Estimates for resolvents and functions of operator pencils on tensor products of Hilbert spaces

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Abstract Let $H = X \otimes Y$ be a tensor product of separable Hilbert spaces X and Y. We establish norm estimates for the resolvent and operator-valued functions of the operator $A = \sum_{k=0}^{m} B_k \otimes S^k$, where B_k $(k = 0, \dots, m)$ are bounded operators acting in Y, and S is a self-adjoint operator acting in X. By these estimates we investigate spectrum perturbations of A. The abstract results are applied to the nonself-adjoint differential operators in Hilbert and Euclidean spaces. Our main tool is a combined use of some properties of operators on tensor products of Hilbert spaces and the recent estimates for the norm of the resolvent of a nonself-adjoint operator.

1. Introduction and preliminaries

Let X and Y be separable Hilbert spaces with scalar products $\langle \cdot,\cdot \rangle_X$ and $\langle \cdot,\cdot \rangle_Y$, respectively, and norms $\|\cdot\|_X = \sqrt{\langle \cdot,\cdot \rangle_X}$, $\|\cdot\|_Y = \sqrt{\langle \cdot,\cdot \rangle_Y}$. Let $H = X \otimes Y$ be the tensor product of X and Y with the scalar product defined by

$$\langle x \otimes y, x_1 \otimes y_1 \rangle_H = \langle y, y_1 \rangle_Y \langle x, x_1 \rangle_X \quad (y, y_1 \in Y; x, x_1 \in X),$$

and the cross norm $\|\cdot\|_H = \sqrt{\langle \cdot,\cdot \rangle_H}$. This means that H is the closure in the norm $\|\cdot\|_H$ of the collection of all formal sums of the form

$$h = \sum_j x_j \otimes y_j \quad (x_j \in X, y_j \in Y)$$

with the understanding that

$$\lambda(x \otimes y) = (\lambda y) \otimes x = y \otimes (\lambda x), \qquad (x + x_1) \otimes y = x \otimes y + x_1 \otimes y$$
$$x \otimes (y + y_1) = x \otimes y + x \otimes y_1 \quad (\lambda \in \mathbb{C}).$$

From the theory of tensor products we only need the basic definition and elementary facts which can be found in [2]. Operators on tensor products of Hilbert spaces arise in various problems of pure and applied mathematics (cf. [17], [18]). The classical results on operators on tensor products of Hilbert spaces are presented in [1] and [2] and in the above-mentioned books; recent results can be

Kyoto Journal of Mathematics, Vol. 51, No. 3 (2011), 673–686
DOI 10.1215/21562261-1299927, © 2011 by Kyoto University
Received July 14, 2010. Revised November 19, 2010. Accepted January 12, 2011.
2010 Mathematics Subject Classification: Primary 47A80, 47E05, 34L15.

found, in particular, in [11]–[13] and references therein. In [11] the author investigates the invariant subspaces of operators on multiple tensor products. In [13] the authors prove that weak (strong, uniform) convergence of sequences of Hilbert space operators is preserved by tensor products. In the case of convergence to zero, it is shown that the boundedness of one sequence and the weak (strong, uniform) convergence to zero of the other one suffice to ensure the convergence of their tensor products to zero in the same topology and that the converse holds for power sequences. They also show that a tensor product of operators is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry. In [12] the authors investigate the problem of transferring Weyl and Browder's theorems from operators to their tensor product. In [5]–[7] spectrum perturbations and resolvents of some classes of operators in H are investigated. Papers [8] and [9] are devoted to regular functions of operators in H. In this paper we establish norm estimates for the resolvent and operator-valued functions of an operator pencil on H. These estimates are applied to differential operators in Hilbert and Euclidean spaces. The spectrum of ordinary differential operators was considered in many papers and books (see [2], [14], [15], [19]) and references therein. In particular, in [19], the author studies the problem of localization of the spectrum for a class of a scalar differential operator on a finite segment. However, the resolvents and bounds for the spectrum of the nonself-adjoint differential operators with matrix and operator coefficients are still not sufficiently investigated in the available literature.

We need to say a few words about the contents. The paper consists of six sections. In Section 2 we formulate and prove the main result of the paper, Theorem 2.1, on the resolvent of the considered operators. In Section 3, we make Theorem 2.1 sharper in the case when Y is a Euclidean space. Sections 4 and 5 are devoted to differential operators. In Section 6 we consider the operator-valued functions.

For a linear operator A, $\sigma(A)$ is the spectrum, $\mathrm{Dom}(A)$ is the domain, A^* is adjoint to A, and $\mathrm{Im}\, A := \frac{1}{2i}(A - A^*)$. By SN_p $(1 \le p < \infty)$ we denote the ideal of the Schatten–von Neumann operators K in Y with the finite Schatten–von Neumann norm $N_p(K) := [\mathrm{Trace}(KK^*)^{p/2}]^{1/p}$. I_X denotes the identity operator in a space X.

