Special values of the Hurwitz zeta function via generalized Cauchy variables

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Abstract As a continuation of the work of Bourgade, Fujita, and Yor, we show how to recover the extension of the Euler formulae concerning some special values of the Hurwitz zeta function from the product of two, and then N, independent generalized Cauchy variables. Meanwhile, we consider the ratio of two independent generalized Cauchy variables and give another proof of the partial fraction expansion of the cotangent function.

1. Introduction

The special value of the Riemann zeta function $\zeta(2)$ is well known:

$$\zeta(2)=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}. \tag{1.1}$$

To find the value of this series is known as the Basel problem. The Basel problem was originally solved by Euler in 1735. Since then many ways to solve the problem have been proposed. Bourgade, Fujita, and Yor [3] have suggested another approach to solve the Basel problem and recovered the Euler formulae as below for the Riemann zeta function in a probabilistic way. We recall the Euler formulae:

$$\Big(1-\frac{1}{2^{2n+2}}\Big)\zeta(2n+2)=\frac{1}{2}\Big(\frac{\pi}{2}\Big)^{2n+2}\frac{A_{2n}}{\Gamma(2n+2)}, \tag{1.2}$$

for $n=0,1,2,\dots$. Here the coefficients A_n stand for the tangent numbers which appear in the series expansion:

$$\frac{1}{\cos^2\theta}=\sum_{k=0}^{\infty}\frac{A_k}{k!}\theta^k,\quad |\theta|<\frac{\pi}{2}. \tag{1.3}$$

The authors in [3] have discovered that the product of independent Cauchy variables could be used to solve the Basel problem and the Euler formulae. More precisely, let $\mathcal{C}$ be a Cauchy variable; that is, $\mathcal{C}$ has the following probability

Kyoto Journal of Mathematics, Vol. 52, No. 3 (2012), 465–477
DOI 10.1215/21562261-1625163,
Received April 5, 2011. Revised October 5, 2011. Accepted October 5, 2011.
2010 Mathematics Subject Classification: Primary 60E05; Secondary 60G52, 11M35.
Yano's work supported by Grant-in-Aid for Scientific Research (KAKENHI) 23740073.

density function:

$$f_{\mathcal{C}}(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad x \in \mathbb{R}. \tag{1.4}$$

Let $\hat{\mathcal{C}}$ be an independent copy of $\mathcal{C}$. Then the density function of $|\mathcal{C}\hat{\mathcal{C}}|$ is given by

$$f_{|\mathcal{C}\hat{\mathcal{C}}|}(x) = \frac{4}{\pi^2}\frac{\log x}{x^2-1}, \quad x > 0. \tag{1.5}$$

Since $f_{|\mathcal{C}\hat{\mathcal{C}}|}(x)$ is a probability density function, it holds that

$$\int_0^\infty f_{|\mathcal{C}\hat{\mathcal{C}}|}(x)\,\mathrm{d}x = 1,$$

that is,

$$\frac{\pi^2}{4} = \int_0^\infty \frac{\log x}{x^2-1}\,\mathrm{d}x. \tag{1.6}$$

By some basic computation involving Taylor expansions, we have

$$\text{the RHS of (1.6)} = 2\sum_{k=1}^\infty \frac{1}{(2k+1)^2}.$$

Meanwhile, we have

$$\zeta(2) = \sum_{k=1}^\infty \frac{1}{k^2} = \sum_{k=0}^\infty \frac{1}{(2k+1)^2} + \sum_{k=1}^\infty \frac{1}{(2k)^2} \tag{1.7}$$

$$= \sum_{k=0}^\infty \frac{1}{(2k+1)^2} + \frac{1}{4}\zeta(2); \tag{1.8}$$

therefore we obtain the desired result (1.1). Considering the moment of $\log|\mathcal{C}\hat{\mathcal{C}}|$ or product of N independent Cauchy random variables, we can obtain the Euler formulae (for the details, see Fujita [5]).

On the other hand, a Cauchy random variable appears as the law of the first hitting time for planar Brownian motion. Yano, Yano, and Yor [9] have studied the hitting time of a single point for symmetric α-stable Lévy processes with indices $\alpha \in (1,2]$ and shown the following result.

