

# On the gauge action of a Leavitt path algebra

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**Abstract** We introduce a revised notion of gauge action in relation to Leavitt path algebras. This notion is based on the Laurent polynomial algebra and captures the full information of the grading on the algebra as it is the case of the gauge action of the graph  $C^*$ -algebra.

## 1. Notations and preliminaries

The Laurent polynomial algebra is playing an increasingly important role in the theory of Leavitt path algebras. For instance it arises in the description of the prime spectrum of  $L_K(E)$  (see [5]) and recently in the study of the graded Grothendieck group  $K_0^{\text{gr}}$  of a Leavitt path algebra, where  $K_0$  is seen as a module over  $K[x, x^{-1}]$  (see [8]). In the present work, the Laurent polynomial algebra helps us to understand the canonical  $\mathbb{Z}$ -grading, which seems to have its origin in the fact that  $K[x, x^{-1}]$  is the representing Hopf algebra of the diagonalizable group  $\mathbf{Diag}(\mathbb{Z})$ . In any case, this work has the same flavor as those mentioned above in what concerns the ubiquity of  $K[x, x^{-1}]$  in the theory of Leavitt path algebras.

For a directed graph  $E$  denote by  $C^*(E)$  the graph  $C^*$ -algebra (see for instance [11]), and given any commutative unitary ring  $K$ , denote by  $L_K(E)$  the Leavitt path algebra associated to  $E$  (see [12, Definition 2.5] or [1, p. 90] for the case of a ground field of scalars). For a graph  $E$  we will denote by  $E^0$  the set of vertices and  $E^1$  the set of edges of  $E$ . The notation  $\text{path}(E)$  will be reserved for the set of all paths in the graph. As usual, given edges  $f_1, \dots, f_n \in E^1$  and the path  $\lambda = f_1 \dots f_n$ , we will denote by  $s(\lambda) = s(f_1)$  the source of  $f_1$  and by  $r(\lambda) = r(f_n)$  the range of  $f_n$ . Also recall that a vertex  $v \in E^0$  is said to be regular when  $s^{-1}(v)$  is a nonempty finite set. The set of all regular vertices of  $E$  will be denoted by  $\text{Reg}(E)$ . In [3, Corollary 1.5.11], a basis is constructed for

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$L_K(E)$  when  $K$  is a field. As in [3], the basis can be described by taking, for each  $v \in \text{Reg}(E)$ , an enumeration of  $s^{-1}(v)$  in the form  $\{e_1^v, \dots, e_{n_v}^v\}$ . Then, the basis is given by

$$\{\lambda\mu^* : \lambda, \mu \in \text{path}(E), r(\lambda) = r(\mu)\} \setminus \{\lambda e_{n_v}^v (e_{n_v}^v)^* \nu^* : r(\lambda) = r(\nu) = v \in \text{Reg}(E)\}.$$

Observe that the vertices belong to this basis. As the reader can check, this set is also a basis of  $L_K(E)$  in the general case, that is, for any commutative unitary ring  $K$ . In this way the Leavitt path algebra  $L_K(E)$  is always a free  $K$ -module. Now, some easy observations are convenient for further quotation.

REMARK 1

Let  $K$  be a commutative unitary ring, and let  $U, V$  be modules over  $K$ .

(a) If  $U$  is a free  $K$ -module and  $u \in U$  is an element of some basis of  $U$ , then for any  $k \in K$  we have that  $ku = 0$  implies  $k = 0$ .

(b) If  $U$  and  $V$  are free  $K$ -modules, then  $U \otimes_K V$  is also a free  $K$ -module. In particular, if  $u$  and  $v$  are basic elements of  $U$  and  $V$ , respectively, and  $r \in K$ , then the equality  $ru \otimes v = 0$  in  $U \otimes_K V$  implies  $r = 0$ .

(c) If  $E$  and  $F$  are graphs and  $(u, v) \in E^0 \times F^0$ ,  $r \in K$ , then when  $ru \otimes v = 0$  in  $L_K(E) \otimes L_K(F)$  we have  $r = 0$ .

As usual for any ring  $K$  we will denote by  $K^\times$  the group of invertible elements of  $K$ . Also denote by  $\mathbb{T} := S^1$  the unit circle in  $\mathbb{R}^2$ . In this work we shall have the occasion to deal with  $\mathbb{Z}$ -graded algebras.

The notion of a  $\mathbb{Z}$ -graded algebra  $A$  (in a purely algebraic context) is clear: the algebra  $A$  splits as a direct sum  $A = \bigoplus_{n \in \mathbb{Z}} A_n$  of submodules  $A_n$  verifying  $A_n A_m \subset A_{n+m}$  for any  $n, m \in \mathbb{Z}$ . However, a  $\mathbb{Z}$ -grading on a  $C^*$ -algebra  $A$  must be understood as defined in [7, Definition 3.1]:  $A$  is the closure of a direct sum  $\bigoplus_{n \in \mathbb{Z}} A_n$  of closed (linear) subspaces  $A_n$  of  $A$  such that  $A_n^* = A_{-n}$  and  $A_n A_m \subset A_{n+m}$  for any  $n, m \in \mathbb{Z}$ . We will denote this fact by writing  $A = \overline{\bigoplus_{n \in \mathbb{Z}} A_n}$ .

