullet; y_{0,\bullet})\big) \pmod{I_X}.$$

Next, we consider the k-linear maps

$$T(y_\bullet) : k[X]_{\ell-\deg_{X_\bullet} u_1} \oplus \cdots \oplus k[X]_{\ell-\deg_{X_\bullet} u_h} \to k[X]_\ell,$$
$$\big(f_1(X_\bullet), \ldots, f_h(X_\bullet)\big) \mapsto \sum_i u_i(X_\bullet; y_\bullet) f_i(X_\bullet)$$

defined for $y_\bullet = (y_0 : \cdots : y_m) \in \mathbb{P}^m$. By fixing a basis for the above k-vector spaces, we can represent $T(y_\bullet)$ by a matrix whose entries are homogeneous polynomials of $y_\bullet$ of degree less than or equal to q. By the choice of ℓ, we can see that there exists a certain $\dim_k k[X]_\ell \times \dim_k k[X]_\ell$-minor of the representation matrix of $T(y_\bullet)$ whose determinant is nonzero (see [8, Theorem 2.23]). Then the image $\mathrm{pr}_2(Z)$ is contained in the hypersurface defined by the nonzero determinant, which is homogeneous of degree less than or equal to $(\dim_k k[X]_\ell) \cdot q$. Since

$$\dim_k k[X]_\ell \leq \varphi_X(\ell) \leq \varphi_X(\deg X \cdot p^{\dim X+1}),$$

we have the result. □

REMARK 1.2

For example, we consider the case where $X = \mathbb{P}^n$. Then $\dim_k k[X]_l = \binom{l+n}{n} \leq (l+n)^n/n!$. Thus, the bound in the above lemma becomes less than or equal to $(p^{n+1}+n)^n q/n!$. Moreover, by applying the theory of resultants (see [8, page 35]) to $\mathrm{pr}_2 : \mathbb{P}^n \times_k \mathbb{A}^m \to \mathbb{A}^m$, one can obtain a weaker bound less than or equal to $(2p)^{2^n-1}q + 1$ in the above lemma (where the added 1 is for the hyperplane at infinity).

Let A be an effective line bundle on X, and let $R_\bullet$ be a subgraded ring of $\bigoplus_{m\geq 0} \mathrm{H}^0(X, mA)$ with Kodaira–Iitaka dimension $\kappa(R_\bullet) := \mathrm{tr.deg}_k R_\bullet - 1$. Suppose that R_1 is base point free. Let $\phi_m : X \to \mathbb{P}(R_m)$ be a k-morphism associated to R_m, and set

$$N_m := \dim_k R_m - 1 \tag{1.2}$$

for $m \geq 1$. We recall that the rational function field $k(X)$ of X is given by

$$k(X) = \left\{ \frac{u \pmod{I_X}}{v \pmod{I_X}} \;\middle|\; \begin{array}{l} u, v \in k[X_0, \dots, X_n] \text{ are homogeneous} \\ \text{of the same degree and } v \notin I_X \end{array} \right\}.$$

Given a nonzero section $e \in R_1$, we define the *degree* of a nonzero section $s \in \mathrm{H}^0(X, mA)$ for $m \geq 1$ with respect to e by

$$\deg_{X_\bullet, e} s := \min \left\{ \deg_{X_\bullet} u = \deg_{X_\bullet} v \;\middle|\; \begin{array}{l} \operatorname{div} s = (u/v \pmod{I_X}) + m \operatorname{div} e, \\ u/v \pmod{I_X} \in k(X)^\times \end{array} \right\}.$$

(Compare the definition with Jelonek's in [3, Section 2].) Then, for any other nonzero section $s' \in \mathrm{H}^0(X, m'A)$, we have that

$$\deg_{X_\bullet, e}(s \otimes s') \leq \deg_{X_\bullet, e} s + \deg_{X_\bullet, e} s'.$$

THEOREM 1.3

Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety over k, and let A be a line bundle on X. Let $R_\bullet$ be a graded linear series belonging to A with Kodaira–Iitaka dimension $\kappa(R_\bullet)$. Suppose that the following three conditions are satisfied.