THEOREM 1.1 ([9, THEOREM 5.3]; SEE ALSO [4, PROPOSITION 1.11])
Let $X = (X_t)$ be a symmetric α-stable Lévy process of index $\alpha \in (1,2]$, and let $\hat{X}$ be an independent copy of X. For $a \in \mathbb{R}$, let $T_{\{a\}}(X)$ be the first hitting time at a of X. Then

$$\hat{X}\big(T_{\{a\}}(X)\big) \overset{\text{law}}{=} |a|\mathcal{C}_\alpha, \tag{1.9}$$

where

$$P(\mathcal{C}_\alpha \in \mathrm{d}x) = \frac{\sin(\pi/\alpha)}{2\pi/\alpha}\frac{\mathrm{d}x}{1+|x|^\alpha}, \quad x \in \mathbb{R}. \tag{1.10}$$

The authors in [9] call the random variable $\mathcal{C}_\alpha$ an *α-Cauchy random variable.* We note that, in the case $\alpha = 2$, X is Brownian motion (up to a multiplicative constant) and $\mathcal{C}_2$ is equal in law to the Cauchy random variable, that is, $\mathcal{C}_2 \stackrel{\text{law}}{=} \mathcal{C}$. We remark that

$$|\mathcal{C}_\alpha|^\alpha \stackrel{\text{law}}{=} \frac{\mathcal{G}_\gamma}{\hat{\mathcal{G}}_{1-\gamma}}, \tag{1.11}$$

where $\gamma = 1/\alpha \in [1/2, 1)$, and $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_{1-\gamma}$ are independent Gamma random variables with parameters γ and $1-\gamma$, respectively (see [9, (2.21)]). Here $\mathcal{G}_a$ is a Gamma random variable with parameter $a > 0$ if it has the following probability density function:

$$P(\mathcal{G}_a \in \mathrm{d}x) = \frac{1}{\Gamma(a)} x^{a-1} \mathrm{e}^{-x}\, \mathrm{d}x, \quad x > 0. \tag{1.12}$$

REMARK 1.2

In [5], Fujita introduced (m,n)-generalized Cauchy random variables $\mathcal{C}_{m,n}$ ($m, n \in \mathbb{R}$ with $n > m+1 > 0$) which are defined by

$$P(\mathcal{C}_{m,n} \in \mathrm{d}x) = C_{m,n}^2 \frac{x^m}{1+x^n}\, \mathrm{d}x, \quad x > 0, \tag{1.13}$$

where

$$C_{m,n} = \frac{\sin((m+1)/n)\pi}{\pi/n}. \tag{1.14}$$

The connection between $\mathcal{C}_{m,n}$ and $\mathcal{C}_\alpha$ is as follows:

$$\mathcal{C}_{m,n} \stackrel{\text{law}}{=} \Big(\frac{\mathcal{G}_{(m+1)/n}}{\hat{\mathcal{G}}_{1-(m+1)/n}}\Big)^{1/n} = \Big(\Big(\frac{\mathcal{G}_\gamma}{\hat{\mathcal{G}}_{1-\gamma}}\Big)^\gamma\Big)^{1/(m+1)} \stackrel{\text{law}}{=} |\mathcal{C}_\alpha|^{1/(m+1)}, \tag{1.15}$$

where $\gamma = (m+1)/n$ and $\alpha = 1/\gamma$.

In the present paper we consider some generalizations of Bourgade, Fujita, and Yor's results. Using the law of the product of two, then N, independent α-Cauchy random variables, we can obtain the following formulae, which are well-known generalizations of the Euler formulae concerning some special values of the Hurwitz zeta function (see, e.g., [1, (6.4.7), (6.4.10)]).

THEOREM 1.3

For $\gamma \in (0,1)$ and $n = 0, 1, 2, \dots,$ it holds that

$$\zeta(2n+2, \gamma) + \zeta(2n+2, 1-\gamma) = \frac{\pi^{2n+2}}{\Gamma(2n+2)} A_{2n}(\gamma\pi), \tag{1.16}$$

where $\zeta(s,a)$ stands for the Hurwitz zeta function

$$\zeta(s,a) = \sum_{k=0}^\infty \frac{1}{(k+a)^s}, \quad 0 < a \le 1, \tag{1.17}$$

and $A_n(a)$ are the coefficients of the following series expansion:

$$\frac{1}{\sin^2(a+\theta)} = \sum_{k=0}^\infty \frac{A_k(a)}{k!} \theta^k, \quad 0 < a+\theta < \pi. \tag{1.18}$$

REMARK 1.4

(i) We show the Euler formulae (1.2) from (1.16) when $\gamma = 1/2$.