When we speak of the canonical  $\mathbb{Z}$ -grading on  $C^*(E)$ , we will mean the  $\mathbb{Z}$ -grading (in the  $C^*$ -sense) such that the component of degree  $n$  is formed by those elements  $x$  satisfying  $\rho(z)(x) = z^n x$  for any  $z$  (see below). Roughly speaking, this means that the vertices “are of degree 0” and the element  $f_1 \cdots f_n g_1^* \cdots g_m^*$  “is homogeneous of degree  $n - m$ ” (for any collection of edges  $f_i$  and  $g_j$ ). On the other hand the canonical  $\mathbb{Z}$ -grading on  $L_K(E)$  is the one for which the vertices are of degree 0 and the element  $f_1 \cdots f_n g_1^* \cdots g_m^*$  is homogeneous of degree  $n - m$  (for any collection of edges  $f_i$  and  $g_j$ ). Now let  $A = L_K(E)$ , and consider the canonical  $\mathbb{Z}$ -grading on  $A$ . Then for any  $n$  we consider the canonical epimorphism  $p: \mathbb{Z} \rightarrow \mathbb{Z}_n$  and the grading on  $A$  whose component of degree  $i$  is the sum  $\bigoplus_{p(n)=i} A_n$ . This is a coarsening of the canonical  $\mathbb{Z}$ -grading, and since it is a  $\mathbb{Z}_n$ -grading, we call it the canonical  $\mathbb{Z}_n$ -grading on  $A$ .

## 2. Drawbacks of the conventional definition

The gauge action of the  $C^*$ -algebra  $A := C^*(E)$  of a graph  $E$  is defined as the group homomorphism  $\rho: \mathbb{T} \rightarrow \text{aut}(A)$  such that  $\rho(z)(p_u) = p_u$  for each vertex  $u$  of the graph and  $\rho(z)(s_f) = z s_f$ ,  $\rho(z)(s_f^*) = z^{-1} s_f^*$  for any arrow  $f$  and any  $z \in \mathbb{T}$  (see [11, Proposition 2.1]). With this definition of the gauge action we can recover the homogeneous components of the canonical  $\mathbb{Z}$ -grading on  $A$  easily, since for any integer  $n$  we have that  $A_n$  is just the set of all  $a \in A$  such that for any  $z \in \mathbb{T}$  we have  $\rho(z)(a) = z^n a$ . Thus, when we are given the gauge action on  $A$ , we reconstruct immediately the canonical  $\mathbb{Z}$ -grading. Since the gauge action of  $A$  codifies all the information of the graded algebra  $A$ , all the notions related to this graded structure can be defined in terms of the action. The gauge action is omnipresent in the theory of graph  $C^*$ -algebras for the same reason that the canonical grading on Leavitt path algebras appears in many of the contributions on the subject. Most of the research works on graph  $C^*$ -algebras involve their gauge action. By contrast, most works on Leavitt path algebras miss the gauge action in the terms in which it has been defined in the literature.

Let us think about the “official” definition of the gauge action of a Leavitt path algebra  $B := L_K(E)$  over the commutative (and unitary) ring  $K$  (see [2]). This is nothing but the group homomorphism  $\tau: K^\times \rightarrow \text{aut}(B)$  such that  $\tau(z)(u) = u$ ,  $\tau(z)(f) = z f$ , and  $\tau(z)(f^*) = z^{-1} f^*$  for any vertex  $u$ , any edge  $f$ , and any  $z \in K^\times$ . Though, in the original definition,  $K$  is a field, we have allowed  $K$  to be a unital commutative ring so as to cover the general notion of a Leavitt path algebra. Thus  $K^\times$  in the above definition must be understood as the group of invertible elements of the ring  $K$ .

### REMARK 2

Let  $\tau: K^\times \rightarrow \text{aut}(A)$  be any representation of the group  $K^\times$  by automorphisms on the  $K$ -algebra  $A$ . The action of  $\tau$  on an element  $t \in K^\times$  will be denoted by  $\tau(t)$  or  $\tau_t$  depending on the typographical convenience.

Let us analyze now some peculiarities of the definition above.

### DRAWBACK 1

The gauge action  $\tau$  does not capture all the information of the graded algebra  $L_K(E)$ .

In fact in some extreme cases  $\tau$  contains no information at all simply because  $\tau$  is trivial. For instance take  $K = \mathbb{F}_2$  to be the field of two elements. Then  $K^\times$  is the trivial group  $K^\times = \{1\}$  and  $\tau$  is the trivial group homomorphism, so in this case  $\tau$  gives no information at all of the grading on  $B = L_K(E)$ . There are also examples of other rings and even fields for which the gauge action does not allow the recovery of the canonical  $\mathbb{Z}$ -grading of  $L_K(E)$ .

## DRAWBACK 2

For the gauge action  $\rho$  of  $A = C^*(E)$  the notion of graded ideal is equivalent to that of  $\rho$ -invariant ideal. This is not the case for the gauge action  $\tau$  of  $B = L_K(E)$ .

Indeed, if the ring of scalars  $K$  has a trivial group of invertibles (as in the case  $K = \mathbb{F}_2$ ), then the gauge action  $\rho$  is trivial and so any ideal is  $\rho$ -invariant. Since not every ideal is graded in general we conclude that the equivalence between gauge invariant ideals and graded ideals does not hold for Leavitt path algebras.

## DRAWBACK 3

Consider two graph  $C^*$ -algebras  $A_i$  ( $i = 1, 2$ ) with associated gauge actions  $\rho_i$ . Define a homomorphism  $f: A_1 \rightarrow A_2$  to be a gauge homomorphism when for any  $z \in \mathbb{T}$  the following square is commutative

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \rho_1(z) \downarrow & & \downarrow \rho_2(z) \\ A_1 & \xrightarrow{f} & A_2 \end{array}$$

In a similar fashion we can define the notion of a gauge homomorphism of Leavitt path algebras. However, while the notion of gauge homomorphism is equivalent to that of graded homomorphism in the setting of graph  $C^*$ -algebras, it is not the case that for Leavitt path algebras both notions agree.