- *R_1 is base point free.*
- *$R_\bullet$ is generated by R_1.*
- *(i) $\mathrm{char}(k) = 0$ or (ii) $\mathrm{char}(k) \neq 0$ and $\phi_m : X \to \mathbb{P}(R_m)$ is unramified for every $m \geq 1$.*

Then one can find a polynomial function $P(m)$ and hypersurfaces $Z_m \subseteq \mathbb{P}(R_m^\vee)$ for $m = 1, 2, \dots$ having the following two properties.

(a) $\deg P \leq \dim X(\dim X + 1)(\kappa(R_\bullet) + 1)$.
(b) *For every $m \geq 1$, the hypersurface $Z_m \subseteq \mathbb{P}(R_m^\vee)$ contains the set*

$$\left\{ H \in \mathbb{P}(R_m^\vee) \;\middle|\; \phi_m(X) \subseteq H \text{ or } \phi_m^{-1}(H) \text{ is not smooth} \right\}$$

and the homogeneous degree of Z_m in $\mathbb{P}(R_m^\vee)$ is less than or equal to $P(m)$.

REMARK 1.4

Throughout this paper, we assume that the empty set $\emptyset$ is smooth, so that if $H \notin Z_m$, then $\phi_m^{-1}(H)$ is empty or smooth of pure dimension $\dim X - 1$.

Proof

Let $I_X \subseteq k[X_0, \dots, X_n]$ denote the homogeneous prime ideal defining X. We consider the universal hyperplane section

$$W_m := \left\{ (x, H) \in X \times_k \mathbb{P}(R_m^\vee) \;\middle|\; \phi_m(x) \in H \right\} \tag{1.3}$$

endowed with the reduced induced scheme structure, and consider the restriction of the second projection $\mathrm{pr}_2 : X \times_k \mathbb{P}(R_m^\vee) \to \mathbb{P}(R_m^\vee)$ to W_m, which we denote by

$$\pi_m : W_m \to \mathbb{P}(R_m^\vee). \tag{1.4}$$

Note that W_m is the inverse image of the canonical bilinear hypersurface in $\mathbb{P}(R_m)\times_k\mathbb{P}(R_m^\vee)$ via $\phi_m\times\mathrm{id}: X\times_k\mathbb{P}(R_m^\vee)\to\mathbb{P}(R_m)\times_k\mathbb{P}(R_m^\vee)$. Since the restriction of the first projection to W_m, $W_m\to X$, is surjective with fiber a projective space of dimension N_m-1, W_m is irreducible. The set-theoretic image of the singular locus of π_m is given by

$$\pi_m\bigl(\mathrm{Sing}(\pi_m)\bigr)=\bigl\{H\in\mathbb{P}(R_m^\vee)\bigm|\phi_m(X)\subseteq H\text{ or }\phi_m^{-1}(H)\text{ is not smooth}\bigr\}.$$

We fix a basis $e_0,\dots,e_{N_1}$ for R_1. From now on, we explain a method to construct an equation w_0 that vanishes along W_m from the section e_0. First, we set

$$D_{1,e_0}:=\max_{1\le i\le N_1}\{\deg_{X_\bullet,e_0}e_i\} \tag{1.5}$$

and take rational functions $u_1^{(1)}/v_1^{(1)},\dots,u_{N_1}^{(1)}/v_{N_1}^{(1)}\in k(X_0,\dots,X_n)^\times$ such that

$$\operatorname{div}e_i=\Bigl(\frac{u_i^{(1)}}{v_i^{(1)}}\ (\mathrm{mod}\,I_X)\Bigr)+\operatorname{div}e_0\qquad\text{and}\qquad\deg_{X_\bullet}u_i^{(1)}=\deg_{X_\bullet}v_i^{(1)}\le D_{1,e_0}$$

for $i=1,\dots,N_1$. Next, for $m\ge2$, we can choose sections $e_1^{(m)},\dots,e_{N_m}^{(m)}\in R_m$ such that

$$e_i^{(m)}\in\{e_0^{\otimes\alpha_0}\otimes\cdots\otimes e_{N_1}^{\otimes\alpha_{N_1}}\mid\alpha_0+\cdots+\alpha_{N_1}=m\}$$

