Threefold extremal contractions of type (IA)

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To the memory of Professor Masaki Maruyama

Abstract Let (X,C) be a germ of a threefold X with terminal singularities along an irreducible reduced complete curve C with a contraction $f:(X,C)\to(Z,o)$ such that $C=f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is ample. Assume that a general member $F\in|-K_X|$ meets C only at one point P, and furthermore assume that (F,P) is Du Val of type A if index $(X,P)=4$. We classify all such germs in terms of a general member $H\in|\mathscr{O}_X|$ containing C.

Contents

1. Introduction

DEFINITION 1.1

Let (X,C) be a germ of a threefold with terminal singularities along a reduced complete curve. We say that (X,C) is an *extremal curve germ* if there is a contraction $f:(X,C)\to(Z,o)$ such that $C=f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is f-ample.

Furthermore, if f is birational, then (X,C) is said to be an *extremal neighborhood* (see [Mor2]). In this case f is called *flipping* if its exceptional locus coincides with C (and then (X,C) is called *isolated*). Otherwise, the exceptional locus of f is 2-dimensional and f is called *divisorial.* If f is not birational, then $\dim Z=2$ and (X,C) is said to be a $\mathbb{Q}$-*conic bundle germ* (see [MP1]).

In this paper, unless explicitly stated otherwise, we assume that C is *irreducible.*

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1.2

Let (X, C) be an extremal curve germ as above. For each singular point P of X with $P \in C$, consider the germ $(P \in C \subset X)$. All such germs (or all such singular points for simplicity) are classified into types: (IA), (IC), (IIA), (IIB), (III), $(\mathrm{IA}^\vee)$, $(\mathrm{II}^\vee)$, $(\mathrm{ID}^\vee)$, $(\mathrm{IE}^\vee)$, as for whose definitions we refer the reader to [Mor2] and [MP1]. The possible configurations of such points are also classified in [Mor2] and [MP1]. Moreover, it is known that a general member $F \in |-K_X|$ has only Du Val singularities and all possibilities for F are described (see [Mor2, Theorem 7.3, 9.10], [KM, Theorem 2.2], [MP1, Proposition 1.3.7], [MP2, paragraphs 2.1-2.2]). The next step in the classification is to study a general hyperplane section, that is, a general divisor H of $|\mathscr{O}_X|_C$, the linear subsystem of $|\mathscr{O}_X|$ consisting of sections $\supset C$. Roughly speaking, the importance of this divisor can be explained as follows. Once we have this H, the total threefold can be considered as a one-parameter deformation of H. Then one can apply the deformation theory to construct X starting from two-dimensional data $H \supset C$.

Recall that $\mathbb{Q}$-conic bundles having only points of types (III), $(\mathrm{IA}^\vee)$ – $(\mathrm{IE}^\vee)$, as well as points of type (IA) over singular base, are classified in [MP1]. In this paper we start our classification of $\mathbb{Q}$-conic bundles and divisorial contractions which are not treated in earlier papers. To be more precise, we classify extremal curve germs of type (IA) or $(\mathrm{IA}^\vee)$ in terms of a general member $H \in |\mathscr{O}_X|_C$. An extremal curve germ (X, C) is said to be of *type* (IA) (resp., $(\mathrm{IA}^\vee)$) if it contains exactly one non-Gorenstein point P and it is of type (IA) (resp., $(\mathrm{IA}^\vee)$). For readers' convenience, we note the following characterization (cf. [KM, Theorem 2.2]) for an extremal curve germ (X, C) with a point P of index $m > 1$ to be of type (IA) or $(\mathrm{IA}^\vee)$ in terms of a general member $F \in |-K_X|$: (X, C) is of type (IA) or $(\mathrm{IA}^\vee)$ if and only if

(i) $F \cap C = \{P\}$ as a set and
(ii) (F, P) is Du Val of type A if $m = 4$.

1.3

Throughout this paper, if we do not specify otherwise, we assume that (X, C) is of type (IA) or $(\mathrm{IA}^\vee)$. More precisely, X contains a unique non-Gorenstein terminal point $P \in X$, which is of type (IA) or $(\mathrm{IA}^\vee)$.

A point $(X \supset C \ni P)$ of index $m > 1$ is said to be of type (IA) if there exists an embedding $X \subset \mathbb{C}^4_{x_1,\dots,x_4}/\boldsymbol{\mu}_m(a_1, a_2, -a_1, 0)$ such that

$$C = \{x_1^{a_2} - x_2^{a_1} = x_3 = x_4 = 0\}/\boldsymbol{\mu}_m(a_1, a_2, -a_1, 0)$$

for some positive integers a_1, a_2 with $\gcd(a_1 a_2, m) = 1$ and $m \in a_1\mathbb{Z}_{>0} + a_2\mathbb{Z}_{>0}$, and X is given by an invariant vanishing along C (see [Mor2, Summary A.3]). If f is a $\mathbb{Q}$-conic bundle, then $a_2 = 1$ by [MP1, Proposition 8.5]. Points of type $(\mathrm{IA}^\vee)$ are described similarly (see [Mor2, Summary A.3]).

For a normal surface S and a curve $V \subset S$, we use the usual notation of graphs $\Delta(S, V)$ of the minimal resolution of S near V: each $\diamond$ corresponds to an irreducible component of V, and each $\circ$ corresponds to an exceptional divisor on the minimal resolution of S. We may use $\bullet$ instead of $\diamond$ if we want to emphasize that it is a complete (-1)-curve. A number attached to a vertex denotes the minus self-intersection number. For short, we may omit 2 if the self-intersection is -2.

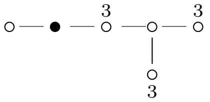
1.4

For a triple (X,C,P) of type (IA) or (IA$^\vee$), the singularity (X,P) is either cA/m, cD/3, or of index 2. Extremal neighborhoods of index 2 are classified in [KM, Section 4]. Flipping extremal neighborhoods containing a terminal singular point of type cD/3 (see [Mor1], [Rei2]) are classified in [KM, Theorems 6.2, 6.3]. Thus the following theorem completes the treatment of extremal curve germs containing a (cD/3)-point.

THEOREM 1.5

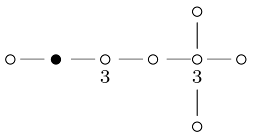
Let (X,C) be an extremal curve germ. Assume that (X,C) is of type (IA), *and let $P \in X$ be the non-Gorenstein point. Assume, furthermore, that (X,P) is of type* cD/3. *Then f is a birational contraction, not a $\mathbb{Q}$-conic bundle. The general member $H \in |\mathscr{O}_X|_C$ and its image $T = f(H) \in |\mathscr{O}_Z|$ are normal and have only rational singularities. Moreover, if f is not a flipping contraction, then the following are the only possibilities for the dual graphs of (H,C) and T:*

(1.5.1) $\Delta(H,C)$:



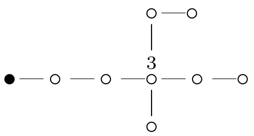
and T is of type A_2*; here (X,P) is a simple* (cD/3)*-point (see Section 4.1);*

(1.5.2) $\Delta(H,C)$:



and T is of type D_4*; here (X,P) is a double* (cD/3)*-point;*

(1.5.3) $\Delta(H,C)$:



and T is of type E_6*; here (X,P) is a triple* (cD/3)*-point.*

In all the cases above the right-hand side of the graph for (H,C) corresponds to the non-Gorenstein point $P \in H$. The left-hand side corresponds to either a type (III) *point or a smooth point of X.*

This is shown in Examples 4.14.1 and 4.14.2.

Note that $\mathbb{Q}$-conic bundles of type (IA$^\vee$) are completely classified in [MP1]. The following two theorems cover the $\mathbb{Q}$-conic bundles of type (IA).

THEOREM 1.6

Let $(X, C \simeq \mathbb{P}^1)$ *be a* $\mathbb{Q}$*-conic bundle germ of index* $m > 2$ *and of type* (IA). *Let* $P \in X$ *be the non-Gorenstein point. Then* (X, P) *is a point of type* cA/m *and a general member* $H \in |\mathscr{O}_X|_C$ *is not normal. Furthermore, the dual graph of* (H', C')*, the normalization* H'*, and the inverse image* C' *of* C *is of the form*

$$\underbrace{\overset{a_r}{\circ} - \cdots - \overset{a_1}{\circ}}_{\Delta_1} - \bullet - \underbrace{\overset{b_1}{\circ} - \cdots - \overset{b_s}{\circ}}_{\Delta_2}$$

(*in particular,* C' *is irreducible*). *Here the chain* Δ_1 (*resp.,* Δ_2) *corresponds to the singularity of type* $1/m(1, a)$ (*resp.,* $1/m(1, -a)$) *for some integer* a $(\in [1, m])$ *relatively prime to* m. *The germ* (H, C) *is analytically isomorphic to the germ along the line* $y = z = 0$ *of the hypersurface given by the following weighted polynomial of degree* $2m$ *in variables* x, y, z, u:

$$\phi := x^{2m-2a}y^2 + x^{2a}z^2 + yzu$$

in $\mathbb{P}(1, a, m-a, m)$. *Furthermore,* (X, C) *is given as an analytic germ of a subvariety of* $\mathbb{P}(1, a, m-a, m) \times \mathbb{C}_t$ *along* $C \times 0$ *given by*

$$\phi + \alpha_1 x^{2m-a}y + \alpha_2 x^{m-a}uy + \alpha_3 x^{2m} + \alpha_4 x^m u + \alpha_5 u^2 = 0$$

for some $\alpha_1, \ldots, \alpha_5 \in t\mathscr{O}_{0,\mathbb{C}_t}$, *and there is a* $\mathbb{Q}$*-conic bundle structure* $X \to \mathbb{C}^2$ *through which the second projection* $X \to \mathbb{C}_t$ *factors. The* $\mathbb{Q}$*-conic bundle structure is given as deformation of the fibration in Definition 6.8.1, which is explained in Lemma 6.8.2.*

An explicit example is given in Example 6.8.4.

THEOREM 1.7 ([Pro1, SECTION 3], [MP1, THEOREM 12.1])

Let $(X, C \simeq \mathbb{P}^1)$ *be a* $\mathbb{Q}$*-conic bundle germ of index* 2 *and type* (IA). *Let* $f\colon (X, C) \to (Z, o)$ *be the corresponding contraction. Then* (Z, o) *is smooth. Let* u, v *be local coordinates on* (Z, o). *Then there is an embedding*

$$\begin{tikzcd} f\colon\ X \arrow[r, hook] & \mathbb{P}(1,1,1,2) \times Z \arrow[r, "p"] & Z \end{tikzcd}$$

such that X *is given by two equations*

$$q_1(y_1, y_2, y_3) = \psi_1(y_1, \ldots, y_4; u, v),$$
$$q_2(y_1, y_2, y_3) = \psi_2(y_1, \ldots, y_4; u, v),$$

where ψ_i *and* q_i *are weighted quadratic in* $y_1, \ldots, y_4$ *with respect to* $\mathrm{wt}(y_1, \ldots, y_4) = (1, 1, 1, 2)$ *and* $\psi_i(y_1, \ldots, y_4; 0, 0) = 0$. *The only non-Gorenstein point of* X *is* $(0, 0, 0, 1; 0, 0)$. *Up to projective transformations, the following are the only possibilities for* q_1 *and* q_2.

(i) *We have* $q_1 = y_1^2$, $q_2 = y_2^2 - y_1 y_3$: *here a general member* $H \in |\mathscr{O}_X|_C$ *is normal.*

(ii) *We have* $q_1 = y_1^2$, $q_2 = y_2^2$: *here every member* $H \in |\mathscr{O}_X|_C$ *is nonnormal.*

In both cases, C *is given by* $u = v = y_1 = y_2 = 0$.

Explicit examples are given in Section 7 (see also Remark 6.7.1).

1.8

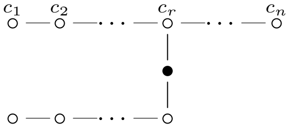
The next theorem completes the remaining case by Section 1.4.

THEOREM 1.9 (SEE [Tzi1])

Let (X, C) *be an extremal neighborhood of type* (IA) *or* $(\mathrm{IA}^\vee)$. *Let* $P \in X$ *be the non-Gorenstein point. Assume, furthermore, that* (X, P) *is of type* cA/m. *Let* $F \in |-K_X|$ *be a general member. Then there exists a member* $H \in |\mathscr{O}_X|_C$ *such that the pair* $(X, H+F)$ *is log canonical (LC).*

(1.9.1) *If* H *is normal, then* H *has only log terminal singularities of type* T. *The graph* $\Delta(H, C)$ *is of the form*

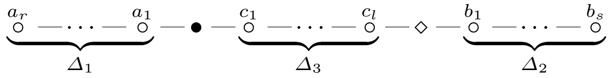
(1.9.1.1)



Here the chain $[c_1, \ldots, c_n]$ *corresponds to the non–Du Val singularity* (H, P) *of type* T. *The chain of* (-2)*-vertices in the last line corresponds to a Du Val point* (H, Q). *It is possible that this chain is empty* (*i.e.,* (H, Q) *is smooth*). *Cases* $r = 1$ *and* $r = n$ *are also not excluded.*

(1.9.2) *If every member of* $|\mathscr{O}_X|_C$ *is nonnormal, then the dual graph of the normalization* (H', C') *is of the form*

(1.9.2.1)



(*in particular,* C' *is reducible*). *The chain* $\varDelta_1$ (*resp.,* $\varDelta_2$) *corresponds to the singularity of type* $1/m(1, a)$ (*resp.,* $1/m(1, -a)$) *for some* a *with* $\gcd(m, a) = 1$, *and the chain* $\varDelta_3$ *corresponds to the point* (H', Q'), *where* $Q' = C'_1 \cap C'_2$. *The strings* $[a_1, \ldots, a_r]$ *and* $[b_1, \ldots, b_s]$ *are conjugate* (*cf. Definition 2.1.2*). *Moreover,*

$$\sum (c_i - 2) \le 2 \qquad \text{and} \qquad \widetilde{C}_1^2 + \widetilde{C}_2^2 + 5 - \sum (c_i - 2) \ge 0,$$

where $\widetilde{C} = \widetilde{C}_1 + \widetilde{C}_2$ *is the proper transform of* C *on the minimal resolution* $\widetilde{H}$. *Both components of* $\widetilde{C}$ *are contracted on the minimal model of* $\widetilde{H}$. *In this case, the triple* (X, C, P) *is analytically isomorphic to* $(\{\alpha = 0\}, x_1\text{-axis}, 0)/\boldsymbol{\mu}_m(1, a, -a, 0)$, *where* $\gcd(m, a) = 1$ *and* $\alpha(x_1, \ldots, x_4) = 0$ *is the equation of a terminal* (cA/m)*-point in* $\mathbb{C}^4/\boldsymbol{\mu}_m(1, a, -a, 0)$. (*In particular,* (X, C) *is of type* (IA)).

Conversely, for any germ $(H, C \simeq \mathbb{P}^1)$ *of the form in Sections 1.9.1 or 1.9.2 admitting a birational contraction* $(H, C) \to (T, o)$, *there exists a threefold birational contraction* $f : (X, C) \to (Z, o)$ *as in Definition 1.1 of type* (IA) *such that* $H \in |\mathscr{O}_X|_C$.

REMARK 1.9.3

Basically this result is proved in [Tzi1]. However, [Tzi1] treated only *divisorial* contractions that contract a divisor to a *smooth* curve. Under these assumptions the result of [Tzi1] is much stronger.

REMARK 1.9.4

Note that in Theorem 1.9, H is not assumed to be a general element of $|\mathscr{O}_X|_C$. If H is chosen general, then cases (1.9.1) and (1.9.2) cover all the cases under Theorem 1.9. Proposition 6.3 gives a criterion for a general member of $|\mathscr{O}_X|_C$ to be nonnormal, and Proposition 6.6 gives, under some additional assumptions, a criterion, for a given H to be general.

To check divisoriality one can use the following criterion, which is an immediate consequence of Theorem 3.1.

THEOREM 1.10

Let $f : (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a 3-dimensional birational extremal curve germ. Then f is divisorial if and only if (Z, o) is a terminal singularity.

One of our technical tools is the deformation of extremal curve germs. In particular, we prove Theorem 3.2, which shows that for every extremal curve germ $f : (X, C) \to (Z, o)$ the contraction f deforms with X. Combined with Theorem 1.10, it allows us to run the minimal model program for every deformation of an extremal curve germ which may not be $\mathbb{Q}$-factorial.

CONVENTIONS 1.11

We work over the complex number field $\mathbb{C}$. Notation and techniques of [Mor2] are used freely. In particular, for a terminal singularity (X, P) the index-one cover is denoted by $(X^\sharp, P^\sharp) \to (X, P)$, and for a subvariety $V \subset X$ its preimage is denoted by $V^\sharp$.

2. Preliminaries

2.1. Some facts about 2-dimensional toric singularities

NOTATION 2.1.1

A continued fraction

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_r}}} \qquad (a_1, \dots, a_r \geq 2)$$

is denoted by $[a_1, \dots, a_r]$ and called a *string*. Write $m/q = [a_1, \dots, a_r]$, where $\gcd(m, q) = 1$. Given m and q, this expression is unique. It is well known that the minimal resolution of the cyclic quotient singularity $1/m(1, q)$ is a chain of smooth rational curves whose self-intersection numbers are $-a_1, \dots, -a_r$.

DEFINITION 2.1.2

We say that a string $[b_1,\dots,b_s]$ is *conjugate* to $[a_1,\dots,a_r]$ if $[b_1,\dots,b_s]=m/(m-q)$.

LEMMA 2.1.3

(i) *If the strings* $[a_1,\dots,a_r]$ *and* $[b_1,\dots,b_s]$ *are conjugate, then either* $a_1=2$ *or* $b_1=2$.

(ii) *The strings* $[a_1,\dots,a_r]$ *and* $[b_1,\dots,b_s]$ *with* $a_1=2$ *and* $r>1$ *are conjugate if and only if* $[a_2,\dots,a_r]$ *and* $[b_1-1,\dots,b_s]$ *are conjugate.*

(iii) *The strings* $[a_1,\dots,a_r]$ *and* $[b_1,\dots,b_s]$ *are conjugate if and only if* $[a_r,\dots,a_1]$ *and* $[b_s,\dots,b_1]$*are conjugate.*

2.2. T-singularities

DEFINITION 2.2.1 (SEE [KSB])

A normal surface singularity is said to be of *type* T if it is log terminal and admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing.

PROPOSITION 2.2.2 ([LW, PROPOSITION 5.9], [KSB, PROPOSITION 3.10])

A surface singularity is of type T *if and only if it is either Du Val or a cyclic quotient of type* $1/n(a,b)$, *where* $\gcd(n,a)=\gcd(n,b)=1$ *and* $(a+b)^2\equiv 0 \mod n$.

By 2.1, any non–Du Val T-singularity is represented by some string $[a_1,\dots,a_r]$. Then we say that $[a_1,\dots,a_r]$ is a T-*string* or a *string of type* T.

PROPOSITION 2.2.3 ([KSB, PROPOSITION 3.11])

(i) *The strings* $[4]$, $[3,3]$, *and* $[3,2,\dots,2,3]$ *are of type* T.

(ii) *If the string* $[a_1,\dots,a_r]$ *is of type* T, *then so are* $[2,a_1,\dots,a_{r-1},a_r+1]$ *and* $[a_1+1,a_2,\dots,a_r,2]$.