Of course this drawback and the previous one do not exist for Leavitt path algebras over infinite fields but we would like to give a notion of gauge action which overcomes these difficulties and does not depend so much on the ground ring of scalars.

## DRAWBACK 4

The gauge-invariant uniqueness theorem is stated in [11] in the following terms.

## THEOREM 1 ([11, THEOREM 2.2, P. 16])

*Let  $E$  be a row-finite graph, and suppose that  $\{T, Q\}$  is a Cuntz–Krieger  $E$ -family in a  $C^*$ -algebra  $B$  with each  $Q_v \neq 0$ . If there is a continuous action  $\beta: \mathbb{T} \rightarrow \text{aut}(B)$  such that  $\beta(z)(T_e) = zT_e$  for every  $e \in E^1$  and  $\beta(z)(Q_v) = Q_v$  for every  $v \in E^0$ , then  $\pi_{T,Q}$  is an isomorphism onto  $C^*(T, Q)$ .*

As far as we know the best version of the previous theorem for Leavitt path algebras is given in [2, Theorem 1.8, p. 6] and it claims the following.

## THEOREM 2 (THE ALGEBRAIC GAUGE-INVARIANT UNIQUENESS THEOREM)

*Let  $E$  be a row-finite graph,  $K$  an infinite field, and  $A$  a  $K$ -algebra. Denote by*

$\tau^E$  the gauge action of  $L_K(E)$ . Suppose that

$$\phi: L_K(E) \rightarrow A$$

is a  $K$ -algebra homomorphism such that  $\phi(v) \neq 0$  for every  $v \in E^0$ . If there exists a group action  $\sigma: K^\times \rightarrow \text{Aut}_K(A)$  such that  $\phi \circ \tau_t^E = \sigma_t \circ \phi$  for every  $t \in K^\times$ , then  $\phi$  is injective.

The hypothesis on the infiniteness of the ground field cannot be removed as the following example shows: take  $K = \mathbb{Z}_2$  and  $A = L_K(E)/I$  where  $I$  is an ideal which does not contain any vertex. For instance  $E$  could be the one-petal rose (where  $E^0$  and  $E^1$  have cardinal 1). Then  $L_K(E) \cong K[T, T^{-1}]$ , the Laurent polynomial algebra on the indeterminate  $T$ , and the ideal  $I$  generated by the non-invertible element  $1+T$  does not contain any vertex. So with these ingredients the canonical epimorphism  $p: L_K(E) \rightarrow A$  satisfies  $p(v) \neq 0$  for (the unique)  $v \in E^0$  and it is not injective. Furthermore the gauge action of  $L_K(E)$  is trivial (since  $K^\times = \{1\}$ ) and we can consider the trivial group action  $\sigma: K^\times \rightarrow \text{aut}(A)$  and the requirement  $p \circ \tau_t^E = \sigma_t \circ p$  is trivially satisfied.

As we shall see, the hypothesis on the infiniteness of  $K$  in Theorem 2 is not necessary if we use the new version of the gauge action.

**DRAWBACK 5**

The natural translation of the crossed product of  $C^*$ -algebras to a purely algebraic setting must be made carefully.

We recall the definition of the crossed product of  $C^*$ -algebras. Assume that  $A$  and  $B$  are  $C^*$ -algebras, and assume that  $G$  is a compact abelian group with actions  $\mu: G \rightarrow \text{aut}(A)$  and  $\nu: G \rightarrow \text{aut}(B)$ . Consider next the action  $\lambda: G \rightarrow \text{aut}(A \otimes B)$  defined by  $\lambda(g)(a \otimes b) = \mu(g)(a) \otimes \nu(g^{-1})(b)$ . Define now the *crossed product*  $A \otimes_G B$  as the fixed-point algebra under the action  $\lambda$ .

The gauge action of a graph  $C^*$ -algebra has been successfully applied to certain interesting constructions in [6] and [10]. Take two row-finite graphs  $E$  and  $F$  and define their product  $E \times F := (E^0 \times F^0, E^1 \times F^1, r, s)$  where  $s(f, g) = (s(f), s(g))$  and  $r(f, g) = (r(f), r(g))$  for any  $(f, g) \in E^1 \times F^1$ . Though this is not the usual definition of the product of two graphs, this notion is interesting for us since it allows the description of the crossed product of graph  $C^*$ -algebras. Indeed, it is proved in [6, Proposition 4.1, p. 62] that if  $E$  and  $F$  are row-finite graphs with no sinks, then there is an isomorphism

$$(1) \quad C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F),$$

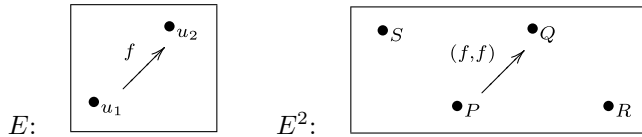
where the crossed product on the right-hand side is the one induced by the gauge actions  $\mathbb{T} \rightarrow \text{aut}(C^*(E))$  and  $\mathbb{T} \rightarrow \text{aut}(C^*(F))$ . We can try to mimic this definition of crossed product in a purely algebraic context. So assume that  $G$  is an abelian group action by automorphisms in the  $K$ -algebras  $A$  and  $B$ . Let  $\mu: G \rightarrow \text{aut}(A)$  and  $\tau: G \rightarrow \text{aut}(B)$  be two representations of  $G$  in  $A$  and  $B$ , respectively, and we define  $\lambda: G \rightarrow \text{aut}(A \otimes B)$  as above. Then the fixed-point algebra under the

action  $\lambda$  will be denoted by  $A \otimes_G B$ . Thus,  $A \otimes_G B$  is a subalgebra of the usual tensor product algebra  $A \otimes_K B$ . Observe that when the group  $G$  is trivial we get  $A \otimes_G B = A \otimes_K B$ .

