Semigroups preserving a convex set in a Banach space

Ichiro Shigekawa

Abstract We discuss semigroups that preserve a convex set in a Banach space or a Hilbert space. We give sufficient conditions for which a semigroup preserves a convex set. Using this, we show that various issues can be treated in a unified way. We also discuss the problem in the Hilbert space setting, in which we use the sesquilinear form associated with a semigroup.

1. Introduction

Markovian properties or positivity-preserving properties are a fundamental notion in probability theory. There are many criteria for them. Among them, Brezis and Pazy [1] and Ouhabaz [7] gave a unified method in the framework of convex set–preserving properties. Let $\{T_t\}$ be a C_0-semigroup in a Hilbert space H. Suppose that we are given a convex closed set C. If $T_tC \subseteq C$ for all $t \geq 0$, we say that the $\{T_t\}$ preserves the convex set C or that C is stable under the semigroup $\{T_t\}$. Markovian property is characterized in this framework. In fact, taking $C = \{f; 0 \leq f \leq 1\}$, the semigroup $\{T_t\}$ is Markovian if and only if $\{T_t\}$ preserves C. The positivity-preserving property and others are also characterized in this framework.

Ouhabaz [7] and Brezis and Pazy [1] discussed this issue on Hilbert space. In this paper, we generalize it to the Banach space setting. To get a condition for the convex set–preserving property, the shortest points to C and the duality mapping play an important role. In Banach space, the set of shortest points is not a single point, and the duality mapping is multivalued in general. To get over this difficulty, we introduce the notion of good selection. Using this, we can give a condition for the convex set–preserving property.

The organization of the paper is as follows. In Section 2, we consider semigroups in a Banach space and give a condition for which the semigroup preserves a convex set. To do this, we introduce the notion of good selection. In Section 3, we give some examples of semigroups that preserve a convex set in a Banach space. We show that the following issues can be treated in a unified way:

(i) positivity-preserving property,

Kyoto Journal of Mathematics, Vol. 51, No. 3 (2011), 647–672
DOI 10.1215/21562261-1299918,
Received November 1, 2010. Accepted January 12, 2011.
2010 Mathematics Subject Classification: 47D06, 47D07, 60J45.

(ii) Markovian property,
(iii) L^1 contraction property,
(iv) excessive functions,
(v) invariant sets.

In Section 4, we deal with the same problem in the framework of the Hilbert space setting. We discuss it in terms of bilinear form. This approach was already adopted by Ouhabaz [7], but he assumed that the semigroups are contractive. We remove this restriction. We also give some sufficient condition for which the semigroup is contractive. We have to use different types of criteria to distinguish contractive semigroups and noncontractive ones.

2. Semigroups that preserve a convex set in a Banach space

Let B be a real or complex Banach space, and suppose that we are given a C_0-semigroup $\{T_t\}$. We emphasize that we do not assume that the semigroup is contractive. This is a main point of this paper. We denote the generator by $\mathfrak{A}$ and its domain by $\mathrm{Dom}(\mathfrak{A})$.

In this section, we consider the case when a semigroup preserves a closed convex set in a Banach space. So we are given a closed convex set C as well. We say that the semigroup $\{T_t\}$ preserves the closed convex set if $T_tC \subseteq C$ for all $t \geq 0$. A similar notion can be defined for any family of operators. We rewrite this condition in terms of resolvents. The resolvent $\{G_\alpha\}$ is defined as

$$G_\alpha = \int_0^\infty e^{-\alpha t} T_t \, dt. \tag{2.1}$$

Since we do not assume that $\{T_t\}$ is a contraction semigroup, we may need α to be large. The following conditions are equivalent to each other:

(i) $\{T_t\}$ preserves C,
(ii) $\{\alpha G_\alpha\}$ preserves C.

In fact, it suffices to notice (2.1) and

$$T_t x = \lim_{\alpha \to \infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n x.$$

Let us give examples. Suppose that B is a function space on E, for example, $B = L^p(E)$. Take $C = \{f \geq 0\}$, that is, a set of all nonnegative functions. Then the semigroup is called *positivity preserving* if it preserves the closed convex set C. If we take $C = \{0 \leq f \leq 1\}$, then the semigroup is called *Markovian* when it preserves C. It is easy to see that the following are equivalent to each other.

(i) $\{T_t\}$ preserves $\{0 \leq f \leq 1\}$.
(ii) $\{T_t\}$ preserves $\{f \leq 1\}$.

Many properties are formulated in this closed convex set–preserving property. We give other examples later.

We recall some notions. For a Banach space B, we denote its dual space by B^*. For any $x \in B$, we define

$$F(x) := \{\varphi \in B^*; \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}.$$

Here $\langle\ ,\ \rangle$ denotes the pairing of B and B^*. $F(x)$ is a (multivalued) function called the *duality map* of B. The following fact plays a fundamental role in the later argument. Take any $x \in B$. Then $\|x + \lambda y\| \geq \|x\|$ for small $\lambda > 0$ if and only if there exists $\varphi \in B^*$ such that

$$\Re\langle y, \varphi \rangle \geq 0$$

(see, e.g., Goldstein [3, Lemma I.3.4, p. 26]). Here $\Re$ stands for the real part.

Let C be a closed convex set. Take any $x \in B$. We denote by $P(x)$ the set of all shortest points from x to C. $P(x)$ is possibly an empty set or infinite set. We have the following.

PROPOSITION 2.1

Take any $x \in B$ and $z \in C$. Then, for any $y \in P(x)$, there exists $\varphi \in F(x - y)$ such that

$$\Re\langle z - y, \varphi \rangle \leq 0. \tag{2.2}$$

Proof

Let x, y, and z be as above. Since C is convex, $y + \lambda(z - y) \in C$ for any $\lambda \in [0, 1]$. The minimality of $\|x - y\|$ implies

$$\|x - y\| \leq \bigl\|x - \bigl(y + \lambda(z - y)\bigr)\bigr\| = \|x - y + \lambda(y - z)\|.$$

Now, by using the above remark, there exists $\varphi \in F(x - y)$ such that

$$\Re\langle y - z, \varphi \rangle \geq 0,$$

which shows (2.2). □

From now on, we *assume* that $P(x) \neq \emptyset$ for any $x \in B$.

THEOREM 2.2

Take any $\gamma \in \mathbb{R}$, and fix it. Assume that for any $x \in \mathrm{Dom}(\mathfrak{A})$, there exists $y \in P(x)$ such that for all $\varphi \in F(x - y)$,

$$\Re\langle \mathfrak{A}x, \varphi \rangle \leq \gamma\|x - y\|^2. \tag{2.3}$$

Then $\{T_t\}$ preserves C.

Conversely, assume that $\{T_t\}$ preserves C and, moreover, that $\{e^{-\gamma t}T_t\}$ is a contraction semigroup. Then, for any $x \in \mathrm{Dom}(\mathfrak{A})$ and $y \in P(x)$, there exists $\varphi \in F(x - y)$ such that (2.3) holds.

Proof

Assuming (2.3), we show that the resolvents preserve C. Take any $z \in C$, and set $x = \alpha G_\alpha z$. Note that $\mathfrak{A}x = \alpha(x - z)$. Now from the assumptions, we can take

$y \in P(x)$ so that

$$\Re\langle \mathfrak{A}x, \varphi\rangle \le \gamma\|x-y\|^2, \quad \forall \varphi \in F(x-y).$$

By Proposition 2.1, we can choose $\varphi \in F(x-y)$ so that

$$\Re\langle z-y, \varphi\rangle \le 0. \tag{2.4}$$

Then, for this φ, we have

$$\begin{aligned} 0 &\ge \Re\langle \mathfrak{A}x, \varphi\rangle - \gamma\|x-y\|^2 \\ &= \alpha\Re\langle x-z, \varphi\rangle - \gamma\|x-y\|^2 \\ &= \alpha\Re\langle x-y+y-z, \varphi\rangle - \gamma\|x-y\|^2 \\ &= (\alpha-\gamma)\|x-y\|^2 + \alpha\Re\langle y-z, \varphi\rangle \\ &\ge (\alpha-\gamma)\|x-y\|^2 \quad (\because (2.4)). \end{aligned}$$

By taking $\alpha > \gamma$, we get $x = y$, which means that $x \in C$.