(iii) *Every non–Du Val string of type* T *can be obtained by starting with one described in (i) and iterating the steps described in (ii).*

COROLLARY 2.2.4

Let (X,P) *be a* $\mathbb{Q}$-*Gorenstein isolated threefold singularity, and let* $H\subset X$ *be a surface such that* H *is a Cartier divisor. If the singularity* (H,P) *is log terminal, then* (H,P) *is a* T-*singularity and the point* (X,P) *is terminal of type* $\mathrm{cA/n}$ *or isolated* cDV.

Proof

The only thing we have to prove is the last statement. By the inversion of adjunction (see [Sho, Section 3], [Kol, Chapter 16]), the pair (X,H) is purely log terminal (PLT). Since H is Cartier and (X,P) is isolated, it is terminal. Clearly, we may assume that (H,P) is not Du Val. Let $F\in|-K_X|$ be a general member. Then $F|_H$ is a general member of $|-K_H|$. Since (H,P) is cyclic quotient (by Proposition 2.2.2), $(H,F|_H)$ is LC. Again by the inversion of adjunction, the pair $(X, H+F)$ is also LC. But this means that (F,P) is of type A, and so (X,P) is of type cA/n. □

2.3. Two-dimensional contractions

The following fact is easy and well known (see, e.g., [Pro2, Lemma 7.1.11]).

LEMMA 2.3.1

Let $\upsilon : S \to R \ni o$ be a rational curve fibration germ over a smooth curve and let $C := \upsilon^{-1}(o)_{\mathrm{red}}$. If the pair (S,C) is PLT, then there is an analytic isomorphism

$$S \simeq (\mathbb{P}^1 \times \mathbb{C})/\boldsymbol{\mu}_m(1,a),$$

where $\gcd(a,m)=1$. *The graph $\Delta(S,C)$ is of the form*

$$\overset{a_r}{\circ} \;\text{—}\cdots\text{—}\; \overset{a_1}{\circ} \;\text{—}\; \bullet \;\text{—}\; \overset{b_1}{\circ} \;\text{—}\cdots\text{—}\; \overset{b_s}{\circ}$$

where $[a_1,\dots,a_r]$ and $[b_1,\dots,b_r]$ are conjugate strings.

LEMMA 2.3.2 ([Sho, THEOREM 6.9], [Kol, PROPOSITION 12.3.1, 2])

Let $\upsilon : S \to R$ be a rational curve fibration germ over a smooth curve, and let Δ be an effective $\mathbb{Q}$-divisor on S such that $K_S + \Delta \equiv 0$ over R. Assume that the locus of log canonical singularities $LCS(S,\Delta)$ of (S,Δ) is not connected near a fiber $\upsilon^{-1}(o)$, $o \in R$. Then near $\upsilon^{-1}(o)$, the pair (S,Δ) is PLT and $\lfloor \Delta \rfloor$ is a disjoint union of two sections.

LEMMA 2.3.3

Let C be a smooth complete curve contained in a normal surface H. Assume that the pair (H,C) is not PLT at some point, say, $P \in C$, and that $(K_H + C)\cdot C < 0$. Then

(i) *H has at most two singular points on C;*

(ii) *if H is singular at a point $Q \in C$ and $Q \neq P$, then the pair (H,C) is PLT at Q. The dual graph $\Delta(H,C)$ for the minimal resolution of (H,C) at Q is of the form*

$$\underset{C}{\bullet} \;\text{—}\; \underbrace{\overset{b_1}{\circ} \;\text{—}\cdots\text{—}\; \overset{b_r}{\circ}}_{(H,Q)}$$

If, moreover, (H,Q) is a Gorenstein point, then it is Du Val.

Proof

By the inversion of adjunction (see [Sho, Section 3], [Kol, Chapter 16]), one has $(K_H + C)|_C = K_C + \mathrm{Diff}_C(0)$, where $\mathrm{Diff}_C(0)$ is a $\mathbb{Q}$-divisor with support at $C \cap \mathrm{Sing}(H)$. Moreover, the multiplicity of $\mathrm{Diff}_C(0)$ at every point of $C \cap \mathrm{Sing}(H)$ is at least $1/2$, and its multiplicity at P is at least 1. Since $\deg \mathrm{Diff}_C(0) \le -\deg K_C = 2$, the assertion of (i) follows. As for (ii), we see that the multiplicity of $\mathrm{Diff}_C(0)$ at Q is less than 1. Again by the inversion of adjunction the pair

(H,C) is PLT at Q. The rest follows from the classification of surface PLT pairs (see, e.g., [Kol, Chapter 3]). □

LEMMA 2.4

Let (X,C) *be an extremal curve germ, and let* $f:(X,C)\to(Z,o)$ *be the corresponding contraction. Assume that a member* $H\in|\mathscr{O}_X|_C$ *is normal. If* (X,C) *is a* $\mathbb{Q}$*-conic bundle germ, then* H *has only rational singularities.*

Proof

The assertion follows from the observation that $H\to f(H)$ is a rational curve fibration. □

THEOREM 2.5 ([Mor2, THEOREM 7.3], [MP1, PROPOSITION 1.3.7])

Let (X,C) *be an extremal curve germ of type* (IA) *or* (IA$^\vee$), *and let* $P\in X$ *be the non-Gorenstein point. Then a general member* $F\in|-K_X|$ *does not contain* C *and has only Du Val singularity of type* A *at* P.

PROPOSITION 2.6

Let $f:(X,C)\to(Z,o)$ *be a contraction from a threefold with only terminal singularities such that* C *is a (not necessarily irreducible) curve and* $-K_X$ *is ample. Let* $F\in|-K_X|$ *be a general member. Assume that* $F\cap C$ *is a point* P *such that* (F,P) *is a Du Val singularity of type* A. *Then, for a general member* $H\in|\mathscr{O}_X|_C$, *the pair* $(X,F+H)$ *is LC.*

If f *is birational, then so is the pair* (Z,F_Z+T), *where* $F_Z=f(F)\in|-K_Z|$ *and* $T:=f(H)\in|\mathscr{O}_Z|$. *In this case,* (T,o) *is a cyclic quotient singularity.*

Proof

First, we consider the case where f is birational. (This case was considered in [Tzi1].) Then $(F_Z,o)\simeq(F,P)$ is a Du Val singularity of type A. Let T be a general hyperplane section of (Z,o). Then $T\cap F_Z$ is general hyperplane section of (F_Z,o). Clearly, $T\cap F_Z=\Gamma_1+\Gamma_2$ for some irreducible curves Γ_i, and the pair $(F_Z,\Gamma_1+\Gamma_2)$ is LC. By the inversion of adjunction, so is the pair (Z,F_Z+T). Hence $(T,\Gamma_1+\Gamma_2)$ is LC and (T,o) is a cyclic quotient singularity (see, e.g., [Kol, Chapter 3]). Take $H:=f^*T$. Then $K_X+F+H=f^*(K_Z+F_Z+T)$; that is, the contraction f is (K_X+F+H)-crepant. Hence the pair $(X,F+H)$ is LC.

Now consider the case where Z is a surface. First, we claim that $(X,F+H)$ is LC near F. Consider the restriction $\varphi=f_F:(F,P)\to(Z,o)$. Let $\Xi\subset Z\simeq\mathbb{C}^2$ be the branch divisor of φ. By the Hurwitz formula, we can write $K_F=\varphi^*\big(K_Z+(1/2)\Xi\big)$. Hence,

$$K_F+H|_F=\varphi^*\Big(K_Z+\frac{1}{2}\Xi+T\Big).$$

Using this and the inversion of adjunction, we get the following equivalences: $(X,F+H)$ is LC near F $\iff$ $(F,H|_F=\varphi^*T)$ is LC $\iff$ $(Z=\mathbb{C}^2,(1/2)\Xi+T)$ is LC. Thus it is sufficient to show that $(Z,(1/2)\Xi+T)$ is LC.

Let $\xi(u,v)=0$ be the equation of $\Xi\subset\mathbb{C}^2$. Then (F,P) is given by the equation $w^2=\xi(u,v)$ in $\mathbb{C}^3_{u,v,w}$. By the classification of Du Val singularities, we can choose coordinates u, v so that

$$\xi=u^2+v^{n+1}.$$

Take $T:=\{v-u=0\}$. Then $\operatorname{ord}_0\xi(u,v)|_T=2$. By the inversion of adjunction, the pair $(Z,T+(1/2)\Xi)$ is LC.

Thus we have shown that $(X,F+H)$ is LC near F. Assume that $(X,F+H)$ is not LC at some point $Q\in C$. By the above, $Q\notin F$. Note that H is smooth outside of C by Bertini's theorem.

If H is normal, then we have an immediate contradiction by Lemma 2.3.2 applied to $(H,F|_H)$. Assume that H is not normal. Let $\nu:H'\to H$ be the normalization, and let $C':=\nu^{-1}(C)_{\mathrm{red}}$. Write

$$K_{H'}+\operatorname{Diff}_H(F)=\nu^*(K_X+H+F)\sim 0.$$

Here $\operatorname{Diff}_H(F)=C'+\nu^{-1}(F|_H)$, where $C'=\nu^{-1}(C)$. By the inversion of adjunction, C' is reduced and $(H',C'+\nu^{-1}(F|_H))$ is not LC at $\nu^{-1}(Q)$. Now we can apply Lemma 2.3.2 to $(H',C'+\nu^{-1}(F|_H)-\varepsilon v^*(o))$. □

COROLLARY 2.6.1

Under the assumptions of Proposition 2.6, if H is not normal, then there is an analytic isomorphism $(H,P)\simeq\{x_1'x_2'=0\}/\boldsymbol{\mu}_m(a,-a,1)$.

Proof

Let $\pi:(X^\sharp,P^\sharp)\to(X,P)$ be the index-one cover, and let $H^\sharp:=\pi^*H$, $F^\sharp:=\pi^*F$. Then the pair $(X^\sharp,H^\sharp+F^\sharp)$ is LC.

Assume that (X,P) is not a cyclic quotient singularity. One can choose a $\boldsymbol{\mu}_m$-equivariant embedding $X^\sharp\subset\mathbb{C}^4_{x_1,\dots,x_4}$ so that $\operatorname{wt}(x_1,\dots,x_4)\equiv(a,-a,1,0)\bmod m$ and $X^\sharp$ is given by the equation $x_1x_2=\phi(x_3^m,x_4)$, where $\operatorname{ord}_0\phi\ge 2$. For some hypersurfaces $D=\{\xi=0\}$ and $S=\{\psi=0\}$ in $\mathbb{C}^4_{x_1,\dots,x_4}$, we have $H^\sharp=D\cap X^\sharp$ and $F^\sharp=S\cap X^\sharp$. By the inversion of adjunction, the pair $(\mathbb{C}^4,X^\sharp+D+S)$ is LC. On the other hand, by blowing up the origin we get an exceptional divisor of discrepancy

$$a(E,X^\sharp+D+S)=3-2-\operatorname{ord}_0\xi-\operatorname{ord}_0\psi\ge -1.$$

Hence, $\operatorname{ord}_0\xi=1$. Since ξ is an $\boldsymbol{\mu}_m$-invariant, it contains the term x_4. Thus $\xi=x_4-\xi'$, where $\operatorname{ord}_0\xi'\ge 2$. Then $H^\sharp$ is given by two equations $x_1x_2=\phi(x_3^m,\xi')$ and $x_4=\xi'$. By changing coordinates, we get what we need.

Now assume that (X,P) is a cyclic quotient singularity. Then $X^\sharp\simeq\mathbb{C}^3$. Again one can choose a coordinate system x_1,x_2,x_3 in $\mathbb{C}^3$ so that $\operatorname{wt}(x_1,x_2,x_3)\equiv(a,-a,1)\mod m$. Let ξ be the equation of $H^\sharp$. By blowing up the origin, we get $\operatorname{ord}_0\xi\le 2$. On the other hand, ξ is an invariant. Hence, ξ contains the term x_1x_2 (possibly up to permutations of coordinates if $a\equiv\pm 1$). □

3. Deformations of 3-dimensional divisorial contractions

In this section we recall and set up deformation tools to study extremal curve germs.

THEOREM 3.1

Let $f:(X,C)\to(Z,o)$ be a 3-dimensional divisorial extremal curve germ, where C is not necessarily irreducible, and let E be its exceptional locus. Then the divisorial part of E is a $\mathbb{Q}$-Cartier divisor. If, furthermore, C is irreducible, then E is $\mathbb{Q}$-Cartier and (Z,o) is a terminal singularity.

THEOREM 3.2 (CF. [KM, (11.4)], [MP1, (6.2)])

Let $f:(X,C)\to(Z,o)$ be an extremal divisorial (resp., flipping, $\mathbb{Q}$-conic bundle) curve germ, where C is not necessarily irreducible. Let $\pi:\mathcal{X}\to(\mathbb{C}^1_\lambda,0)$ be a flat deformation of $X=\mathcal{X}_0:=\pi^{-1}(0)$ over a germ $(\mathbb{C}^1_\lambda,0)$ with a flat closed subspace $\mathcal{C}\subset\mathcal{X}$ such that $C=\mathcal{C}_0$. Then there exist a flat deformation $\mathcal{Z}\to(\mathbb{C}^1_\lambda,0)$ and a proper $\mathbb{C}^1_\lambda$-morphism $\mathfrak{f}:\mathcal{X}\to\mathcal{Z}$ such that $f=\mathfrak{f}_0$ and $\mathfrak{f}_\lambda:(\mathcal{X}_\lambda,\mathfrak{f}_\lambda^{-1}(o_\lambda)_{\mathrm{red}})\to(\mathcal{Z}_\lambda,o_\lambda)$ is a divisorial (resp., flipping, $\mathbb{Q}$-conic bundle) extremal curve germ for every small λ, where $o_\lambda:=\mathfrak{f}_\lambda(\mathcal{C}_\lambda)$.

COROLLARY 3.2.1

Let $f:(X,C)\to(Z,o)$ be an extremal divisorial curve germ, where C is not necessarily irreducible. Let $P^{(1)},\dots,P^{(r)}\in X$ be singular points. Let $(X_\lambda,P^{(i)}_\lambda)\supset(C_\lambda,P^{(i)}_\lambda)$ be a set of local one-parameter analytic deformations of $(X,P^{(i)})\supset(C,P^{(i)})$. Then it extends to a one-parameter analytic deformation $X_\lambda\supset C_\lambda\supset\{P^{(1)}_\lambda,\dots,P^{(r)}_\lambda\}$ of global $X\supset C\supset\{P^{(1)},\dots,P^{(r)}\}$ in the sense that there exist a flat deformation $\mathcal{Z}\to(\mathbb{C}^1_\lambda,0)$ and a proper $\mathbb{C}^1_\lambda$-morphism $\mathfrak{f}:\mathcal{X}\to\mathcal{Z}$ such that $f=\mathfrak{f}_0$ and $\mathfrak{f}_\lambda:(\mathcal{X}_\lambda,\mathfrak{f}_\lambda^{-1}(o_\lambda)_{\mathrm{red}})\to(\mathcal{Z}_\lambda,o_\lambda)$ is a divisorial extremal curve germ for every small λ, where $o_\lambda:=\mathfrak{f}_\lambda(\mathcal{C}_\lambda)$.

We need the following easy lemma, which can be found in [Bin, (9.3)] (without proof).

LEMMA 3.3

Let $p:\mathcal{D}\to\mathcal{X}\supset\ell$ be an arbitrary analytic morphism, and let $\ell\subset X$ be a compact subset such that $p^{-1}(\ell)$ is compact. Then there exist open subsets $W\supset p^{-1}(\ell)$ of $\mathcal{D}$ and $V\supset p(W)$ of $\mathcal{X}$ such that $p|_W:W\to V$ is proper and $p(W)$ is an analytic subset of V.

Proof

There is an open subset $U\supset p^{-1}(\ell)$ of $\mathcal{D}$ such that $\bar{U}$ is compact (and U is open and closed in $\mathcal{D}\setminus\partial U$). Since $p(\partial U)$ is a closed set disjoint from ℓ, there is an open set $V\supset\ell$ such that $\bar{V}$ is disjoint from $p(\partial U)$. Then $p^{-1}(\bar{V})$ is disjoint from ∂U. Hence $W:=U\cap p^{-1}(V)$ is an open and closed subset of $p^{-1}(V)$ and is $\bar{W}\subset U$

is compact. Hence $p|_W : W \to V$ is proper. This means that $p(W)$ is an analytic subset of V. □

The following is the key step in the proof of Theorems 3.1 and 3.2.

PROPOSITION 3.4

Let $f:(X,C)\to(Z,o)$ *be a divisorial extremal curve germ, where* C *is not necessarily irreducible. Let* $\bar{\pi}:\bar{\mathscr{X}}\to(\mathbb{C}^1_\lambda,0)$ *be a flat deformation of* $X=\bar{\mathscr{X}}_0:=\bar{\pi}^{-1}(0)$ *over a germ* $(\mathbb{C}^1_\lambda,0)$.

(i) *Let* $\bar{\mathscr{X}}^\wedge$ *be the completion of* $\bar{\mathscr{X}}$ *along* $\lambda=0$. *Then* $f:X\to Z$ *extends to a contraction* $\mathfrak{f}^\wedge:\bar{\mathscr{X}}^\wedge\to\mathscr{Z}^\wedge$.

(ii) *Let* n *be an arbitrary positive integer. Then there exist flat deformations* $\pi:\mathscr{X}\to(\mathbb{C}^1_\lambda,0)$ *and* $\mathscr{Z}\to(\mathbb{C}^1_\lambda,0)$ *and a proper* $\mathbb{C}^1_\lambda$*-morphism* $\mathfrak{f}:\mathscr{X}\to\mathscr{Z}$ *such that* $\pi_{(n)}\simeq\bar{\pi}_{(n)}$, $f=\mathfrak{f}_0$, *and* $\mathfrak{f}_\lambda:\mathscr{X}_\lambda\to\mathscr{Z}_\lambda$ *is a divisorial contraction (which contracts a divisor to a curve) for every small* λ, *where* $\mathcal{A}_{(i)}:=\mathcal{A}\times_{\mathbb{C}^1_\lambda}\operatorname{Spec}\mathbb{C}[[\lambda]]/(\lambda^{i+1})$ *for any object* $\mathcal{A}$ *over* $\mathbb{C}^1_\lambda$ *and* $i\geq 0$.

Proof

Let $\phi\in H^0(X,\mathscr{O}_X)$ be a general section vanishing on C, and let H (resp., H_Z) be the member of $|\mathscr{O}_X|$ (resp., $|\mathscr{O}_Z|$) defined by ϕ (resp., $f_*\phi$). We note that H (resp., H_Z) is smooth outside C (resp., o) and f induces an isomorphism $H\setminus C\simeq H_Z\setminus\{o\}$.