If we use this naive interpretation of the crossed product of algebras, then a similar property to (1) for Leavitt path algebras does not hold. For instance, consider  $L_K(E)$  for a ring such that  $K^\times$  is trivial. Then  $L_K(E) \otimes_{K^\times} L_K(E) = L_K(E) \otimes_K L_K(E)$  and we would have  $L_K(E \times F) \cong L_K(E) \otimes_K L_K(F)$ . But there are two ways to see that this is not true:

(1) Apply the results in [4] in which the impossibility of this isomorphism is studied. More concretely assume that  $E$  is the one-petal rose ( $|E^0| = |E^1| = 1$ ) and  $F$  is the two-petal rose ( $|F^0| = 1, |F^1| = 2$ ). Then  $E \times F \cong F$  (a graph isomorphism), and applying [4, Theorem 5.1, p. 2635] for  $n = 2, E_1 = E_2 = E, m = 1, F_1 = F$ , we conclude that  $L_K(E) \otimes_K L_K(F)$  is not isomorphic to  $L_K(F)$ .

(2) A simple example also proves the impossibility of the isomorphism  $L_K(E \times F) \cong L_K(E) \otimes_K L_K(F)$ . Consider the graph  $E$  on the left-hand side of



Then  $E^2 := E \times E$  is the graph on the right-hand side of the above figure. Thus  $L_K(E) \cong \mathcal{M}_2(K)$  and  $L_K(E^2) = K \oplus K \oplus \mathcal{M}_2(K)$ , which has dimension 6. However,  $L_K(E) \otimes L_K(E)$  has dimension 16. Hence the cross product of the Leavitt path algebras does not agree with the Leavitt path algebra of  $E^2$ .

Now that we have realized some handicaps of the gauge action of Leavitt path algebras, we propose a different approach.

### 3. Redefining the gauge action

In this section our goal is to define a new notion of the gauge action of a Leavitt path algebra which overcomes the difficulties that we have mentioned in previous sections.

**DEFINITION 1**

For a Leavitt path algebra  $A = L_K(E)$  over a ring  $K$  define the gauge action as a representation of the group  $K[x, x^{-1}]^\times$  on  $L_K(E)_{K[x, x^{-1}]} := L_K(E) \otimes_K K[x, x^{-1}]$  by automorphisms given by  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(L_K(E)_{K[x, x^{-1}]})$  where, for any  $z \in K[x, x^{-1}]^\times$  and any  $u \in L_K(E)$  of degree  $n$ , we have  $\rho(z)(u \otimes 1) = u \otimes z^n$ .

**REMARK 3**

Since  $K^\times$  is a subgroup of  $K[x, x^{-1}]$  we can consider the restriction

$$\rho: K^\times \rightarrow \text{aut}(A_{K[x, x^{-1}]})$$

where  $A = L_K(E)$ . Thus, if  $z \in K^\times$ , then the automorphism  $\rho(z)$  satisfies  $\rho(z)(A \otimes 1) \subset A \otimes 1$ ; hence (after identifying  $L_K(E) \otimes_K K$  with  $L_K(E)$ ) we may consider  $\rho(z)|_{L_K(E)} \in \text{aut}(L_K(E))$ . So the restriction  $\rho: K^\times \rightarrow \text{aut}(L_K(E))$  such that  $z \mapsto \rho(z)|_{L_K(E)}$  agrees with the official gauge action of  $L_K(E)$ . In this way we recover the “official” definition from this new one. For some rings, passing from  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(A_{K[x, x^{-1}]})$  to  $\rho: K^\times \rightarrow \text{aut}(A)$  implies a loss of information as we have seen before. By what we have discussed previously, in the particular case of an infinite field  $K$  both representations encode the same information; furthermore if we are given  $\rho: K^\times \rightarrow \text{aut}(A)$ , then we recover the canonical  $\mathbb{Z}$ -grading on  $A$  by  $A_n = \{a \in A: \rho(z)(a) = z^n a, \forall z \in K^\times\}$  and define  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(A_{K[x, x^{-1}]})$  such that  $\rho(\lambda x^t)(a_n \otimes 1) := \rho(\lambda)(a_n) \otimes x^{tn}$  for any  $\lambda \in K^\times$ ,  $t, n \in \mathbb{Z}$ , and  $a_n \in A_n$ .

**REMARK 4**

The reader familiar with the representation theory of diagonalizable groups schemes may recognize here a representation  $\sigma: \mathbf{Diag}(\mathbb{Z}) \rightarrow \mathbf{aut}(L_K(E))$  whose particularization to  $K[x, x^{-1}]$  (the group  $K$ -algebra of  $\mathbb{Z}$ ) is precisely the gauge action  $\rho$  introduced in Definition 1. Here  $\mathbf{Diag}(\mathbb{Z})$  is the diagonalizable affine group scheme whose representing Hopf algebra is  $K[x, x^{-1}] = K\mathbb{Z}$  while  $\mathbf{aut}(L_K(E))$  is the  $K$ -group functor of automorphisms of  $L_K(E)$ .