Next we show the converse implication. So we assume that $\{T_t\}$ preserves C. In this case, we additionally assume the contraction property of $\{e^{-\gamma t}T_t\}$. Take any $x \in \mathrm{Dom}(\mathfrak{A})$ and $y \in P(x)$. Then $T_t y \in C$. From Proposition 2.1 we can take $\varphi_t \in F(x-y)$ so that

$$\langle T_t y - y, \varphi_t\rangle \le 0.$$

Using this, we have

$$\begin{aligned} \Re\langle T_t x - x, \varphi_t\rangle &= \Re\langle T_t(x-y) + (T_t y - y) + (y-x), \varphi_t\rangle \\ &\le \Re\langle T_t(x-y), \varphi_t\rangle - \|x-y\|^2 \\ &\le e^{\gamma t}\|x-y\|^2 - \|x-y\|^2 \qquad (\because e^{-\gamma t}T_t \text{ is contractive}) \\ &\le (e^{\gamma t}-1)\|x-y\|^2. \end{aligned}$$

We can take a sequence $\{t_n\}$ so that $\{\varphi_{t_n}\}$ converges *-weakly. Then

$$\Re\Big\langle \frac{T_{t_n}x - x}{t_n}, \varphi_{t_n}\Big\rangle \le \frac{e^{\gamma t_n}-1}{t_n}\|x-y\|^2.$$

Now letting $n \to \infty$, we have $\Re\langle \mathfrak{A}x, \varphi\rangle \le \gamma\|x-y\|^2$, which is the desired result. $\square$

In the sufficiency part, we assumed that $\varphi \in F(x-y)$ for all φ. In applications, this condition is rather difficult to check, so we give another formulation. To do this, we need the notion of good selection. For $x \in B$, we call $Q(x) \in P(x)$ and $G(x) \in F(x - Q(x))$ a good selection if

$$\Re\langle z - Q(x), G(x)\rangle \le 0, \quad \forall z \in C. \tag{2.5}$$

Of course, good selections do not always exist. We give examples of good selections later. In addition, when a good selection $(Q(x), G(x))$ can be taken for all $x \in B$, the function $x \mapsto (Q(x), G(x))$ is called a good selection function. The theorem above can be rewritten as follows by using this concept.

THEOREM 2.3

Suppose that $\gamma \in \mathbb{R}$ *is given. If there exists a good selection function* $(Q(x), G(x))$ *such that for all* $x \in \mathrm{Dom}(\mathfrak{A})$,

$$\Re\langle \mathfrak{A}x, G(x)\rangle \le \gamma \|x - Q(x)\|^2, \tag{2.6}$$

then the semigroup $\{T_t\}$ *preserves* C.

Conversely, assume that the semigroup $\{T_t\}$ *preserves* C. *We additionally assume that* $\{e^{-\gamma t}T_t\}$ *is a contraction semigroup. Then, for any good selection function* $(Q(x), G(x))$ *if it exists, (2.6) holds for all* $x \in \mathrm{Dom}(\mathfrak{A})$.

Proof

Assuming (2.6), we show that the resolvent preserves C. Take any $z \in C$, and set $x = \alpha G_\alpha z$. Then $\mathfrak{A}x = \alpha(x - z)$. Since $(Q(x), G(x))$ is a good selection, we have

$$\Re\langle z - Q(x), G(x)\rangle \le 0. \tag{2.7}$$

Hence

$$\begin{aligned} 0 &\ge \Re\langle \mathfrak{A}x, G(x)\rangle - \gamma\|x - Q(x)\|^2 \\ &= \alpha\Re\langle x - z, G(x)\rangle - \gamma\|x - Q(x)\|^2 \\ &= \alpha\Re\langle x - Q(x) + Q(x) - z, G(x)\rangle - \gamma\|x - Q(x)\|^2 \\ &= (\alpha - \gamma)\|x - Q(x)\|^2 + \alpha\Re\langle Q(x) - z, G(x)\rangle \\ &\ge (\alpha - \gamma)\|x - Q(x)\|^2 \quad (\because (2.7)). \end{aligned}$$

We can take $\alpha > \gamma$, and so $x = Q(x) \in C$ follows.

Conversely, assume that $\{T_t\}$ preserves C. Take any $x \in \mathrm{Dom}(\mathfrak{A})$ and any good selection $(Q(x), G(x))$. Then $Q(x) \in C$, and hence $T_tQ(x) \in C$ from the assumption. By (2.5), we have

$$\langle T_tQ(x) - Q(x), G(x)\rangle \le 0.$$

Using this, we have

$$\begin{aligned} \Re\langle T_tx - x, G(x)\rangle &= \Re\big\langle T_t\big(x - Q(x)\big) + \big(T_tQ(x) - Q(x)\big) + \big(Q(x) - x\big), G(x)\big\rangle \\ &\le \Re\big\langle T_t\big(x - Q(x)\big), G(x)\big\rangle - \|x - Q(x)\|^2 \\ &\le e^{\gamma t}\|x - Q(x)\|^2 - \|x - Q(x)\|^2 \qquad (\because e^{-\gamma t}T_t \text{ is contractive}) \\ &\le (e^{\gamma t} - 1)\|x - Q(x)\|^2. \end{aligned}$$

Dividing both sides by t and letting $t \to 0$, it follows that

$$\Re\langle \mathfrak{A}x, G(x)\rangle \le \gamma\|x - Q(x)\|^2,$$

which is what we want. □

Let us give some examples of good selection. Let E be a locally compact Hausdorff space with countable basis. We consider a Banach space $C_\infty(E)$, a set of all continuous functions that vanish at infinity. We take $\mathbb{R}$ as a scalar field. Let C

be a set of all nonnegative functions. C is a closed convex set in $C_\infty(E)$. We can easily find a good selection as follows. For any $f\in C_\infty(E)$, we can take $f_+=f\vee 0$ as an element of $P(f)$. Here $a\vee b=\max\{a,b\}$. Take any $\varphi\in F(f-f_+)$. Then φ is a nonpositive Radon measure with a support contained in $\{t\in E; f(t)=-\|f\|\}$. Therefore $\langle\varphi,f_+\rangle=0$ and

$$\Re\langle h-f_+,\varphi\rangle\le 0,\quad \forall h\in C.$$

This means that (f_+,φ) becomes a good selection for any $\varphi\in F(f-f_+)$.

Next, let (M,μ) be a measure space, and take $L^1(\mu)$ as a Banach space. Again C is a set of all nonnegative functions in $L^1(\mu)$. Then, for any $f\in L^1(\mu)$, $P(f)=\{f_+\}$. So P is a single-valued function. This time, define $\varphi\in L^\infty(\mu)$ as

$$\varphi(t)=\begin{cases}-\|f_-\|_1 & \text{if } f(t)<0,\\ 0 & \text{if } f(t)\ge 0.\end{cases}\tag{2.8}$$

Then $\varphi\in F(f-f_+)$, and it is easy to see that $\langle h-f_+,\varphi\rangle\le 0$ for any $h\in C$. This means that (f_+,φ) is a good selection.

If we assume a stronger assumption of a Banach space B, then the good selection function $(G(x),Q(x))$ is uniquely determined. In fact, let us assume that B is uniformly convex and that the dual space B^* is also uniformly convex. Then it is known that $P(x)$ and $F(x)$ are sets of a single point. Hence $Q(x)=P(x)$ and $G(x)=F(x-P(x))$ are uniquely determined, and $(Q(x),P(x))$ becomes a good selection. If B is a Hilbert space, then it is uniformly convex. In this case, $F(x)=x$ and (2.6) becomes $\Re\langle\mathfrak{A}x,x-P(x)\rangle\le\gamma\|x-P(x)\|^2$. This criterion is proved in Brezis and Pazy [1], whereas they formulated the theorem in terms of the subdifferential. So our result is a generalization to Banach space.

3. Examples of convex set–preserving semigroups

We give examples of semigroups which preserve a closed convex set. In this section, we always assume that Banach spaces are real. Applying Theorem 2.3, we see that many issues can be treated in a unified way. Changing a closed convex set, we can get various criteria. We treat the following issues:

(i) positivity-preserving property,
(ii) Markovian property,
(iii) L^1 contraction property,
(iv) excessive functions,
(v) invariant sets.

These are individually well discussed, but the point of this paper is that they can be treated in a unified way.

Suppose that we are given a semigroup $\{T_t\}$ generated by $\mathfrak{A}$ on a Banach space B. For any closed convex set C, we are interested in when $\{T_t\}$ preserves a closed convex set C.

3.1. Positivity-preserving property

We first discuss the positivity-preserving property. In this case, we take a closed convex set C as $C = \{f \geq 0\}$. In the case $B = C_\infty(E)$ with E being a locally compact Hausdorff space, we have the following.

THEOREM 3.1

Take $\gamma \in \mathbb{R}$. Assume that for any $f \in \mathrm{Dom}(\mathfrak{A})$ taking negative minimum, there exists a minimum point x_0 of f such that

$$(\mathfrak{A} - \gamma) f(x_0) \geq 0. \tag{3.1}$$

Then the semigroup $\{e^{-\gamma t} T_t\}$ is a positivity-preserving contraction semigroup.

Conversely, if $\{e^{-\gamma t} T_t\}$ is a positivity-preserving contraction semigroup, then (3.1) holds for any $f \in \mathrm{Dom}(\mathfrak{A})$ and the negative minimum point x_0 of f.

Proof

Take $f \in \mathrm{Dom}(\mathfrak{A})$ taking negative minimum. Let x_0 be the minimum point so that (3.1) holds. Then, defining $G(f) = -\delta_{x_0}$ and $Q(f) = f_+$, $(Q(f), G(f))$ becomes a good selection. We also have to consider the case when $f \in \mathrm{Dom}(\mathfrak{A})$ takes nonnegative minimum. In this case, $Q(f) = f_+ = f$ and so $f - Q(f) = 0$. This means that $G(f) = 0$ and $(Q(f), G(f))$ is also a good selection and satisfies (2.6) of Theorem 2.3.