(ii) Formula (1.16) can be rewritten as

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\gamma)^{2n+2}} = \frac{\pi^{2n+2}}{\Gamma(2n+2)} A_{2n}(\gamma\pi). \tag{1.19}$$

Thus, when $n = 0$, we can see the well-known formula (see, e.g., [2, (1.2.9)])

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\gamma)^{2}} = \Big(\frac{\pi}{\sin\gamma\pi}\Big)^2. \tag{1.20}$$

Recently Fujita [6] succeeded in giving another proof of the Basel problem and the Euler formulae: the author considered the ratio of two independent arc-sine random variables. In the present paper we consider the ratio of two independent α-Cauchy random variables. Then we can obtain a simple proof for the well-known formula for the partial fraction expansion of the cotangent function:

$$\pi\cot\gamma\pi = \frac{1}{\gamma} + 2\gamma\sum_{k=1}^{\infty}\frac{1}{\gamma^2-k^2} = \frac{1}{\gamma} + \sum_{k=1}^{\infty}\Big(\frac{1}{\gamma+k}+\frac{1}{\gamma-k}\Big). \tag{1.21}$$

We note that formula (1.20) can be obtained from (1.21) by differentiating both sides in γ.

The organization of this paper is as follows. In Section 2, we consider the product of two independent α-Cauchy random variables. We first show the special case of (1.16) for $n = 0$ and then give a proof of Theorem 1.3. In Section 3, we consider the product of N independent α-Cauchy random variables and give another proof of Theorem 1.3. In Section 4, we consider the ratio of two independent α-Cauchy random variables. We give a new approach to show the partial fraction expansion of the cotangent function (1.21).

2. Product of two independent α-Cauchy random variables and some special values of the Hurwitz zeta function

We first present the probability density function of the law of $|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|$ where $\mathcal{C}_\alpha$ and $\hat{\mathcal{C}}_\alpha$ are independent.

PROPOSITION 2.1

Let $\mathcal{C}_\alpha$ be an α-Cauchy random variable with $\alpha\in(1,2]$, and let $\hat{\mathcal{C}}_\alpha$ be an independent copy of $\mathcal{C}_\alpha$. Then

$$f_{|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|}(x) = C_\alpha^2\frac{\log x}{x^\alpha-1}, \quad x>0, \tag{2.1}$$

where $C_\alpha = (\sin(\pi/\alpha))/(\pi/\alpha)$.

Proof

Formula (2.1) can be obtained by easy computations as follows. We have

$$\begin{aligned}
f_{|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}(x) &= \int_0^\infty f_{|\mathcal{C}_\alpha|}(u) f_{|\hat{\mathcal{C}}_\alpha|}\Big(\frac{x}{u}\Big)\frac{1}{u}\,\mathrm{d}u \\
&= C_\alpha^2 \int_0^\infty \frac{1}{1+u^\alpha}\frac{u^{\alpha-1}}{u^\alpha+x^\alpha}\,\mathrm{d}u \\
&= \frac{C_\alpha^2}{\alpha}\int_0^\infty \frac{1}{1+v}\frac{1}{v+x^\alpha}\,\mathrm{d}v \quad \text{(by change of variables } u^\alpha = v) \\
&= \frac{C_\alpha^2}{\alpha}\int_0^\infty \frac{1}{x^\alpha-1}\Big(\frac{1}{1+v}-\frac{1}{v+x^\alpha}\Big)\,\mathrm{d}v \\
&= C_\alpha^2 \frac{\log x}{x^\alpha - 1},
\end{aligned}$$

which is our desired result. □

Using this, we obtain the following result.

THEOREM 2.2 (CF. [5, THEOREM 5.1])
For $0<\gamma<1$, it holds that

$$\zeta(2,\gamma)+\zeta(2,1-\gamma) = \Big(\frac{\pi}{\sin\gamma\pi}\Big)^2, \tag{2.2}$$

or equivalently,

$$\sum_{k=-\infty}^{\infty}\frac{1}{(k+\gamma)^2} = \Big(\frac{\pi}{\sin\gamma\pi}\Big)^2, \tag{2.3}$$

which is stated as (1.20) above.

Proof
Since the function $f_{|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}$ is a probability density function, we have

$$1 = \int_0^\infty f_{|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}(x)\,\mathrm{d}x = C_\alpha^2 \frac{\log x}{x^\alpha-1}.$$

Hence we have

$$\begin{aligned}
\frac{1}{C_\alpha^2} &= \int_0^\infty \frac{\log x}{x^\alpha-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\log x}{x^\alpha-1}\,\mathrm{d}x + \int_1^\infty \frac{\log x}{x^\alpha-1}\,\mathrm{d}x \\
&= \int_0^\infty \frac{z\mathrm{e}^{-z}}{1-\mathrm{e}^{-\alpha z}}\,\mathrm{d}z + \int_0^\infty \frac{w\mathrm{e}^{-(\alpha-1)w}}{1-\mathrm{e}^{-\alpha w}}\,\mathrm{d}w \\
&\qquad \text{(by change of variables } x=\mathrm{e}^{-z}, y=\mathrm{e}^{-z}) \\
&= \int_0^\infty z\mathrm{e}^{-z}\,\mathrm{d}z \sum_{k=0}^\infty \mathrm{e}^{-k\alpha z} + \int_0^\infty w\mathrm{e}^{-(\alpha-1)w}\,\mathrm{d}w \sum_{k=0}^\infty \mathrm{e}^{-k\alpha w}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} z e^{-(\alpha k+1)z}\, dz + \sum_{k=0}^{\infty} \int_0^{\infty} w e^{-(\alpha(k+1)-1)w}\, dw$$