It is a standard result in the aforementioned theory that the representation  $\sigma$  is fully determined by its particularization to the group  $K$ -algebra of  $\mathbb{Z}$ . In an early version of this work we introduced the gauge action by using  $\sigma$  rather than  $\rho$ ; however, given that both objects contain exactly the same information, it seemed convenient to adhere to the referee’s suggestion of introducing the new gauge action as in Definition 1.

**REMARK 5**

We can define such an action for any  $K$ -algebra  $A$  endowed with a  $\mathbb{Z}$ -grading  $A = \bigoplus_{n \in \mathbb{Z}} A_n$ : just define the map  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(A_{K[x, x^{-1}]})$  given by  $\rho(z)(a_n \otimes 1) := a_n \otimes z^n$  for any  $n \in \mathbb{Z}$  and  $a_n \in A_n$ .

Reciprocally, if we have a representation  $\rho: K[x, x^{-1}] \rightarrow \text{aut}(A)$ , where  $A$  is a  $K$ -algebra, then one can induce a  $\mathbb{Z}$ -grading on  $A$  such that for each integer  $n$  we have

$$(2) \quad A_n = \{a \in A: \rho(z)(a \otimes 1) = a \otimes z^n \text{ for all } z \in K[x, x^{-1}]^\times\}.$$

The fastest way to see this is to define  $\sigma: \mathbf{Diag}(\mathbb{Z}) \rightarrow \mathbf{aut}(A)$  by  $\sigma_R(f)(a \otimes 1) = (1 \otimes f)\rho(x)(a \otimes 1)$  for any  $a \in A$ ,  $f \in \text{hom}(K[x, x^{-1}], R)$ , and  $R$  any commutative, associative unital  $K$ -algebra. Then apply [9, Paragraph 2.11, Formulas (2) and (3), p. 35]. However in the next theorem we will not need such a general result.

**THEOREM 3**

*The gauge action in the new sense encloses all the information of the canonical*

$\mathbb{Z}$ -grading on  $A := L_K(E)$ . We can recover the homogeneous components from the proposed gauge action. So Drawback 1 no longer holds with this new definition.

*Proof*

We know that the group of invertibles of the Laurent polynomial algebra  $K[x, x^{-1}]^\times$  contains the set of elements  $\{x^k : k \in \mathbb{Z}\}$  (which is infinite independently of the nature of the ring  $K$ ). Then the representation  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(A \otimes K[x, x^{-1}])$  suffices to describe the homogeneous components  $A_n$  ( $n \in \mathbb{Z}$ ). Indeed, we are proving that

$$A_n = \{a \in A : \rho(z)(a \otimes 1) = a \otimes z^n, z \in (K[x, x^{-1}])^\times\}.$$

If  $a \in A$  is homogeneous of degree  $n$ , then it is immediate that  $\rho(z)(a \otimes 1) = a \otimes z^n$  for any  $z \in K[x, x^{-1}]^\times$ . Reciprocally take  $a \in A$  such that  $\rho(z)(a \otimes 1) = a \otimes z^n$  for any invertible  $z \in K[x, x^{-1}]$ . Decompose  $a = \sum a_q$  with  $a_q \in A_q$ ; then  $\sum_q \rho(x^k)(a_q \otimes 1) = \sum_q a_q \otimes x^{kq}$ . So  $\sum_q a_q \otimes x^{kq} = \sum_q a_q \otimes x^{kn}$  and given the linear independence of the powers of  $x$  in the  $K$ -module  $K[x, x^{-1}]$  we conclude that  $a_q = 0$  for any  $q \neq n$ . Thus  $a \in A_n$ .  $\square$

Let us go now to the notion of  $\rho$ -invariant ideal of  $A := L_K(E)$ . So we assume given  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(A_{K[x, x^{-1}]})$  the gauge action that we propose.

#### DEFINITION 2

An ideal  $I$  of  $A$  is said to be  $\rho$ -invariant when for any  $z \in K[x, x^{-1}]^\times$  we have  $\rho(z)(I \otimes 1) \subset I \otimes K[x, x^{-1}]$ .

Clearly, if  $I$  is a graded ideal of  $A$ , then  $I$  is  $\rho$ -invariant: indeed take any  $z \in R^\times$  and  $a \in I$ ; then  $a = \sum a_n$  where each  $a_n \in I \cap A_n$ . Thus  $\rho(z)(a \otimes 1) = \sum_n \rho(z)(a_n \otimes 1) = \sum_n a_n \otimes z^n \in I \otimes K[x, x^{-1}]$ . Consequently graded ideals of  $A$  are  $\rho$ -invariant. But the reciprocal is also true.

#### THEOREM 4

An ideal  $I$  of  $A = L_K(E)$  is graded if and only if it is  $\rho$ -invariant. Thus Drawback 2 no longer holds.

*Proof*

Let  $R := K[x, x^{-1}]$  be the Laurent polynomial algebra in the indeterminate  $x$  over the commutative unitary ring  $K$ . Then  $\{x^n : n \in \mathbb{Z}\}$  is a linearly independent set. Define the  $K$ -modules homomorphism  $f_n: R \rightarrow K$  by  $f_n(T^m) = \delta_{nm}$  (Kronecker's delta). Consider the  $K$ -bilinear map  $A \times R \rightarrow A$  such that  $(a, r) \mapsto f_n(r)a$  and the  $K$ -modules homomorphism  $\Phi_n: A \otimes R \rightarrow A$  such that  $\Phi_n(a \otimes r) = f_n(r)a$ . If  $I$  is an ideal in  $A$ , then  $\Phi_n(I \otimes R) \subset I$ . Consider now  $\rho: R^\times \rightarrow \text{aut}(A)$  the gauge action, take  $a \in I$ , and decompose it as  $a = \sum_m a_m$ , where  $a_m \in A_m$ . Assume that  $I$  is  $\rho$ -invariant. Then  $\rho(x)(a \otimes 1) = \sum_m a_m \otimes x^m \in I \otimes R$ . So  $\Phi_n(\sum_m a_m \otimes$