Then as in [KM, (11.3), (11.4)], the miniversal deformation spaces $\operatorname{Def}(H)$ and $\operatorname{Def}(H_Z)$ exist as analytic spaces, and f induces a complex analytic morphism $\operatorname{Def}(f,H):\operatorname{Def}(H)\to\operatorname{Def}(H_Z)$. Let $\phi:X\to\mathbb{C}^1_s$ be the morphism defined by $s=\phi$. This morphism is a flat family of H over $\mathbb{C}^1_s$. Thus we have an induced morphism $\bar{w}:\mathbb{C}^1_s\to\operatorname{Def}(H)$, that is, an element $\bar{w}\in\operatorname{Hom}(\mathbb{C}^1_s,\operatorname{Def}(H))$. Furthermore, X, Z, and f can be reconstructed by the morphism $\bar{w}:(\mathbb{C}^1_s,0)\to\operatorname{Def}(H)$. Our goal is to construct the following morphism extending $\bar{w}$:

$$w:(\mathbb{C}^2_{s,\lambda},0)\longrightarrow\operatorname{Def}(H).$$

Since $R^1f_*\mathscr{O}_X=0$, the section ϕ extends to a formal section $\hat{\phi}$ on the completion $\bar{\mathscr{X}}^\wedge$ of $\bar{\mathscr{X}}$ along X. This proves (i). We thus see that $\bar{w}\in\operatorname{Hom}(\mathbb{C}^1_s,\operatorname{Def}(H))$ extends to $\hat{w}\in\operatorname{Hom}((\mathbb{C}^2_{s,\lambda},0)^\wedge,\operatorname{Def}(H))$, where $(\mathbb{C}^2_{s,\lambda},0)^\wedge$ is the completion of $(\mathbb{C}^2_{s,\lambda},0)$ along $\{\lambda=0\}$. Then by [Art, Theorem 1.5(i)], $\hat{w}$ can be approximated by an analytic extension $w\in\operatorname{Hom}((\mathbb{C}^2_{s,\lambda},0),\operatorname{Def}(H))$ of $\bar{w}$. This gives us a flat family $\mathscr{X}$ over $\mathbb{C}^1_\lambda$ approximating $\bar{\mathscr{X}}$.

It remains to settle divisoriality. Arbitrarily close to C there is an f-exceptional curve $\ell\simeq\mathbb{P}^1$ such that $N_{\ell/X}\simeq\mathscr{O}_\ell\oplus\mathscr{O}_\ell(-1)$, which sweep out an f-exceptional divisor of X. Hence, $N_{\ell/\mathscr{X}}\simeq\mathscr{O}_\ell^{\oplus 2}\oplus\mathscr{O}_\ell(-1)$, and there are no obstructions to deforming these ℓ out to $\mathscr{X}_\lambda$. Hence, $\mathfrak{f}_\lambda$ contracts a divisor. This proves statement (ii) of our proposition. □

Proof of Theorem 3.1
Let $P^{(1)}, \dots, P^{(r)} \in X$ be singular points. As in [Mor2, Appendix 1b], one can see that every local deformation of singularities extends to a deformation of global X. For every terminal singularity $(X, P^{(i)})$ we take a $\mathbb{Q}$-smoothing, a deformation whose general member has only cyclic quotient singularities (see [Rei2, (6.4)]). By the above, there exists a one-parameter deformation $\bar{\mathcal{X}}$ over a disk in $\mathbb{C}^1_\lambda$ such that $\bar{\mathcal{X}}_0 \simeq X$ and, for small $\lambda \neq 0$, the fiber $\bar{\mathcal{X}}_\lambda$ has only terminal cyclic quotient singularities. Then we apply Proposition 3.4(ii). In notation of Proposition 3.4, there exists a divisorial contraction $\mathfrak{f} : \mathcal{X} \to \mathcal{Z}$ contracting a divisor $\mathcal{E}$ (the divisorial part of the exceptional locus) to a surface on $\mathcal{Z}$, and, for small $\lambda \neq 0$, the fiber $\mathcal{X}_\lambda$ also has only terminal cyclic quotient singularities because at every singular point P of X the local germ of $\bar{\mathcal{X}}$ at P can be approximated by one of $\mathcal{X}$ to an arbitrarily high order of λ.

Let $P \in X = \mathcal{X}_0$ be a singular point, and let $(X^\sharp, P^\sharp)$ be the index-one cover. Then the local deformation $(\mathcal{X}, P)$ is induced by a deformation $(\mathcal{X}^\sharp, P^\sharp)$ of $(X^\sharp, P^\sharp)$ (cf. [Ste, Section 6, last paragraph]). Since the germ $(X^\sharp, P^\sharp)$ is a hypersurface singularity (see [Rei1]), so is $(\mathcal{X}^\sharp, P^\sharp)$. Moreover, the singularity $(\mathcal{X}^\sharp, P^\sharp)$ is isolated. Hence, by [Gro, Exp. XI, Corollary 3.14], the variety $\mathcal{X}^\sharp$ is factorial at $P^\sharp$, and so $\mathcal{X}$ is $\mathbb{Q}$-factorial at P. In particular, $\mathcal{E}$ is a $\mathbb{Q}$-Cartier divisor. Thus $\mathcal{E}|_X = E$ on $X \setminus C$. If, moreover, C is irreducible, then $\rho(X) = 1$ (see [Mor2, (1.3)]), and so $K_X \sim_{\mathbb{Q}} \mathcal{E}|_X$. Hence, $\mathcal{E}|_X$ is negative on C and $\mathcal{E}|_X \supset C$. This implies that $E = \mathcal{E}|_X$, and it is also $\mathbb{Q}$-Cartier. □

Proof of Theorem 3.2
The flipping case follows from [KM, (11.4)], and the $\mathbb{Q}$-conic bundle case from [MP1, (6.2)]. So we assume that f is divisorial. Let $E \subset X$ be the exceptional divisor of f, and let the E_i's be its irreducible components. Then, for each i, $B_i := f(E_i) \subset Z$ is an irreducible curve passing through o.

First, we treat the case where C is irreducible. Then by Theorem 3.1, E is a $\mathbb{Q}$-Cartier divisor and $Z \ni o$ is a terminal singularity.

For each E_i, choose a smooth fiber ℓ'_i of $E_i \to B_i$, and let $[\ell'_i]$ degenerate to $[\ell_i]$ lying over o in the Douady space of X/Z. We assume that each $[\ell'_i]$ is chosen arbitrarily close to $[\ell_i]$. Consider the closed subspace A' of the Douady space of X/Z parameterizing all compact subspaces $F \subset X$ with $\operatorname{Supp} F \subset C$. Then each irreducible component of A' is compact (see [Fuj]), and we let A be the smallest open and closed subset of A' containing all $[\ell_i]$. Thus A is also compact. Then we work on a sufficiently small neighborhood $\mathcal{D}'$ of A in the Douady space of $\mathcal{X}/\mathbb{C}^1_\lambda$ such that $\mathcal{D}' \ni [\ell'_i]$ for each i.

We note that $\mathcal{X}$ is smooth along each ℓ'_i and that $N_{\ell'_i/\mathcal{X}} \simeq \mathscr{O}_{\ell'_i}^{\oplus 2} \oplus \mathscr{O}_{\ell'_i}(-1)$. Hence, $\mathcal{D}'$ is smooth of dimension 2 at each $[\ell'_i]$. Let $\mathcal{D} \subset \mathcal{D}'$ be the smallest one among the union of the irreducible components of $\mathcal{D}'$ such that $\mathcal{D} \ni [\ell'_i]$ for all i. Then $\mathcal{D}$ is a 2-dimensional closed subspace of $\mathcal{D}'$.

Let $\mathcal{T} \subset \mathcal{X} \times_{\mathbb{C}^1_\lambda} \mathcal{D}$ be the universal closed subspace parameterized by $\mathcal{D}$ with two projections $\pi : \mathcal{T} \to \mathcal{D}$ and $p : \mathcal{T} \to \mathcal{X}$.

We note that $p^{-1}(C) \subset A$, and it is compact because the variety $\mathcal{X}_0 = X$ has a divisorial contraction to Z, C is the fiber over $o \in Z$, and $\pi^{-1}(t)$ does not intersect C for $t \notin A$.

Let $\mathcal{E} := p(\mathcal{T}) \subset \mathcal{X}$ be the image of the proper morphism p and it is an analytic subset by Lemma 3.3. We also denote by $p : \mathcal{T} \to \mathcal{E}$ the morphism induced by p and let $p : \mathcal{T} \xrightarrow{p'} \bar{\mathcal{T}} \xrightarrow{\bar{p}} \mathcal{E}$ be the Stein factorization of p so that $p'_* \mathscr{O}_{\mathcal{T}} = \mathscr{O}_{\bar{\mathcal{T}}}$.

CLAIM 3.4.1

$\mathcal{E}$ is a $\mathbb{Q}$-Cartier divisor.

Proof

Let $\mathcal{X}^\wedge$ be the completion of $\mathcal{X}$ along $\lambda = 0$. By Proposition 3.4(i), the morphism $f : X \to Z$ extends to a contraction $\mathfrak{f}^\wedge : \mathcal{X}^\wedge \to \mathcal{Z}^\wedge$, where $\mathcal{Z}^\wedge$ is $\mathbb{Q}$-Gorenstein (see [Ste, Corollary 10]) because Z is terminal. Comparing $K_{\mathcal{X}^\wedge}$ and $\mathfrak{f}^{\wedge *} K_{\mathcal{Z}^\wedge}$, we see that there is an effective $\mathbb{Q}$-Cartier divisor $\mathcal{F}^\wedge \sim_{\mathbb{Q}} K_{\mathcal{X}^\wedge} - \mathfrak{f}^{\wedge *} K_{\mathcal{Z}^\wedge}$ on $\mathcal{X}^\wedge$ such that $\mathcal{F}^\wedge|_{X^\wedge} = E^\wedge$ and $\mathcal{F}^\wedge = \mathcal{E}^\wedge$ outside of $C^\wedge$. Hence $\mathcal{F}^\wedge = \mathcal{E}^\wedge$. $\square$

Now we define a morphism $q : \mathcal{D} \to \mathcal{B}$ such that $q(p^{-1}(C))$, is one point as follows. Take a general point ζ of C, and take a small 3-dimensional disk $(\Delta^3, 0)$ centered at ζ and transversal to C at ζ. Then the Cartier divisor Δ^3 in a neighborhood of C induces a Cartier divisor of $\mathcal{T}$ finite and flat over $\mathcal{D}$. Let d be the degree of $p^{-1}(\Delta^3)/\mathcal{D}$. Then $x \in \mathcal{D} \mapsto \pi^{-1}(x) \cap p^{-1}(\Delta^3)$ associates to x a zero-cycle of degree d on Δ^3 and we have thus a required morphism $q : \mathcal{D} \to \mathcal{B} := S^d(\Delta^3)$ such that $q(p^{-1}(C))$ is the zero-cycle $d \cdot [0]$.

We claim that we have a proper morphism $r : \bar{\mathcal{T}} \to \mathcal{B}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "p'"] \arrow[d, "\pi"] & \bar{\mathcal{T}} \arrow[r, "\bar{p}"] \arrow[d, "r"] & \mathcal{E} \\
\mathcal{D} \arrow[r, "q"] & \mathcal{B} &
\end{tikzcd}$$

Indeed, since $q\bigl(\pi(p^{-1}(C))\bigr)$ is one-point $d \cdot [0]$, we can shrink $\mathcal{E}$ so that $q(\mathcal{D})$ is contained in a Stein open neighborhood of $d \cdot [0]$. Hence the morphism $\mathcal{T} \to \mathcal{B}$ factors through $p' : \mathcal{T} \to \bar{\mathcal{T}}$, and the claim is proved.

We claim that p, p', $\bar{p}$ are isomorphisms over every ℓ'_i, and, in particular, $\bar{p}$ is finite and bimeromorphic. Indeed, by $N_{\ell'_i/\mathcal{X}} \simeq \mathscr{O}_{\ell'_i}^{\oplus 2} \oplus \mathscr{O}_{\ell'_i}(-1)$, p is an isomorphism near $\pi^{-1}([\ell'_i])$, and by the divisorial contraction on $X = \{\lambda = 0\} \subset \mathcal{X}$, one has $p^{-1}(\ell'_i) = \pi^{-1}([\ell'_i])$. These settle the claim.

Let $\mathfrak{c} := \mathscr{H}om_{\mathscr{O}_{\mathcal{T}}}(\mathscr{O}_{\bar{\mathcal{T}}}, \mathscr{O}_{\mathcal{T}})$ be the conductor of $\bar{p}$, and let $V(\mathfrak{c}) \subset \bar{\mathcal{T}}$ be the locus defined by $\mathfrak{c}$. Then we claim that $r(V(\mathfrak{c}))$ is finite over $\mathbb{C}^1_\lambda$. Indeed, this is obvious since $r(V(\mathfrak{c})) \not\ni q([\ell_i'])$ and the fiber of $r(V(\mathfrak{c}))$ over $\{\lambda = 0\}$ is a finite set.

Let $J \subset \mathscr{O}_{\mathcal{B}}$ be an arbitrary sheaf of ideals such that $J\mathscr{O}_{\bar{\mathcal{T}}} \subset \mathfrak{c}$ and $V(J)$ is finite over $\mathbb{C}^1_\lambda$. By [Bin, Theorem (6.1)], we have the following diagram:

$$\begin{tikzcd}
V(J) \arrow[r] \arrow[d, hook] & \mathbb{C}^1_\lambda \arrow[d] \\
\mathcal{B} \arrow[r, "q'"] & \mathcal{E}'
\end{tikzcd}$$

where $\mathcal{E}' := \mathcal{B} \coprod_{V(J)} \mathbb{C}^1_\lambda$ is the amalgamated sum (coproduct) of $\mathcal{B}$ and $\mathbb{C}^1_\lambda$ over $V(J)$ and $q' : \mathcal{B} \to \mathcal{E}'$ is a bimeromorphic finite morphism. Since $\mathfrak{c}$ is the conductor of $\bar{p}$, we have

$$\mathcal{E} = \bar{\mathcal{T}} \coprod\nolimits_{V_{\bar{\mathcal{T}}}(\mathfrak{c})} V_{\mathcal{E}}(\mathfrak{c})$$

and the following commutative diagram:

$$\begin{tikzcd}
V_{\bar{\mathcal{T}}}(\mathfrak{c}) \arrow[r] \arrow[d, hook] & V_{\mathcal{E}}(\mathfrak{c}) \arrow[d, hook] \\
\bar{\mathcal{T}} \arrow[r, "\bar{p}"] & \mathcal{E}
\end{tikzcd}$$

These two diagrams fit into a big one, which allows us to define an induced morphism $\eta : \mathcal{E} \to \mathcal{E}'$:

$$\begin{tikzcd}
V_{\bar{\mathcal{T}}}(\mathfrak{c}) \arrow[r] \arrow[d, hook] \arrow[drr] & V_{\mathcal{E}}(\mathfrak{c}) \arrow[d, hook] \arrow[drr, bend left, "r"] & & \\
\bar{\mathcal{T}} \arrow[r] \arrow[drr, bend right] & \mathcal{E} \arrow[drr, dashed, "\eta"] & V(J) \arrow[r] \arrow[d, hook] & \mathbb{C}^1_\lambda \arrow[d] \\
 & & \mathcal{B} \arrow[r, "q'"'] & \mathcal{E}'
\end{tikzcd}$$

Finally, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "p'"] \arrow[d, "\pi"] & \bar{\mathcal{T}} \arrow[r, "\bar{p}"] \arrow[d, "r"] & \mathcal{E} \arrow[d, "\eta"] \\
\mathcal{D} \arrow[r, "q"] & \mathcal{B} \arrow[r, "q'"] & \mathcal{E}'
\end{tikzcd}$$

For any $i, j \geq 0$ the sheaf $\mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})$ denotes the quotient $\mathscr{O}_{\mathcal{X}}(-j\mathcal{E})/\mathscr{O}_{\mathcal{X}}(-(i+j)\mathcal{E})$.

CLAIM 3.4.2

For any $i,j\geq 0$, we have $R^1\eta_*\mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})=0$. Therefore, the sequence

$$0\longrightarrow \eta_*\mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})\longrightarrow \eta_*\mathscr{O}_{(i+j)\mathcal{E}}\longrightarrow \eta_*\mathscr{O}_{j\mathcal{E}}\longrightarrow 0$$

is exact.

Proof

By the Kawamata-Viehweg vanishing (see [Nak, Theorem 3.6]), $R^1 f_*\mathscr{O}_X(-k\times E)=0$ for $k\geq 0$. Then from the exact sequence

$$0\longrightarrow \mathscr{O}_X\big(-(i+j)E\big)\longrightarrow \mathscr{O}_X(-jE)\longrightarrow \mathscr{O}_{iE}(-jE)\longrightarrow 0$$

we see that $R^1 f_*\mathscr{O}_{iE}(-jE)=0$ for $i,j\geq 0$. Now we assert that the sequence

$$0\longrightarrow \mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})\xrightarrow{\cdot\lambda} \mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})\longrightarrow \mathscr{O}_{iE}(-jE)\longrightarrow 0$$

is exact for $i,j\geq 0$. Recall that the space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein (see [Ste, Section 6, last paragraph]. Consider the index-one cover $\nu:(\mathcal{X}^\sharp,P^\sharp)\to(\mathcal{X},P)$ with respect to $\mathcal{E}$ at an arbitrary point $P\in X$. Since the map ν is étale in codimension two, both $\mathcal{X}^\sharp$ and $X^\sharp:=\nu^{-1}(X)$ are terminal. The induced divisors $\mathcal{E}^\sharp$ and $E^\sharp$ are Cartier on $\mathcal{X}^\sharp$ and $X^\sharp$, respectively, and $E^\sharp=\mathcal{E}^\sharp|_{X^\sharp}$. Hence the assertion on exactness can be readily checked on $\mathcal{X}^\sharp$. Then by Nakayama's lemma we obtain $R^1\eta_*\mathscr{O}_{i\mathcal{E}}(-j\mathcal{E})=0$. □

Fix a positive integer m such that both mE and $m\mathcal{E}$ are Cartier, and define a ringed space $\mathcal{E}''$ as a topological space $\operatorname{Spec}_{\mathcal{E}'}\eta_*\mathscr{O}_{\mathcal{E}}$ with the sheaf of rings $\eta_*\mathscr{O}_{m\mathcal{E}}$. Then $\mathcal{E}''$ is a complex space by Claim 3.4.2 and [Bin, Section 10].

Now we show that $\mathcal{X}$ has a modification, and to do that we check conditions (1) and (2) of Bingener [Bin, Corollary 8.2] for the morphism $\mathcal{X}\supset m\mathcal{E}\to\mathcal{E}''$ induced by η. Condition (1) is obvious because $-\mathcal{E}$ is ample, and condition (2) follows from the exact sequence in Claim 3.4.2 with $j=1$. Thus the desired contraction $\mathfrak{f}:\mathcal{X}\to\mathcal{Z}$ exists by [Bin, Corollary 8.2]. So the proof of the case of irreducible C is completed.

Now we consider the general case; that is, we assume that C is reducible. Run an analytic minimal model program on $\mathcal{X}$ in the following way. Every irreducible K-negative curve on the central fiber of X/Z generates an extremal ray on X. By [KM, (11.7)], flips on X extend to ones on $\mathcal{X}$. So do divisorial contractions by our previous arguments. By Theorem 3.1, we stay in the terminal category. At the end we get $X'\subset\mathcal{X}'/\mathbb{C}^1_\lambda$ such that X' is a minimal model over Z. Moreover, all fibers of $f':X'\to Z$ are of dimension ≤ 1, and $-K_{X'}$ is ample over Z outside of the central fiber. Hence $f':X'\to Z$ is a small contraction. Note that $R^1 f'_*\mathscr{O}_{X'}=R^1 f_*\mathscr{O}_X=0$. By [KM, (11.4)] the contraction $f':X'\to Z$ extends to $\mathfrak{f}':\mathcal{X}'\to\mathcal{Z}$. Thus we have a bimeromorphic map $\mathfrak{f}:\mathcal{X}\dashrightarrow\mathcal{Z}$. By Zariski's main theorem, this map is actually a proper morphism. This proves Theorem 3.2. □

4. Case: $\mathrm{cD}/3$

In this section we prove Theorem 1.5.