Let $m < \infty$ be an integer. The main object of this paper is the operator

$$A = \sum_{k=0}^{m} B_k \otimes S^k \quad \text{with } \mathrm{Dom}(A) = \mathrm{Dom}(S^m) \otimes Y, \tag{1.1}$$

where B_k $(k = 0, \dots, m-1)$ are bounded operators acting in Y, $B_m = I_Y$, and S is an invertible self-adjoint operator acting in X. Below we present the relevant examples. Recall that the polynomial

$$\sum_{k=0}^{m-1} B_k \lambda^k \quad (\lambda \in \mathbb{C})$$

is called the *operator pencil* (of a scalar argument). Following that definition, we call the operator defined by (1.1) *the operator pencil of an operator argument.*

Let E_s $(s \in \sigma(S))$ be the orthogonal expansion of the identity of S, and let

$$B(s) := s^m I_Y + \sum_{k=0}^{m-1} B_k s^k.$$

Then it is not hard to see that

$$A = \int_{\sigma(S)} B(s) \otimes dE_s, \tag{1.2}$$

where the integral for $h = x \otimes y, g = x_1 \otimes y_1$ with $x \in \mathrm{Dom}(S^m), x_1 \in X; y, y_1 \in Y$ is defined by

$$(Ah, g)_H = \int_{\sigma(S)} \big(B(s)y, y_1\big)_Y \, d(E_s x, x_1)_X$$

and is linearly extended to the whole $\mathrm{Dom}(A)$.

Put $S_0 = S \otimes I_Y$.

LEMMA 1.1

Let $\lambda \in \mathbb{C}$ be a regular point of $B(s)$ for all $s \in \sigma(S)$, and let

$$\theta(B, \nu, \lambda) := \sup_{s \in \sigma(S)} \left\| s^\nu \big(B(s) - \lambda I_Y\big)^{-1} \right\|_Y < \infty \tag{1.3}$$

for a $\nu \in [0, m)$. Then λ is a regular point of the operator A defined by (1.1), and

$$(A - \lambda I_H)^{-1} = \int_{\sigma(S)} \big(B(s) - \lambda I_Y\big)^{-1} \otimes dE_s \tag{1.4}$$

and $\|S_0^\nu (A - \lambda I_H)^{-1}\|_H \le \theta(B, \nu, \lambda)$.

The integral in (1.4) is understood as the one in (1.2).

Proof
Put

$$J(\lambda) = \int_{\sigma(S)} \big(B(s) - \lambda I_Y\big)^{-1} \otimes dE_s.$$

Clearly,

$$\begin{aligned}
(A - \lambda I_H) J(\lambda) &= \int_{\sigma(S)} \big(B(s) - \lambda I_Y\big) \otimes dE_s \int_{\sigma(S)} \big(B(s_1) - \lambda I_Y\big)^{-1} \otimes dE_{s_1} \\
&= \int_{\sigma(S)} \big(B(s) - \lambda I_Y\big)\big(B(s) - \lambda I_Y\big)^{-1} \otimes dE_s \\
&= \int_{\sigma(S)} I_Y \otimes dE_s = I_Y \otimes I_X = I_H.
\end{aligned}$$

Similarly, $J(\lambda)(A-\lambda I_H)=I_H$. This proves (1.4). Furthermore,

$$\begin{aligned}&\big(S_0^{\nu}(A-\lambda I)^{-1}h, S_0^{\nu}(A-\lambda I)^{-1}h\big)_H\\ &\quad=\int_{\sigma(S)}\big(s^{\nu}(B(s)-\lambda I_Y)^{-1}y, s^{\nu}\big(B(s)-\lambda I_Y\big)^{-1}y\big)_Y\, d(E_s x,x)_X\end{aligned}$$

for an $h=x\otimes y$, with $x\in X; y\in Y$. Hence,

$$\begin{aligned}\big(S_0^{\nu}(A-\lambda I)^{-1}h, S_0^{\nu}(A-\lambda I)^{-1}h\big)_H &\le \theta^2(B,\nu,\lambda)\|y\|_Y^2\int_{\sigma(S)} d(E_s x,x)_X\\ &= \theta^2(B,\nu,\lambda)\|y\|_Y^2\|x\|_X^2.\end{aligned}$$

Extending this inequality linearly, we prove the lemma. □

We need also the following simple lemma.