Now we can apply Theorem 2.3 and get the desired result.

Taking $-f$ instead of f in (3.1), we can see that the opposite inequality of (3.1) holds at a point x_0 taking positive maximum of f. Combining both of them, we can show that $\mathfrak{A} - \gamma$ is dissipative, which implies the contraction property of the semigroup.

The reversed implication can also be shown by Theorem 2.3. □

We proceed to the case $B = L^1(\mu)$ on a measure space (M, μ). We have the following.

THEOREM 3.2

Let $\gamma \in \mathbb{R}$ be given. Then $\{e^{-\gamma t} T_t\}$ is a positivity-preserving contraction semigroup if and only if for any $f \in \mathrm{Dom}(\mathfrak{A})$, the following inequality holds:

$$-\int_{\{f<0\}} \mathfrak{A} f(t)\, d\mu(t) \leq \gamma \|f_-\|_1. \tag{3.2}$$

Proof

Sufficiency is easily shown by Theorem 2.3. We show the necessity. If we assume (3.2), then from Theorem 2.3 it follows that the semigroup preserves the positivity. Further, taking $-f$ instead of f in (3.2), we have

$$\int_{\{f>0\}} \mathfrak{A} f(t)\, d\mu(t) \leq \gamma \|f_+\|_1. \tag{3.3}$$

Note that $\varphi(t) = \operatorname{sgn} f(t) \in F(f)$. Here sgn is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{for } x < 0, \\ 0 & \text{for } x = 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Combining (3.2) and (3.3), we have

$$\int_M \mathfrak{A} f(t)\varphi(t)\, d\mu(t) \le \gamma \|f\|_1,$$

which means that $\mathfrak{A} - \gamma$ is dissipative. Therefore we have that $\{e^{-\gamma t}T_t\}$ is a contraction semigroup, as desired. $\square$

Under the same setting, let us proceed to the case $B = L^p(\mu)$, $1 < p < \infty$. In this case, B and B^* become uniformly convex, and the duality map F becomes a single-valued map given by $F(f) = |f|^{p-1} \operatorname{sgn} f / \|f\|_p^{p-2}$. Moreover, for any closed convex set C, $P(x)$ is uniquely determined. Concerning the positivity-preserving property, we should take $C = \{f \ge 0\}$ and, in this case, $P(f) = f_+$. Applying Theorem 2.3, we have the following.

THEOREM 3.3

Let $\gamma \in \mathbb{R}$ be given. Suppose that $\{T_t\}$ is a C_0-semigroup in L^p. Then the following three conditions are equivalent to each other.

(i) *$\{T_t\}$ preserves the positivity, and $\{e^{-\gamma t}T_t\}$ is a contraction semigroup.*

(ii) *For any $f \in \operatorname{Dom}(\mathfrak{A})$, we have*

$$\langle \mathfrak{A} f, f_-^{p-1} \rangle \ge -\gamma \|f_-\|_p^p. \tag{3.4}$$

(iii) *For any $f \in \operatorname{Dom}(\mathfrak{A})$, we have*

$$\langle \mathfrak{A} f, f_+^{p-1} \rangle \le \gamma \|f_+\|_p^p. \tag{3.5}$$

Proof

Noting that $f_- = (-f)_+$, we can easily see the equivalence of (ii) and (iii).

We show that (ii) $\Rightarrow$ (i). Since $f - Pf = f - f_+ = -f_-$, Theorem 2.3 implies that $\{T_t\}$ preserves $C = \{f \ge 0\}$. Moreover, since

$$\|f_-\|_p^p = \langle f_-, f_-^{p-1} \rangle = -\langle f, f_-^{p-1} \rangle,$$

(3.4) is equivalent to

$$\langle (\mathfrak{A} - \gamma) f, f_-^{p-1} \rangle \ge 0.$$

Similarly, taking $-f$ instead of f, we have

$$\langle (\mathfrak{A} - \gamma) f, f_+^{p-1} \rangle \le 0.$$

Combining both of them, we have

$$0 \ge \langle (\mathfrak{A} - \gamma) f, f_+^{p-1} - f_-^{p-1} \rangle = \langle (\mathfrak{A} - \gamma) f, |f|^{p-1} \operatorname{sgn} f \rangle.$$

This brings $\langle(\mathfrak{A}-\lambda)f, F(f)\rangle \leq 0$, and by the Lumer-Phillips theorem, we can show that $\mathfrak{A}-\lambda$ generates a contraction semigroup.

Next we show that (i) $\Rightarrow$ (ii). Since we assume that $\{e^{-t\lambda}T_t\}$ is a contraction semigroup, we can have the desired result by using Theorem 2.3. □

3.2. Markovian property

Next we discuss the Markovian property. A semigroup is called Markovian if it preserves the set $\{0 \leq f \leq 1\}$. This is equivalent to the fact that the semigroup preserves the set $\{f \leq 1\}$. Until the end of this subsection, we discuss mainly the case $C = \{f \leq 1\}$. In this case, the shortest point to C is given by

$$Q(f) = f \wedge 1. \tag{3.6}$$

Let us begin with the case $B = C_\infty(E)$, where E is a locally compact Hausdorff space. It is clear that the semigroup is Markovian if and only if it is a positivity-preserving contraction semigroup. So the result is included in the positivity-preserving case. That is, if for any $f \in \mathrm{Dom}(\mathfrak{A})$ we have

$$\mathfrak{A}f(x_0) \geq 0 \tag{3.7}$$

at some point x_0 taking the negative minimum of f, then the semigroup is Markovian. Conversely, if the semigroup is Markovian, then for any $f \in \mathrm{Dom}(\mathfrak{A})$, (3.7) holds at any point x_0 taking the negative minimum.

The second case is $B = L^1(\mu)$ on a measure space (M, μ). Since $Q(f) = f \wedge 1$, we have $f - Q(f) = (f-1)_+$. So we can take a $\varphi \in F((f-1)_+)$ as

$$\varphi(u) = \|(f-1)_+\|_1 1_{\{f>1\}}(u).$$

If we take this φ, then, for any $h \in C$, we have

$$\begin{aligned}\langle h - f\wedge 1, \varphi\rangle &= \int_{\{f>1\}} \|(f-1)_+\|_1 (h - f\wedge 1)\, d\mu \\ &= \|(f-1)_+\|_1 \int_{\{f>1\}} (h-1)\, d\mu \\ &\leq 0 \quad (\because h \leq 1).\end{aligned}$$

Thus $Q(f) = (f-1)_+$, $G(f) = \varphi = \|(f-1)_+\|_1 1_{\{f>1\}}$ becomes a good selection. Hence by using Theorem 2.3, we have the following.

THEOREM 3.4

Let $\gamma \in \mathbb{R}$ be given. Then the following are equivalent to each other.

(i) *$\{T_t\}$ is Markovian, and $\{e^{-\gamma t}T_t\}$ is a contraction semigroup.*
(ii) *For any $f \in \mathrm{Dom}(\mathfrak{A})$, we have*

$$\int_{\{f>1\}} \mathfrak{A}f(u)\, d\mu(u) \leq \gamma \|(f-1)_+\|_1. \tag{3.8}$$

(iii) *For any* $f \in \mathrm{Dom}(\mathfrak{A})$, *we have*

$$\int_{\{f>1\}} \mathfrak{A}f(u)\,d\mu(u) - \int_{\{f<0\}} \mathfrak{A}f(u)\,d\mu(u) \le \gamma \|f - f_+ \wedge 1\|_1. \tag{3.9}$$

Proof

It is enough to use Theorem 2.3. We see only that the contraction property can be deduced from (3.8). Take any constant $c>0$. Taking f/c instead of f in (3.8) and noting that $(f/c-1)_+ = 1/c(f-c)_+$,

$$\int_{\{f>c\}} \mathfrak{A}f(u)\,d\mu(u) \le \gamma \|(f-c)_+\|_1,$$

which bears, by letting $c \downarrow 0$,

$$\int_{\{f>0\}} \mathfrak{A}f(u)\,d\mu(u) \le \gamma \|f_+\|_1.$$

This is equivalent to (3.2) in Theorem 3.2. So the contraction property can be shown in the same way as in Theorem 3.2. □

We can do the same thing in the case $B = L^p(\mu)$ $1<p<\infty$.

THEOREM 3.5

Let $\gamma \in \mathbb{R}$ *be given, and let* $\{T_t\}$ *be a semigroup in* L^p. *Then the following three conditions are equivalent to each other.*

(i) $\{T_t\}$ *is Markovian and* $\{e^{-\gamma t}T_t\}$ *is a contraction semigroup.*

(ii) *For any* $x \in \mathrm{Dom}(\mathfrak{A})$, *we have*

$$\langle \mathfrak{A}f, (f-1)_+^{p-1}\rangle \le \gamma \|(f-1)_+\|_p^p. \tag{3.10}$$

If we replace (3.10) with

$$\langle \mathfrak{A}f, |f - f_+ \wedge 1|^{p-1} \operatorname{sgn} f\rangle \le \gamma \|f - f_+ \wedge 1\|_p^p, \tag{3.11}$$

the same conclusion holds.