SETUP 4.1

Let (X,C) be an extremal curve germ, and let $f:(X,C)\to(Z,o)$ be the corresponding contraction. In particular, f can be flipping. Throughout this section we assume that (X,C) is of type (IA) and the only non-Gorenstein point $P\in(X,C)$ is of type cD/3 (see [Mor1], [Rei2]). Our arguments here are very similar to those in [KM, Section 6]. Note that by Corollary 2.2.4 the point (H,P) is not log terminal for any divisor $H\in|\mathscr{O}_X|_C$. Let $\sigma=(\sigma_1,\dots,\sigma_n)$ be a weight. Below, for a formal power series α in n variables, $\alpha_{\sigma=m}$ means the sum of the monomials in α whose σ-weight is m. Put $\sigma:=(1,1,2,3)$. As in [KM, paragraph 6.5], up to coordinate change the point (X,P) is given by

$$\{\alpha(y_1,y_2,y_3,y_4)=0\}\subset\mathbb{C}^4_{y_1,y_2,y_3,y_4}/\boldsymbol{\mu}_3(1,1,2,0),$$

where

$$\alpha=y_4^2+y_3^3+\delta_3(y_1,y_2)+(\text{terms of degree}\ge 4),$$

$\delta_3(y_1,y_2)=\alpha_{\sigma=3}(y_1,y_2,0,0)\ne 0$, $\operatorname{wt}\alpha\equiv 0 \mod 3$, and $C^\sharp$ is the y_1-axis. If $\delta_3(y_1,y_2)$ is square free (resp., has a double factor, is a cube of a linear form), then (X,P) is said to be a *simple* (resp., *double*, *triple*) (cD/3)-point. The general member $F\in|-K_X|$ modulo a coordinate change is given by the equation $y_1=0$ (see [Rei2]).

LEMMA 4.2

In the above coordinate system there exists a member $H\in|\mathscr{O}_X|_C$ given by the equation $y_4=\xi$, where $\xi=\xi(y_1,y_2,y_3)$ is an invariant in the ideal $(y_2,y_3)^3+y_1(y_2,y_3)$.

Proof

We have the following exact sequence:

$$0\longrightarrow\omega_X\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{O}_F\longrightarrow 0.$$

4.2.1. (cf. [Mor2, Theorem 1.2])

If f is a birational contraction, then $R^1f_*\omega_X=0$ by the Grauert-Riemenschneider vanishing theorem. Hence any section $\bar{s}\in\mathscr{O}_F$ lifts to a section $s\in f_*\mathscr{O}_X$. So the assertion is clear in this case.

4.2.2

Assume that f is a $\mathbb{Q}$-conic bundle. Obviously, $\tau:=f|_F$ is a double cover. Since $R^1f_*\omega_X=\omega_Z$ (see [MP1, Lemma 4.1]) and $\omega_F\simeq\mathscr{O}_F$, we have

$$f_*\mathscr{O}_X\longrightarrow\tau_*\omega_F\longrightarrow\omega_Z\longrightarrow 0.$$

The last map is nothing but the trace map $\mathrm{Tr}_{F/Z}:\tau_*\omega_F\to\omega_Z$. According to [MP2, 2.1, 2.2] the induced map

$$f_*\mathscr{O}_X\longrightarrow\tau_*(\omega_F/\tau^*\omega_Z)$$

is surjective.

We may assume that the equation of F in $\mathbb{C}^3_{y_2,y_3,y_4}$ is as follows:

$$\beta(y_2,y_3,y_4):=\alpha(0,y_2,y_3,y_4)=y_4^2+y_3^3+\delta_3(0,y_2)+(\text{terms of degree}\geq 4).$$

Locally, near $P^\sharp$, the sheaf $\omega_{F^\sharp}$ is generated by

$$\eta:=\operatorname{Res}\frac{dy_2\wedge dy_3\wedge dy_4}{\beta}=-\frac{dy_2\wedge dy_3}{\partial\beta/\partial y_4}=\frac{dy_2\wedge dy_4}{\partial\beta/\partial y_3}=-\frac{dy_3\wedge dy_4}{\partial\beta/\partial y_2}.$$

Since η is an invariant, it is also a generator of ω_F near P. Further, since Z is smooth, one has

$$\tau^*\Omega^2_Z=\tau^*\omega_Z\subset\Omega^2_F\longrightarrow\omega_F.$$

The generators of $\mathscr{O}_{F,P}$ are y_4, $w:=y_2y_3$, $u:=y_2^3$, and $v:=y_3^3$ with relations $uv=w^3$ and $y_4^2+v+u+\cdots=0$. Eliminating v we get three generators y_4, w, u and one relation $u(u+y_4^2+\cdots)+w^3=0$. Hence Ω^2_F is generated by the elements

$$\begin{aligned}dw\wedge du&=d(y_2y_3)\wedge d(y_2^3)=3y_2^3dy_3\wedge dy_2,\\ du\wedge dy_4&=d(y_2^3)\wedge dy_4=3y_2^2dy_2\wedge dy_4,\\ dw\wedge dy_4&=d(y_2y_3)\wedge dy_4=y_2dy_3\wedge dy_4+y_3dy_2\wedge dy_4.\end{aligned}$$

Then Ω^2_F is contained in ηI, where

$$I:=\langle y_3^3\partial\beta/\partial y_4,y_2^2\partial\beta/\partial y_3,y_2\partial\beta/\partial y_2,y_3\partial\beta/\partial y_3\rangle\subset(y_2,y_3,y_4)^3.$$

So $\tau^*\omega_Z\subset(\tau_*\omega_F)I$. Therefore, for some $\xi\in I$ the section $\bar{s}=y_4-\xi\in\mathscr{O}_F$ lifts to a section $s\in f_*\mathscr{O}_X$. Since

$$s\equiv y_4\mod(y_2,y_3,y_4)^3+y_1(y_2,y_3,y_4),$$

one can apply Weierstrass's preparation theorem to get Lemma 4.2. □

COROLLARY 4.3

If y_4 is a part of an ℓ-free ℓ-basis of $\mathrm{gr}^1_C\mathscr{O}$, then a general member $H\in|\mathscr{O}_X|_C$ is normal.

4.4

Recall that $\ell(P):=\mathrm{len}_{P^\sharp}I^{\sharp(2)}/I^{\sharp 2}$ (see [Mor2, Corollary-Definition 9.4.7]). According to [Mor2, Lemma 2.16] we have $i_P(1)=\lfloor\ell(P)/3\rfloor+1$, and the coordinate system (y_i) can be chosen so that $\alpha\equiv y_1^{\ell(P)}y_i\mod(y_2,y_3,y_4)^2$, where $i\in\{2,3,4\}$ and $\ell(P)+\mathrm{wt}\,y_i\equiv 0\mod 3$. Since (X,P) is of type cD/3, we have $\ell(P)>1$.

Now we are going to prove Theorem 1.5 by considering cases according to the value of $\ell(P)$. We start with the case $\ell(P)=2$.

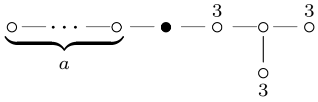
THEOREM 4.5

Let the notation and assumptions be as in Section 4.1. Assume that $\ell(P) = 2$ *or, equivalently,* $i_P(1) = 1$. *Then the following assertions hold.*

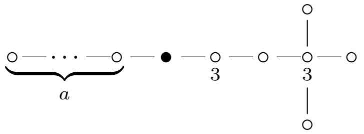
(4.5.1) *The contraction* f *is birational; the general member* $H \in |\mathscr{O}_X|_C$ *and its image* $T = f(H) \in |\mathscr{O}_Z|$ *are normal and have only rational singularities.*

(4.5.2) *If* f *is flipping (resp., divisorial), then* P *is not a triple* (cD/3)*-point and the dual graph of* (H, C) *is given as follows with* $a = 0$ *(resp.,* $a = 1$*):*

(4.5.2.1) *Case of simple* (cD/3)*-point* P:



(4.5.2.2) *Case of double* (cD/3)*-point* P:



(4.5.3) *We have* $\mathrm{gr}^1_C \mathscr{O} = (a) \tilde{\oplus} (-a + P^\sharp)$.

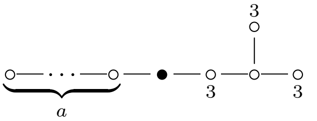
We now start the proof of Theorem 4.5.

Proof

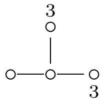
In addition to assuming Section 4.1, we assume that $\ell(P) = 2$. Then by [Mor2, Lemma 2.16], $i_P(1) = 1$ and (in some coordinate system) α satisfies $\alpha \equiv y_1^2 y_2 \mod (y_2, y_3, y_4)^2$. Here $C^\sharp$ is the y_1-axis as above. Hence y_3, y_4 form an ℓ-basis of $\mathrm{gr}^1_C \mathscr{O}$. By Corollary 4.3, H is normal and by Lemma 2.3.3, $H \setminus \{P\}$ can have at most one singular point R which is Du Val. Therefore, X can have at most one type (III) point.

4.5.4. Subcase: $\alpha_{\sigma=3}(y_1, y_2, 0, 0)$ is squarefree (cf. Setup 4.1)

By [KM, case 6.7.1] and Lemma 2.3.3, the graph $\Delta(H, C)$ is of the form



We have $a \leq 1$ since the corresponding matrix is negative semidefinite. But then this matrix is negative definite. Hence the contraction f is birational. If $a = 1$, then H is contracted to a singularity $T = f(H)$ of type A_2. Since T is Gorenstein, f is a divisorial contraction as in Section 1.5.1. If $a = 0$, that is, if P is the only singular point of H, then H is contracted to a singularity $T = f(H)$ with the following dual graph:

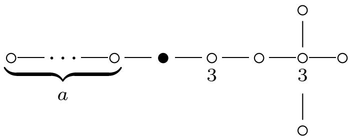


Let $s \in H^0(X, \mathscr{O}_X)$ be the section defining H. Then $s\mathscr{O}_C \subset \mathrm{gr}^1_C \mathscr{O}$ is a subbundle outside P since $H \setminus \{P\}$ is smooth. At $P^\sharp$, $s\mathscr{O}^\sharp_C$ is a subbundle of $\mathrm{gr}^1_C \mathscr{O}^\sharp$ by Lemma 4.2, whence $s\mathscr{O}_C \simeq (0)$ with ℓ-structure. Since $\deg \mathrm{gr}^1_C \mathscr{O} = 0$ by $i_P(1) = 0$, we have $\mathrm{gr}^1_C \mathscr{O} = (0) \tilde{\oplus} (P^\sharp)$. Thus f is flipping by [KM, (6.2.4)]).

By 4.5.4 it remains to consider the case where $\alpha_{\sigma=3}(y_1, y_2, 0, 0)$ has a double factor. Note that y_2 divides $\alpha_{\sigma=3}(y_1, y_2, 0, 0)$ because $C^\sharp = (y_1\text{-axis}) \subset X^\sharp$. Since $\ell(P) = 2$, $y_2 y_1^2 \in \alpha$. Then making a coordinate change $y_1 \mapsto y_1 + cy_2$, we get $\alpha_{\sigma=3}(y_1, y_2, 0, 0) = y_1^2 y_2$ and $C^\sharp$ unchanged.

4.5.5. Subcase: $\alpha_{\sigma=3}(y_1, y_2, 0, 0) = y_1^2 y_2$ and $\alpha_{\sigma=6}(0, y_2, y_3, 0)$ is squarefree

As above, by [KM, 6.7.2] and Lemma 2.3.3 the graph $\Delta(H, C)$ is of the form



with $a \le 1$. Again, if $a = 1$, then T is Du Val of type D_4, so f is a divisorial contraction as in Section 1.5.2. If $a = 0$, then similarly to Section 4.5.4 the contraction f is flipping (cf. [KM, (6.2.3.2)]). Since $s\mathscr{O}^\sharp_C$ is a subbundle of $\mathrm{gr}^1_{C^\sharp} \mathscr{O}^\sharp_C$ at $P^\sharp$, as we saw above, it is easy to see Section 4.5.3.

4.5.6. Subcase: $\alpha_{\sigma=3}(y_1, y_2, 0, 0) = y_1^2 y_2$, and $\alpha_{\sigma=6}(0, y_2, y_3, 0)$ has a multiple factor

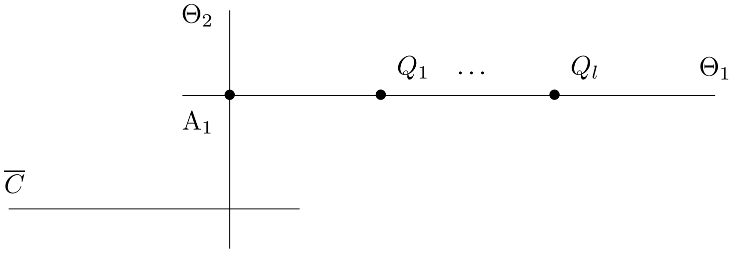
We will show that this case does not occur. Assume that f is birational. Then as in Section 4.2.2 the map $H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_F)$ is surjective. Therefore, for any $\lambda \in \mathbb{C}^*$ there is a semiinvariant δ with $\mathrm{wt}\,\delta = 2$ such that the section $y_4 + \lambda y_2^3 + \delta y_1$ extends to some element $H' \in |\mathscr{O}_X|_C$. After the coordinate change $y_4' = y_4 + \lambda y_2^3 + \delta y_1$ we see that H' is given by $y_4' = 0$ and $\alpha' = \alpha(y_1, y_2, y_3, y_4' - \lambda y_2^3 - \delta y_1)$. Note that $y_4^2 \in \alpha$, $y_4 \notin \alpha$, and α may contain $y_2^3 y_4$. Thus $\alpha'_{\sigma=3}(y_1, y_2, y_3, 0) = \alpha_{\sigma=3}(y_1, y_2, y_3, 0)$ and $\alpha'_{\sigma=6}(0, y_2, y_3, 0) = \alpha_{\sigma=6}(0, y_2, y_3, 0) + (\lambda^2 + c\lambda) y_2^6$ for some $c \in \mathbb{C}$. Hence we may assume that $\alpha_{\sigma=6}(0, y_2, y_3, 0)$ is squarefree. This contradicts our assumption. (In fact, the above arguments show that the chosen H is not general).

Therefore, f is a $\mathbb{Q}$-conic bundle. By Lemma 2.4, (H, P) is a rational singularity, and by Lemma 4.2, this singularity is analytically isomorphic to

$$\{\gamma(y_1, y_2, y_3) = 0\} / \boldsymbol{\mu}_3(1, 1, 2),$$

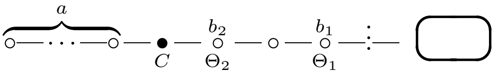
where $\gamma(y_1, y_2, y_3) := \alpha(y_1, y_2, y_3, \xi)$, and $C \subset H$ is the image of y_1-axis. Note that the pair (H, C) is not PLT at P. Indeed, otherwise the singularity $\{\gamma = 0\}$ is log terminal (see [Kol, Corollary 20.4]). Hence it is Du Val. On the other hand, $\operatorname{ord}\gamma > 2$, a contradiction. Let $\sigma' := (1,1,2)$. Note that $\gamma_{\sigma'=6}(0,0,1)\neq 0$ because $y_3^3\in\alpha$. Consider the weighted σ'-blowup $\varsigma\colon \overline{H}\subset\overline{\mathbb{C}^3/\boldsymbol{\mu}_3}\to H\subset\mathbb{C}^3/\boldsymbol{\mu}_3$. Let $\Xi := \varsigma^{-1}(0)_{\mathrm{red}}$. The exceptional divisor $\Theta\subset\overline{H}$ is given in $\Xi\simeq\mathbb{P}(1,1,2)$ by the equation $\gamma_{\sigma'=3}(y_1,y_2,0) = y_1^2y_2 = 0$. Hence $\Theta = 2\Theta_1+\Theta_2$, where Θ_i are irreducible toric divisors in $\mathbb{P}(1,1,2)$. The proper transform $\overline{C}$ of C meets $\Xi\simeq\mathbb{P}(1,1,2)$ at the point $\{y_2 = y_3 = 0\}$. So $\overline{C}\cap\Theta_1=\emptyset$. Since Θ_2 is a smooth reduced component of the Cartier divisor $\Theta = \Xi\cap\overline{H}$ on $\overline{H}$, we see that $\overline{H}$ is smooth at points on $\Theta_2\setminus\Theta_1$.

In the chart $U_3\simeq\mathbb{C}^3/\boldsymbol{\mu}_2(1,1,1)$ over $\{y_3\neq 0\}$ we have a new coordinate system $y_1\mapsto y_1y_3^{1/3}$, $y_2\mapsto y_2y_3^{1/3}$, $y_3\mapsto y_3^{2/3}$. Here the surface $\overline{H}$ is given by the equation $y_1^2y_2+\gamma_{\sigma'=6}(y_1,y_2,1)y_3+(\cdots)y_3^2=0$, where $\gamma_{\sigma'=6}(0,0,1)\neq 0$. The origin $O_3\in\overline{H}\cap U_3$ is a Du Val point of type A_1. Components Θ_1 and Θ_2 of the exceptional divisor meet each other at O_3 and the pair $(\overline{H},\Theta_1+\Theta_2)$ is LC at O_3. Outside of O_3, $\overline{H}$ is a hypersurface and has only rational singularities. Therefore, the singularities of $\overline{H}$ are Du Val. Thus the curves $\overline{C}$, Θ_1, and Θ_2 on $\overline{H}$ look as follows:



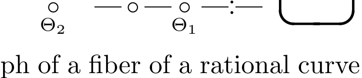
where $Q_1,\dots,Q_l$ are some Du Val points and $\Theta_1\cap\Theta_2$ is a Du Val point of type A_1. By Lemma 2.3.3 the dual graph $\Delta(H,C)$ is of the form

(4.5.6.1)

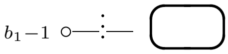


where the vertical dots $\vdots$ mean that one or more curves are attached here; the box on the right-hand side indicates some Du Val graphs, and the number of these Du Val tails is not important. This configuration forms a fiber of a rational curve fibration. Contracting black vertices successively we obtain

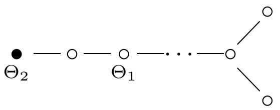
(4.5.6.2)



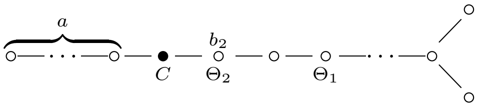
This is again a dual graph of a fiber of a rational curve fibration. Hence $b_2-a-1=1$, and we further obtain



Hence $b_1 = 2$ (because the last graph must contain a (-1)-vertex), and so the graph (4.5.6.2) consists of (-2)- and (-1)-curves. Furthermore, the graph (4.5.6.2) is not a linear chain because the pair (H,C) is not PLT at P. In this situation there is only one possibility (see, e.g., [Pro2, Lemmas 7.1.3, 7.1.12]):



Therefore, the original graph (4.5.6.1) is of the form



But then H has only log terminal singularities (see, e.g., [Kol, Chapter 3]). Hence H has only T-singularities (see Definition 2.2.1), while the right-hand side singularity is not of type T (see Proposition 2.2.2), a contradiction. Thus the case of Section 4.5.6 does not occur.