LEMMA 1.2

Let A be defined by (1.1), and let $\tilde{A}$ be a linear operator in H satisfying

$$\mathrm{Dom}(\tilde{A})=\mathrm{Dom}(A) \quad \textit{and} \quad q_{\nu}:=\|(\tilde{A}-A)S_0^{-\nu}\|_H<\infty. \tag{1.5}$$

Let the conditions (1.3) and $q_{\nu}\theta(B,\nu,\lambda)<1$ hold. Then λ is a regular point for $\tilde{A}$, and

$$\|(\tilde{A}-I_H)^{-1}\|_H \le \frac{\theta(B,0,\lambda)}{1-q_{\nu}\theta(B,\nu,\lambda)}.$$

Proof

Since

$$\begin{aligned}(A-\lambda I)^{-1}-(\tilde{A}-\lambda I)^{-1} &= (A-\lambda I)^{-1}(\tilde{A}-A)(A-\lambda I)^{-1}\\ &= (\tilde{A}-\lambda I)^{-1}(\tilde{A}-A)S_0^{-\nu}S_0^{\nu}(A-\lambda I)^{-1},\end{aligned}$$

it is not hard to check that λ is regular for $\tilde{A}$ if $q_{\nu}\|S_0^{\nu}(A-\lambda I)^{-1}\|_H<1$. In addition,

$$\|(\tilde{A}-\lambda I)^{-1}\|_H \le \frac{\|(A-\lambda I)^{-1}\|_H}{1-q_{\nu}\|S_0^{\nu}(A-\lambda I)^{-1}\|_H}.$$

Now the previous lemma yields the required result. □

2. The main result

Let m_0 $(-1\le m_0\le m-1)$ be the *smallest integer* such that

$$B_k=B_k^* \quad (k=m_0+1,\dots,m-1). \tag{2.1}$$

So if all B_k are self-adjoint, then $m_0=-1$. If all B_k $(k<m)$ are nonself-adjoint, then $m_0=m-1$. Assume that

$$\mathrm{Im}\, B_k\in SN_{2p} \quad (k=0,\dots,m_0) \tag{2.2}$$

for an integer $p \geq 1$. Since S is invertible, we have

$$\xi(S) := \inf_{s \in \sigma(S)} |s| > 0.$$

Put

$$b_p := 2\Big(1 + \frac{2p}{\exp(2/3)\ln 2}\Big),$$

and

$$v(m_0) = b_p \sum_{k=0}^{m_0} N_{2p}(\operatorname{Im} B_k)\xi^{k-m_0}(S) \quad \text{for } m_0 \geq 0 \text{ and } v(-1) = 0.$$

Finally, for a $\lambda \in \mathbb{C}$, set $\rho(B(s),\lambda) := \inf_{z \in \sigma(B(s))} |z - \lambda|$, and let

$$\Phi_p(y) := \sum_{j=0}^{p-1} y^j \exp\Big[\frac{1}{2}(1 + y^{2p})\Big] \quad (y > 0), \text{ and } \Phi_p(0) = 1.$$

Now we are in a position to formulate our main result.

THEOREM 2.1

Under condition (2.2), let

$$\rho(A,\lambda) := \inf_{s \in \sigma(A)} \rho\big(B(s),\lambda\big) > 0. \tag{2.3}$$

Then λ is a regular point for the operator A defined by (1.1), and relation (1.4) holds. Moreover, for any nonnegative $\nu < m$, we have

$$\gamma_\nu(\lambda) := \sup_{s \in \sigma(S)} \frac{|s|^\nu}{\rho(B(s),\lambda)} < \infty, \tag{2.4}$$

and

$$\|S_0^\nu (A - \lambda I)^{-1}\|_H \leq \gamma_\nu(\lambda)\Phi_p\big(v(m_0)\gamma_{m_0}(\lambda)\big), \tag{2.5}$$

where $\gamma_{m_0}(\lambda)$ $(m_0 \geq 0)$ is defined by (2.4) with $\nu = m_0$.

Proof

For a bounded linear operator C acting in Y, assume that

$$\operatorname{Im} C \in SN_{2p} \quad \text{for some integer } p \geq 1. \tag{2.6}$$

We need the following result: let $\rho(C,\lambda) = \inf_{s \in \sigma(C)} |s - \lambda| > 0$ and condition (2.6) hold. Then

$$\|(C - \lambda I_Y)^{-1}\|_Y \leq \sum_{j=0}^{p-1} \frac{(b_p N_{2p}(\operatorname{Im} C))^j}{\rho^{j+1}(C,\lambda)} \exp\Big[\frac{1}{2} + \frac{(b_p N_{2p}(\operatorname{Im} C))^{2p}}{2\rho^{2p}(C,\lambda)}\Big].$$

For the proof, see [4, Theorem 7.9.1]. Hence it follows that

$$\big\|\big(B(s) - \lambda I_Y\big)^{-1}\big\|_Y \leq \sum_{j=0}^{p-1} \frac{\hat{g}_p^j(B(s))}{\rho^{j+1}(B(s),\lambda)} \exp\Big[\frac{1}{2} + \frac{\hat{g}_p^{2p}(B(s))}{2\rho^{2p}(B(s),\lambda)}\Big]$$

with

$$\hat{g}_p\big(B(s)\big) := b_p \sum_{k=0}^{m_0} N_{2p}(\operatorname{Im} B_k)|s|^k.$$