Proof

We first show that (ii) $\Rightarrow$ (i). Assume (3.10). Then, by Theorem 2.3, it follows that $\{T_t\}$ preserves $\{f \le 1\}$, and hence $\{T_t\}$ is Markovian.

We now take any $c>0$ and substitute f/c in (3.10). By noting that $(f/c-1)_+ = 1/c(f-c)_+$, we have

$$\langle \mathfrak{A}f, (f-c)_+^{p-1}\rangle \le \gamma \|(f-c)_+\|_p^p.$$

Now, letting $c \to 0$, we can see that (3.5) holds, and hence $\{T_t\}$ becomes a contraction semigroup by Theorem 2.3.

To show that (i) $\Rightarrow$ (ii), we just apply Theorem 2.3.

A similar result holds when we assume (3.11). □

When $\gamma = 0$, the above result was discussed in [4, Section 4.6] and [2]. The contraction property is assumed there, but it is not necessary. In the L^2-case, Ma and Röckner [6] called an operator satisfying (3.10) with $\gamma = 0$ a *Dirichlet* operator. By this definition, the operator generates a contraction semigroup. It may be better to remove the restriction of contraction.

3.3. L^1-contraction property

The Markovian property is equivalent to L^∞-contraction and positivity preserving. Its dual notion is L^1-contraction and positivity preserving. We discuss it here.

Let (M, μ) be a measure space. We take a Banach space B as $L^p(\mu)$ $(p \in [1, \infty))$. The L^1-contraction property means that for any $f \in B$,

$$\int_M |T_t f| \, d\mu \le \int_M |f| \, d\mu. \tag{3.12}$$

In addition, if we assume the positivity preserving, then the above property is equivalent to (3.12) with nonnegative f. So we take a convex set C as

$$C = \left\{ f; f \ge 0, \int_M f \, d\mu \le 1 \right\}. \tag{3.13}$$

C is clearly closed. The semigroup is L^1-contractive and positivity preserving if and only if it preserves C. To see this, assume that the semigroup preserves C. Take any $f \ge 0$. Then, for any $\varepsilon > 0$, we can find $\delta > 0$ so that $(f - \varepsilon)_+ \delta \in C$. Since the semigroup preserves C, we have $T_t (f - \varepsilon)_+ \ge 0$. Letting $\varepsilon \to 0$, we are lead to $T_t f \ge 0$, which means the positivity preserving. Now the L^1-contraction property easily follows. The converse is much easier.

We need to get the shortest point to C. We show that it is given by $(f - c)_+$, where $c \ge 0$ is chosen so that

$$\int_M (f - c)_+ \, d\mu = 1,$$

whereas, when

$$\int_M f_+ \, d\mu \le 1,$$

then we set $c = 0$. If f satisfies

$$\int_M f_+ \, d\mu \le 1,$$

then it is clear that f_+ is the shortest point. If f does not satisfy the condition above, we need the following.

PROPOSITION 3.6

Suppose that f, $g \in L^p(\mu)$ $(p > 1)$ satisfy $0 \le g \le f$ and that a constant $c > 0$ satisfies

$$\int_M (f - c)_+ \, d\mu = \int_M (f - g) \, d\mu. \tag{3.14}$$

Then we have

$$\int_M (f \wedge c)^p \, d\mu \le \int_M g^p \, d\mu, \tag{3.15}$$

and the identity holds only when $g = f \wedge c$.

Proof
Note that

$$\int_M f^p \, d\mu = \frac{1}{p} \int_0^\infty x^{p-1} \mu(f \ge x) \, dx.$$

Using this, (3.14) can be rewritten as

$$\int_0^\infty \bigl\{\mu(f \ge x) - \mu(g \ge x)\bigr\} \, dx = \int_c^\infty \mu(f \ge x) \, dx.$$

Hence we have

$$\int_0^c \bigl\{\mu(f \ge x) - \mu(g \ge x)\bigr\} \, dx = \int_c^\infty \mu(g \ge x) \, dx.$$

Therefore

$$\begin{aligned}\int_0^c \Bigl(\frac{x}{c}\Bigr)^{p-1} \bigl\{\mu(f \ge x) - \mu(g \ge x)\bigr\} \, dx &\le \int_0^c \bigl\{\mu(f \ge x) - \mu(g \ge x)\bigr\} \, dx \\ &= \int_c^\infty \mu(g \ge x) \, dx \\ &\le \int_c^\infty \Bigl(\frac{x}{c}\Bigr)^{p-1} \mu(g \ge x) \, dx.\end{aligned}$$

Multiplying c^{p-1} to both hands, we have

$$\int_0^c x^{p-1} \bigl\{\mu(f \ge x) - \mu(g \ge x)\bigr\} \, dx \le \int_c^\infty x^{p-1} \mu(g \ge x) \, dx,$$

which is the desired result. If the equality holds in the above equation, then all the inequalities above must be equalities, and hence $g = f \wedge c$ would hold. □

The result above holds even when $p = 1$ and the last inequality should be equality, whereas we cannot have $g = f \wedge c$ in general.

Using the result above, we can see that the shortest point is given by $(f - c)_+$. Now, if we assume that the semigroup preserves C and that $\{e^{-\gamma t} T_t\}$ is a contraction semigroup, then by Theorem 2.3, we have

$$\int_M \mathfrak{A} f \operatorname{sgn}(f) |f \wedge c|^{p-1} \, d\mu \le \gamma \|f \wedge c\|_p^p. \tag{3.16}$$

Here c is a constant that satisfies

$$\int_M (f - c)_+ \, d\mu = 1.$$

But the constant $c \geq 0$ can be arbitrary. In fact, for any $\lambda > 0$, take λf instead of f. Then (3.16) becomes

$$\int_M \lambda \mathfrak{A} f \operatorname{sgn}(f)\big(\lambda f \wedge c(\lambda f)\big)^{p-1}\, d\mu \leq \gamma \|\lambda f \wedge c(\lambda f)\|_p^p,$$

which leads to

$$\int_M \mathfrak{A} f \operatorname{sgn}(f)\Big(f \wedge \frac{c(\lambda f)}{\lambda}\Big)^{p-1}\, d\mu \leq \gamma \|f \wedge \frac{c(\lambda f)}{\lambda}\|_p^p.$$

The constant $c(\lambda f)/\lambda$ is characterized by

$$\int_M \Big(f - \frac{c(\lambda f)}{\lambda}\Big)_+ d\mu = \frac{1}{\lambda}.$$

So it can take all positive values by varying $\lambda > 0$. We take $c = 1$ for simplicity. Thus we have the following theorem.

THEOREM 3.7

Let $\gamma \in \mathbb{R}$ be given. Then the following conditions are equivalent to each other.

(i) *$\{T_t\}$ preserves C and $\{e^{-\gamma t} T_t\}$ is contractive.*

(ii) *For any $f \in \operatorname{Dom}(\mathfrak{A})$, we have*

$$\int_M \mathfrak{A} f \operatorname{sgn}(f) |f \wedge 1|^{p-1}\, d\mu \leq \gamma \|f \wedge 1\|_p^p. \tag{3.17}$$

3.4. Excessive functions

We show that we can deal with excessive functions in our framework. The excessive functions are defined as follows. If a nonnegative function u satisfies

$$e^{-\alpha t} T_t u \leq u \tag{3.18}$$

for any $t \geq 0$, then we call it an α-excessive function. In the sequel, we always assume that u is nonnegative. We usually assume that the $\{T_t\}$ is Markovian, but we do not need this. Being excessive is a property of a function. But we change the viewpoint. It can be thought to be a property of the semigroup. We take this viewpoint. We assume that $\{T_t\}$ is positivity preserving, and we define a convex set C by $C = \{f; f \leq u\}$. Then it is easily verified that u is α-excessive if and only if $\{e^{-\alpha t} T_t\}$ preserves C. So we can apply our theorem. We note that for any f, the shortest point to C is given by $P(f) = f \wedge u$.

We start with the case when E is locally compact and $B = C_\infty(E)$. For any f, we can take $Q(f) = f \wedge u \in P(f)$. Then

$$f - Q(f) = (f - u)_+.$$

Assume that $(f - u)_+ \neq 0$, and take any maximum point x_0 of $(f - u)_+$. Now define

$$\varphi = \|(f - u)_+\|_\infty \delta_{x_0}.$$

Then, for any $h \in C$, we have

$$\langle h - (f \wedge u), \varphi\rangle = \big(h(x_0) - u(x_0)\big)\|(f - u)_+\|_\infty \leq 0.$$

This means that $f \wedge u$, $\|(f-u)_+\|_\infty \delta_{x_0}$ is a good selection. So we can get the following.

THEOREM 3.8

Let $\gamma \in \mathbb{R}$ be given. Assume that u is α-excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is contractive. Then for any $f \in \mathrm{Dom}(\mathfrak{A})$ and any x_0 taking positive maximum of $(f-u)_+$, we have

$$(\mathfrak{A}-\alpha)f(x_0) \le \gamma\bigl(f(x_0)-u(x_0)\bigr). \tag{3.19}$$

Conversely, assume that for any $f \in \mathrm{Dom}(\mathfrak{A})$, (3.19) holds at some point x_0 taking positive maximum of $(f-u)_+$. Then u is excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is Markovian.