$x^m) \in I$ , but then  $I \ni \Phi_n(\sum_m a_m \otimes x^m) = \sum_m f_n(x^m)a_m = a_n$  for any  $n$ . Thus  $I$  is graded.  $\square$

Let us deal with Drawback 3 now. Consider two Leavitt path  $K$ -algebras  $A_1$  and  $A_2$  with their respective gauge actions  $\rho_i: R^\times \rightarrow \text{aut}(A_i)$ ,  $i = 1, 2$ , and  $R = K[x, x^{-1}]$ , then we can define the following.

**DEFINITION 3**

A homomorphism  $f: A_1 \rightarrow A_2$  is said to be a gauge homomorphism if the following square is commutative

$$\begin{array}{ccc} A_1 \otimes R & \xrightarrow{f \otimes 1} & A_2 \otimes R \\ \rho_1(z) \downarrow & & \downarrow \rho_2(z) \\ A_1 \otimes R & \xrightarrow{f \otimes 1} & A_2 \otimes R \end{array}$$

for any  $z \in R^\times$ .

It is easy to prove that, when  $f: A_1 \rightarrow A_2$  is a graded homomorphism, it is a gauge homomorphism: take  $a \in A_1$  homogeneous of degree, say,  $n$ . Then  $\rho_2(z)(f \otimes 1)(a \otimes 1) = \rho_2(z)(f(a) \otimes 1)$ , and since  $f(a)$  is a homogeneous element of  $A_2$  of degree  $n$ ,  $\rho_2(z)(f \otimes 1)(a \otimes 1) = f(a) \otimes z^n = (f \otimes 1)(a \otimes z^n) = (f \otimes 1)\rho_1(z)(a \otimes 1)$ . Thus  $f$  is a gauge homomorphism. But we have also the reciprocal.

**THEOREM 5**

*The homomorphism  $f: A_1 \rightarrow A_2$  is graded if and only if it is a gauge-homomorphism. Thus Drawback 3 disappears.*

*Proof*

Assume that  $f$  is a gauge homomorphism and take  $a$  in the homogeneous component of degree  $n$  of  $A_1$ . Recall that such a component agrees with the submodule of all the elements  $a \in A_1$  such that  $\rho_1(z)(a \otimes 1) = a \otimes z^n$  for any  $z \in K[x, x^{-1}]^\times$ . Consequently we must prove that  $\rho_2(z)(f(a) \otimes 1) = a \otimes z^n$  for any  $z$  as before. But this is a direct corollary of the commutativity of the square in Definition 3.  $\square$

**REMARK 6**

Theorem 5 can be generalized in the following sense. Let  $A_i$  ( $i = 1, 2$ ) be  $K$ -algebras endowed with representations  $\rho_i: K[x, x^{-1}]^\times \rightarrow \text{aut}(A_i)$  ( $i = 1, 2$ ). Consider now the  $\mathbb{Z}$ -grading induced by  $\rho_i$  in  $A_i$  as in Remark 5: the homogeneous component of degree  $n$  is just the  $K$ -submodule of those elements  $a \in A_i$  such that  $\rho_i(z)(a \otimes 1) = a \otimes z^n$  for any  $z \in K[x, x^{-1}]^\times$ . Take then a  $K$ -algebra homomorphism  $f: A_1 \rightarrow A_2$ . With the same proof as above, we have that  $f$  is graded if and only if it is a gauge homomorphism in the sense that the squares in Definition 3 are commutative.

Concerning the statement of the algebraic gauge-invariant uniqueness theorem for Leavitt path algebras, by using the definition we propose, we can restate it in the following form.

**THEOREM 6 (THE NEW ALGEBRAIC GAUGE-INVARIANT UNIQUENESS THEOREM)**

Let  $E$  be a graph, let  $K$  be any commutative unitary ring, and let  $A$  be any  $K$ -algebra. Denote by  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(L_K(E))$  the gauge action of  $L_K(E)$ . Assume that

$$\phi: L_K(E) \rightarrow A$$

is a  $K$ -algebra homomorphism such that  $\phi(rv) \neq 0$  for every  $v \in E^0$  and  $r \in K \setminus \{0\}$ . If there exists an action  $\sigma: K[x, x^{-1}]^\times \rightarrow \text{aut}(A)$  such that  $(\phi \otimes 1)\rho(z) = \sigma(z)(\phi \otimes 1)$  for any  $z \in R^\times$ , then  $\phi$  is injective.

The proof is straightforward since by Remark 6 the homomorphism  $\phi$  is graded relative to the grading induced by  $\sigma$  in  $A$ . Then we can apply [12, Theorem 5.3, p. 476].

When  $K$  is a field, we have the following.

**COROLLARY 1**

Let  $E$  be a graph, let  $K$  be any field, and let  $A$  be any  $K$ -algebra. Denote by  $\rho: K[x, x^{-1}]^\times \rightarrow \text{aut}(L_K(E))$  the gauge action of  $L_K(E)$ . Assume that

$$\phi: L_K(E) \rightarrow A$$

is a  $K$ -algebra homomorphism such that  $\phi(v) \neq 0$  for every  $v \in E^0$ . If there exists an action  $\sigma: K[x, x^{-1}] \rightarrow \text{aut}(A)$  such that  $(\phi \otimes 1)\rho(z) = \sigma(z)(\phi \otimes 1)$  for any  $z \in R^\times$ , then  $\phi$  is injective.