$$= \frac{1}{\alpha^2}\Big(\sum_{k=0}^{\infty} \frac{1}{(k+1/\alpha)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+1-1/\alpha)^2} \Big).$$

Putting $\gamma = 1/\alpha \in [1/2, 1)$, we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(k+\gamma)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+(1-\gamma))^2} = \frac{\pi^2}{\sin^2(\gamma\pi)}. \tag{2.4}$$

Equation (2.4) above is symmetric with respect to $\gamma = 1/2$, so that this holds for every $\gamma \in (0,1)$. Thus we obtain (2.2). □

Next we show Theorem 1.3 in Section 1 by using the even moments of $\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|$. Before the proof, we compute the even moments of $\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|$.

PROPOSITION 2.3

Let $\mathcal{C}_\alpha$ and $\hat{\mathcal{C}}_\alpha$ be independent α-Cauchy random variables. Then, for $n = 0, 1, 2, \ldots,$

$$\begin{aligned} &E\big[(\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|)^{2n}\big] \\ &\quad = C_\alpha^2 \Gamma(2n+2) \Big\{ \sum_{k=0}^{\infty} \frac{1}{(\alpha k+1)^{2n+2}} + \sum_{k=0}^{\infty} \frac{1}{(\alpha(k+1)-1)^{2n+2}} \Big\}. \end{aligned} \tag{2.5}$$

This can be obtained easily by using Proposition 2.1, so we omit the details.

Proof of Theorem 1.3

For $p \in \mathbb{R}, p > 0$, and $\alpha \in (1,2]$, we have

$$E[|\mathcal{C}_\alpha|^p] = E\Big[\Big(\frac{\mathcal{G}_{1/\alpha}}{\hat{\mathcal{G}}_{1-1/\alpha}}\Big)^{p/\alpha}\Big] = \frac{\sin(\pi/\alpha)}{\sin((p+1)\pi/\alpha)}, \tag{2.6}$$

by (1.11). Here the second identity of (2.6) is obtained from

$$E[\mathcal{G}_a^b] = \frac{1}{\Gamma(a)} \int_0^{\infty} x^b x^{a-1} e^{-x}\, dx = \frac{\Gamma(a+b)}{\Gamma(a)}. \tag{2.7}$$

Thus we obtain

$$E[e^{p\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}] = E[|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|^p] = \Big(\frac{\sin(\pi/\alpha)}{\sin((p+1)\pi/\alpha)}\Big)^2. \tag{2.8}$$

By the series expansion (1.18), we have

$$E[e^{p\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}] = \Big(\sin\frac{\pi}{\alpha}\Big)^2 \sum_{k=0}^{\infty} \frac{A_k(\pi/\alpha)}{k!} \Big(\frac{p\pi}{\alpha}\Big)^k. \tag{2.9}$$

On the other hand, we have

$$E[e^{p\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|}] = \sum_{k=0}^{\infty} \frac{p^k}{k!} E[\log|\mathcal{C}_\alpha \hat{\mathcal{C}}_\alpha|^k]. \tag{2.10}$$

We note that $E[\mathrm{e}^{p\log|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|}]$ is analytic in $p\in(-1,\alpha-1)$ by (2.8). Combining (2.10) with (2.9), we obtain

$$E\big[(\log|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|)^{2n}\big]=\Big(\sin\frac{\pi}{\alpha}\Big)^2\Big(\frac{\pi}{\alpha}\Big)^{2n}A_{2n}\Big(\frac{\pi}{\alpha}\Big). \tag{2.11}$$

Together with Proposition 2.3 this implies

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{(\alpha k+1)^{2n+2}}+\sum_{k=0}^{\infty}\frac{1}{(\alpha(k+1)-1)^{2n+2}}\\&\qquad=\frac{1}{\Gamma(2n+2)}\Big(\frac{\pi}{\alpha}\Big)^{2n+2}A_{2n}\Big(\frac{\pi}{\alpha}\Big),\end{aligned} \tag{2.12}$$

and hence we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(k+\gamma)^{2n+2}}+\sum_{k=0}^{\infty}\frac{1}{(k+1-\gamma)^{2n+2}}=\frac{\pi^{2n+2}}{\Gamma(2n+2)}A_{2n}(\gamma\pi), \tag{2.13}$$

where $\gamma=1/\alpha\in[1/2,1)$. We note that

$$A_k\big((1-\gamma)\pi\big)=(-1)^kA_k(\gamma\pi), \tag{2.14}$$

so that (2.13) is symmetric with respect to $\gamma=1/2$. Therefore the desired result is obtained. □

3. Product of N independent α-Cauchy random variables and another proof of Theorem 1.3

In this section we present the density function of the product of N independent α-Cauchy random variables, which is a generalization of [3, Proposition 2], and then we give another proof of Theorem 1.3.