Proof

The first part follows from Theorem 2.3.

In the converse part, we show only that $\{e^{-(\alpha+\gamma)t}T_t\}$ is Markovian. Take any $f \in \mathrm{Dom}(\mathfrak{A})$, and let x_0 be a point taking the positive maximum of $(f-u)_+$. Then we have

$$(\mathfrak{A}-\alpha)f(x_0) \le \gamma\bigl(f(x_0)-u(x_0)\bigr).$$

Now, for any $c>0$, take f/c instead of f. Note that $(f/c-u)_+ = (1/c)(f-cu)_+$. Therefore we have

$$(\mathfrak{A}-\alpha)f(x_c) \le \gamma\bigl(f(x_c)-cu(x_c)\bigr). \tag{3.20}$$

Here x_c is a point taking the positive maximum of $(f-cu)_+$. When $c \downarrow 0$, we can take a convergent subsequence from x_c. We set the limit by y_0. It is clear that f_+ takes the positive maximum at y_0. Moreover, taking the limit in (3.20) along a subsequence, we have

$$(\mathfrak{A}-\alpha)f(y_0) \le \gamma f(y_0).$$

The Markovian property of $\{e^{-(\alpha+\gamma)t}T_t\}$ follows from this. □

We proceed to the case $B = L^1(\mu)$ with (M,μ) a measure space. For $C = \{f : f \le u\}$, $Q(f) = f \wedge u$. Moreover, we can take $\varphi = \|(f-u)_+\|_1 1_{\{f>u\}}$ as an element of $F(f-Q(f)) = F((f-u)_+)$; $(f \wedge u, \varphi)$ is a good selection. In fact, for any $h \in C$, we have

$$\begin{aligned}\langle h-Q(f),\varphi\rangle &= \|(f-u)_+\|_1 \int_{\{f>u\}} (h - f\wedge u)\, d\mu \\ &= \|(f-u)_+\|_1 \int_{\{f>u\}} (h-u)\, d\mu \le 0.\end{aligned}$$

Now we have the following theorem.

THEOREM 3.9

Let $\gamma \in \mathbb{R}$ be given. Then the following conditions are equivalent to each other:

(i) *u is α-excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is a contraction semigroup;*

(ii) *for any $f \in \mathrm{Dom}(\mathfrak{A})$, we have*

$$\int_{\{f>u\}} (\mathfrak{A}-\alpha)f\,d\mu \le \gamma\|(f-u)_+\|_1. \tag{3.21}$$

Proof

The implication that (i) $\Rightarrow$ (ii) follows from Theorem 2.3.

We show only that (ii) implies that $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity-preserving contraction semigroup. For any $c>0$, take f/c instead of f. Then (3.21) is written as

$$\int_{\{f>cu\}} (\mathfrak{A}-\alpha)f\,d\mu \le \gamma\|(f-cu)_+\|_1.$$

Now letting $c \downarrow 0$, we have

$$\int_{\{f>0\}} (\mathfrak{A}-\alpha)f\,d\mu \le \gamma\|f_+\|_1.$$

By using Theorem 3.2, we can show that $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity-preserving contraction semigroup. □

Last, we discuss the case L^p $(1<p<\infty)$.

THEOREM 3.10

Let $\gamma \in \mathbb{R}$ be given. Then the following are equivalent to each other:

(i) *u is α-excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity-preserving contraction semigroup;*

(ii) *for $f \in \mathrm{Dom}(\mathfrak{A})$, we have*

$$\langle(\mathfrak{A}-\alpha)f,(f-u)_+^{p-1}\rangle \le \gamma\|(f-u)_+\|_p^p. \tag{3.22}$$

Proof

We show only that (ii) implies that the semigroup is positivity preserving and contractive. So take f/c instead of f in (3.22). Then

$$\langle(\mathfrak{A}-\alpha)f,(f-cu)_+^{p-1}\rangle \le \gamma\|(f-cu)_+\|_p^p.$$

Letting $c \downarrow 0$, we have

$$\langle(\mathfrak{A}-\alpha)f,f_+^{p-1}\rangle \le \gamma\|f_+\|_p^p.$$

Applying Theorem 3.3, we can get the desired result. □

3.5. Invariant sets

A measurable set $D \subseteq E$ is called *weakly invariant* if for any $t \ge 0$,

$$1_{D^c}T_t1_D = 0. \tag{3.23}$$

We want to give a characterization of a weakly invariant set. To do this, define a convex set C by

$$C = \{f; 1_{D^c} f = 0\}. \tag{3.24}$$

In this case, we define $Q(f)$ by

$$Q(f) = 1_D f.$$

Hence we have $f - Q(f) = 1_{D^c} f$.

We first consider the case $B = C_\infty(E)$, where E is a locally compact Hausdorff space. We assume that D is open and closed. Let x_0 be a point where $|f - Q(f)|$ takes its positive maximum. Set $\varphi = \|1_{D^c} f\|_\infty \operatorname{sgn}(f(x_0))\delta_{x_0}$. Then $\varphi \in F(f - Q(f))$, and for any $h \in C$,

$$\langle h - 1_D f, \varphi\rangle = \big(h(x_0) - f(x_0)\big)\|1_{D^c} f\|_\infty \operatorname{sgn}\big(f(x_0)\big) = -\|1_{D^c} f\|_\infty |f(x_0)| \le 0.$$

Thus $(Q(f), \varphi)$ is a good selection. Now the following theorem can be obtained from Theorem 2.3.

THEOREM 3.11

Let $\gamma \in \mathbb{R}$ be given. Assume that for any $f \in \operatorname{Dom}(\mathfrak{A})$, we have

$$\mathfrak{A} f(x_0) \operatorname{sgn} f(x_0) \le \gamma |f(x_0)| \tag{3.25}$$

at some x_0 taking the positive maximum of $|f|$ in D^c. Then D is a weakly invariant set.

Conversely, if D is a weakly invariant set and the $\{e^{-\gamma t} T_t\}$ is contractive, then for any $f \in \operatorname{Dom}(\mathfrak{A})$, (3.25) holds for all x_0, where $|f|$ takes positive maximum in D.

In the case $B = L^1(\mu)$ with (M, μ) a measure space, we can take $\varphi = 1_{D^c} \operatorname{sgn} f \|1_{D^c} f\|$ from $F(1_{D^c} f)$. To see that $(Q(f), \varphi)$ is a good selection, note that for any $h \in C$,

$$\langle h - 1_D f, \varphi\rangle = \int_{D^c} (h - 1_D f)\|1_{D^c} f\|_1 \operatorname{sgn} f \, d\mu \le 0.$$

From Theorem 2.3, we can have the following.

THEOREM 3.12

Let $\gamma \in \mathbb{R}$ be given. Assume that for any $f \in \operatorname{Dom}(\mathfrak{A})$,

$$\int_{D^c} \mathfrak{A} f \operatorname{sgn}(f)\, d\mu \le \gamma \|1_{D^c} f\|_1. \tag{3.26}$$

Then D is a weakly invariant set.

Conversely, if D is weakly invariant and $\{e^{-\gamma t} T_t\}$ is a contraction semigroup, then (3.26) holds for any $f \in \operatorname{Dom}(\mathfrak{A})$.

Similarly, we have the following theorem in the case when $B = L^p(\mu)$ $(1 < p < \infty)$.

THEOREM 3.13

Let γ be given. Assume that for any $f \in \mathrm{Dom}(\mathfrak{A})$,

$$\langle \mathfrak{A} f, 1_{D^c}|f|^{p-1} \operatorname{sgn} f\rangle \leq \gamma \|1_{D^c} f\|_p^p. \tag{3.27}$$

Then D is a weakly invariant set.

Conversely, if D is weakly invariant and the semigroup $\{e^{-\gamma t}T_t\}$ is contractive, then (3.27) holds for any $f \in \mathrm{Dom}(\mathfrak{A})$.

4. Convex set–preserving semigroups in Hilbert space

In this section, we consider conditions for which a semigroup in a Hilbert space preserves a convex set. Of course, Hilbert spaces are Banach spaces, so the previous result in Section 2 holds. In the Hilbert space case, we consider a semigroup associated with a sesquilinear form. We describe conditions in terms of sesquilinear forms. This kind of problem was discussed by Ouhabaz [7], but he always assumed that semigroups are contractive. Our aim here is to remove the restriction of the contraction property. We mainly follow his argument, but sometimes we need modifications.