and $e_0^{\otimes m},e_1^{(m)},\dots,e_{N_m}^{(m)}$ form a basis for R_m. By identifying $\mathbb{P}(R_m^\vee)$ with $\mathbb{P}^{N_m}$ via the dual basis of $e_0^{\otimes m},e_1^{(m)},\dots,e_{N_m}^{(m)}$, we can write $\phi_m: X\to\mathbb{P}(R_m^\vee)$ as

$$\phi_m: X_{e_0}\to\mathbb{P}^{N_m},\qquad x\mapsto\Bigl(1:\frac{u_1^{(m)}(x)}{v_1^{(m)}(x)}:\cdots:\frac{u_{N_m}^{(m)}(x)}{v_{N_m}^{(m)}(x)}\Bigr)$$

over $X_{e_0}:=\{x\in X\mid e_0(x)\neq0\}$, where $u_i^{(m)}/v_i^{(m)}\in k(X_0,\dots,X_n)^\times$ satisfies

$$\operatorname{div}e_i^{(m)}=\bigl(u_i^{(m)}/v_i^{(m)}(\mathrm{mod}\,I_X)\bigr)+m\operatorname{div}e_0$$

and

$$\deg_{X_\bullet}u_i^{(m)}=\deg_{X_\bullet}v_i^{(m)}\le D_{1,e_0}m.$$

We set

$$w_0:=v_1^{(m)}\cdots v_{N_m}^{(m)}Y_0+u_1^{(m)}v_2^{(m)}\cdots v_{N_m}^{(m)}Y_1+\cdots+v_1^{(m)}\cdots v_{N_m-1}^{(m)}u_{N_m}^{(m)}Y_{N_m}, \tag{1.6}$$

which is homogeneous in $X_\bullet$ (resp., in $Y_\bullet$) of degree less than or equal to $D_{1,e_0}mN_m$ (resp., 1). Then $w_0\ (\mathrm{mod}\,I_X)$ vanishes along W_m and defines W_m in $X_{e_0}\times_k\mathbb{P}^{N_m}$.

By the same method, starting from $e_j\in R_1$, we can construct an equation

$$w_j=\sum(\text{homogeneous in }X_\bullet\text{ of degree at most }D_{1,e_j}mN_m)\times(\text{linear in }Y_\bullet)$$

that vanishes along W_m and defines W_m in $X_{e_j}\times_k\mathbb{P}^{N_m}$. Let $w_{N_1+1},\dots,w_h\in k[X_0,\dots,X_n]$ be homogeneous polynomials that generate I_X. Notice that the bihomogeneous ideal

$$(w_0,\dots,w_{N_1},w_{N_1+1},\dots,w_h)\subseteq k[X_0,\dots,X_n;Y_0,\dots,Y_m] \tag{1.7}$$

may not be prime but the closed subscheme defined by $(w_0, \dots, w_h)$ in $\mathbb{P}^n \times_k \mathbb{P}^{N_m}$ coincides with W_m.

Set

$$D_1 := \max_{0 \le i \le N_1} \{D_{1,e_i}\}, \qquad D_2 := \max_{N_1+1 \le j \le h} \{\deg_{X_\bullet} w_j\}, \tag{1.8}$$

which does not depend on m. By the Euler rule together with the Jacobian criterion in the affine case, we conclude that the singular locus $\mathrm{Sing}(\pi_m) \subseteq X \times_k \mathbb{P}(R_m^\vee)$ is defined by the determinants of certain $(n - \dim X + 1) \times (n - \dim X + 1)$-minors of the Jacobian matrix $(\frac{\partial w_i}{\partial X_j})$, whose degrees in $X_\bullet$ (resp., in $Y_\bullet$) are all bounded from above by $(N_1 + 1)(D_1 m N_m - 1) + (n - \dim X)(D_2 - 1)$ (resp., by $N_1 + 1$). We choose a positive constant $D' > 0$ such that

$$(N_1 + 1)(D_1 m N_m - 1) + (n - \dim X)(D_2 - 1) \le D' m^{\kappa(R_\bullet)+1}$$

for all $m \ge 1$. Let $\varphi_X(l)$ be as in (1.1), and set

$$P(m) := \varphi_X\bigl(\deg X (D' m^{\kappa(R_\bullet)+1})^{\dim X + 1}\bigr) \cdot (N_1 + 1). \tag{1.9}$$