Now the assertion of Theorem 4.5 follows from Sections 4.5.4, 4.5.5, and 4.5.6. This completes our treatment of the case $\ell(P) = 2$. □

COROLLARY 4.6

In the notation of Section 4.1, X has at most one type (III) *point.*

Proof

If X has two type (III) points R_1 and R_2, then by [Mor2, (2.3.3)] and [MP1, (3.1.5)] we have $i_P(1) = i_{R_1}(1) = i_{R_2}(1) = 1$. Then by [Mor2, Lemma 2.16], $\ell(P) = 2$. This contradicts Theorem 4.5. □

LEMMA 4.7 (CF. [KM, LEMMA 6.12])

If, in the notation of Section 4.1, X has a type (III) *point, then $\ell(P) \leq 4$ and $i_P(1) \leq 2$.*

Proof

Assume $\ell(P) \geq 5$. As in Section 4, take a coordinate system so that $\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$, where $i \in \{2,3,4\}$ and $\ell(P) + \operatorname{wt} y_i \equiv 0 \mod 3$. Similarly to the proof of [KM, Lemma 6.12], we use the deformation $\alpha_\lambda = \alpha + \lambda y_1^{\ell(P)-3} y_i$ (see Theorem 3.2) and get a germ (X_λ, C_λ) with two type (III) points and a point of type cD/3. This contradicts Corollary 4.6. □

For the case $\ell(P) \geq 3$, we are going to prove the following, which settles Theorem 1.5.

THEOREM 4.8

Let the notation and assumptions be as in Section 4.1. Assume $\ell(P) \geq 3$ or, equivalently, $i_P(1) \geq 2$. Then the following assertions hold.

(4.8.1) *We have* $\ell(P)=3$ *or* 4 *(i.e.,* $i_P(1)=2$*), and* f *is birational.*

(4.8.2) P *is a double (resp., triple)* (cD/3)*-point if* (X,C) *is isolated (resp., divisorial).*

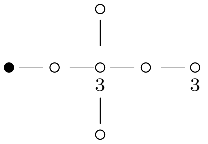
(4.8.3) X *is smooth outside of* P*, and there is an* ℓ*-isomorphism*

$$\operatorname{gr}_C^1 \mathscr{O} = \bigl((4-\ell(P))P^\sharp\bigr) \mathbin{\tilde{\oplus}} (-1+2P^\sharp). \tag{4.8.3.1}$$

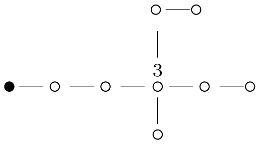
(4.8.4) *For general members* $D \in |K_X|$ *and* $D' \in |K_X|$ *(resp.,* $D' \in |\mathscr{O}_X|_C$*),* $D \cap D'$ *is equal to* $4C$ *(resp.,* $3C$*) as a* 1*-cycle.*

(4.8.5) *The general member* $H \in |\mathscr{O}_X|_C$ *and its image* $T = f(H) \in |\mathscr{O}_Z|$ *are normal and have only rational singularities. The dual configuration of* (H,C) *is as follows:*

(4.8.5.1) *Case of isolated* (X,C):



(4.8.5.2) *Case of divisorial* (X,C):



(4.8.6) *Conversely, if* (X,C) *is an arbitrary germ of a threefold along* $C \simeq \mathbb{P}^1$ *with a double (resp., triple)* (cD/3)*-point* $P \in C$*. If* (X,C) *satisfies the statement 4.8.3, then* (X,C) *is an isolated (resp., a divisorial) extremal curve germ.*

Proof
In the hypothesis of Section 4.1 we additionally assume that $\ell(P) \geq 3$.

LEMMA 4.9
Under the notation of Theorem 4.8*,* X *has no type* (III) *points.*

Proof
Assume that X has a type (III) point R. We derive a contradiction. By Lemma 4.7, $\ell(P)=3$ or 4.

4.9.1. Case $\ell(P)=3$

We claim that $H^1(\operatorname{gr}_C^2 \omega) \neq 0$. By [Mor2, Lemma 2.16], $i_P(1)=2$, and (in some coordinate system) α satisfies $\alpha \equiv y_1^3 y_4 \mod (y_2,y_3,y_4)^2$ (and $C^\sharp$ is the y_1-axis). If α contains the term $y_1^k y_2 y_3$, then $k \geq 3$ and this term can be removed by the coordinate change $y_4 \mapsto y_4 - y_1^{k-3} y_2 y_3$. Hence we may assume that

$$\alpha \equiv y_1^3 y_4 + \lambda y_1 y_2^2 + \mu y_1^2 y_3^2 \mod (y_2, y_3)^3 + y_4(y_2, y_3, y_4) \quad (\subset I_C^{(3)\sharp})$$

for some $\lambda, \mu \in \mathscr{O}_X \mod I_C$. The functions y_2, y_3 form an ℓ-basis of $\operatorname{gr}_C^1 \mathscr{O}$ at P. Since

$$\deg \operatorname{gr}_C^1 \mathscr{O} = 1 - i_P(1) - i_R(1) = -2$$

and $H^1(\operatorname{gr}_C^1 \mathscr{O}) = 0$, we have $\operatorname{gr}_C^1 \mathscr{O} = \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. Furthermore, by [KM, Lemma 2.8], one has

$$\operatorname{gr}_C^1 \mathscr{O} = (-1 + P^\sharp) \mathbin{\tilde{\oplus}} (-1 + 2P^\sharp),$$

where y_3 (resp., y_2) is an ℓ-free ℓ-basis of $(-1 + P^\sharp)$ (resp., of $(-1 + 2P^\sharp)$) at P. Let σ be an ℓ-basis of ω. By the above, one has

$$\omega \mathbin{\tilde{\otimes}} \widetilde{S}^2 \operatorname{gr}_C^1 \mathscr{O} = (-2 + P^\sharp) \mathbin{\tilde{\oplus}} (-2 + 2P^\sharp) \mathbin{\tilde{\oplus}} (-1),$$

where $y_3^2 \sigma$ (resp., $y_3 y_2 \sigma$, $y_2^2 \sigma$) is an ℓ-free ℓ-basis of $(-2 + P^\sharp)$ (resp., $(-2 + 2P^\sharp)$, (-1)) at P. There is an injection of coherent sheaves

$$\iota : \omega \mathbin{\tilde{\otimes}} \widetilde{S}^2 \operatorname{gr}_C^1 \mathscr{O} \longrightarrow \operatorname{gr}_C^2 \omega.$$

As an abstract sheaf, $\omega \mathbin{\tilde{\otimes}} \widetilde{S}^2 \operatorname{gr}_C^1 \mathscr{O}$ at P is generated by sections $y_3^2 y_1 \sigma$, $y_3 y_2 y_1^2 \sigma$, $y_2^2 \sigma$. Further, it is easy to see that $I_C^{(2)\sharp} / I_C^{(3)\sharp}$ at P is generated by elements y_4, y_3^2, $y_2 y_3$, y_2^2. Hence $\operatorname{gr}_C^2 \omega$ at P is generated by $y_4 y_1^2 \sigma$, $y_3^2 y_1 \sigma$, $y_2 y_3 y_1^2 \sigma$, $y_2^2 \sigma$. On the other hand, $y_4 \in I_C^{(2)}$ and $y_1^2 y_2 y_4, y_1 y_3 y_4 \in I_C^{(3)}$. By our expression of α,

$$(y_1^2 y_4 + \lambda y_2^2 + \mu y_1 y_3^2)\sigma = 0 \quad \text{in } \operatorname{gr}_C^2 \omega \text{ at } P.$$

Hence $\operatorname{gr}_C^2 \omega$ at P is generated by the elements $y_3^2 y_1 \sigma$, $y_2 y_3 y_1^2 \sigma$, $y_2^2 \sigma$. This means that ι is an isomorphism at P.

Since $i_R(1) = 1$, by [Mor2, Lemma 2.16], $\ell(R) = 1$, and in some coordinate system the local equation $\beta(z_1, \dots, z_4) = 0$ of (X, R) satisfies $\beta \equiv z_1 z_2 \mod (z_2, z_3, z_4)^2$, where C is the z_1-axis. Then locally near R we have $I_C^{(2)} = (z_3^2, z_3 z_4, z_4^2, z_2)$, so

$$\operatorname{gr}_C^1 \mathscr{O} = \mathscr{O} z_3 \oplus \mathscr{O} z_4 \qquad \text{and} \qquad S^2 \operatorname{gr}_C^1 \mathscr{O} = \mathscr{O} z_3^2 \oplus \mathscr{O} z_4^2 \oplus \mathscr{O} z_3 z_4.$$

Furthermore, $\operatorname{gr}_C^2 \mathscr{O}$ is generated by z_2, z_3^2, z_4^2, $z_3 z_4$. Hence z_2 generates $\operatorname{Coker} \iota$, and so $\operatorname{len}_R \operatorname{Coker} \iota \le 1$. In this case, $\dim H^0(\operatorname{Coker} \iota) \le 1$ and $\dim H^1(\operatorname{Im} \iota) = 2$. Therefore, $H^1(\operatorname{gr}_C^2 \omega) \ne 0$ as claimed.

Now from $H^0(\operatorname{gr}_C^j \omega) = 0$, where $j = 0, 1$ and the exact sequences

$$0 \longrightarrow \operatorname{gr}_C^n \omega \longrightarrow \omega_X / F^{n+1} \omega_X \longrightarrow \omega_X / F^n \omega_X \longrightarrow 0, \quad n = 1, 2,$$

we have $H^1(\omega_X / F^3 \omega_X) \ne 0$. If f is birational, then by [Mor2, Theorem 1.2, Remark 1.2.1], we get a contradiction. Assume that f is a $\mathbb{Q}$-conic bundle. Put $V := \operatorname{Spec}_X \mathscr{O}_X / I_C^{(3)}$. By [MP1, Theorem 4.4], $V \supset f^{-1}(o)$. Since

$$-K_X \cdot V = -6 K_X \cdot C = 2 = -K_X \cdot f^{-1}(o),$$

we have $V = f^{-1}(o)$. Let $P \in C$ be a general point. Then in a suitable coordinate system (x, y, z) near P we may assume that C is the z-axis. So $I_C = (x, y)$ and

$I_C^{(3)}=(x^3,x^2y,xy^2,y^3)$. But then $V=f^{-1}(o)$ is not a local complete intersection near P, a contradiction. This disproves case 4.9.1.

4.9.2. Case $\ell(P)=4$

By deformation $\alpha_\lambda=\alpha+\lambda y_1^3y_4$ at (X,P), we get a germ (X_λ,C_λ) with a point P_λ of type cD/3 with $\ell(P_\lambda)=3$ (see Theorem 3.2). Moreover, X_λ has a point R_λ of type (III). This is impossible by case 4.9.1.

This proves Lemma 4.9. □

From now on we treat the case where P is the only singular point of X and $\ell(P)\geq 3$.

LEMMA 4.10 (CF. [KM, LEMMA 6.12])

In the notation of Section 4.1 we have $\ell(P)\leq 4$ *and* $i_P(1)\leq 2$.

Proof

Assume that $\ell(P)\geq 5$. Similarly to [KM, Lemma 6.12] and Lemma 4.7 we write $\alpha\equiv y_1^{\ell(P)}y_j \mod (y_2,y_3,y_4)^2$, where $j\in\{2,3,4\}$ and $\ell(P)+\operatorname{wt}y_j\equiv 0 \mod 3$, and we use deformation $\alpha_\lambda=\alpha+\lambda y_1^{\ell(P)-3}y_j$ (see Theorem 3.2). We get a germ (X_λ,C_λ) with a type (III) point R_λ and a point P_λ of type cD/3 with $\ell(P_\lambda)=\ell(P)-3$. If $\ell(P)\geq 6$, we get a contradiction by Lemma 4.9 considered above.

Hence $\ell(P)=5$, and $X\setminus\{P\}$ is smooth by Lemma 4.7. Then $\alpha\equiv y_1^5y_2 \mod (y_2,y_3,y_4)^2$, $\deg\operatorname{gr}_C^1\mathscr{O}_X=-1$, and y_4,y_3 form an ℓ-basis for $\operatorname{gr}_C^1\mathscr{O}_X$. Thus H is normal at P by Corollary 4.3, and we see that $\operatorname{gr}_C^1\mathscr{O}_X=(0)\mathbin{\tilde\oplus}(-1+P^\sharp)$, H is smooth outside P, y_3 is an ℓ-basis of $\operatorname{gr}_C^1\mathscr{O}_H$, and $\operatorname{gr}_C^1\mathscr{O}_H=(-1+P^\sharp)$. We also see that

$$\operatorname{gr}_C^0\omega_H=\operatorname{gr}_C^0\omega_X=(-1+2P^\sharp)\qquad\text{and}\qquad\operatorname{gr}_C^1\omega_H=(-1).$$

We note that $C^\sharp=y_1\text{-axis}\subset H^\sharp\subset\mathbb{C}^3_{y_1,y_2,y_3}$, and $H^\sharp=\{\beta=0\}$, where

$$\beta\equiv y_1^5y_2+cy_1^2y_3^2 \mod (y_2^2,y_2y_3,y_3^3),$$

and $c\in\mathbb{C}$. We claim $c\neq 0$. Indeed, otherwise we have $y_2\in\mathscr{O}_H(-3C)^\sharp$, whence $\operatorname{gr}_C^2\mathscr{O}_H^\sharp=\mathscr{O}_{C^\sharp}y_3^2$ and $\operatorname{gr}_C^2\mathscr{O}_H=(\operatorname{gr}_C^1\mathscr{O}_H)^{\tilde\otimes 2}=(-2+2P^\sharp)$. Thus $H^1(H,\mathscr{O}_H)\neq 0$, a contradiction. Hence $c\neq 0$.

Since P is a (cD/3) point, we have $y_2y_3\notin\alpha$ and $y_3^3\in\alpha$, and hence $y_2y_3\notin\beta$ and $y_3^3\in\beta$. Since $c\neq 0$, the terms $\gamma(y_1)y_1^3y_2y_3$ can be killed by a $\boldsymbol{\mu}_3$-coordinate change $y_3\mapsto y_3-\gamma(y_1)y_1y_2/(2c)$, and we may further assume

$$\beta\equiv y_1^5y_2+cy_1^2y_3^2+y_3^3 \mod (y_2^2,y_2y_3^2,y_3^4). \tag{4.10.1}$$

We claim that $\operatorname{gr}_C^2\mathscr{O}_H=(-1+2P^\sharp)$ and $\operatorname{gr}_C^3\mathscr{O}_H=(-1)$. First, by $y_2\in\mathscr{O}_H(-2C)^\sharp$, one has $y_1^2(y_1^3y_2+cy_3^2)\in\mathscr{O}_H(-3C)^\sharp$. Hence if we set $z:=y_1^3y_2+cy_3^2$, then $z\in\mathscr{O}_H(-3C)^\sharp$ and $y_3^2\equiv -y_1^3y_2/c \mod (z)$. Thus by $\mathscr{O}_H(-2C)^\sharp=(y_2,y_3^2)$, we see

$$\mathscr{O}_H(-2C)^\sharp/(y_2^2, y_2y_3, z) = \mathscr{O}_{C^\sharp}y_2 \simeq \mathscr{O}_{C^\sharp} \qquad \text{and} \qquad \mathscr{O}_H(-3C)^\sharp = (y_2^2, y_2y_3, z).$$

We also have $y_1^2 z + y_3^3 \in (y_2^2, y_2y_3^2, y_3^4)$ by (4.10.1), whence $z \equiv y_1y_2y_3/c \mod (y_2^2, y_2y_3^2, zy_3)$. Thus

$$\mathscr{O}_H(-3C^\sharp)/(y_2^2, y_2y_3^2, zy_3) = \mathscr{O}_{C^\sharp}y_2y_3 \simeq \mathscr{O}_{C^\sharp} \qquad \text{and}$$
$$\mathscr{O}_H(-4C)^\sharp = (y_2^2, y_2y_3^2, zy_3).$$

From these follows the claim:

$$\operatorname{gr}^2_C \mathscr{O}_H = (\operatorname{gr}^1_C \mathscr{O}_H)^{\tilde{\otimes} 2}(3P^\sharp) = (-1 + 2P^\sharp)$$

and

$$\operatorname{gr}^3_C \mathscr{O}_H = \operatorname{gr}^1_C \mathscr{O}_H \mathbin{\tilde{\otimes}} \operatorname{gr}^2_C \mathscr{O}_H = (-1).$$

We then claim that $H^1(\omega_H/\omega_H(-4C)) \neq 0$. Indeed, this follows from

$$\operatorname{gr}^2_C \omega_H = \operatorname{gr}^0_C \omega_H \mathbin{\tilde{\otimes}} \operatorname{gr}^2_C \mathscr{O}_H = (-1 + P^\sharp)$$

and

$$\operatorname{gr}^3_C \omega_H = \operatorname{gr}^0_C \omega_H \mathbin{\tilde{\otimes}} \operatorname{gr}^3_C \mathscr{O}_H = (-2 + 2P^\sharp).$$

Since $\omega_H = \omega_X \mathbin{\tilde{\otimes}} \mathscr{O}_H$, the nonvanishing $H^1(\omega_H/\omega_H(-4C)) \neq 0$ means that f is a $\mathbb{Q}$-conic bundle (see [Mor2, Remark 1.2.1]) and the subscheme $4C$ of H contains the scheme-theoretic fiber $f^{-1}(o)$ (see [MP1, Theorem 4.4]). However,

$$-K_X \cdot 4C = 4/3 < 2 = -K_X \cdot f^{-1}(o),$$

a contradiction. The case $\ell(P) = 5$ is thus disproved. □

4.11. Case $\ell(P) = 3$ and no type (III) points

By [Mor2, Lemma 2.16], $i_P(1) = 2$ and (in some coordinate system) α satisfies $\alpha \equiv y_1^3 y_4 \mod (y_2, y_3, y_4)^2$ (and $C^\sharp$ is the y_1-axis). Hence y_2, y_3 form an ℓ-basis of $\operatorname{gr}^1_C \mathscr{O}$. Since $\deg \operatorname{gr}^1_C \mathscr{O} = 1 - i_P(1) = -1$ and $H^1(\operatorname{gr}^1_C \mathscr{O}) = 0$, $\operatorname{gr}^1_C \mathscr{O} = \mathscr{O} \oplus \mathscr{O}(-1)$. Further, by [KM, (2.8)], there are only two possibilities:

$$\operatorname{gr}^1_C \mathscr{O} = \begin{cases} (2P^\sharp), & \tilde{\oplus}\,(-1 + P^\sharp), \\ (P^\sharp), & \tilde{\oplus}\,(-1 + 2P^\sharp). \end{cases}$$

Consider the first case, that is, $\operatorname{gr}^1_C \mathscr{O} = (2P^\sharp) \mathbin{\tilde{\oplus}} (-1 + P^\sharp)$. Then the arguments in the first part of the proof of ([KM, (6.13)]) can be applied. Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O} = (2P^\sharp)$. Then we conclude that $H^1(\omega/F^4(\omega, J)) \neq 0$ (see [KM, pp. 599–600]). If the contraction f is birational, we get a contradiction by [Mor2, Theorem 1.2, Remark 1.2.1]. Let f be a $\mathbb{Q}$-conic bundle. Put $V := \operatorname{Spec}_X \mathscr{O}_X/F^4(\mathscr{O}, J)$. Then $V \equiv mC$ for some m. By [MP1, Theorem 4.4], $V \supset f^{-1}(o)$. Hence $m/3 = -K_X \cdot V < 2 = -K_X \cdot f^{-1}(o)$. On the other hand, near a general point $S \in C$, J is generated by (z_2, z_3^2), where (z_1, z_2, z_3) are some local coordinates such that C is the z_1-axis. Hence

$F^4(\omega, J) = J^2 = (z_2, z_3^2)^2$ near S. So $m = \operatorname{len}(\mathbb{C}[z_2, z_3]/F^4(\omega, J)) = 6$. Therefore, $f^{-1}(o) = V$, and its ideal sheaf coincides with $F^4(\omega, J)$. However, $F^4(\omega, J)$ is not generated by two elements near S, so $f^{-1}(o)$ is not a locally complete intersection, a contradiction.