PROPOSITION 3.1

For $N=0,1,2,\ldots,$ one has

$$\begin{aligned}f_{2N+1}(x)&=f_{|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(2N+1)}|}(x)\\&=\Big(\frac{\pi}{\alpha}\Big)^{2N}\frac{C_\alpha^{2N+1}}{(2N)!}\frac{1}{1+x^\alpha}\prod_{k=1}^{N}\Big(\Big(\frac{\alpha\log x}{\pi}\Big)^2+(2k-1)^2\Big),\quad x>0.\end{aligned} \tag{3.1}$$

For $N=1,2,\ldots,$ one has

$$\begin{aligned}f_{2N}(x)&=f_{|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(2N)}|}(x)\\&=\Big(\frac{\pi}{\alpha}\Big)^{2N-2}\frac{C_\alpha^{2N}}{(2N-1)!}\frac{\log x}{x^\alpha-1}\prod_{k=1}^{N-1}\Big(\Big(\frac{\alpha\log x}{\pi}\Big)^2+(2k)^2\Big),\quad x>0.\end{aligned} \tag{3.2}$$

Proof

The results follow by induction on N. For $N=0$, $f_1(x)$ is precisely the density function of the absolute value of an α-Cauchy random variable. For $N=1$, $f_2(x)$ has been computed in Theorem 2.1.

We recall formula (2.6):

$$E[|\mathcal{C}_\alpha|^p] = \frac{\sin(\pi/\alpha)}{\sin((p+1)\pi/\alpha)}.$$

Consequently, for $N \in \mathbb{N}$, we have

$$\begin{aligned} E[|\mathcal{C}_\alpha^{(1)}\mathcal{C}_\alpha^{(2)}\cdots\mathcal{C}_\alpha^{(N)}|^p] &= \Big(\frac{\sin(\pi/\alpha)}{\sin((p+1)\pi/\alpha)}\Big)^N \\ &= C_\alpha^N\Big(\frac{\pi}{\alpha}\Big)^N\Big(\sin\frac{(p+1)\pi}{\alpha}\Big)^{-N}. \end{aligned} \tag{3.3}$$

Differentiating twice (3.3) in p, we have

$$\begin{aligned} &E\big[|\mathcal{C}_\alpha^{(1)}\mathcal{C}_\alpha^{(2)}\cdots\mathcal{C}_\alpha^{(N)}|^p\{\log|\mathcal{C}_\alpha^{(1)}\mathcal{C}_\alpha^{(2)}\cdots\mathcal{C}_\alpha^{(N)}|\}^2\big] \\ &\quad= C_\alpha^N\Big(\frac{\pi}{\alpha}\Big)^{N+2}N(N+1)\Big(\sin\frac{(p+1)\pi}{\alpha}\Big)^{-(N+2)}\Big(\cos\frac{(p+1)\pi}{\alpha}\Big)^2 \\ &\qquad+ C_\alpha^N\Big(\frac{\pi}{\alpha}\Big)^{N+2}N\Big(\sin\frac{(p+1)\pi}{\alpha}\Big)^{-N} \end{aligned} \tag{3.4}$$

$$\begin{aligned} &= C_\alpha^N\Big(\frac{\pi}{\alpha}\Big)^{N+2}N(N+1)\Big(\sin\frac{(p+1)\pi}{\alpha}\Big)^{-(N+2)} \\ &\quad- C_\alpha^N\Big(\frac{\pi}{\alpha}\Big)^{N+2}N^2\Big(\sin\frac{(p+1)\pi}{\alpha}\Big)^{-N}. \end{aligned} \tag{3.5}$$