But $\hat{g}_p(B(s)) \le v(m_0)|s|^{m_0}$. So

$$\begin{aligned}\big\|\big(B(s)-\lambda I_Y\big)^{-1}\big\|_Y &\le \sum_{j=0}^{p-1} \frac{(v(m_0)|s|^{m_0})^j}{\rho^{j+1}(B(s),\lambda)} \exp\Big[\frac{1}{2} + \frac{(v(m_0)|s|^{m_0})^{2p}}{2\rho^{2p}(B(s),\lambda)}\Big] \\ &= \frac{1}{\rho(B(s),\lambda)} \Phi_p\Big(\frac{v(m_0)|s|^{m_0}}{\rho(B(s,\lambda))}\Big).\end{aligned} \tag{2.7}$$

Put

$$F(s) = B(s) - s^m I_Y = \sum_{k=0}^{m-1} B_k s^k.$$

Let $\mu(B(s)) \in \sigma(B(s))$. Then $\mu(B(s)) = s^m + \mu(F(s))$, where $\mu(F(s)) \in \sigma(F(s))$. So

$$\big|\mu\big(B(s)\big)\big| \ge |s|^m - \sum_{k=0}^{m-1} \|B_k\||s|^k.$$

Hence,

$$\frac{|s|^\nu}{\rho(B(s),\lambda)} \le \frac{|s|^\nu}{|s-\lambda|^m - \sum_{k=0}^{m-1} \|B_k\||s-\lambda|^k}$$

for all sufficiently large $|s|$. Therefore, $\gamma_\nu(\lambda) < \infty$, provided that condition (2.3) holds. So by (2.7),

$$\big\|s^\nu\big(B(s)-\lambda I_Y\big)^{-1}\big\|_Y \le \gamma_\nu(\lambda)\Phi_p\big(v(m_0)\gamma_{m_0}(\lambda)\big).$$

Now Lemma 1.1 implies the required result. □

Note that $\gamma_0(\lambda) = 1/\rho(A,\lambda)$. Theorem 2.1 and Lemma 1.2 imply the following.

COROLLARY 2.2

Under the hypotheses of Theorem 2.1, let $\tilde{A}$ be a linear operator in H satisfying (1.5). In addition, let $q_\nu\gamma_\nu(\lambda)\Phi_p(v(m_0)\gamma_{m_0}(\lambda)) < 1$. Then λ is a regular point of $\tilde{A}$, and

$$\|(\tilde{A} - \lambda I_H)^{-1}\|_H \le \frac{\gamma_0(\lambda)\Phi_p(v(m_0)\gamma_{m_0}(\lambda)}{1 - q_\nu\gamma_\nu(\lambda)\Phi_p(v(m_0)\gamma_{m_0(\lambda)})}.$$

REMARK 2.3

In the case $p = 2^{n-1}, n = 1, 2, \ldots,$ one can take sharper values for b_p. Namely, $b_1 = \sqrt{2}$ (see [4, Theorem 7.7.2]), and $b_p = 2(1 + ctg(\frac{\pi}{4p}))$ if $p = 2^n, n = 1, 2, \ldots,$ (see [4, Theorem 7.7.2]).

3. The case $Y = \mathbb{C}^n$

In this section we improve Theorem 2.1 in the case $Y = \mathbb{C}^n$. That is, $B_k (k = 0, \dots, n-1)$ are $(n \times n)$-matrices. Again, m_0 $(-1 \le m_0 \le m-2)$ is the smallest integer such that (2.1) holds. Put

$$w(m_0) = \sqrt{2} \sum_{k=0}^{m_0} N_2(\operatorname{Im} B_k) \xi^{k-m_0}(S) \quad \text{for } m_0 \ge 0 \text{ and } w(-1) = 0.$$

In the considered case, we have

$$\rho\big(B(s), \lambda\big) = \min_{j=1,\dots,n} \big|\lambda_j\big(B(s)\big) - \lambda\big|,$$

where $\lambda_j(B(s))$ are the eigenvalues of $B(s)$ counted with their algebraic multiplicities. In addition, condition (2.3) takes the form

$$\rho(A, \lambda) = \inf_{s \in \sigma(S)} \min_{j=1,\dots,n} \big|\lambda_j\big(B(s)\big) - \lambda\big| > 0. \tag{3.1}$$

Set also

$$\Psi_n(y) := \sum_{j=0}^{n-1} \frac{y^j}{\sqrt{k!}} \quad (y > 0) \text{ and } \Psi_n(0) = 1.$$

THEOREM 3.1

Let $H = X \otimes \mathbb{C}^n$, and for a $\lambda \in \mathbb{C}$, let condition (3.1) hold. Then λ is a regular point of the operator A defined by (1.1), and relation (2.4) holds for any nonnegative $\nu < m$. Moreover,

$$\|S_0^\nu (A - \lambda I_H)^{-1}\|_H \le \gamma_\nu(\lambda) \Psi_n\big(w(m_0)\gamma_{m_0}(\lambda)\big). \tag{3.2}$$