Also by using the gauge action in the sense that we propose, the hypothesis on the infiniteness of the ground field  $K$  in [2, Proposition 1.6] can be dropped. Since the notions of graded ideal and of gauge-invariant ideal agree when we use the new version of gauge action, such exceptionalities as the ones observed in [2, Proposition 1.7] are no longer present.

### 3.1. Cross product of Leavitt path algebras by their gauge actions

Let  $K$  be a commutative unitary ring, let  $R := K[x, x^{-1}]$  be the Laurent polynomial algebra over  $K$ , and let  $A$  be a  $K$ -algebra with an action  $\rho: R^\times \rightarrow \text{aut}(A)$ .

**DEFINITION 4**

The fixed subalgebra  $A^\rho$  of  $A$  under  $\rho$  is the one whose elements are the elements  $a \in A$  such that  $\rho(z)(a \otimes 1) = a \otimes 1$  for any  $z \in R^\times$ .

If  $A$  and  $B$  are  $K$ -algebras provided with actions  $\rho: R^\times \rightarrow \text{aut}(A)$  and  $\sigma: R^\times \rightarrow \text{aut}(B)$ , then there is an action  $\rho \otimes \sigma: R^\times \rightarrow \text{aut}(A \otimes B)$  such that for any  $z \in R^\times$

we have  $(\rho \otimes \sigma)(z)$  given by the composition

$$\begin{array}{ccccc}
 (A \otimes B)_R = A \otimes B \otimes R & \xrightarrow{1 \otimes \delta} & A \otimes B \otimes R \otimes R & \xrightarrow{\theta} & A_R \otimes B_R \\
 (\rho \otimes \sigma)(z) \downarrow \dashv & & & & \downarrow \rho(z) \otimes \sigma(z^{-1}) \\
 (A \otimes B)_R = A \otimes B \otimes R & \xleftarrow{1 \otimes \mu} & A \otimes B \otimes R \otimes R & \xleftarrow{\theta^{-1}} & A_R \otimes B_R
 \end{array}$$

where

- $\delta: R \rightarrow R \otimes R$  is given by  $\delta(z) = z \otimes 1$ ,
- $\theta$  is the isomorphism  $a \otimes b \otimes r \otimes r' \mapsto a \otimes r \otimes b \otimes r'$ , and
- $\mu: R \otimes R \rightarrow R$  is the multiplication  $\mu(r \otimes r') = rr'$ .

In summary,

$$(\rho \otimes \sigma)(z) = (1 \otimes \mu)\theta^{-1}(\rho(z) \otimes \sigma(z^{-1}))\theta(1 \otimes \delta).$$

Now a direct (but not short) computation reveals that

$$(\rho \otimes \sigma)(zz') = (\rho \otimes \sigma)(z)(\rho \otimes \sigma)(z')$$

for any  $z$  and  $z'$ . So any  $(\rho \otimes \sigma)(z)$  is invertible with inverse  $(\rho \otimes \sigma)(z^{-1})$ . Moreover, since  $(\rho \otimes \sigma)(z)$  is a composition of  $R$ -algebra homomorphisms,  $(\rho \otimes \sigma)(z) \in \text{aut}((A \otimes B)_R)$ .

**DEFINITION 5**

The action  $\rho \otimes \sigma: R^\times \rightarrow \text{aut}(A \otimes B)$  will be called the tensor product action of  $\rho$  and  $\sigma$ . The fixed-point subalgebra  $(A \otimes B)^{\rho \otimes \sigma}$  of  $A \otimes B$  under  $\rho \otimes \sigma$  will be denoted  $A_{\rho \otimes \sigma} B$  and called the cross product of  $A$  and  $B$  by the actions  $\rho$  and  $\sigma$ . If there is no ambiguity with respect to the actions involved we could shorten the notation to  $A \otimes_{R^\times} B$ .

Consider now two Leavitt path algebras  $L_K(E)$  and  $L_K(F)$  of the graphs  $E$  and  $F$ , respectively. We assume the gauge action of each algebra and ask about the cross product algebra  $L_K(E) \otimes_{R^\times} L_K(F)$ . With not much effort one can prove that it consists of the elements of the form  $\sum_{n \in \mathbb{Z}} a_n \otimes b_n$ , where  $a_n \in L_K(E)$  with  $\text{deg}(a) = n$  while  $b_n \in L_K(F)$  also has degree  $n$ . Of course we can define on this algebra also an action  $\tau: R^\times \rightarrow \text{aut}(L_K(E) \otimes_{R^\times} L_K(F))$  by declaring for each  $z \in R^\times$  that  $\tau(z)(a_n \otimes r \otimes b_n \otimes s) = a_n \otimes z^n r \otimes b_n \otimes s$ . By Remark 5, this action induces a grading on  $L_K(E) \otimes_{R^\times} L_K(F)$  in which the homogeneous component of degree  $n$  is the  $K$ -submodule generated by the elements of the form  $a \otimes b$  where  $a$  and  $b$  are homogeneous of degree  $n$ .

Our next goal is to prove the following.