Then $\deg P = \dim X(\dim X + 1)(\kappa(R_\bullet) + 1)$. Since $\pi_m(\mathrm{Sing}(\pi_m))$ is properly contained in $\mathbb{P}(R_m^\vee)$ due to Kleiman [4, Corollaries 5 and 12], we can apply Lemma 1.1 to this situation by setting

$$p = D' m^{\kappa(R_\bullet)+1} \qquad \text{and} \qquad q = N_1 + 1.$$

Then we conclude that there exists a hypersurface $Z_m \subseteq \mathbb{P}(R_m^\vee)$ having degree less than or equal to $P(m)$ and containing $\pi_m(\mathrm{Sing}(\pi_m))$. □

By applying Theorem 1.3 to the image of R_m via $\mathrm{H}^0(X, mA) \to \mathrm{H}^0(Y, mA|_Y)$, we have the following.

COROLLARY 1.5

Under the same assumptions as in Theorem 1.3, let Y be a smooth closed subvariety of X, and let $y_1, \dots, y_q$ be closed points on X. Then one can find a polynomial function $P(m)$ and hypersurfaces $Z_m \subseteq \mathbb{P}(R_m^\vee)$ for $m = 1, 2, \dots$ having the following two properties.

(a) $\deg P \le \dim Y(\dim Y + 1)(\kappa(R_\bullet) + 1) + q$.
(b) *For every $m \ge 1$, the hypersurface $Z_m \subseteq \mathbb{P}(R_m^\vee)$ contains the set*

$$\left\{ H \in \mathbb{P}(R_m^\vee) \;\middle|\; \begin{array}{l} \phi_m(Y) \subseteq H,\ \phi_m^{-1}(H) \cap Y \text{ is not smooth,} \\ \text{or } H \text{ contains one of } y_1, \dots, y_q \end{array} \right\},$$

and the homogeneous degree of Z_m in $\mathbb{P}(R_m^\vee)$ is less than or equal to $P(m)$.

2. Proofs

In this section, we turn to the arithmetic case and give proofs of Theorem A and Corollaries B and C. To prove Theorem A, we use Lemmas 2.1, 2.2, and 2.4.

LEMMA 2.1 (COMBINATORIAL NULLSTELLENSATZ [5, LEMMA 5.2], [1, THEOREM 1.2])
Let V be a finite-dimensional vector space over a field k, and let

$$u : V \to k$$

be a nonzero polynomial function with maximal total degree $\deg u$. *Let* $e_1, \ldots, e_N$ *be generators of V over k, and let $S_1, \ldots, S_N$ be subsets of k. If* $\operatorname{Card}(S_j) \geq \deg u + 1$ *for every j, then there exist $a_1 \in S_1, \ldots, a_N \in S_N$ such that*

$$u(a_1 e_1 + \cdots + a_N e_N) \neq 0.$$

LEMMA 2.2
Let X be a projective arithmetic variety, let A be a line bundle on X, and let $R_\bullet$ be a graded linear series belonging to A. Suppose that R_1 is base point free. Let $y_1, \ldots, y_l \in X$ be distinct closed points on X such that $\operatorname{char}(k(y_i)) \neq 0$ *for every i, and let $e_1^{(m)}, \ldots, e_{N_m}^{(m)} \in R_m$ be generators of the $\mathbb{Z}$-module R_m. Set $F := \prod_{\substack{p:\ \text{prime} \\ \exists i, p | \operatorname{char}(k(y_i))}} p$. Then, for every sufficiently large m, there exist integers $a_1, \ldots, a_{N_m}$ such that $0 \leq a_j < F$ for every j, and*

$$(a_1 + F b_1) e_1^{(m)}(y_i) + \cdots + (a_{N_m} + F b_{N_m}) e_{N_m}^{(m)}(y_i) \neq 0$$

for every integer $b_1, \ldots, b_{N_m}$ and for every i.

Proof
First, we need the following claim.

CLAIM 2.3
For every sufficiently large m, there exists an $s \in R_m$ such that $s(y_i) \neq 0$ for every i.