4.1. Convex set–preserving property

Let a complex or a real Hilbert space H be given. We denote its inner product by $(\ |\ )_H$ and the norm by $|\cdot|$. Suppose that we are given a closed sesquilinear form $\mathcal{E}$. For any $\gamma \in \mathbb{R}$, we define $\mathcal{E}_\gamma$ by

$$\mathcal{E}_\gamma(x,y) = \mathcal{E}(x,y) + \gamma(x \mid y)_H.$$

We assume that $\mathcal{E}$ is bounded from below and satisfies the sector condition: there exist constants ξ and K such that

$$\mathcal{E}_\xi(x,x) \geq 0,$$
$$\mathcal{E}_\xi(x,y) \leq K\mathcal{E}_{\xi+1}(x,x)^{1/2}\mathcal{E}_{\xi+1}(y,y)^{1/2}.$$

We denote the associated semigroup and the generator by $\{T_t\}$ and $\mathfrak{A}$, respectively. We also denote the resolvent by G_α. G_α is defined at least for $\alpha > \xi$. Let a closed convex set C be given. As before, we denote the shortest point from x to C by Px. Since H is uniformly convex, Px is a single-valued function, and the duality map $F(x)$ is just $F(x) = x$. As was mentioned in Section 3, $(Px, x - Px)$ is a good selection; that is, we have, for any $y \in C$,

$$\Re(y - Px \mid x - Px)_H \leq 0. \tag{4.1}$$

Now we show the following.

THEOREM 4.1

Let $\gamma \in \mathbb{R}$ and $\theta \in [0,1]$ be given. We consider the following conditions.

(i) *For any $x \in \mathrm{Dom}(\mathcal{E})$, we have $Px \in \mathrm{Dom}(\mathcal{E})$ and*

$$\Re\mathcal{E}\big((1-\theta)x + \theta Px, x - Px\big) \geq -(1-\theta)\gamma|x - Px|^2, \quad \forall x \in \mathrm{Dom}(\mathcal{E}). \tag{4.2}$$

(ii) *The semigroup* $\{T_t\}$ *preserves* C.

(iii) *For any* $x \in \mathrm{Dom}(\mathcal{E})$, *we have* $Px \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\Re\mathcal{E}(Px, x - Px) \geq 0, \quad \forall x \in \mathrm{Dom}(\mathcal{E}). \tag{4.3}$$

Then the implications that (i) ⇒ (ii) ⇒ (iii) hold; (iii) is nothing but (i) with $\theta = 1$, *so (ii) and (iii) are equivalent to each other.*

If, in addition, $\{e^{-\gamma t}T_t\}$ *is contractive, then the three conditions are equivalent and, moreover, they are equivalent to the following condition (iv).*

(iv) *For any* $\eta \in [0,1]$ *and for any* $x \in \mathrm{Dom}(\mathcal{E})$, *we have* $Px \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\Re\mathcal{E}\big((1-\eta)x + \eta Px, x - Px\big) \geq -(1-\eta)\gamma|u - Pu|^2, \quad \forall x \in \mathrm{Dom}(\mathcal{E}). \tag{4.4}$$

If $\mathcal{E}$ *is Hermitian (we do not assume that* $e^{-\gamma t}T_t$ *is contractive), then (ii) follows from the following condition (v).*

(v) *For any* $x \in \mathrm{Dom}(\mathcal{E})$, *we have* $Px \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(Px, Px) \leq \mathcal{E}(x, x) + \gamma|x - Px|^2, \quad \forall x \in \mathrm{Dom}(\mathcal{E}). \tag{4.5}$$

If we assume that $\{e^{-\gamma t}T_t\}$ *is contractive in addition to the Hermitian property, then all five conditions are equivalent to each other.*

Proof

We first show that (i) ⇒ (ii). It suffices to show that $\alpha G_\alpha x \in C$ for any $x \in C$. Set $y = \alpha G_\alpha x$. Since $\mathcal{E}(G_\alpha x, z) = (x - \alpha G_\alpha x \mid z)_H$, we have

$$\mathcal{E}(y, z) = \alpha(x - y \mid z)_H. \tag{4.6}$$

From (i), (4.2) holds, whereas we take a larger γ. So we may assume that $\mathcal{E}_\gamma(f, f) \geq 0$ for all f. Moreover, take α such that $\alpha \geq \gamma$. Then

$$\begin{aligned}
0 &\geq -\Re\mathcal{E}\big((1-\theta)y + \theta Py, y - Py\big) - (1-\theta)\gamma|u - Pu|^2 - \theta\Re\mathcal{E}_\gamma(y - Py, y - Py) \\
&= -\Re\mathcal{E}(y, y - Py) + \theta\Re\mathcal{E}(y - Py, y - Py) - (1-\theta)\gamma|y - Pu|^2 \\
&\quad - \theta\Re\mathcal{E}(y - Py, y - Py) - \theta\gamma|y - Py|^2 \\
&= -\Re\mathcal{E}(y, y - Py) - \theta\gamma|y - Py|^2 \\
&= -\alpha\Re(x - y \mid y - Py)_H - \gamma|y - Py|^2 \quad (\because (4.6)) \\
&= -\alpha\Re(x - Py \mid y - Py)_H + \alpha\Re(y - Py \mid y - Py)_H - \gamma|y - Py|^2 \\
&\geq (\alpha - \gamma)|y - Py|^2 \quad (\because (4.2)),
\end{aligned}$$

which leads to $y = Py \in C$, as we wanted.

Ouhabaz [7, Theorem 2.1] proved that (ii) ⇒ (iii).

Assuming that $\{e^{-\gamma t}T_t\}$ is contractive, that is, that $\mathcal{E}_\gamma(f, f) \geq 0$ for any $f \in \mathrm{Dom}(\mathcal{E})$, let us show that (iii) ⇒ (iv). Since $\mathcal{E}_\gamma$ is nonnegative,

$$\begin{aligned}
0 &\leq \Re\mathcal{E}(Px, x - Px) + (1-\eta)\mathcal{E}_\gamma(x - Px, x - Px) \\
&= \Re\mathcal{E}\big((1-\eta)x - \eta P, x - Px\big) + (1-\eta)\gamma|x - Px|^2,
\end{aligned}$$

which is the desired result.

If $\mathcal{E}$ is Hermitian, then we have

$$\begin{aligned}
&\Re\mathcal{E}(x+Px, x-Px)+\gamma|x-Px|^2 \\
&\quad=\mathcal{E}(x,x)+\Re\{\mathcal{E}(Px,x)-\mathcal{E}(x,Px)\}-\mathcal{E}(Px,Px)+\gamma|x-Px|^2 \\
&\quad=\mathcal{E}(x,x)+\Re\{\mathcal{E}(Px,x)-\overline{\mathcal{E}(Px,x)}\}-\mathcal{E}(Px,Px)+\gamma|x-Px|^2 \\
&\quad=\mathcal{E}(x,x)-\mathcal{E}(Px,Px)+\gamma|x-Px|^2.
\end{aligned}$$

The left-hand side is (4.2) with $\theta=1/2$, and so (ii) follows from (v). If, in addition, we assume the positivity of $\mathcal{E}_\gamma$, it is easy to see that all conditions are equivalent to each other. □

Now we discuss examples. We take a measure space (M,m) and consider a real Hilbert $H=L^2(m)$. We denote the inner product by $(\ |\ )_2$ and the norm by $\|\ \|_2$. So far, we denote the Hilbert norm by $|\ |$, but we reserve it for the absolute value of a function. Elements of L^2 are denoted by f, g, $h,\ldots$. Now we proceed to individual cases, as in Section 3.

4.2. Positivity-preserving property

The convex set is given as $C=\{f\geq 0\}$; recall that $Pf=f_+$. From Theorem 4.1, we have the following theorem.

THEOREM 4.2

The following two conditions are equivalent to each other.

(i) $\{T_t\}$ *is positivity preserving.*

(ii) *For any* $f\in\operatorname{Dom}(\mathcal{E})$*, we have* $|f|\in\operatorname{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(f_+,f_-)\leq 0. \tag{4.7}$$

Under (i) or (ii), the following condition (iii) holds.

(iii) *For any* $f\in\operatorname{Dom}(\mathcal{E})$*, we have* $|f|\in\operatorname{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(|f|,|f|)\leq\mathcal{E}(f,f). \tag{4.8}$$

If $\mathcal{E}$ *is symmetric, the three conditions are equivalent to each other.*

Proof

The equivalence between (i) and (ii) is a direct consequence from Theorem 4.1. On the other hand,

$$\begin{aligned}
\mathcal{E}(|f|,|f|)-\mathcal{E}(f,f)&=\mathcal{E}(f_++f_-,f_++f_-)-\mathcal{E}(f_+-f_-,f_+-f_-)\\
&=\mathcal{E}(f_+,f_+)+2\mathcal{E}(f_+,f_-)+\mathcal{E}(f_-,f_-)\\
&\quad-\mathcal{E}(f_+,f_+)+2\mathcal{E}(f_+,f_-)-\mathcal{E}(f_-,f_-)\\
&=2\mathcal{E}(f_+,f_-)+2\mathcal{E}(f_-,f_+).
\end{aligned}$$

Now we can see that (iii) follows from (ii). If $\mathcal{E}$ is symmetric, (ii) follows from (iii) by the identity above. □

In connection to the contraction property of the semigroup, we have the following.

THEOREM 4.3
Let $\gamma \in \mathbb{R}$ and $\theta \in [0,1)$ be given. Then the following three conditions are equivalent to each other.