Consider the second case, that is, $\operatorname{gr}^1_C \mathscr{O} = (P^\sharp) \mathbin{\tilde{\oplus}} (-1 + 2P^\sharp)$. If (X, P) is a double (cD/3)-point, then f is a flipping contraction by [KM, Theorem 6.3], whence we get the configuration (4.8.5.1). Thus we assume that the term $y_1 y_2^2$ does not appear in α. Further, we use arguments from the proof of [KM, Lemma 6.13, p. 600]. Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O} = (P^\sharp)$. Modulo a $\boldsymbol{\mu}_3$-equivariant change of coordinates, we may further assume that y_3 (resp., y_2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp., $(-1 + 2P^\sharp)$) in $\operatorname{gr}^1_C \mathscr{O}$ and that $\alpha \equiv y_1^3 y_4 \mod I^\sharp J^\sharp$. Whence $J^\sharp = (y_2^2, y_3, y_4)$ at $P^\sharp$ and $y_4 \in F^3(\mathscr{O}, J)^\sharp$. Let K be the ideal such that $J \supset K \supset F^3(\mathscr{O}, J)$ and $K/F^3(\mathscr{O}, J) = (P^\sharp)$ in

$$\operatorname{gr}^2(\mathscr{O}, J) = \operatorname{gr}^{2,0}(\mathscr{O}, J) \mathbin{\tilde{\oplus}} \operatorname{gr}^{2,1}(\mathscr{O}, J) = (P^\sharp) \mathbin{\tilde{\oplus}} (-1 + P^\sharp).$$

Here we may assume that y_3 (resp., y_2^2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp., $(-1 + P^\sharp)$) in the above ℓ-splitting modulo a coordinate change $y_3 \mapsto y_3 + (\cdots) y_2^2$. We then have $K^\sharp = (y_2^3, y_3, y_4)$ at $P^\sharp$ and

$$\operatorname{gr}^1(\mathscr{O}, K) = (-1 + 2P^\sharp), \qquad \operatorname{gr}^2(\mathscr{O}, K) = (-1 + P^\sharp).$$

We have $\operatorname{gr}^{3,0}(\mathscr{O}, K) \simeq \operatorname{gr}^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$ and

$$\alpha \equiv y_1^3 y_4 + c y_2^3 \mod I^\sharp K^\sharp$$

for some unit $c \in \mathscr{O}_X^\times$ because $I^\sharp J^\sharp = I^\sharp K^\sharp + (y_2^3)$ and $y_2^3 \in \alpha$. Whence we have an ℓ-isomorphism

$$\operatorname{gr}^{3,1}(\mathscr{O}, K) \simeq \operatorname{gr}^1(\mathscr{O}, K)^{\tilde{\otimes} 3}(3P^\sharp) \simeq (0)$$

as in [KM, p. 600] and an ℓ-splitting

$$\operatorname{gr}^3(\mathscr{O}, K) = \operatorname{gr}^{3,0}(\mathscr{O}, K) \mathbin{\tilde{\oplus}} \operatorname{gr}^{3,1}(\mathscr{O}, K),$$

in which y_3 (resp., y_4) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp., (0)) modulo a coordinate change $y_3 \mapsto y_3 + (\cdots) y_1^2 y_4$. For any $l > 0$ there is a natural exact sequence

$$0 \longrightarrow F^{l+1}(\mathscr{O}, K) \longrightarrow F^l(\mathscr{O}, K) \longrightarrow \operatorname{gr}^l(\mathscr{O}, K) \longrightarrow 0. \tag{4.11.1}$$

We claim that the sections $y_1 y_3, y_4 \in \operatorname{gr}^3(\mathscr{O}, K)$ can be extended to sections of $F^3(\mathscr{O}, K) = F^1(K)$. By (4.11.1) it is sufficient to show that $H^1(F^4(\mathscr{O}, K)) = 0$. There are injections of coherent sheaves

$$\begin{aligned}
\operatorname{gr}^{3n}(\mathscr{O}, K) &\hookleftarrow S^n \operatorname{gr}^3(K), \\
\operatorname{gr}^{3n+1}(\mathscr{O}, K) &\hookleftarrow S^n \operatorname{gr}^3(K) \mathbin{\tilde{\otimes}} \operatorname{gr}^1(\mathscr{O}, K), \\
\operatorname{gr}^{3n+2}(\mathscr{O}, K) &\hookleftarrow S^n \operatorname{gr}^3(K) \mathbin{\tilde{\otimes}} \operatorname{gr}^2(\mathscr{O}, K)
\end{aligned}$$

with cokernels of finite length. Therefore, for any $l > 0$, the degree of each component in a decomposition of $\operatorname{gr}^l(\mathscr{O}, K)$ in a direct sum is at least -1. Then

$H^1(\mathrm{gr}^l(\mathscr{O},K))=0$, and from (4.11.1) we get surjections

$$H^1\big(F^{l+n}(\mathscr{O},K)\big)\to H^1\big(F^l(\mathscr{O},K)\big)\quad \text{for } l,n>0.$$

Hence $H^1(F^l(\mathscr{O},K)/F^{l+n}(\mathscr{O},K))=0$. Note that for any $m>0$ there is $n>0$ such that $I_C^m F^l(\mathscr{O},K)\supset F^{l+n}(\mathscr{O},K)$. By the formal function theorem we have

$$\begin{aligned} H^1\big(F^l(\mathscr{O},K)\big)^{\wedge} &= H^1\big(F^l(\widehat{\mathscr{O}},K)\big)=\varprojlim H^1\big(F^l(\mathscr{O},K)/I_C^m F^l(\mathscr{O},K)\big)\\ &=\varprojlim H^1\big(F^l(\mathscr{O},K)/F^{l+n}(\mathscr{O},K)\big)=0. \end{aligned}$$

Hence $H^1(F^l(\mathscr{O},K))=0$ for $l>0$, and there are surjections

$$H^0\big(F^l(\mathscr{O},K)\big)\longrightarrow H^0\big(\mathrm{gr}^l(\mathscr{O},K)\big)\longrightarrow 0.$$

This proves our claim. Therefore, near P a general member $H\in|\mathscr{O}_X|_C$ is given by equations $\alpha(y_1,\dots,y_4)=0$ and $\beta(y_1,\dots,y_4)=0$, where $\alpha=y_4^2+y_3^3+y_2^3+$ (terms of degree ≥ 4) (recall that $\alpha\not\ni y_1^2y_2, y_1y_2^2$), $\beta\equiv\lambda y_3y_1+y_4 \mod F^4(\mathscr{O},K)$, and $\lambda\in\mathscr{O}_{\mathbb{C}^4}$ such that $\lambda(P)\in\mathbb{C}$ can be chosen arbitrarily. Hence we can eliminate y_4 and get

$$(H,P)=\big\{\gamma(y_1,y_2,y_3)=0\big\}/\boldsymbol{\mu}_3(1,1,2)\supset C=y_1\text{-axis}/\boldsymbol{\mu}_3,$$

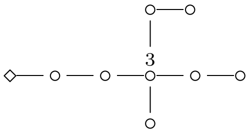
where γ is a $\boldsymbol{\mu}_3$-invariant convergent power series such that, for $\sigma=(1,1,2)$, $\gamma_{\sigma=3}=y_2^3$ and the term $\gamma_{\sigma=6}(y_1,0,y_3)$ is squarefree. Hence we are done by Computation 4.12.

COMPUTATION 4.12

Let (D,P) be a normal surface singularity

$$(D,P)=\{\gamma=0\}/\boldsymbol{\mu}_3\subset\mathbb{C}^3/\boldsymbol{\mu}_3(1,1,2),$$

where $\gamma=\gamma(y_1,y_2,y_3)$ is $\boldsymbol{\mu}_3$-invariant, and let $C:=(y_1\text{-axis})/\boldsymbol{\mu}_3$. Let σ be the weight $(1,1,2)$. Assume that $\gamma_{\sigma=3}=y_2^3$, and assume that $\gamma_{\sigma=6}(y_1,0,y_3)$ is squarefree. Then D has only rational singularities, and $\Delta(D,C)$ is as follows:



Sketch of the proof

We note that $\gamma_{\sigma=6}(y_1,0,y_3)$ contains y_3^3 since it is squarefree. Consider the weighted blowup $\hat{H}\to H$ with weights $1/3(1,1,2)$. The exceptional divisor Λ is given by $\gamma_{\sigma=3}=y_2^3=0$ in the weighted projective plane $\mathbb{P}(1,1,2)$. Hence Λ is a smooth rational curve. Clearly, $\mathrm{Sing}(\hat{H})$ is contained in Λ. In the chart $U_1:=\{y_1\neq 0\}$ the surface $\hat{H}$ is given by

$$y_2^3+y_1\gamma_{\sigma=6}(1,y_2,y_3)+y_1^2\gamma_{\sigma=9}(1,y_2,y_3)+\cdots=0.$$

Hence $\mathrm{Sing}(\hat{H})\cap U_1$ is given by $y_1=y_2=\gamma_{\sigma=6}(1,0,y_3)=0$. Since $\gamma_{\sigma=6}(1,0,y_3)$ is a cubic polynomial without multiple factors, $\mathrm{Sing}(\hat{H})\cap U_1$ consists of three points: $P_0 := (0,0,0)$, P_1, P_2. In particular, this shows that $\hat{H}$ is normal. Further, $\gamma_{\sigma=6}(1,y_2,y_3)$ contains the term y_3. Hence at the origin $\hat{H}$ has a Du Val singularity of type A_2, and the pair

$$(\hat{H},\Lambda+\hat{C})\simeq(\{y_2^3+y_1y_3=0\},\{y_2=0\})$$

is LC, where $\hat{C}$ is the proper transform of C. This gives us the left-hand side of the graph. Similarly, from P_1 and P_2 we get the upper and the right-hand side of the graph. The vertex $\circ$ in the bottom comes from the chart $y_3\neq 0$. The computation of the self-intersection number of the central vertex is an easy exercise. □

4.13. Case $\ell(P)=4$ and no type (III) points

By [Mor2, Lemma 2.16], $i_P(1)=2$ and (in some coordinate system) α satisfies $\alpha\equiv y_1^4y_3 \mod (y_2,y_3,y_4)^2$ (and $C^\sharp$ is the y_1-axis). Hence y_2, y_4 form an ℓ-basis of $\mathrm{gr}^1_C\mathscr{O}$.

We prove claim 4.8.3. Since it has been proved that a type (III) point does not occur, it remains to settle the ℓ-isomorphism (4.8.3.1). If it does not hold, then we have $\mathrm{gr}^1_C\mathscr{O}=(2P^\sharp)\mathbin{\tilde{\oplus}}(-1)$ and $\mathrm{gr}^1_C\omega=(P^\sharp)\mathbin{\tilde{\oplus}}(-2+2P^\sharp)$, whence $H^1(\mathrm{gr}^1_C\omega)\neq 0$. Thus we get a contradiction as in case 4.11, and claim 4.8.3 is proved.

If (X,C) is flipping, then claims 4.8.2, 4.8.4, and 4.8.5 are already proved in [KM, (6.3)]. Since $\ell(P)>2$, P is a double or triple (cD/3)-point, claim 4.8.6 is proved in [KM, (6.3.4)] if P is a double (cD/3)-point.

Assume that (X,C) is not isolated. Then P, as a (cD/3)-point, is triple by $\ell(P)>2$ and [KM, (6.3.4)]. This proves Claim 4.8.2.

Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O}=(0)$ in the ℓ-splitting (4.8.3.1). Up to coordinate change we may assume that y_4 (resp., y_2) is an ℓ-free ℓ-basis of (0) (resp., $(-1+2P^\sharp)$) in $\mathrm{gr}^1_C\mathscr{O}$ and that $\alpha\equiv y_1^4y_3 \mod I^\sharp_CJ^\sharp$. Whence $y_3\in F^3(\mathscr{O},J)^\sharp$. We note that $y_1y_2^2\notin\alpha$ in the new coordinates since P is a triple (cD/3)-point.

Since we have ℓ-isomorphisms

$$\begin{aligned}\mathrm{gr}^{2,0}(\mathscr{O},J)&\simeq\mathrm{gr}^0(\mathscr{O},J)\simeq(0),\\ \mathrm{gr}^{2,1}(\mathscr{O},J)&\simeq\mathrm{gr}^1(\mathscr{O},J)^{\tilde{\otimes}2}\simeq(-1+P^\sharp),\end{aligned}$$

the ℓ-exact sequence

$$0\to\mathrm{gr}^{2,1}(\mathscr{O},J)\to\mathrm{gr}^2(\mathscr{O},J)\to\mathrm{gr}^{2,0}(\mathscr{O},J)\to 0$$

is ℓ-split. Let K be the ideal such that $J\supset K\supset F^3(\mathscr{O},J)$ and $K/F^3(\mathscr{O},J)=(0)$ in

$$\mathrm{gr}^2(\mathscr{O},J)\simeq(0)\mathbin{\tilde{\oplus}}(-1+P^\sharp).$$

Here we may assume that y_4 (resp., y_2^2) is an ℓ-free ℓ-basis of (0) (resp., $(-1+P^\sharp)$) modulo a coordinate change $y_4\mapsto y_4+(\cdots)y_1y_2^2$.

We have thus $K^\sharp = (y_2^3, y_3, y_4)$ and

$$\operatorname{gr}^1(\mathscr{O}, K) = (-1 + 2P^\sharp), \qquad \operatorname{gr}^2(\mathscr{O}, K) = (-1 + P^\sharp).$$

We have $\operatorname{gr}^{3,0}(\mathscr{O}, K) \simeq \operatorname{gr}^{2,0}(\mathscr{O}, J) \simeq (0)$ and

$$\alpha \equiv y_1^4 y_3 + c y_2^3 \mod I^\sharp K^\sharp$$

for some unit $c \in \mathscr{O}_X^\times$ because $I^\sharp J^\sharp = I^\sharp K^\sharp + (y_2^3)$ and $y_2^3 \in \alpha$. Whence we have an ℓ-isomorphism

$$\operatorname{gr}^{3,1}(\mathscr{O}, K) \simeq \operatorname{gr}^1(\mathscr{O}, K)^{\tilde{\otimes} 3}(4P^\sharp) \simeq (P^\sharp).$$

Thus we have an ℓ-splitting

$$\operatorname{gr}^3(\mathscr{O}, K) \simeq \operatorname{gr}^{3,0}(\mathscr{O}, K) \,\tilde{\oplus}\, \operatorname{gr}^{3,1}(\mathscr{O}, K) \simeq (0) \,\tilde{\oplus}\, (P^\sharp).$$

By a change of coordinate $y_4 \mapsto y_4 + (\cdots) y_1 y_3$, we may further assume that y_4 (resp., y_3) is an ℓ-free ℓ-basis of (0) (resp., $(P^\sharp)$). By the same computation as in case 4.11, we get the configuration (4.8.5.2). This contracts to a Du Val point of type E_6, and hence f is a divisorial contraction, which proves Claim 4.8.1.

Finally, we note that [KM, (6.15) and (6.20)] settled Claim 4.8.4 for isolated (X, C) and Claim 4.8.6 for a double, $(cD/3)$-point. We omit the proofs of Claims 4.8.4 and 4.8.6 in other cases since the arguments are similar. This completes our treatment of the case $\ell(P) > 2$. □

EXAMPLE 4.14

To show that all the possibilities in cases (1.5.1), (1.5.2), and (1.5.3), occur, we use deformation arguments. Consider the surface contraction $f_H : H \to T$ with dual graph of the form in cases (1.5.1) or (1.5.2). By [KM, Proposition 11.4] the natural map from the deformation space of H to the product of deformation spaces of singularities $P, R \in H$ is smooth, in particular, surjective. Moreover, the total deformation space $\mathfrak{X}$ of H has a morphism $\mathfrak{f}$ to the total deformation space $\mathfrak{X}_Z$ of T so that $\mathfrak{f}|_H = f_H$. This means in particular that any $\mathbb{Q}$-Gorenstein deformation of singularities of H can be globalized. Now assume that (H, P) and (H, R) can be obtained as hyperplane sections of some terminal singularities (X, P) and (X, R), respectively. Regard (X, P) and (X, R) as deformation spaces of (H, P) and (H, R), respectively. By the above there is a globalization $f : X \supset H \to Z \supset T$.

EXAMPLE 4.14.1

Consider the surface contraction $f_H : H \to T$ with dual graph (1.5.1), and consider the following terminal singularities:

$$\begin{aligned} (X, P) &= \{y_4^2 + y_3^3 + y_1 y_2 (y_1 + y_2) = 0\} / \boldsymbol{\mu}_3(1, 1, 2, 0), \\ (X, R) &= \{z_1 z_2 + z_3^2 + z_4^m = 0\}, \quad m \geq 1. \end{aligned}$$

Let $H \subset (X, P)$ be given by $y_4 = 0$, and let $H \subset (X, R)$ be given by $z_4 = 0$. By [KM, (6.7.1)] the dual graph of the minimal resolution of (H, P) is the same as that in 1.5.1. By Section 4.14 one obtains the corresponding birational contraction $f : X \supset H \to Z \supset T$. Here (X,P) is a simple (cD/3)-singularity (see [Rei2]). Therefore, this f is a divisorial contraction of type in case (1.5.1). The point $R \in X$ is smooth if $m = 1$ and is a cA_1-singularity if $m > 1$.

EXAMPLE 4.14.2

Similarly to Example 4.14.1, take

$$(X,P) = \{y_4^2 + y_1^2 y_2 + y_2^6 + y_3^3 = 0\}/\boldsymbol{\mu}_3(1,1,2,0).$$

By [KM, (6.7.2)] we get an example of a divisorial contraction as in case (1.5.2).

EXAMPLE 4.14.3

As above, take

$$(X,P) = \{y_2^3 + y_3^3 + y_3 y_1^4 + y_4^2\}/\boldsymbol{\mu}_3(1,1,2,0),$$

where H is cut out by $y_4 = y_1 y_3$. We get an example of a divisorial contraction as in case (1.5.3).

5. Case: P is of type $\mathrm{cA/m}$ and H is normal

In this section we prove Theorems 1.6 and 1.9 in the case where a general $H \in |\mathscr{O}_X|_C$ is normal. Thus throughout this section we assume that (X,C) is an extremal curve germ of type (IA) or ($\mathrm{IA}^\vee$) such that the only non-Gorenstein point $P \in X$ is of type cA/m (see Sections 1.4, 1.8). Let $F \in |-K_X|$ be a general member. Take $H \in |\mathscr{O}_X|_C$ so that the pair $(X, F+H)$ is LC (see Proposition 2.6). Assume that H is normal. Let $f : (X,C) \to (Z,o)$ be the corresponding contraction.