Proof

We need the following result: for a linear operator C in $\mathbb{C}^n$, let $\lambda \notin \sigma(S)$; that is, $\rho(C, \lambda) = \min_j |\lambda_j(C) - \lambda| > 0$. Then

$$\|(C - \lambda I_Y)^{-1}\|_{C^n} \le \sum_{j=0}^{n-1} \frac{(\sqrt{2} N_2(\operatorname{Im} C))^j}{\sqrt{k!}\rho^{j+1}(C, \lambda)}.$$

For the proof, see [4, Corollary 2.1.2]. Hence it follows that

$$\big\|\big(B(s) - \lambda I_Y\big)^{-1}\big\|_{C^n} \le \sum_{j=0}^{n-1} \frac{(\sqrt{2} N_2(\operatorname{Im} B(s)))^j}{\sqrt{k!}\rho^{j+1}(B(s), \lambda)}.$$

But

$$\sqrt{2} N_2\big(\operatorname{Im}(B(s))\big) \le w(m_0)|s|^{m_0},$$

and thus

$$\big\|\big(B(s) - \lambda I_Y\big)^{-1}\big\|_{C^n} \le \frac{1}{\rho(B(s), \lambda)} \Psi_n\Big(\frac{w(m_0)|s|^{m_0}}{\rho(B(s), \lambda)}\Big). \tag{3.3}$$

As it was shown in the proof of Theorem 2.1, we have $\gamma_\nu(\lambda) < \infty$, provided that condition (3.1) holds. So by (3.3),

$$\left\| s^\nu \big(B(s) - \lambda I_Y\big)^{-1} \right\|_{C^n} \le \gamma_\nu(\lambda)\Psi_n\big(w(m_0)\gamma_{m_0}(\lambda)\big).$$

Now Lemma 1.1 yields the required result. □

Theorem 3.1 and Lemma 1.2 imply the following.

COROLLARY 3.2
Let $H = X \otimes \mathbb{C}^n$, let A be defined by (1.1), and let $\tilde{A}$ be a linear operator in H satisfying (1.5). In addition, for a $\lambda \in \mathbb{C}$, let

$$q_\nu \gamma_\nu(\lambda)\Psi_n\big(w(m_0)\gamma_{m_0}(\lambda)\big) < 1.$$

Then λ is a regular point of $\tilde{A}$, and

$$\|(\tilde{A} - \lambda I_Y)^{-1}\|_H \le \frac{\gamma_0(\lambda)\Psi_n(w(m_0)\gamma_{m_0}(\lambda))}{1 - q_\nu\gamma_\nu\Psi_n(w(m_0)\gamma_{m_0}(\lambda))}.$$

4. Second-order differential operators in a Hilbert space

Let $X = L^2[0,1]$, and let $H = X \otimes Y$ with an arbitrary separable Hilbert space Y. Put $Su = u''$

$$\operatorname{Dom}(S) = \big\{u \in L^2[0,1] : u'' \in L^2[0,1]; u(0) = u(1) = 0\big\}, \tag{4.1}$$

and consider the operator

$$\tilde{A} = -\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x) \quad \big(x \in (0,1); \operatorname{Dom}(\tilde{A}) = \operatorname{Dom}(S) \otimes \mathbb{C}^n\big), \tag{4.2}$$

where $a_1(x), a_0(x)$ are continuous functions defined on $[0,1]$ whose values are bounded operators in Y. That is, the Dirichlet boundary conditions hold.

For example, let $Y = L^2(a,b)$, and let

$$(\tilde{A}u)(x,y) = -\frac{\partial^2 u(x,y)}{\partial x^2} + \int_a^b K_1(x,y,s)\frac{\partial u(x,s)}{\partial x}\,ds + \int_a^b K_0(x,y,s)u(x,s)\,ds$$

$(x \in (0,1); y \in (a,b))$ with the corresponding kernels K_0 and K_1.

Furthermore, take $A = S \otimes I_Y + I_X \otimes B_0$ with a constant operator B_0 satisfying the condition $\operatorname{Im} B_0 \in SN_{2p}$ for an integer $p \ge 1$. We have $\sigma(S) = \{(\pi k)^2 : k = 1, 2, \ldots\}$. So S is invertible and $\xi(S) = \pi^2$, and $\|S^{-1/2}\|_X = 1/\pi$. Let $e_k(x) = \sqrt{2}\sin(\pi k x)$. Taking $\nu = 1/2$, we obtain

$$S^{-1/2}h = \frac{1}{\pi}\sum_{k=1}^\infty \frac{1}{k}(h, e_k)_X e_k \quad (h \in X). \tag{4.3}$$

Hence,

$$\Big(\frac{d}{dx}S^{-1/2}h\Big)(x) = \sqrt{2}\sum_{k=1}^\infty \frac{1}{k}(h, e_k)_X \cos(\pi k x).$$