Proof
Let $\phi : X \to \mathbb{P}_{\mathbb{Z}}^{N_1}$ be the morphism associated to R_1 such that $\phi^* X_j = e_j^{(1)}$ for every j, and let $\mathcal{O}(1)$ be the hyperplane line bundle on $\mathbb{P}_{\mathbb{Z}}^{N_1}$. Then, for every sufficiently large m, the homomorphism

$$\mathrm{H}^0\big(\mathbb{P}_{\mathbb{Z}}^{N_1}, \mathcal{O}(m)\big) \to \bigoplus_i \mathcal{O}(m)\big(\phi(y_i)\big)$$

is surjective. Let $t \in \mathrm{H}^0(\mathbb{P}_{\mathbb{Z}}^{N_1}, \mathcal{O}(m))$ be a section such that $t(\phi(y_i)) \neq 0$ for every i. Then $s := \phi^* t$ has the desired property. $\square$

Next, let $s \in R_m$ as above. Since $F e_j^{(m)}(y_i) = 0$ for every i, j, we have that

$$(s + Ft)(y_i) = s(y_i) \neq 0$$

for every $t \in R_m$ and for every i. Thus we conclude the claim. $\square$

LEMMA 2.4 (ZHANG–MORIWAKI [7, THEOREM A AND COROLLARY B])
Under the same assumptions as in Theorem A, take an $m_0 \gg 1$, and fix $e_1, \ldots,$

$e_N \in R_{m_0}$ *such that*

$$\left\{x \in X_{\mathbb{Q}} \mid e_1(x) = \cdots = e_N(x) = 0\right\} = \emptyset$$

and such that $\|e_j\|_{m_0} < 1$ *for every* j*. Then there exists a positive constant* $C > 0$ *such that, for every sufficiently large* m*, one can find a* $\mathbb{Z}$*-basis* $e_1^{(m)}, \ldots, e_{N_m}^{(m)}$ *for* R_m *such that*

$$\max_i\left\{\|e_i^{(m)}\|_m\right\} \leq C m^{(\dim X+2)(\dim X-1)}\left(\max_j\left\{\|e_j\|_{m_0}\right\}\right)^{m/m_0}.$$

Proof of Theorem A
Let $r := [K : \mathbb{Q}]$, and let $X(\mathbb{C}) = X_1 \cup \cdots \cup X_r$ be the decomposition into connected components. Let $R_{m,\alpha}$ be the image of $R_m \otimes_{\mathbb{Z}} \mathbb{C}$ via $\mathrm{H}^0(X, A) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{H}^0(X_\alpha, A_{\mathbb{C}}|_{X_\alpha})$, and let $\phi_{m,\alpha} : X_\alpha \to \mathbb{P}_{\mathbb{C}}^{M_m}$ be a morphism associated to $R_{m,\alpha}$, where we set $M_m := \mathrm{rk}_{\mathbb{Z}} R_m/r$. By Lemma 2.4, there exist constants C, Q with $C > 0$ and $0 < Q < 1$ such that there exists a $\mathbb{Z}$-basis $e_1^{(m)}, \ldots, e_{rM_m}^{(m)}$ for R_m consisting of the sections with supremum norms less than or equal to

$$C m^{(\dim X+2)(\dim X-1)} Q^m. \tag{2.1}$$

For each Y^j, there exists a unique component $X_{\alpha(j)}$ that contains Y^j. Suppose that $\mathrm{char}(x_i) = 0$ for $i = 1, \ldots, q_1$ and $\mathrm{char}(x_i) \neq 0$ for $i = q_1 + 1, \ldots, q = q_1 + q_2$, and let y_i be a closed point in $\overline{\{x_i\}}$. By applying Corollary 1.5 to $X_{\alpha(j)}$, Y^j, $y_1, \ldots, y_{q_1}$, and $R_{\bullet,\alpha(j)}$, one can find a polynomial function $P_j(m)$ of degree less than or equal to $\dim Y^j(\dim Y^j - 1)(\kappa(R_{\bullet,\alpha(j)}) + 1) + q_1$ and hypersurfaces $Z_{m,j} \subseteq \mathbb{P}(R_{m,\alpha(j)}^\vee)$ defined by homogeneous polynomials $u_{m,j}$ of degree less than or equal to $P_j(m)$, respectively, such that $Z_{m,j}$ contains all the hyperplanes H in $\mathbb{P}(R_{m,\alpha(j)}^\vee)$ such that $\phi_{m,\alpha(j)}(Y^j) \subseteq H$, $\phi_{m,\alpha(j)}^{-1}(H) \cap Y^j$ is not smooth, or $\phi_{m,\alpha(j)}^{-1}(H)$ contains one of $y_1, \ldots, y_{q_1}$. Set