PROPOSITION 5.1

In the above notation, H has only log terminal singularities of type T. *Furthermore, the pair (H,C) is PLT outside of P and $H \setminus \{P\}$ has at most one singular point, which if it exists is Du Val of type* $\mathrm{A_n}$. *If, moreover, f is birational, then $\Delta(H,C)$ is as in (1.9.1.1). If, moreover, f is a $\mathbb{Q}$-conic bundle, then $\Delta(H,C)$ is of the form*

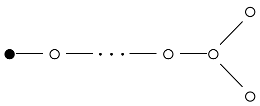
$$\circ - \circ - \circ - \bullet - \overset{4}{\circ} \tag{5.1.1}$$

In particular, $m = 2$ and (X,P) is either a cyclic quotient singularity $1/2(1,1,1)$ *or a singularity of the form* $\{xy + z^2 + t^k = 0\}/\boldsymbol{\mu}_2(1,1,1,0)$.

Proof

First, we claim that H has only log terminal singularities. Write $K_H + F|_H = (K_X + H + F)|_H \sim 0$. Recall that $F \cap C = \{P\}$. So $(H, F|_H)$ is not klt at P and klt at a general point of C. We see that $(H, F|_H)$ is klt outside of P by the connectedness lemma (if f is birational, see [Sho, 5.7], [Kol, 17.4]) and by Lemma 2.3.2 (if f is a $\mathbb{Q}$-conic bundle). On the other hand, by our assumptions and the adjunction formula, the pair $(H, F|_H)$ is LC near $F \cap H$, so the surface H has at worst log terminal singularities. Further, since H is a Cartier divisor in X, the singularities of H are of type T (see Definition 2.2.1).

Now we claim that the pair (H, C) is PLT outside of P. Assume that $K_H + C$ is not PLT at some point $Q \neq P$. Take c so that $(H, F|_H + cC)$ is maximally LC. By the connectedness lemma and Lemma 2.3.2, we have $c = 1$, so $(H, F|_H + C)$ is LC. Therefore, H has a log terminal singularity at Q, and the point (H, Q) is Du Val. From the classification of log canonical pairs (see, e.g., [Kol, Chapter 3]) we obtain that the part of the dual graph $\Delta(H, C)$ which represents H near the singularity Q is of the form



But then the corresponding matrix of this subgraph is not negative definite, a contradiction. Thus (H, C) is PLT outside of P. Since any point $Q \in H \setminus \{P\}$ is Gorenstein, it is Du Val of type $\mathrm{A_n}$ or smooth. Near each such point the dual graph $\Delta(H, C)$ is of the form

$$\bullet\!-\!\!-\circ-\!\!-\cdots-\!\!-\circ$$

If (H, C) contains two such points, we get a contradiction with negative definiteness of the corresponding matrix. Thus we obtain (1.9.1.1).

Now consider the case where f is a $\mathbb{Q}$-conic bundle. If (H, C) is PLT also at P, then H has two singularities of types $1/n(1, q)$ and $1/n(1, n-q)$ (see Lemma 2.3.1). Since they are of type T, we see the following by Proposition 2.2.2:

$$(q+1)^2 \equiv 0 \mod n, \qquad (n-q+1)^2 \equiv 0 \mod n.$$

This gives us $4 \equiv 0 \mod n$. Since X is not Gorenstein, the singularities of H are worse than Du Val. Hence $n = 4$. We get the graph (5.1.1).

Finally, assume that (H, C) is not PLT at P. Then $\Delta(H, C)$ is of the form (1.9.1.1) with $r \neq 1$, $r \neq n$, and $c_1 c_n \geq 6$ by Proposition 2.2.3. Contracting black vertices successively, on some step we get a subgraph

(5.1.2) $$\overset{c_1}{\circ}-\!\!-\cdots-\!\!-\overset{c_{r-1}}{\circ}-\!\!-\bullet-\!\!-\overset{c_{r+1}}{\circ}-\!\!-\cdots-\!\!-\overset{c_n}{\circ}$$

Hence strings $[c_{r-1}, \ldots, c_1]$ and $[c_{r+1}, \ldots, c_n]$ are conjugate. This contradicts the following claim because $c_1 c_n \geq 6$. □

CLAIM 5.1.3

Let $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ be conjugate strings. If, for some $c \geq 2$, the string of the form

(5.1.3.1) $$[a_r, \ldots, a_1, c, b_1, \ldots, b_s]$$

is of type T, then it is Du Val.

Proof
Assume that the string (5.1.3.1) is not Du Val. Take it so that $r+s$ is minimal. Since $[a_1,\dots,a_r]$ and $[b_1,\dots,b_s]$ are conjugate, either $a_r=2$ or $b_s=2$. Assume that $a_r=2$. If $r=1$, then $s=1$ and $b_1=2$, which is a contradiction by Proposition 2.2.3(iii). Hence, $r>1$, $b_s>2$, and $[a_{r-1},\dots,a_1,c,b_1,\dots,b_{s-1},b_s-1]$ is again a non–Du Val T-string (see Proposition 2.2.2), and the strings $[a_1,\dots,a_{r-1}]$ and $[b_1,\dots,b_{s-1},b_s-1]$ are conjugate. This contradicts our minimality assumption. □

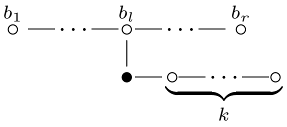
Thus Theorem 1.7(i) exhausts all $\mathbb{Q}$-conic bundles with normal H. Explicit examples are given in Section 7.

5.2

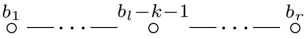
In the birational case, similarly to Section 4.14, any $\mathbb{Q}$-Gorenstein deformations of singular points of H can be globalized by [KM, Proposition 11.4].

EXAMPLE 5.2.1

Let $[b_1,\dots,b_r]$ be any T-string, and let $b_l>2$. Then the configuration



where $k\le b_l-3$, determines a surface germ (H,C) which is contracted to (T,o) with the dual graph



For example, for $[b_1,\dots,b_r]=[4]$ and $k=0$, this gives Francia's flip (see [KM, Theorem 4.7]). For $[b_1,\dots,b_r]=[3,2,\dots,2,3]$, $l=r$, and $k=1$, this gives examples of divisorial extremal neighborhoods of index two (see [KM, case 4.7.3.1.1]).

6. Case: P is of type $\mathrm{cA/m}$ and H is not normal

6.1

In this section we prove Theorems 1.6 and 1.9 in the case where a general $H\in|\mathscr{O}_X|_C$ is not normal. Thus throughout this section we assume that (X,C) is an extremal curve germ of type (IA) or (IA$^\vee$); the only non-Gorenstein point $P\in X$ is of type cA/m. Let $F\in|-K_X|$ be a general member. Let $H\in|\mathscr{O}_X|_C$ be a nonnormal member such that the pair $(X,H+F)$ is LC (see Proposition 2.6). Let $f:(X,C)\to(Z,o)$ be the corresponding contraction.

SETUP 6.2

Let $\nu:H'\to H$ be the normalization, and let $\mu:\widetilde{H}\to H'$ be the minimal resolution. Let $C'=\nu^{-1}(C)$ (with reduced structure), and let $\widetilde{C}\subset\widetilde{H}$ be the proper transform of C'. If C' is reducible, components of C' (resp., $\widetilde{C}$) are denoted by C'_i (resp., $\widetilde{C}_i$). Let $\overline{H}$ be a minimal model over T (so that $\overline{H}$ is smooth and has no (-1)-curves on fibers over T). Thus we have the following diagram:

$$\begin{tikzcd}
 & \widetilde{H} \arrow[rr, "\mu"] \arrow[dl, "\widetilde{\upsilon}"'] \arrow[ddr, "\upsilon"] & & H' \arrow[dr, "\nu"] & \\
\overline{H} \arrow[drr, "\overline{\upsilon}"'] & & & & H \arrow[dll, "f_H"] \\
 & & T & &
\end{tikzcd}$$

Let $\Upsilon := \nu^{-1}(F \cap H)$. By Section 6.1 and Corollary 2.6.1, we have the following.

COROLLARY 6.2.1

The pair $(H', C' + \Upsilon)$ *is LC, and the restriction map* $\nu|_{C'} : C' \to C$ *is of degree* 2.

COROLLARY 6.2.2

The pullback $C^\sharp$ *of* C *to the index-one cover* $(X^\sharp, P^\sharp) \to (X, P)$ *is smooth. In particular,* (X, C) *is of type* (IA).

Note that $\Delta(H', C')$ is the dual graph of the 1-cycle $\upsilon^{-1}(o) \subset \widetilde{H}$. Hence $\Delta(H', C')$ is negative semidefinite, and its *fundamental cycle* is defined as usual.

PROPOSITION 6.3

Under the assumptions of Section 6.1 the following are equivalent:

(i) *every member of* $|\mathscr{O}_X|_C$ *is nonnormal,*

(ii) *each component of* $\widetilde{C}$ *appears with coefficient* > 1 *in the fundamental cycle* G *of* $\Delta(H', C')$.

In particular, if every member of $|\mathscr{O}_X|_C$ *is nonnormal, then all the components of* $\widetilde{C}$ *are contracted by* $\widetilde{\upsilon} : \widetilde{H} \to \overline{H}$.

Proof

Assume that (ii) does not hold; that is, a component $\widetilde{C}_1 \subset \widetilde{C}$ appears with coefficient 1 in G. Then there is a function $\psi \in \mathfrak{m}_{o,T}$ such that $\upsilon^*\psi$ has a simple zero along $\widetilde{C}_1$. Note that the map $H^0(Z, \mathscr{O}_Z) \to H^0(T, \mathscr{O}_T)$ is surjective. Hence $\psi = \phi|_T$ for some $\phi \in \mathscr{O}_Z$. Pick a general point $S \in C$. If $f^*\phi = 0$ is singular along C, then $f^*\phi \in I_C^2$ at S. By the commutativity of the above diagram, we have $\upsilon^*\psi = \mu^*\nu^*(f^*\phi)|_H \in I^2_{\widetilde{C}_1}$ at a point above S. This contradicts the construction of ψ. So $f^*\phi = 0$ is smooth along C and a general member of $|\mathscr{O}_X|_C$ is normal, so (i) does not hold.

Conversely, assume that (i) does not hold. Then there is a normal member $L \in |\mathscr{O}_X|_C$. Regard X as an analytic neighborhood of a general point $Q \in C$. Then $H = H_1 + H_2$, where H_1, H_2 are smooth surfaces intersecting transversely along C. Hence L intersects transversely at least one of H_1, H_2 along C. This means that $\nu^* L|_H$ is reduced along at least one component of C'. Thus (ii) does not hold.

As for the last statement, we note that (T, o) is either a cyclic quotient singularity (see Proposition 2.6) or a smooth curve. In both cases, $\widetilde{v}(G)$ is reduced. $\square$

PROPOSITION 6.4

Under the assumptions of Section 6.1, there are only two possibilities for the dual graph $\Delta(H', C' + \Upsilon)$*:*

6.4.1. C' *has two irreducible components:* $C' = C'_1 + C'_2$*.*

$$\square - \underbrace{\overset{a_r}{\circ} - \cdots - \overset{a_1}{\circ}}_{\Delta_1} - \diamond - \underbrace{\overset{c_1}{\circ} - \cdots - \overset{c_l}{\circ}}_{\Delta_3} - \diamond - \underbrace{\overset{b_1}{\circ} - \cdots - \overset{b_s}{\circ}}_{\Delta_2} - \square$$

6.4.2. C' *is irreducible:*

$$\square - \underbrace{\overset{a_r}{\circ} - \cdots - \overset{a_1}{\circ}}_{\Delta_1} - \diamond - \underbrace{\overset{b_1}{\circ} - \cdots - \overset{b_s}{\circ}}_{\Delta_2} - \square$$

Here $\square$ *corresponds to an irreducible component of* Υ*,* $\diamond$ *corresponds to an irreducible component of* C'*, the chain* Δ_1 *(resp.,* Δ_2*) corresponds to the singularity of type* $1/m(1, a)$ *(resp.,* $1/m(1, -a)$*), and in case 6.4.1, the chain* Δ_3 *corresponds to the point* (H', Q')*, where* $Q' = C'_1 \cap C'_2$*. The strings* $[a_1, \ldots, a_r]$ *and* $[b_1, \ldots, b_s]$ *are conjugate. If* f *is birational, then at least one of the vertices* $\diamond$ *corresponds to a* (-1)*-curve under the extra assumption that every member of* $|\mathscr{O}_X|_C$ *is nonnormal. If* f *is a* $\mathbb{Q}$*-conic bundle, then all the vertices* $\diamond$ *correspond to* (-1)*-curves.*

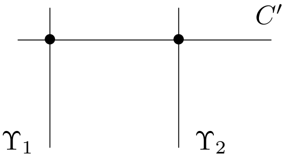
Proof

Note that C' is a fiber of a contraction $H' \to T \ni o$, where (T, o) is either a cyclic quotient singularity (see Lemma 2.6) or a curve germ. Hence $p_a(C') = 0$, and all components of C' are smooth rational curves. By Corollary 6.2.1, C' has at most two components. So either $C' \simeq \mathbb{P}^1$ or C' is a union of two $\mathbb{P}^1$'s meeting each other at one point, say, Q'.

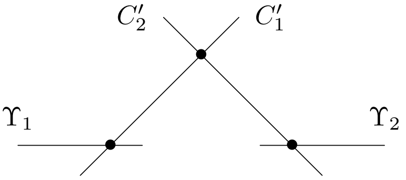
By the classification of log canonical pairs (see, e.g., [Kol, Chapter 3]), Υ is smooth at any point $\Upsilon \cap C'$. On the other hand, $\Upsilon = \nu^{-1}(F \cap H)$, where H is Cartier and the pair $(F, H \cap F)$ is LC. Hence Υ has exactly two components Υ_1, Υ_2, and these components are smooth.

Further, since $(H', \Upsilon + C')$ is LC, through any point of H' pass at most two components of $\Upsilon + C'$. Thus for the configuration of $\Upsilon + C'$ on H' we have only the following two possibilities:

(a)



(b)



Since the pair $(H', \Upsilon + C')$ is LC, from the classification of log canonical pairs (see, e.g., [Kol, Chapter 3]) we get the desired graphs 6.4.1 and 6.4.2.

It remains to prove the last statements about (-1)-curves. If f is birational, then by Proposition 6.3 at least one of the components of C' is a (-1)-curve. Assume that f is a $\mathbb{Q}$-conic bundle. Clearly, the fiber $\upsilon^{-1}(o)$ of a rational curve fibration υ contains a (-1)-curve, and this curve must coincide with a component of C'. So we are done if C' is irreducible. Consider case 6.4.1. By the above, one of the $\diamond$-vertices corresponds to a (-1)-curve. Hence the chain Δ_1—$\bullet$—Δ_3—$\diamond$—Δ_2 forms a fiber of a rational curve fibration, and we may assume that $\bullet$ is the only (-1)-vertex. In this case, the chain Δ_1 is conjugate to both Δ_2 and Δ_3—$\diamond$—Δ_2 (see Lemma 2.3.1), a contradiction. □

LEMMA 6.5

Let $Q \in H \setminus \{P\}$ be any point, and let $Q' \in \nu^{-1}(Q)$. Then $4 \geq \operatorname{emb}\dim(H, Q) \geq \operatorname{emb}\dim(H', Q') - 1$.

Proof

By Corollary 6.2.1 the conductor ideal coincides with the ideal sheaf $I_{C'}$. The natural map $\mathscr{O}_H \to \nu_*\mathscr{O}_{H'}$ induces an isomorphism $I_C \simeq \nu_* I_{C'}$. (Any regular function on H' that vanishes on C' descends to H.) From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \nu_* I_{C'} \arrow[r] & \nu_*\mathscr{O}_{H'} \arrow[r] & \nu_*\mathscr{O}_{C'} \arrow[r] & 0 \\
0 \arrow[r] & I_C \arrow[r] \arrow[u, equal] & \mathscr{O}_H \arrow[r] \arrow[u, hook] & \mathscr{O}_C \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

we have $\nu_*\mathscr{O}_{H'}/\mathscr{O}_H \simeq \nu_*\mathscr{O}_{C'}/\mathscr{O}_C$. Note that $\nu_*\mathscr{O}_{C'}$ is a locally free $\mathscr{O}_C$-module and there is a local splitting $\nu_*\mathscr{O}_{C'} = \mathscr{O}_C \oplus \mathscr{O}_C t$ for some $t \in \nu_*\mathscr{O}_{C'}$. Thus $\nu_*\mathscr{O}_{H'}/\mathscr{O}_H \simeq \mathscr{O}_C t$. Therefore, $\mathfrak{m}_{Q',H'}/\mathfrak{m}_{Q',H'}^2$ is generated by $1 + \dim \mathfrak{m}_{Q,H}/\mathfrak{m}_{Q,H}^2$ elements as an $\mathscr{O}_{Q,H}$-module. □

COROLLARY 6.5.1 (SEE [Tzi1])

The chain $\varDelta_3$ in case 6.4.1 satisfies the inequality

$$\operatorname{emb\,dim}(H',Q') - 3 = \sum (c_i - 2) \le 2. \tag{6.5.2}$$

The proof of this statement is contained in [Tzi1, proof of Theorem 5.6], which is rather computational and uses the classification of degenerate cusp singularities. Here is a much shorter proof.

Proof

By Lemma 6.5, we have $\operatorname{emb\,dim}(H',Q') \le \operatorname{emb\,dim}(H,Q) + 1 \le 5$. On the other hand, since (H',Q') is a cyclic quotient singularity,

$$\begin{aligned}\operatorname{emb\,dim}(H',Q') &= -\Bigl(\sum E_i\Bigr)^2 + 1 \\ &= 1 + \sum c_i - 2\sum_{i\neq j} E_i \cdot E_j = 3 + \sum (c_i - 2),\end{aligned}$$

where the E_i's are exceptional divisors on the minimal resolution. This immediately gives the desired inequality. □

PROPOSITION 6.6

Assume that we are in case 6.4.1 under Section 6.1. Furthermore, assume that every member of $|\mathscr{O}_X|_C$ is nonnormal and that $\sum(c_i - 2) = 2$ (whence $\operatorname{emb\,dim}(H,Q) = 4$). Let G (resp., G') be the fundamental cycle of $\Delta(H',C')$ (resp., $\varDelta_3$). Then $G \ge 2G'$ if and only if $\operatorname{emb\,dim}(M,Q) = 4$ for general $M \in |\mathscr{O}_X|_C$.

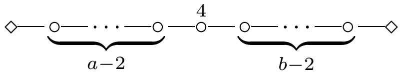
Proof

We have an analytic isomorphism $(H',Q') \simeq \mathbb{C}^2_{u,v}/\boldsymbol{\mu}_n(1,q)$ for some n, q with $\gcd(n,q) = 1$. By Proposition 6.3, the graph $\widetilde{C}_1$—$\varDelta_3$—$\widetilde{C}_2$ is contracted on $\overline{H}$. Note that G is $\overline{v}$-numerically trivial. Thus there is a function $\psi \in \mathscr{O}_H$ such that $\mu^*\nu^*\psi = 0$ defines G near $\mu^{-1}\nu^{-1}(Q)$. Hence the lifting of $\nu^*\psi$ to $\mathbb{C}^2_{u,v}$ is given by an invariant monomial λ multiplied by a unit.