By (3.3), we have

$$(3.5) = \frac{N(N+1)}{C_\alpha^2}E[|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(N+2)}|^p] - N^2\Big(\frac{\pi}{\alpha}\Big)^2E[|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(N)}|^p], \tag{3.6}$$

and hence we have

$$\begin{aligned} &E[|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(N+2)}|^p] \\ &\quad= \frac{C_\alpha^2}{N(N+1)}\Big\{E\big[|\mathcal{C}_\alpha^{(1)}\mathcal{C}_\alpha^{(2)}\cdots\mathcal{C}_\alpha^{(N)}|^p\{\log|\mathcal{C}_\alpha^{(1)}\mathcal{C}_\alpha^{(2)}\ldots\mathcal{C}_\alpha^{(N)}|\}^2\big] \\ &\qquad+ N^2\Big(\frac{\pi}{\alpha}\Big)^2E[|\mathcal{C}_\alpha^{(1)}\cdots\mathcal{C}_\alpha^{(N)}|^p]\Big\}, \end{aligned} \tag{3.7}$$

that is,

$$\int_0^\infty x^p f_{N+2}(x)\,\mathrm{d}x = \frac{C_\alpha^2}{N(N+1)}\Big(\frac{\pi}{\alpha}\Big)^2\int_0^\infty x^p\Big(\Big(\frac{\alpha\log x}{\pi}\Big)^2 + N^2\Big)f_N(x)\,\mathrm{d}x. \tag{3.8}$$

By uniqueness of the Mellin transform, we obtain

$$f_{N+2}(x) = \frac{C_\alpha^2}{N(N+1)}\Big(\frac{\pi}{\alpha}\Big)^2\Big(\Big(\frac{\alpha\log x}{\pi}\Big)^2 + N^2\Big)f_N(x), \tag{3.9}$$

which completes the proof. □

Now we give another proof of Theorem 1.3 by using formula (3.2). We start with the following recurrence relation for the Hurwitz zeta function.

PROPOSITION 3.2

Let $\alpha \in (1,2]$, and let $N \in \mathbb{N}$. Let $p_{N,k}$ and $q_{N,k}$ be the coefficients in the following expansion, respectively:

$$\prod_{j=1}^{N}\big(x+(2j)^2\big) = \sum_{k=0}^{N} p_{N,k}x^k, \tag{3.10}$$

$$\prod_{j=1}^{N}\big(x+(2j-1)^2\big) = \sum_{k=0}^{N} q_{N,k}x^k. \tag{3.11}$$

Then

$$\begin{aligned}&\frac{(\sin(\pi/\alpha))^{2N+2}}{(2N+1)!}\sum_{n=0}^{N}\frac{p_{N,n}}{\pi^{2n+2}}(2n+1)!\Big\{\zeta\Big(2n+2,\frac{1}{\alpha}\Big)+\zeta\Big(2n+2,1-\frac{1}{\alpha}\Big)\Big\}\\ &\quad = 1,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\frac{(\sin(\pi/\alpha))^{2N+1}}{(2N)!}\sum_{n=0}^{N}\frac{q_{N,n}}{\pi^{2n+1}}(2n)!\Big\{\sum_{k=0}^{\infty}\Big(\frac{(-1)^k}{(k+1/\alpha)^{2n+1}}+\frac{(-1)^k}{(k+1-1/\alpha)^{2n+1}}\Big)\Big\}\\ &\quad = 1.\end{aligned} \tag{3.13}$$

Proof

We first show (3.12). We have

$$1 = \int_0^\infty f_{2N+2}(x)\,\mathrm{d}x \tag{3.14}$$

$$= \frac{\pi^{2N}C_\alpha^{2N}}{\alpha^{2N}(2N+1)!}\int_0^\infty f_{|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|}(x)\prod_{n=1}^{N}\Big(\Big(\frac{\alpha\log x}{\pi}\Big)^2+(2n)^2\Big)\,\mathrm{d}x \tag{3.15}$$

$$= \frac{\pi^{2N}C_\alpha^{2N}}{\alpha^{2N}(2N+1)!}\int_0^\infty f_{|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|}(x)\Big(\sum_{n=0}^{N}p_{N,n}\Big(\frac{\alpha\log x}{\pi}\Big)^{2n}\Big)\,\mathrm{d}x \tag{3.16}$$

$$= \frac{\pi^{2N}C_\alpha^{2N}}{\alpha^{2N}(2N+1)!}\sum_{n=0}^{N}\Big(\frac{\alpha}{\pi}\Big)^{2n}p_{N,n}\int_0^\infty(\log x)^{2n}f_{|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|}(x)\,\mathrm{d}x \tag{3.17}$$

$$= \frac{\pi^{2N}C_\alpha^{2N}}{\alpha^{2N}(2N+1)!}\sum_{n=0}^{N}\Big(\frac{\alpha}{\pi}\Big)^{2n}p_{N,n}E\big[(\log|\mathcal{C}_\alpha\hat{\mathcal{C}}_\alpha|)^{2n}\big]. \tag{3.18}$$