**THEOREM 7**

*If  $E$  and  $F$  are row-finite graphs with no sinks, then there is an isomorphism  $L_K(E) \otimes_{R^\times} L_K(F) \cong L_K(E \times F)$ , where the product of the graph is the one described in Section 2.*

*Proof*

For a graph  $E$  denote by  $\hat{E}$  the extended graph of  $E$ : the vertices of  $\hat{E}$  are those of  $E$  and the arrows of  $\hat{E}^1$  are those of  $E^1$  plus a family  $\{f^*: f \in E^1\}$  of new edges such that  $s(f^*) = r(f)$  and  $r(f^*) = s(f)$  for any  $f \in E^1$ . The path algebra  $KE$  is the associative  $K$ -algebra with basis the set of all paths of  $E$  (so it is free as a  $K$ -module). There is a well-known relation  $L_K(E) \cong K\hat{E}/I$ , where  $I$  is the ideal of  $K\hat{E}$  generated by the Cuntz–Krieger relations.

We consider the path algebra  $K(\widehat{E \times F})$ ; then there is a canonical homomorphism of  $K$ -algebras  $K(\widehat{E \times F}) \rightarrow L_K(E) \otimes_{R^\times} L_K(F)$  such that for any  $(u, v) \in E^0 \times F^0$  and  $(f, g) \in E^1 \times F^1$  we have

$$(u, v) \mapsto u \otimes v, \quad (f, g) \mapsto f \otimes g, \quad (f^*, g^*) \mapsto f^* \otimes g^*.$$

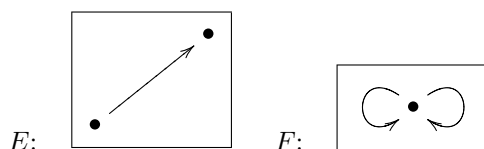
This homomorphism induces one  $\phi: L_K(E \times F) \rightarrow L_K(E) \otimes_{R^\times} L_K(F)$  such that  $\phi(r(u, v)) = ru \otimes v \neq 0$  for each  $(u, v) \in E^0 \times F^0$  and  $r \in K \setminus \{0\}$  (see Remark 1). Furthermore if we take the action  $\tau: R^\times \rightarrow \text{aut}(L_K(E) \otimes_{R^\times} L_K(F))$  defined above, we see that  $(\phi \otimes 1)\rho(z) = \tau(z)(\phi \otimes 1)$ , where  $\rho$  is the gauge action of  $L_K(E \times F)$ . Thus applying Theorem 6 we conclude that  $\phi$  is a monomorphism. To see that it is also an epimorphism we need the hypothesis that the graphs have no sinks. Since  $L_K(E) \otimes_{R^\times} L_K(F)$  is generated by elements of the form  $a \otimes b$  where  $\text{deg}(a) = \text{deg}(b)$  it suffices to show that these elements are in the image of  $\phi$ . First we prove that if  $\mu$  and  $\tau$  are paths of the same length (say,  $n$ ) and  $u$  is a vertex, then  $\mu\tau^* \otimes u$  is in the image of  $\phi$ . Indeed,  $\mu\tau^* \otimes u = \mu\tau^* \otimes \sum_i g_i g_i^*$  (since  $F$  is row-finite and has no sink). If  $\mu = f\mu'$ , where  $f \in E^1$  and  $\mu'$  is a path, then  $\mu\tau^* \otimes u = \sum_i (f \otimes g_i)(\mu'\tau^* \otimes g_i^*)$ , and if  $\tau = h\tau'$  with  $h \in E^1$  and  $\tau'$  a path, then  $\mu\tau^* \otimes u = \sum_i (f \otimes g_i)(\mu'\tau'^* h^* \otimes g_i^*) = \sum_i (f \otimes g_i)(\mu'\tau'^* \otimes r(g_i))(h^* \otimes g_i^*)$ . By applying a suitable induction hypothesis this proves that  $\mu\tau^* \otimes u$  is in the image of  $\phi$ . Symmetrically it can be proved that the image of  $\phi$  contains the elements of the form  $v \otimes \sigma\delta^*$  with  $v \in E^0$  and  $\sigma, \delta$  being paths of  $F$  of the same degree. Now any generator of  $L_K(E) \otimes_{R^\times} L_K(F)$ , say,  $\mu\tau^* \otimes \sigma\delta^*$ , such that  $\text{deg}(\mu) - \text{deg}(\tau) = \text{deg}(\sigma) - \text{deg}(\delta)$  can be written as a product of elements which obey some of the followings patterns:

- $f \otimes g$  with  $f \in E^1$  and  $g \in F^1$ ,
- $f^* \otimes g^*$  with  $f \in E^1$  and  $g \in F^1$ ,
- $\mu\tau^* \otimes u$  with  $u \in F^0$  and  $\mu$  and  $\tau$  being paths of  $E$  of the same length,
- $v \otimes \sigma\delta^*$  with  $v \in E^0$  and  $\sigma$  and  $\delta$  being paths of  $F$  of the same length.

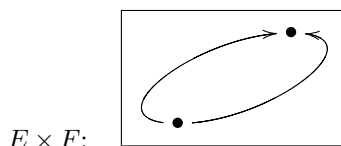
Since any of these elements is in the image of  $\phi$ , this proves that  $\phi$  is an epimorphism. □

**REMARK 7**

The hypothesis on the absence of sinks in the graphs of Theorem 7 cannot be removed. To show this, take  $K$  to be a field, and consider the graphs  $E$  and  $F$  below:



Then  $E \times F$  is the graph



which is isomorphic to  $\mathcal{M}_3(K)$ , the algebra of  $3 \times 3$  matrices with entries in  $K$ . However, the cross product algebra  $L_K(E \otimes_R E)$  is easily shown to be infinite-dimensional; hence no isomorphism between  $L_K(E \times F)$  and  $L_K(E) \otimes_R L_K(F)$  can be expected in this case.

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