So

$$\left(\frac{d}{dx}S^{-1/2}h, v\right)_X = \sum_{k=1}^{\infty}(h, e_k)_X\left(\sqrt{2}\cos(\pi kx), v\right)_X \quad (h, v \in X)$$

and by the Schwarz inequality,

$$\left|\left(\frac{d}{dx}S^{-1/2}h, v\right)_X\right|^2 \le \sum_{k=1}^{\infty}|(h, e_k)_X|^2\sum_{k=1}^{\infty}\left|\left(\sqrt{2}\cos(\pi kx), v\right)_X\right|^2 \le \|h\|_X^2\|v\|_X^2.$$

Consequently,

$$\left\|\frac{d}{dx}S_0^{-1/2}\right\|_H \le 1.$$

Thus,

$$q_{1/2} = \|(A - \tilde{A})S_0^{-1/2}\|_H \le \hat{q}, \tag{4.4}$$

where

$$\hat{q} := \frac{1}{\pi}\sup_x \|a_0(x) - B_0\|_Y + \sup_x \|a_1(x)\|_Y.$$

In the considered case, $m_0 = 0$, $v(m_0) = b_p N_{2p}(\operatorname{Im} B_0)$, $\rho(B(s), \lambda) = \inf_{\mu\in\sigma(B_0)}|\pi^2k^2 + \mu - \lambda|$, and

$$\gamma_{1/2}(\lambda) = \sup_{k=1,2,\ldots}\frac{\pi k}{\rho(B(k), \lambda)} = \sup_{k=1,2,\ldots}\frac{\pi k}{\inf_{\mu\in\sigma(B_0)}|\pi^2k^2 + \mu - \lambda|}.$$

In addition, $\gamma_0(\lambda) = \frac{1}{\rho(A,\lambda)}$ with

$$\rho(A, \lambda) = \inf_{k=1,2,\ldots}\inf_{\mu\in\sigma(B_0)}|\pi^2k^2 + \mu - \lambda|.$$

Now Corollary 2.2 at once implies the following.

COROLLARY 4.1

Let $\operatorname{Im} B_0 \in SN_{2p}$ $(p = 1, 2, \ldots)$, $\lambda \notin \sigma(A)$, *and let*

$$\hat{q}\gamma_{1/2}(\lambda)\Phi_p\big(b_pN_{2p}(\operatorname{Im} B_0)/\rho(A, \lambda)\big) < 1.$$

Then λ *is regular for the operator* $\tilde{A}$ *defined by (4.2), and*

$$\|(\tilde{A}^{-1} - \lambda I_H)^{-1}\|_H \le \frac{\gamma_0(\lambda)\Phi_p(b_pN_{2p}(\operatorname{Im} B_0)/\rho(A, \lambda))}{1 - \hat{q}\gamma_{1/2}(\lambda)\Phi_p(b_pN_{2p}(\operatorname{Im} B_0)/\rho(A, \lambda))}.$$

5. Second-order matrix differential operator on a segment

Let $X = L^2[0, 1]$, and let $H = X \otimes \mathbb{C}^n := L^2([0, 1], \mathbb{C}^n)$. Define $\operatorname{Dom}(S)$ by (4.1), and consider the operator defined by (4.2) except that now $a_1(x), a_0(x)$ are bounded measurable $(n \times n)$-matrix-valued functions defined on $[0, 1]$. So again $Su = u''$ and the Dirichlet boundary conditions hold. Take $A = S \otimes I_{C^n} + I_X \otimes B_0$ with a constant $(n \times n)$-matrix B_0 having the eigenvalues $\lambda_j(B_0)$ $(j = 1, 2, \ldots, n)$.

For example, $B_0 = a_0(0)$. In the case considered, inequality (4.4) is also valid with $Y = \mathbb{C}^n$ and

$$\hat{q} := \frac{1}{\pi} \sup_x \|a_0(x) - B_0\|_{C^n} + \sup_x \|a_1(x)\|_{C^n}.$$

In addition, $m_0 = 0, w(m_0) = \sqrt{2} N_2(\operatorname{Im} B_0)$, $\rho(B(k), \lambda) = \min_{j=1,2,\ldots,n} |\pi^2 k^2 + \lambda_j(B_0) - \lambda|$, and

$$\gamma_{1/2}(\lambda) = \sup_{k=1,2,\ldots} \frac{\pi k}{\min_{j=1,\ldots,n} |\pi^2 k^2 + \lambda_j(B_0) - \lambda|}, \tag{5.1}$$

and $\gamma_0 = \frac{1}{\rho(A,\lambda)}$ with

$$\rho(A, \lambda) = \inf_{k=1,2,\ldots} \min_{j=1,\ldots,n} |k^2 + \lambda_j(B_0) - \lambda|. \tag{5.2}$$

Now from Corollary 3.2, at once we get our next result.