Proposition 2.3 implies

$$\begin{aligned}(3.18) &= \frac{\pi^{2N}C_\alpha^{2N+2}}{\alpha^{2N+2}(2N+1)!}\sum_{n=0}^{N}\Big(\frac{\alpha}{\pi}\Big)^{2n}p_{N,n}\Gamma(2n+2)\\ &\quad\times\Big\{\sum_{k=0}^{\infty}\frac{1}{(\alpha k+1)^{2n+2}}+\sum_{k=0}^{\infty}\frac{1}{(\alpha(k+1)-1)^{2n+2}}\Big\}\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&= \frac{(\sin(\pi/\alpha))^{2N+2}}{(2N+1)!}\sum_{n=0}^{N}\frac{p_{N,n}}{\pi^{2n+2}}\Gamma(2n+2) \\
&\quad\times\Big\{\sum_{k=0}^{\infty}\frac{1}{(k+1/\alpha)^{2n+2}}+\sum_{k=0}^{\infty}\frac{1}{(k+1-1/\alpha)^{2n+2}}\Big\},
\end{aligned}
\tag{3.20}
$$

and hence we obtain the desired result.

Similarly equation (3.13) is proved by the fact that

$$
\int_0^{\infty} f_{2N+1}(x)\,\mathrm{d}x = 1. \tag{3.21}
$$

We complete the proof. □

To prove Theorem 1.3, we should show

$$
\frac{(\sin(\pi/\alpha))^{2N+2}}{(2N+1)!}\sum_{n=0}^{N}p_{N,n}A_{2n}\Big(\frac{\pi}{\alpha}\Big) = 1. \tag{3.22}
$$

We recall the following relations for the function $f(\theta)=1/\sin^2\theta$:

$$
\Big\{\prod_{j=1}^{n}\Big((2j)^2+\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Big)\Big\}f(\theta)=(2n+1)!f(\theta)^{1+n} \tag{3.23}
$$

(see equation in [3, p. 77] in the case where $t=2$). Thus we have

$$
\sum_{n=0}^{N}p_{N,n}A_{2n}\Big(\frac{\pi}{\alpha}\Big)=(2N+1)!\Big(\frac{1}{\sin(\pi/\alpha)}\Big)^{2N+2}, \tag{3.24}
$$

and hence we obtain (3.22) by induction on N. Therefore we have proved Theorem 1.3.

REMARK 3.3

Recall the relations for the function $g(\theta)=1/\sin\theta$:

$$
\Big\{\prod_{j=1}^{n}\Big((2j-1)^2+\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Big)\Big\}g(\theta)=(2n)!g(\theta)^{1+2n} \tag{3.25}
$$

(see equation in [3, p. 77] in the case where $t=1$); thus we obtain

$$
\sum_{k=0}^{\infty}\Big(\frac{(-1)^k}{(k+1/\alpha)^{2n+1}}+\frac{(-1)^k}{(k+1-1/\alpha)^{2n+1}}\Big)=\frac{\pi^{2n+1}}{\Gamma(2n+1)}A_{2n}^{(1)}\Big(\frac{\pi}{\alpha}\Big), \tag{3.26}
$$

where $A_n^{(1)}$ are the coefficients of the following series expansion:

$$
\frac{1}{\sin(a+\theta)}=\sum_{k=0}^{\infty}\frac{A_k^{(1)}(a)}{k!}\theta^k,\quad 0<a+\theta<\pi, \tag{3.27}
$$

by the same steps.

4. Ratio of two independent α-Cauchy random variables and the partial fraction expansion of the cotangent function

Considering the ratio of two independent α-Cauchy random variables gives the partial fraction expansion of the cotangent function.

PROPOSITION 4.1
Let $\mathcal{C}_\alpha$ and $\hat{\mathcal{C}}_\alpha$ be independent α-Cauchy random variables. Then

$$f_{|\mathcal{C}_\alpha/\hat{\mathcal{C}}_\alpha|}(x) = \frac{-\tan(\pi/\alpha)}{(\pi/\alpha)}\frac{1-x^{\alpha-2}}{x^\alpha-1}, \quad x>0. \tag{4.1}$$