$$u_{m,\alpha} := \prod_{\alpha(j)=\alpha} u_{m,j},$$

and consider the homogeneous polynomial function

$$u : R_m \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{\alpha=1}^r R_{m,\alpha} \xrightarrow{\prod_\alpha u_{m,\alpha}} \mathbb{C}$$

of degree less than or equal to

$$P(m) := P_1(m) + \cdots + P_p(m). \tag{2.2}$$

Set $F := \prod_{\substack{q:\ \mathrm{prime} \\ \exists i, q | \mathrm{char}(y_i)}} q$. Since $e_1^{(m)}, \ldots, e_{rM_m}^{(m)} \in R_m$ generate $R_m \otimes_{\mathbb{Z}} \mathbb{C}$ over $\mathbb{C}$, one can find integers $a_1, \ldots, a_{rM_m}$ and $b_1, \ldots, b_{rM_m}$ such that $0 \leq a_i < F$ for every i, $0 \leq b_j \leq P(m)$ for every j, and

$$u\left((a_1 + F b_1) e_1^{(m)} + \cdots + (a_{rM_m} + F b_{rM_m}) e_{rM_m}^{(m)}\right) \neq 0$$

by use of Lemmas 2.2 and 2.1. Hence, for each $m \gg 1$, there exists a section $t_m \in R_m$ such that $t_m|_{X_\alpha}$ is not contained in any of $Z_{m,j}$ and

$$\|t_m\|_m \leq CFrm^{(\dim X+2)(\dim X-1)} M_m\big(1+P(m)\big)Q^m.$$

Since the right-hand side tends to zero as $m \to \infty$, we conclude the proof. □

Corollary B is a direct consequence of Theorem A.

Proof of Corollary C

We can take $a_0 \gg 1$ such that $\mathrm{Bs}\,\mathrm{F}^{0+}(X, a_0\overline{L}) = \mathrm{SBs}^{0+}(\overline{L})$. Let $\mathfrak{b}^{0+}(a_0\overline{L}) := \mathrm{Image}(\mathrm{F}^{0+}(X, a_0\overline{L}) \otimes_{\mathbb{Z}} (-a_0L) \to \mathcal{O}_X)$, let $\mu : X' \to X$ be a blowup such that X' is generically smooth and such that $\mu^{-1}\mathfrak{b}^{0+}(a_0\overline{L}) \cdot \mathcal{O}_{X'}$ is Cartier, and let E be an effective Cartier divisor on X' such that $\mathcal{O}_{X'}(-E) = \mu^{-1}\mathfrak{b}^{0+}(a_0\overline{L}) \cdot \mathcal{O}_{X'}$. We can assume that μ is isomorphic over $X \setminus \mathrm{SBs}^{0+}(\overline{L})$ (see [2]). Set $x_i' := \mu^{-1}(x_i) \in X' \setminus E$ for $i = 1, \ldots, q$. Let $B := \mathcal{O}_{X'}(E)$, and let 1_B be the canonical section.

LEMMA 2.5

(a) *We can endow B with a continuous Hermitian metric $|\cdot|_{\overline{B}}$ such that*

$$|1_B|_{\overline{B}}(x) = \max_{\substack{e \in \mathrm{H}^0(X, a_0L) \\ 0 < \|e\|_{\sup}^{(a_0)} < 1}} \left\{ \frac{|e|_{a_0\overline{L}}(\mu(x))}{\|e\|_{\sup}^{(a_0)}} \right\} \leq 1$$

for all $x \in X'(\mathbb{C})$.