COROLLARY 5.1

Let $\tilde{A}$ be the operator defined by (4.2) with bounded measurable matrix coefficients, and let

$$\hat{q}\gamma_{1/2}(\lambda) \Psi_n\big(\sqrt{2} N_2(\operatorname{Im} B_0)/\rho(A, \lambda)\big) < 1.$$

Then λ is a regular point of $\tilde{A}$, and

$$\|(\tilde{A} - \lambda I_H)^{-1}\|_H \le \frac{\gamma_0(\lambda) \Psi_n(N_2(\sqrt{2} \operatorname{Im} B_0)/\rho(A, \lambda))}{1 - \hat{q}\gamma_{1/2}(\lambda) \Psi_n(\sqrt{2} N_2(\operatorname{Im} B_0)/\rho(A, \lambda))}.$$

We say that an operator C is stable *if* $\inf \operatorname{Re} \sigma(C) > 0$.

Assume that

$$\operatorname{Re} \sigma(B_0) \ge 0. \tag{5.3}$$

Then for any λ with $\operatorname{Re} \lambda \le 0$, by (5.1) and (5.2), we have

$$\rho(A, \lambda) \ge \pi^2 \quad \text{and} \quad \gamma_{1/2}(\lambda) \le \sup_{k=1,2,\ldots} \frac{1}{\pi k} = \frac{1}{\pi}.$$

If

$$\frac{\hat{q}}{\pi} \Psi_n\big(\sqrt{2} N_2(\operatorname{Im} B_0)/\pi^2\big) < 1, \tag{5.4}$$

then thanks to Corollary 5.1, the closed left half-plane is regular for $\tilde{A}$. We thus arrive at the following result.

COROLLARY 5.2

Under the hypothesis of Corollary 5.1, let conditions (5.3) and (5.4) hold. Then $\tilde{A}$ is a stable operator.

Under the hypothesis of the previous corollary, the stability means that the semigroup $e^{-\tilde{A}t}$ generated by $-\tilde{A}$ is exponentially stable.

6. Operator-valued functions

6.1. The Hirsch functional calculus

Let μ be a real nondecreasing function defined on $[0,\infty)$. Consider the function

$$h(z)=\int_0^\infty \frac{d\mu(t)}{z+t} \quad (z\notin(-\infty,0)),\ \text{assuming that}\ \int_0^\infty \frac{d\mu(t)}{\epsilon+t}<\infty \tag{6.1}$$

for any sufficiently small $\epsilon>0$. The Hirsch function $h(A)$ of an operator A acting in H is defined as

$$h(A)=\int_0^\infty (A+tI_H)^{-1}\,d\mu(t) \quad \big(\sigma(A)\cap(-\infty,0]=0\big), \tag{6.2}$$

provided that the integral converges in the sense of the strong topology. The important example of the Hirsch function is the fractional power

$$A^{-\tau}=\frac{\sin(\pi\nu)}{\pi}\int_0^\infty t^{-\tau}(A+It)^{-1}\,dt \quad (0<\tau<1).$$

For other examples, see [16, Section 4.1]. We restrict ourselves by the simple but important case

$$A=S_0+I_X\otimes B_0, \tag{6.3}$$

assuming that

$$c_0:=\inf\sigma(S)+\operatorname{Re}\sigma(B_0)>0. \tag{6.4}$$

Then

$$\rho\big(B(s),-t\big)\ge c_0+t \quad (t\ge 0).$$

By Theorem 2.1 with $\nu=m_0=0$ and $v_0=b_pN_{2p}(\operatorname{Im}B_0)$, we arrive at the following result.

COROLLARY 6.1

Let $H=X\otimes Y$, where Y is an arbitrary separable Hilbert space, let A be defined by (6.3), and let $h(A)$ be defined by (6.2). If the conditions (6.4) and $B_0\in SN_{2p}$ hold for an integer $p\ge 1$, then

$$\|h(A)\|_H\le\int_0^\infty \frac{1}{c_0+t}\Phi_p\Big(\frac{v_0}{c_0+t}\Big)\,d\mu(t). \tag{6.5}$$

Note that according to (6.1), the integral in (6.5) converges. In particular,

$$\|A^{-\tau}\|_H\le\frac{\sin(\pi\tau)}{\pi}\int_0^\infty \frac{t^{-\tau}}{c_0+t}\Phi_p\Big(\frac{v_0}{c_0+t}\Big)\,dt \quad (0<\tau<1).$$

In the case $Y=\mathbb{C}^n$, the previous corollary can be improved. Namely, by Theorem 3.1 with $\nu=m_0=0$ and $w_0=\sqrt{2}N_2(\operatorname{Im}B_0)$, we arrive at the following corollary.