Proof
For $0<x<1$, we have

$$\begin{aligned}
f_{|\mathcal{C}_\alpha/\hat{\mathcal{C}}_\alpha|}(x) &= \int_0^\infty f_{|\mathcal{C}_\alpha|}(u) f_{|\hat{\mathcal{C}}_\alpha|}(ux) u \,\mathrm{d}u = C_\alpha^2 \int_0^\infty \frac{1}{1+u^\alpha}\frac{1}{1+(ux)^\alpha} u\,\mathrm{d}u \\
&= \frac{C_\alpha^2}{\alpha}\int_0^\infty \frac{1}{1+v}\frac{1}{1+vx^\alpha} v^{(2-\alpha)/\alpha}\,\mathrm{d}v \quad \text{(by change of variables } u^\alpha = v) \\
&= \frac{C_\alpha^2}{\alpha}\frac{1}{1-x^\alpha}\int_0^\infty v^{(2/\alpha)-2}\Big(\frac{1}{1+vx^\alpha} - \frac{1}{1+v}\Big)\,\mathrm{d}v \\
&= \frac{C_\alpha^2}{\alpha}\Gamma\Big(1-\Big(\frac{2}{\alpha}-1\Big)\Big)\Gamma\Big(\frac{2}{\alpha}-1\Big)\frac{x^{\alpha-2}-1}{1-x^\alpha} \\
&= \frac{C_\alpha^2}{\alpha}\frac{\pi}{\sin((2/\alpha)-1)\pi}\frac{x^{\alpha-2}-1}{1-x^\alpha} \\
&= \frac{1}{\alpha}\Big(\frac{\sin(\pi/\alpha)}{(\pi/\alpha)}\Big)^2 \frac{\pi}{-2\sin(\pi/\alpha)\cos(\pi/\alpha)}\frac{x^{\alpha-2}-1}{1-x^\alpha} \\
&= \frac{-\tan(\pi/\alpha)}{(2\pi)/\alpha}\frac{x^{\alpha-2}-1}{1-x^\alpha}.
\end{aligned}$$

For $x>1$, the desired result can be obtained immediately from the result for $0<x<1$ by change of variables. □

Now it holds that

$$1 = \int_0^\infty f_{|\mathcal{C}_\alpha/\hat{\mathcal{C}}_\alpha|}(x)\,\mathrm{d}x. \tag{4.2}$$

Then we have

$$\frac{(2\pi/\alpha)}{-\tan(\pi/\alpha)} = \int_0^1 \frac{x^{\alpha-2}-1}{1-x^\alpha}\,\mathrm{d}x + \int_1^\infty \frac{1-x^{\alpha-2}}{x^\alpha-1}\,\mathrm{d}x = 2\int_0^1 \frac{x^{\alpha-2}-1}{1-x^\alpha}\,\mathrm{d}x; \tag{4.3}$$

that is,

$$\frac{(\pi/\alpha)}{-\tan(\pi/\alpha)} = \int_0^1 \frac{x^{\alpha-2}-1}{1-x^\alpha}\,\mathrm{d}x. \tag{4.4}$$

We can compute the right-hand side of (4.4) as follows:

$$\begin{aligned}
&\text{the RHS of (4.4)} \\
&= \int_0^\infty \frac{\mathrm{e}^{-(\alpha-2)y}-1}{1-\mathrm{e}^{-\alpha y}}\mathrm{e}^{-y}\,\mathrm{d}y \quad (\text{by change of variables } x=\mathrm{e}^{-y}) && (4.5)\\
&= \int_0^\infty (\mathrm{e}^{-(\alpha-1)y}-\mathrm{e}^{-y})\,\mathrm{d}y \sum_{k=0}^\infty \mathrm{e}^{-k\alpha y} && (4.6)\\
&= \sum_{k=0}^\infty \int_0^\infty (\mathrm{e}^{-(\alpha(k+1)-1)y}-\mathrm{e}^{-(\alpha k+1)y})\,\mathrm{d}y && (4.7)\\
&= \sum_{k=0}^\infty \Big(\frac{1}{\alpha(k+1)-1}-\frac{1}{\alpha k+1}\Big) && (4.8)\\
&= \gamma\sum_{k=0}^\infty \Big(\frac{1}{k+1-\gamma}-\frac{1}{k+\gamma}\Big), && (4.9)
\end{aligned}$$

where $\gamma = 1/\alpha \in (1/2,1)$. The equation

$$\frac{\pi}{-\tan\gamma\pi} = \sum_{k=0}^\infty \Big(\frac{1}{k+1-\gamma}-\frac{1}{k+\gamma}\Big) \tag{4.10}$$

is symmetric with respect to $\gamma = 1/2$, so that it holds for every $\gamma \in (0,1/2)\cup(1/2,1)$. Thus we obtain

$$\frac{\pi}{\tan\gamma\pi} = \frac{1}{\gamma} + \sum_{k=1}^\infty \Big(\frac{1}{\gamma+k}+\frac{1}{\gamma-k}\Big). \tag{4.11}$$

Acknowledgments. The authors would like to thank Professor Marc Yor for fruitful discussions and helpful comments. We are grateful to the referees for their very careful reading of our manuscript.

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