(i) *The semigroup $\{e^{-\gamma t}T_t\}$ is a positivity-preserving contraction semigroup.*

(ii) *For any $f \in \mathrm{Dom}(\mathcal{E})$, we have $|f| \in \mathrm{Dom}(\mathcal{E})$ and*

$$\mathcal{E}\big((1-\theta)f + \theta f_+, f - f_+\big) \geq -\gamma(1-\theta)\|f_-\|_2^2. \tag{4.9}$$

(iii) *For any $f \in \mathrm{Dom}(\mathcal{E})$, we have $|f| \in \mathrm{Dom}(\mathcal{E})$ and, for any $\eta \in [0,1)$,*

$$\mathcal{E}\big((1-\eta)f + \eta f_+, f - f_+\big) \geq -\gamma(1-\eta)\|f_-\|_2^2. \tag{4.10}$$

If, in addition, $\mathcal{E}$ is symmetric, then the following conditions are also equivalent to previous ones.

(iv) *For any $f \in \mathrm{Dom}(\mathcal{E})$, we have $|f| \in \mathrm{Dom}(\mathcal{E})$ and*

$$\mathcal{E}(f_+, f_+) \leq \mathcal{E}(f,f) + \gamma\|f_-\|^2. \tag{4.11}$$

(v) *For any $f \in \mathrm{Dom}(\mathcal{E})$, we have $|f| \in \mathrm{Dom}(\mathcal{E})$ and*

$$0 \leq \mathcal{E}_\gamma(|f|,|f|) \leq \mathcal{E}_\gamma(f,f). \tag{4.12}$$

Proof
We show only that (ii) ⇒ (i). The others easily follow from Theorem 4.1.

Since $\mathcal{E}$ is bounded from below, there exists a constant $\lambda \geq 0$ such that $\mathcal{E}_{\gamma+\lambda}(f,f) \geq 0$. On the other hand, from (4.9),

$$-\gamma(1-\theta)\|f_-\|_2^2 \leq \mathcal{E}\big((1-\theta)f + \theta f_+, f - f_+\big) \leq -\mathcal{E}(f_+, f_-) + (1-\theta)\mathcal{E}(f_-, f_-),$$

which leads to

$$\mathcal{E}(f_+, f_-) - (1-\theta)\mathcal{E}_\gamma(f_-, f_-) \leq 0.$$

Using this inequality, let us compute $\mathcal{E}_{\gamma+\theta\lambda}$:

$$\begin{aligned}
\mathcal{E}_{\gamma+\theta\lambda}(f, f_-) &= \mathcal{E}_\gamma(f_+ - f_-, f_-) - \theta\lambda(f \mid f_-)_2 \\
&= \mathcal{E}_\gamma(f_+, f_-) - \mathcal{E}_\gamma(f_-, f_-) - \theta\lambda(f \mid f_-)_2 \\
&= \mathcal{E}(f_+, f_-) - (1-\theta)\mathcal{E}_\gamma(f_-, f_-) \\
&\quad - \theta\mathcal{E}_\gamma(f_-, f_-) - \theta\lambda(f_- \mid f_-)_2 \\
&= \mathcal{E}(f_+, f_-) - (1-\theta)\mathcal{E}_\gamma(f_-, f_-) \\
&\quad - \theta\mathcal{E}_{\gamma+\lambda}(f_-, f_-) \leq 0.
\end{aligned}$$

Here, in the third line, we used $\mathcal{E}_\gamma(f_+,f_-)=\mathcal{E}(f_+,f_-)$. Now, taking $-f$ instead of f, we have

$$\mathcal{E}_{\gamma+\theta\lambda}(f,f_+)\geq 0.$$

Combining both of them, we have

$$\mathcal{E}_{\gamma+\theta\lambda}(f,f)=\mathcal{E}_{\gamma+\theta\lambda}(f,f_+)-\mathcal{E}_{\gamma+\theta\lambda}(f,f_-)\geq 0.$$

Thus we have deduced $\mathcal{E}_{\gamma+\theta\lambda}(f,f)\geq 0$ from $\mathcal{E}_{\gamma+\lambda}(f,f)\geq 0$. Repeating this procedure, we have $\mathcal{E}_{\gamma+\theta^n\lambda}(f,f)\geq 0$. Letting $n\to\infty$, we eventually get $\mathcal{E}_\gamma(f,f)\geq 0$, as desired.

If $\mathcal{E}$ is symmetric, it is enough to note that condition (iv) is nothing but (i) with $\theta=1/2$. □

4.3. Markovian property

The next issue is the Markovian property. First, we give a definition.

DEFINITION 4.1

A bilinear form $\mathcal{E}$ is called a *semi-Dirichlet form* if the associated semigroup is Markovian. If $\mathcal{E}$ and its dual $\mathcal{E}^*$ are semi-Dirichlet forms, $\mathcal{E}$ is called a *Dirichlet form.*

Let us give the necessary and sufficient conditions for which $\mathcal{E}$ becomes a semi-Dirichlet form.

THEOREM 4.4

The following two conditions are equivalent to each other.

(i) $\{T_t\}$ *is Markovian.*
(ii) *For any* $f\in\mathrm{Dom}(\mathcal{E})$, *we have* $f\wedge 1\in\mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(f\wedge 1, f-f\wedge 1)\geq 0. \tag{4.13}$$

We may change $f\wedge 1$ *with* f_+.

Proof

This is clear from Theorem 4.1. □

We also have the following theorem.

THEOREM 4.5

Let $\gamma\in\mathbb{R}$ *and* $\theta\in[0,1)$ *be given. Then the following three conditions are equivalent to each other.*

(i) $\{T_t\}$ *is Markovian and* $\{e^{-\gamma t}T_t\}$ *is a contraction semigroup.*
(ii) *For any* $f\in\mathrm{Dom}(\mathcal{E})$, *we have* $f\wedge 1\in\mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}\big((1-\theta)f+\theta(f\wedge 1), f-f\wedge 1\big)\geq -\gamma(1-\theta)\|f-f\wedge 1\|_2^2. \tag{4.14}$$

(iii) *For any* $f \in \mathrm{Dom}(\mathcal{E})$*, we have* $f \wedge 1 \in \mathrm{Dom}(\mathcal{E})$*, and for any* $\eta \in [0,1)$*,*

$$\mathcal{E}\big((1-\eta)f + \eta(f \wedge 1), f - f \wedge 1\big) \geq -\gamma(1-\eta)\|f - f \wedge 1\|_2^2. \tag{4.15}$$

If, in addition, $\mathcal{E}$ *is symmetric, then the following condition is equivalent to the others.*

(iv) *For any* $f \in \mathrm{Dom}(\mathcal{E})$*, we have* $f \wedge 1 \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f,f) + \gamma\|f - f \wedge 1\|_2^2. \tag{4.16}$$

We may replace $f \wedge 1$ *with* $f_+ \wedge 1$ *in the equations above.*

Proof
We need to show only that (ii) $\Rightarrow$ (i). Others easily follow from Theorem 4.1.

From Theorem 4.1, we can have that $\{T_t\}$ is Markovian. To show the contraction property, taking f/c $(c > 0)$ in (4.14), we have

$$\mathcal{E}\big((1-\theta)f + \theta(f \wedge c), f - f \wedge c\big) \geq -\gamma(1-\theta)\|f - f \wedge c\|_2^2.$$

Letting $c \to 0$, we have

$$\mathcal{E}\big((1-\theta)f + \theta(f \wedge 0), f - f \wedge 0\big) \geq -\gamma(1-\theta)\|f - f \wedge 0\|^2.$$

Now, by Theorem 4.3, we can get that $\{e^{-\gamma t}T_t\}$ is contractive. □

In Ma and Röckner [6], $\mathcal{E}$ is called a *semi-Dirichlet form* if for any $f \in \mathrm{Dom}(\mathcal{E})$, we have $f_+ \wedge 1 \in \mathrm{Dom}(\mathcal{E})$ and

$$\mathcal{E}(f + f_+ \wedge 1, f - f_+ \wedge 1) \geq 0. \tag{4.17}$$

As was shown, this condition leads to the contraction property of the semigroup, and so the noncontractive semigroups are excluded. They considered only contraction semigroups, and so there is no problem, but if we include noncontractive semigroups, it seems better to adopt the definition that was given in Definition 4.1.