COROLLARY 6.2

Let $H = X \otimes \mathbb{C}^n$, let A be defined by (6.3), and let $h(A)$ be defined by (6.2). If condition (6.4) holds, then

$$\|h(A)\|_H \le \int_0^\infty \frac{1}{c_0+t}\Psi_n\Big(\frac{w_0}{c_0+t}\Big)\,d\mu(t).$$

In particular,

$$\|A^{-\nu}\|_H \le \frac{\sin(\pi\nu)}{\pi}\int_0^\infty \frac{t^{-\nu}}{c_0+t}\Psi_n\Big(\frac{w_0}{c_0+t}\Big)\,dt.$$

6.2. Regular functions

Let A be defined by (6.3), and for all $s \in \sigma(S)$, let $f(z)$ be a function regular on a neighborhood of $\sigma(B_0 + sI_Y)$. Define $f(A)$ by

$$f(A) = \int_{\sigma(S)} f\big(B(s)\big)\otimes dE_s = \int_{\sigma(S)} f(B_0 + sI_Y)\otimes dE_s. \tag{6.6}$$

The regularity of $f(z)$ on $\sigma(B_0 + sI_Y)$ is equivalent to the regularity of $f(z+s)$ on $\sigma(B_0)$. In addition, repeating the arguments of the proof of Lemma 1.1, we obtain the inequality

$$\|S_0^\nu f(A)\|_H \le \sup_{s\in\sigma(S)} \|s^\nu f(B_0 + sI_Y)\|_Y. \tag{6.7}$$

Assume that

$$\operatorname{Im} B_0 \in SN_2, \tag{6.8}$$

and put $g_0 := \sqrt{2}N_2(\operatorname{Im} B_0)$.

THEOREM 6.3

Let A be defined by (6.3). Assume that condition (6.9) holds and that, for all $s \in \sigma(S)$, the function $f(z+s)$ is regular in z on the closed convex hall $co(B_0)$ of $\sigma(B_0)$ and that $f(A)$ is defined by (6.7). If for a $\nu \in [0,1)$ the condition

$$\theta_\nu(f,A) := \sup_{s\in\sigma(S)} \sum_{j=0}^{n-1} \sup_{z\in co(B_0)} |s^\nu f^{(j)}(z+s)|\frac{g_0^j}{(j!)^{3/2}} < \infty \tag{6.9}$$

holds, where $n = \dim rangeB_0 \le \infty$; then $\|S_0^\nu f(A)\|_H \le \theta_\nu(f,A)$.

Proof

We need the following result: for a bounded linear operator C acting in Y, assume that $\operatorname{Im} C \in SN_2$ and f are regular on the closed convex hall $co(C)$ of $\sigma(C)$. Then

$$\|f(C)\|_Y \le \sum_{j=0}^{m-1} \sup_{z\in co(C)} |f^{(j)}(z)|\frac{(\sqrt{2}N_2(\operatorname{Im} C))^j}{(j!)^{3/2}},$$

where $m = \dim rangeC$. For the proof, see [4, Theorem 7.10.1] and [4, Corollary 2.7.2]. But $N_2(\mathrm{Im}(B(s)) = N_2(\mathrm{Im}\, B_0)$. So

$$\left\|f\left(B(s)\right)\right\|_Y = \|f(sI_Y + B_0)\|_Y \le \sum_{j=0}^{n-1} \sup_{z\in co(B(s))} |f^{(j)}(z+s)| \frac{g_0^j}{(j!)^{3/2}}.$$

Hence $\sup_{s\in\sigma(S)} \|s^\nu f(B(s))\|_Y \le \theta_\nu(f, A)$. Now (6.8) implies the required result. $\square$

Theorem 6.2 is sharp. If B_0 is self-adjoint and $\sup_{z\in co(B(s))} |f(z)| = \sup_{z\in\sigma(B(s))} |f(z)|$, then we obtain the equality

$$\|S_0^\nu f(A)\|_H = \sup_{s\in\sigma(S), z\in\sigma(B_0)} |s^\nu f(z+s)|.$$

For example, take $f(z) = e^{-zt}$ $(t \ge 0)$, and assume that $\beta(B_0) := \inf \mathrm{Re}\, \sigma(B_0) > 0$, $\beta(S) = \inf \sigma(B_0) > 0$. Then $f^{(j)}(z+s) = e^{-(z+s)t}(-t)^j$, and

$$\sup_{z\in\sigma(B_0)} |e^{-zt}| = e^{-\beta(B_0)t}.$$

Put

$$\psi_\nu(t) := \sup_{s\in\sigma(S)} s^\nu e^{-st} = \begin{cases} \beta^\nu(S) e^{-\beta(S)t} & \text{if } 0t \ge \beta(S)/\nu, \\ e^{-\nu}(\nu/t)^\nu & \text{if } 0 \le t \le \beta(S)/\nu. \end{cases}$$

Then we obtain

$$\|S_0^\nu e^{-At}\|_H \le \psi_\nu(t) e^{-\beta(B_0)t} \Big(1 + \sum_{j=1}^{n-1} \frac{t^j g_0^j}{\sqrt{j!}}\Big) \quad (t \ge 0).$$

This result enables us to investigate the stability of parabolic equations.

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