(b) *We set $\overline{B} := (B, |\cdot|_{\overline{B}})$ and $\overline{A} := a_0\mu^*\overline{L} - \overline{B}$. Then $\overline{A}$ is a continuous Hermitian line bundle on X' such that*

$$\mathrm{Bs}\,\mathrm{F}^{0+}(X', \overline{A}) = \emptyset \qquad \textit{and} \qquad c_1(\overline{A}) \geq 0$$

as a current.

Proof

Set $\{e \in \mathrm{H}^0(X, a_0L) \setminus \{0\} \mid \|e\|_{\sup}^{(a_0)} < 1\} = \{e_1, \ldots, e_N\}$.

(a) We choose an open covering $\{U_\nu\}$ of $X'(\mathbb{C})$ such that $a_0\mu^*L_{\mathbb{C}}|_{U_\nu}$ is trivial with local frame η_ν, and $E_{\mathbb{C}} \cap U_\nu$ is defined by a local equation g_ν. Then we can write $\mu^*e_j = f_{j,\nu} \cdot g_\nu \cdot \eta_\nu$ on U_ν, where $f_{1,\nu}, \ldots, f_{N,\nu}$ are holomorphic functions on U_ν satisfying $\{x \in U_\nu \mid f_{1,\nu}(x) = \cdots = f_{N,\nu}(x) = 0\} = \emptyset$. Since

$$\max_j \left\{ \frac{|e_j|_{a_0\overline{L}}(\mu(x))}{\|e_j\|_{\sup}^{(a_0)}} \right\} = \max_j \left\{ \frac{|f_{j,\nu}(x)|}{\|e_j\|_{\sup}^{(a_0)}} \right\} \cdot |\eta_\nu|_{a_0\mu^*\overline{L}}(x) \cdot |g_\nu(x)|$$

on $x \in U_\nu$, we have (a).

(b) For each $x_0 \in X'(\mathbb{C})$, we take indices ν and j_0 such that $x_0 \in U_\nu$ and $f_{j_0,\nu}(x_0) \neq 0$. Let ε_j be the section of A such that $\mu^*e_j = \varepsilon_j \otimes 1_B$, and set $h_{j,\nu} := f_{j,\nu}/f_{j_0,\nu}$. Then

$$-\log|\varepsilon_{j_0}|_{\overline{A}}^2(x) = \max_j \left\{ \log|h_{j,\nu}(x)|^2 - \log\left(\|e_j\|_{\sup}^{(a_0)}\right)^2 \right\}$$

is plurisubharmonic near x_0.

We claim that $\|\varepsilon_j\|_{\sup} = \|e_j\|_{\sup}^{(a_0)}$, so that $\varepsilon_j \in \mathrm{F}^{0+}(X', \overline{A})$. The inequality $\|\varepsilon_j\|_{\sup} \geq \|e_j\|_{\sup}^{(a_0)}$ is clear. Since

$$|\varepsilon_j|_{\overline{A}}(x) = |e_j|_{a_0\overline{L}}\big(\mu(x)\big) \cdot \min_i \left\{ \frac{\|e_i\|_{\sup}^{(a_0)}}{|e_i|_{a_0\overline{L}}(\mu(x))} \right\} \leq \|e_j\|_{\sup}^{(a_0)}$$

for all $x \in (X' \setminus E)(\mathbb{C})$, we have $\|\varepsilon_j\|_{\sup} = \|e_j\|_{\sup}^{(a_0)}$. This means that $\mathrm{Bs}\,\mathrm{F}^{0+}(X', \overline{A}) = \emptyset$. □

We apply Corollary B to $\overline{A}$, and we can find an $m \gg 1$ and a $\sigma \in \mathrm{H}^0(X', mA)$ such that $\mathrm{div}(\sigma)_{\mathbb{Q}}$ is smooth, $\sigma(x_i') \neq 0$ for every i, and $\|\sigma\|_{\sup} < 1$. Since X is normal, there exists an $s \in \mathrm{H}^0(X, ma_0L)$ such that $\mu^* s = \sigma \otimes 1_B^{\otimes m}$. Since μ is isomorphic over $X \setminus \mathrm{SBs}^{0+}(\overline{L})$, s has the desired properties. □

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