Last, we give an example of a Markovian semigroup which does not satisfy the contraction property. We take $\mathbb{R}$ to be a state space with a reference measure $\mu(dx) = (1/\sqrt{2\pi})e^{-x^2/2}\,dx$. On this space, the Ornstein-Uhlenbeck operator $L = \frac{d}{dx^2} - x\frac{d}{dx}$ is associated with the following symmetric Dirichlet form:

$$\int_{\mathbb{R}} \frac{df}{dx}\frac{dg}{dx}\,d\mu(x).$$

We consider an operator of the form $L + b$ with $b = 2\beta\frac{d}{dx}$. The Dirichlet form $\mathcal{E}$ associated with $L + b$ is given by

$$\mathcal{E}(f,g) = \int_{\mathbb{R}} \Big(\frac{df}{dx}\frac{dg}{dx} - 2\beta\frac{df}{dx}g(x)\Big)\,d\mu(x).$$

If $-1/4 < \beta < 1/4$, we can show that this form satisfies the sector condition; to be precise, we can find $\gamma > 0$ such that $\mathcal{E}_\gamma$ satisfies the sector condition. We can also check that it is a semi-Dirichlet form. We are now interested in whether $\mathcal{E}$

is nonnegative; that is, $\mathcal{E}(f,f)\geq 0$. Note that

$$\begin{aligned}\mathcal{E}(f,f) &= \int_{\mathbb{R}}\Big\{\Big(\frac{df}{dx}\Big)^2 - 2\beta x\frac{df}{dx}f(x)\Big\}\,d\mu(x)\\ &= \int_{\mathbb{R}}\Big(\frac{df}{dx}\Big)^2\,d\mu(x) - \int_{\mathbb{R}}\beta x\frac{d}{dx}f(x)^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\\ &= \int_{\mathbb{R}}\Big(\frac{df}{dx}\Big)^2\,d\mu(x) + \int_{\mathbb{R}}f(x)^2\frac{d}{dx}\Big(\beta x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\Big)\,dx\\ &= \int_{\mathbb{R}}\Big(\frac{df}{dx}\Big)^2\,d\mu(x) + \int_{\mathbb{R}}f(x)^2(\beta-\beta x^2)\,d\mu(x)\\ &= \int_{\mathbb{R}}(-L+\beta-\beta x^2)f(x)f(x)\,d\mu(x).\end{aligned}$$

Let us find an eigenvalue of $L+\beta x^2-\beta$. We search for an eigenfunction of the form $e^{\alpha x^2}$:

$$\begin{aligned}(L+\beta x^2-\beta)e^{\alpha x^2} &= \big\{(4\alpha^2-2\alpha)x^2+2\alpha\big\}e^{\alpha x^2}+(\beta x^2-\beta)e^{\alpha x^2}\\ &= (4\alpha^2-2\alpha+\beta)x^2e^{\alpha x^2}+(2\alpha-\beta)e^{\alpha x^2}.\end{aligned}$$

Hence if $4\alpha^2-2\alpha+\beta=0$, then $e^{\alpha x^2}$ is an eigenfunction of the eigenvalue $2\alpha-\beta$. Solving $4\alpha^2-2\alpha+\beta=0$, we have $\alpha=(1\pm\sqrt{1-4\beta})/4$, but we should take $\alpha=(1-\sqrt{1-4\beta})/4$ to ensure that $e^{\alpha x^2}\in L^2(\mu)$. The eigenvalue is

$$2\alpha-\beta=\frac{1-\sqrt{1-4\beta}-2\beta}{2},$$

which is positive if $\beta\neq 0$. Therefore $-L-\beta x^2+\beta$ has a negative eigenvalue, and so the associated semigroup does not satisfy the contraction property.

4.4. L^1-contraction property

From Theorem 4.1, the condition for the L^1-contraction and the positivity-preserving property is given as

$$\mathcal{E}\big((f-1)_+,f\wedge 1\big)\geq 0. \tag{4.18}$$

On the other hand, the Markovian property was characterized by $\mathcal{E}(f\wedge 1,(f-1)_+)\geq 0$, which is exactly the dual of (4.18). So one follows from the other.

If we introduce the parameter θ, then the L^1-contraction and the positivity-preserving property are characterized by

$$\mathcal{E}\big(f-\theta(f\wedge 1),f\wedge 1\big)\geq -\gamma(1-\theta)\|f\wedge 1\|_2^2$$

and the Markovian property is characterized by

$$\mathcal{E}\big(f-\theta(f-1)_+,(f-1)_+\big)\geq -\gamma(1-\theta)\|(f-1)_+\|_2^2.$$

In this case, the duality is not so clear.

4.5. Excessive functions

Let us consider the excessive functions. Recall that a nonnegative function u is called *α-excessive* if

$$e^{-\alpha t}T_t u \le u, \quad \forall t \ge 0. \tag{4.19}$$

Let us give a characterization in terms of bilinear form. We remark that these results are basically known (see, e.g., Ma, Overbeck, and Röckner [5]).

THEOREM 4.6

The following two conditions are equivalent to each other:

(i) *u is α-excessive, and $\{T_t\}$ is positivity preserving;*
(ii) *$u \ge 0$, and for any $f \in \mathrm{Dom}(\mathcal{E})$, we have $f \wedge u \in \mathrm{Dom}(\mathcal{E})$ and*

$$\mathcal{E}_\alpha(f \wedge u, f - f \wedge u) \ge 0. \tag{4.20}$$

Proof

(i) $\Rightarrow$ (ii) This follows from Theorem 4.1.
(ii) $\Rightarrow$ (i) This also follows from Theorem 4.1, but we need to show that $\{T_t\}$ is positivity preserving. We take $\lambda \ge \alpha$ large enough so that $\mathcal{E}_\lambda$ is nonnegative. We may assume that u is λ-excessive. We take f/c in (4.20). Then

$$\mathcal{E}_\lambda(f \wedge cu, f - f \wedge cu) \ge 0.$$

Letting $c \downarrow 0$, we have that $f \wedge cu$ converges to $f \wedge 0$ weakly in $\mathcal{E}_\lambda$. Hence

$$\begin{aligned}\mathcal{E}_\lambda(f \wedge 0, f \wedge 0) &\le \liminf_{c\to 0} \mathcal{E}_\lambda(f \wedge cu, f \wedge cu)\\ &\le \liminf_{c\to 0} \mathcal{E}_\lambda(f \wedge cu, f)\\ &= \mathcal{E}_\lambda(f \wedge 0, f).\end{aligned}$$

This means that $\mathcal{E}_\lambda(f_+, f_-) \le 0$, and by using Theorem 4.2, we can have that $\{T_t\}$ is positivity preserving. $\square$

The following theorem can be proved similarly.

THEOREM 4.7

Let $\gamma \in \mathbb{R}$ and $\theta \in [0,1)$ be given. Then the following three conditions are equivalent to each other:

(i) *u is α-excessive, and $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity-preserving contraction semigroup;*
(ii) *for any $f \in \mathrm{Dom}(\mathcal{E})$, we have $f \wedge u \in \mathrm{Dom}(\mathcal{E})$ and*

$$\mathcal{E}_\alpha\big((1-\theta)f + \theta(f \wedge u), f - f \wedge u\big) \ge -\gamma(1-\theta)\|f - f \wedge u\|^2; \tag{4.21}$$

(iii) *for any* $f \in \mathrm{Dom}(\mathcal{E})$, *we have* $f \wedge u \in \mathrm{Dom}(\mathcal{E})$ *and for any* $\eta \in [0,1)$,

$$\mathcal{E}_\alpha\big((1-\eta)f + \eta(f \wedge u), f - f \wedge u\big) \geq -\gamma(1-\eta)\|f - f \wedge u\|^2. \tag{4.22}$$

4.6. Invariant sets

A set B is called *weakly invariant* if

$$1_{B^c} T_t 1_B = 0, \quad \forall t \geq 0.$$

In this case, we have the following criterion.

THEOREM 4.8

The following three conditions are equivalent to each other:

(i) B *is an invariant set;*

(ii) *for any* $f \in \mathrm{Dom}(\mathcal{E})$, *we have* $1_B f \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(1_B f, 1_{B^c} f) \geq 0; \tag{4.23}$$

(iii) *for any* $f \in \mathrm{Dom}(\mathcal{E})$, *we have* $1_B f \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\mathcal{E}(1_B f, 1_{B^c} f) = 0. \tag{4.24}$$

Proof

The equivalence of (i) and (ii) follows from Theorem 4.1; (iii) $\Rightarrow$ (ii) is clear. Let us show that (i) $\Rightarrow$ (iii). By (ii), we have $1_B f \in \mathrm{Dom}(\mathcal{E})$. Further, by the invariance of B,

$$\big((T_t - I)1_B f, 1_{B^c} f\big) = 0.$$

Divide both hands by t, and letting $t \downarrow 0$, we easily get (4.24). □

References

[1] H. Brezis and A. Pazy, *Semigroups of nonlinear contractions on convex sets*, J. Funct. Anal. **6** (1970), 237–281.

[2] W. Farkas, N. Jacob, and R. Schilling, *Feller semigroups, L^p-sub-Markovian semigroups, and applications to pseudo-differential operators with negative definite symbols*, Forum Math. **13** (2001), 51–90.

[3] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Math. Monogr., Oxford Univ. Press, New York, 1985.

[4] N. Jacob, *Pseudo-Differential Operators and Markov Processes, Vol. I: Fourier Analysis and Semigroups*, Imperial College Press, London, 2001.

[5] Z.-M. Ma, L. Overbeck, and M. Röckner, *Markov processes associated with semi-Dirichlet forms*, Osaka J. Math. **32** (1995), 97–119.

[6] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (Nonsymmetric) Dirichlet Forms*, Springer, Berlin, 1992.

[7] E. Ouhabaz, *Invariance of closed convex sets and domination criteria for semigroups*, Potential Anal. **5** (1996), 611–625.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan; ichiro@math.kyoto-u.ac.jp