Since $\sum(c_i - 2) = 2$, we see $\operatorname{emb\,dim}(H',Q') = 5$ and $\operatorname{emb\,dim}(H,Q) = 4$ by Corollary 6.5.1 and Lemma 6.5, and $I_{C'} \subset \mathfrak{m}'_{Q',H'}$ is generated by exactly three invariant monomials in u,v divisible by uv. Thus every minimal generating set of $I_C \subset \mathfrak{m}_{Q,H}$ induces a minimal generating set of $I_{C'} \subset \mathfrak{m}'_{Q',H'}$ (cf. the proof of Lemma 6.5). This means that $\operatorname{emb\,dim}(M,Q) < 4$ for general $M \in |\mathscr{O}_X|_C$ if and only if $\nu^*\psi$ can be a part of a coordinate of (H',Q'). However since the lifting of $\nu^*\psi$ is an invariant monomial (times a unit), this happens if and only if $\nu^*\psi$ equals one of the three monomial generators of $I_{C'}$.

There are only two series of possibilities for $\Delta(H,C)$ near Q:

(∗)



$a,b \ge 2,$

(∗∗) $\diamond - \underbrace{\circ\cdots\circ}_{a-2} - \overset{3}{\circ} - \underbrace{\circ\cdots\circ}_{b-2} - \overset{3}{\circ} - \underbrace{\circ\cdots\circ}_{c-2} - \diamond \qquad a,b,c \geq 2$

Each monomial in $I_{C'}$ corresponds to an effective divisor of $\widetilde{H}$ with support $\varDelta_3 \cup \tilde{C}$ which is μ-trivial (i.e., numerically trivial along $\varDelta_3$). Table 1 gives three such monomials (or divisors) m_A, m_B, m_C for each of (∗) and (∗∗). For instance, the numbers of the row m_A show the coefficient of the curve corresponding to the vertex in the divisor m_A.

Table 1

(∗)	$\diamond$	$\circ$ .... $\circ$	$\overset{4}{\circ}$	$\circ$ .... $\circ$	$\diamond$
m_A	1	 1	1	3	$2b-1$
m_B	$2a-1$	 3	1	1	1
m_C	a	 2	1	2	b

(∗∗)	$\diamond$	$\circ\cdots\circ$	$\overset{3}{\circ}$	$\circ\cdots\circ$	$\overset{3}{\circ}$	$\circ\cdots\circ$	$\diamond$
m_A	1	......	1	......	b	......	$bc+c-1$
m_B	$ab+a-1$	......	b	......	1	......	1
m_C	a	......	1	......	1	......	c

It is clear that none of these monomials belong to $\mathfrak{m}^2_{Q',H'}$ because each vanishes to order 1 at one of the vertices with weight 3 or 4. Hence m_A, m_B, m_C are the monomial generators of $I_{Q'}$. One can also check that the lifting of $\nu^*\psi$ equals one of m_A, m_B, m_C if and only if one of the vertices of weight 3 or 4 appears with coefficient 1 in G if and only if $G \not\geq 2G'$. □

PROPOSITION 6.7

Assume that f is a $\mathbb{Q}$-conic bundle germ such that every member of $|\mathscr{O}_X|_C$ is nonnormal. Assume furthermore that $H \in |\mathscr{O}_X|_C$ is taken to be general. Then C' is irreducible.

REMARK 6.7.1

If in the above assumptions X is of index 2, then $\Delta(H', C' + \Upsilon)$ is of the form

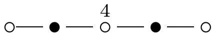
$$\square - \circ - \bullet - \circ - \square.$$

Proof of Proposition 6.7

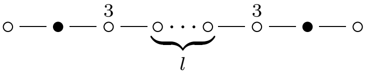
Assume that C' is reducible. Then the dual graph $\Delta(H', C')$ is of the form in case 6.4.1 with $\diamond^2 = -1$. Clearly the chains $\varDelta_1$ and $\varDelta_2$ are not empty. (Otherwise, X is Gorenstein.) Since the matrix corresponding to $\bullet - \varDelta_3 - \bullet$ is negative definite, the subgraph $\varDelta_3$ is not Du Val. We will use the inequality (6.5.2).

6.7.2

Assume that $r = s = 1$. Then $a_1 = b_1 = 2$ and the graph 6.4.1 or 6.4.2 is of the form:

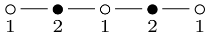


or

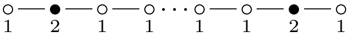


$l \geq 0$

The fundamental cycle G of $\Delta(H', C')$ is given by



or

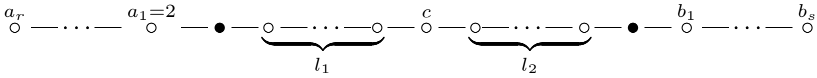


respectively. Then by Proposition 6.6 our H is not general enough, a contradiction.

From now on we assume that $rs > 1$. Since $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ are conjugate, we may assume by symmetry that $a_1 = 2$, $b_1 > 2$, and $r > 1$.

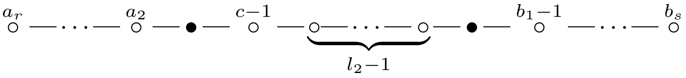
6.7.3

Consider the case where the chain Δ_3 contains exactly one curve with self-intersection < -2. Then graph 6.4.1 has the following form:



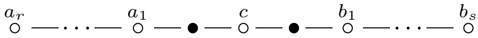
where $c = 3$ or 4. Since $a_1 = 2$, it holds $l_1 = 0$ because the graph ○—●—○ is not negative definite. Choose the above configuration so that c is minimal.

If $l_2 > 0$, then contracting both black vertices we get

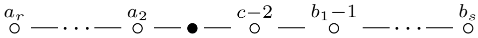


The strings $[a_2, \ldots, a_r]$ and $[b_1 - 1, \ldots, b_s]$ at the ends are again conjugate. This contradicts our minimality assumption because $c' = c - 1 < 4$.

Therefore, $l_1 = l_2 = 0$, and graph 6.4.1 is of the form:



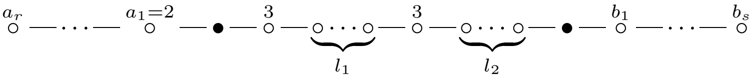
Contracting black vertices, we get



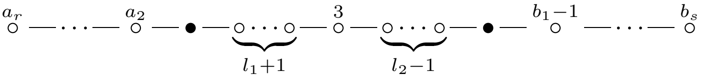
Hence $c = 4$ and $a_2 \geq 3$. Again the string $[a_2, \ldots, a_r]$ is conjugate to both $[b_1 - 1, \ldots, b_s]$ and $[c - 2, b_1 - 1, \ldots, b_s]$, a contradiction.

6.7.4

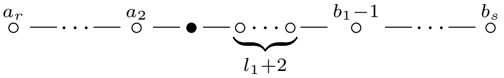
Now we consider the case where Δ_3 contains exactly two (-3)-curves. Then graph 6.4.1 has the following form:



(As above, $c_1 > 2$ since $a_1 = 2$.) If $l_2 > 0$, then contracting both black vertices, we get



Here again the strings $[a_2, \ldots, a_r]$ and $[b_1 - 1, \ldots, b_s]$ are conjugate. This contradicts the case considered above. So $l_2 = 0$. Then contracting both black vertices, we get



As above, the string $[a_2, \ldots, a_r]$ is conjugate to both $[b_1 - 1, \ldots, b_s]$ and $[2, \ldots, 2, b_1 - 1, \ldots, b_s]$, a contradiction. ☐

COROLLARY 6.8

Let f be a $\mathbb{Q}$-conic bundle such that a general member $H \in |\mathscr{O}_X|_C$ is not normal. Then the germ (H, C) is analytically isomorphic to the germ along the line $L := \{y = z = 0\}$ of the hypersurface given by the following weighted polynomial of degree $2m$ in variables x, y, z, u:

$$\phi := x^{2m-2a} y^2 + x^{2a} z^2 + yzu$$

in $\mathbb{P}(1, a, m-a, m)$, for some integers a, m such that $0 < a < m$ and $\gcd(a, m) = 1$.

Proof

By Proposition 6.7, (H, C) is of the type in graph 6.4.2. Then it is easy to see that the pair (H, C) up to analytic isomorphism is uniquely defined by the types of singularities $1/m(1, a)$ and $1/m(1, -a)$. On the other hand, the hypersurface $\phi = 0$ satisfies the conditions of graph 6.4.2. ☐

Note that we are interested only in the *germ* of the hypersurface $\{\phi = 0\}$ along L.

REMARK 6.8.1

Since the germ $(\{\phi = 0\}, L)$ is analytically isomorphic to our (H, C), there is a rational curve fibration on $(\{\phi = 0\}, L)$ whose central fiber is L. One can check that this fibration is given by the rational function

$$s = \frac{y^{m-a} z^a}{x^{2a(m-a)}},$$

which is regular in a neighborhood of L in H.

LEMMA 6.8.2

Let (H, C) be as in Corollary 6.8, and let $s : H \to T$ be the corresponding rational *curve fibration. Let* $t: X \to \mathbb{C}$ *be a one-parameter smoothing of* (H, C) *in a* $\mathbb{Q}$-*Gorenstein family. If* X *has only terminal singularities, then* (X, C) *is a* $\mathbb{Q}$-*conic bundle germ.*

Proof
Let $V := s^{-1}(o)$ (with the scheme structure), and let Z be the component of the Hilbert scheme of X containing the point $o = [V]$ representing V. Let $\mathfrak{X} \subset X \times Z$ be the corresponding universal family. We have the following commutative diagram:

$$\begin{tikzcd}
& V \arrow[r, equal] \arrow[dl, hook'] & W \arrow[dl, hook'] \\
X \arrow[d, "t"] \arrow[dr, dotted] & \mathfrak{X} \arrow[l, "p"'] \arrow[d, "\pi"] & \\
\mathbb{C} & Z \arrow[l] &
\end{tikzcd}$$

where $W := \pi^{-1}(o)$. Both V and W are locally complete intersections. Moreover, $I_V/I_V^2 \simeq \mathscr{O}_V \oplus \mathscr{O}_V$ and $I_W/I_W^2 \simeq \mathscr{O}_W \oplus \mathscr{O}_W$. Since $H^1(V, (I_V/I_V^2)^\vee) = 0$, Z is smooth at o and there is a natural isomorphism $\mathbb{C}^2 \simeq T_{o,Z} \simeq H^0(V, (I_V/I_V^2)^\vee)$. On the other hand, $H^0(V, (I_W/I_W^2)^\vee) \simeq T_{o,Z}$ because W is a fiber of π. Therefore, there is a natural isomorphism $H^0(W, (I_W/I_W^2)^\vee) \simeq H^0(V, (I_V/I_V^2)^\vee)$, and the natural map $(I_W/I_W^2)^\vee \to (I_V/I_V^2)^\vee$ is also an isomorphism. Thus p is an isomorphism in a neighborhood of W. By shrinking $\mathfrak{X}$ and X we may assume that there is a contraction $X \to Z$ such that the whole diagram is commutative. □

The existence of a $\mathbb{Q}$-Gorenstein smoothing follows from [Tzi2]. However, in our particular case we can construct it explicitly.

LEMMA 6.8.3
Let (H, C), m, a *be as in Corollary 6.8. For* $s = (s_1, \dots, s_5) \in \mathbb{C}^5_s$, *hypersurfaces* $H_s \subset \mathbb{P}(1, a, m-a, m)$ *given by the equation*

$$\phi_s := \phi + s_1 x^{2m-a} y + s_2 x^{m-a} u y + s_3 x^{2m} + s_4 x^m u + s_5 u^2 = 0$$

form a miniversal qG-deformation family of the germ $C \subset H$.

Proof
We compute $T^1_{qG}(H)$ from the $\mathbb{Q}$-Gorenstein smoothing $H \subset P := \mathbb{P}(1, a, m-a, m)$ (cf. [Tzi2, Section 3]). By definition, $T^1_{qG}(H)$ has an ℓ-structure and $T^1_{qG}(H)^\sharp = T^1_{qG}(H^\sharp)$. Furthermore, we get an exact sequence

$$\mathscr{H}om_H(\Omega^1_P, \mathscr{O}_H) \longrightarrow \mathscr{H}om_H\bigl(\mathscr{O}_P(-H), \mathscr{O}_H\bigr) \longrightarrow T^1_{qG}(H) \to 0$$

of sheaves with ℓ-structures. So $T^1_{qG}(H) = \mathscr{O}_P(2m)/G$, where G is generated by ϕ and its derivatives. A direct computation shows that $x^{2m-a}y$, $x^{m-a}yu$, x^{2m}, $x^m u$, u^2 form a $\mathbb{C}$-basis of the vector space $T^1_{qG}(H)$; $x^{2m-a}y$, $x^{m-a}yu$ generate the torsion part of $T^1_{qG}(H)$; and x^{2m}, $x^m u$, u^2 generate $T^1_{qG}(H)/(\text{torsion}) \simeq \mathscr{O}_P(2m) \otimes \mathscr{O}_C \simeq \mathscr{O}_{\mathbb{P}^1}(2)$. □

EXAMPLE 6.8.4

Let α, $\beta \in \mathbb{C}$ be some general constants, and let X be the threefold given in $\mathbb{P}(1,a,m-a,m) \times \mathbb{C}_t$ by

$$\phi + (\alpha x^m - u)(\beta x^m - u)t = 0.$$

Then the singularities of X along the curve $C := \{y = z = t = 0\}$ consist of a cyclic quotient singularity of type $1/m(1,a,m-a)$ at $\{x = y = z = t = 0\}$ and two (Gorenstein) ordinary double points at $\{\alpha x^m - u = y = z = t = 0\}$ and $\{\beta x^m - u = y = z = t = 0\}$. The contraction $X \to Z$ exists by Lemma 6.8.2.

Thus Theorem 1.6 is proved. Now assume that f is birational. □

LEMMA 6.9 ([Tzi1, THEOREM 5.6(1A)])

If f is birational, then C' is reducible and the dual graph $\Delta(H', C')$ is of the form in graph 6.4.1, and general H is not normal.

Proof

Assume that $\Delta(H', C')$ is of the form in graph 6.4.2. Then the chain of smooth rational curves corresponding to the graph $\varDelta_1 — \bullet — \varDelta_2$ is contracted by v. On the other hand, $\varDelta_1$ and $\varDelta_2$ are conjugate. By Lemma 2.3.1 this configuration corresponds to a rational curve fibration; that is, v is not birational, a contradiction. □

6.10

The singularity (H, Q) is a so-called *degenerate cusp* (see [SB]). One can define the *fundamental cycle* Γ of (H, Q) and attach an invariant $\zeta = -\Gamma^2$ to (H, Q) such that

$$\begin{aligned}
&\zeta = 1 \iff (H', Q') \text{ is a smooth point } \iff (H, Q) \simeq \{y^2 = x^3 + x^2 z^2\},\\
&\zeta = 2 \iff (H', Q') \text{ is a Du Val point of type } \mathrm{A}_n, n \geq 1\\
&\qquad\quad \iff (H, Q) \simeq \{y^2 = x^2 z^2 + x^{n+3}\},\\
&\zeta = 3 \iff \textstyle\sum (c_i - 2) = 1 \iff (H, Q) \simeq \{xyz = y^{a+3} + z^{b+3}\},\ a, b \geq 0,\\
&\zeta = 4 \iff \textstyle\sum (c_i - 2) = 2 \iff \operatorname{emb}\dim(H, Q) = 4,
\end{aligned}$$

(see [SB, Section 1]). Then by [Tzi2, Theorem 3.1, Proposition 3.4], we have the following.

THEOREM 6.10.1

In the above notation, a one-parameter smoothing of (H, C) with only terminal *singularities exists if and only if*

$$\widetilde{C}_1^2+\widetilde{C}_2^2+1+4\delta_{\zeta,1}+4\delta_{\zeta,2}+3\delta_{\zeta,3}+2\delta_{\zeta,4}\geq 0,$$

where $\delta_{i,j}$ *is Kronecker's delta.*

REMARK 6.10.2

One can see that the last inequality is equivalent to

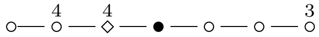
$$\widetilde{C}_1^2+\widetilde{C}_2^2+5-\sum(c_i-2)\geq 0,$$

where we put $\sum(c_i-2)=0$ if Δ_3 is empty.

This completes the proof of Theorem 1.9. $\square$

EXAMPLE 6.10.3

Assume that the configuration in graph 6.4.1 is of the form



Then (X,C) is a divisorial extremal neighborhood. By Proposition 6.3 every member of $|\mathscr{O}_X|_C$ is nonnormal. By Section 6.10 this H is general in $|\mathscr{O}_X|_C$.

7. On index two $\mathbb{Q}$-conic bundles

In this section we show that every type of terminal index two singularity can occur on some index two $\mathbb{Q}$-conic bundle. Let y_1, y_2, y_3, y_4; u, v be as in Theorem 1.7, and let $X\subset\mathbb{P}(1,1,1,2)\times\mathbb{C}^2$ be given by

$$\begin{aligned}0&=\alpha_1y_1^2+\alpha_2u^ey_4+(\beta_2u+v)y_3^2,\\0&=\alpha_3(y_2^2+\beta_1y_1y_3)+\alpha_4uy_3^2+vy_4,\end{aligned}$$

where $\alpha_1,\dots,\alpha_4\in\mathbb{C}$ are general, $\beta_1,\beta_2\in\mathbb{C}$ are either zero or general, and $e=1,2,3$. Furthermore, $C\subset X$ is given by $y_1=y_2=u=v=0$.

By Bertini's theorem, we see that the singular locus, Σ, of X is contained in $\{u=v=0\}$. Hence $\Sigma\subset\{u=v=y_1=y_2=0\}$, and using notation $[y:z]:=(0:0:y:z)\times(0,0)$, we see

$$\begin{aligned}\Sigma&=\left\{[y_3:y_4]\,\middle|\,\operatorname{rank}\begin{pmatrix}0&\alpha_2eu^{e-1}y_4+\beta_2y_3^2&y_3^2\\\beta_1y_3&\alpha_4y_3^2&y_4\end{pmatrix}\leq 1\right\}\cup\{[0:1]\}\\&=\begin{cases}\{[0:1]\}&\text{if }\beta_1\neq 0,\\\{[1:\pm\sqrt{\alpha_4/\alpha_2}],[0:1]\}&\text{if }\beta_1=0,\ \beta_2=0,\text{ and }e=1,\\\{[0:1]\}&\text{if }\beta_1=0,\ \beta_2=0,\text{ and }e>1,\\\{[1:\alpha_4/\beta_2],[0:1]\}&\text{if }\beta_1=0,\ \beta_2\neq 0,\text{ and }e>1.\end{cases}\end{aligned}$$

At $[0:1]$, the singularity $(X,[0:1])$ is a hyperquotient:

$$\{\alpha_1y_1^2+\alpha_2u^e+\beta_2uy_3^2-\alpha_3y_2^2y_3^2-\alpha_3\beta_1y_1y_3^3-\alpha_4uy_3^4=0\}/\boldsymbol{\mu}_2(1,1,1,0).$$

By [Mor1, Corollary 2.1], we see that $(X, [0:1])$ is a terminal singularity of type

- $1/2(1,1,1)$ if $e = 1$,
- cAx/2 if $e = 2$ (cf. [Mor1, Theorem 12(3)]),
- cD/2 if $e = 3$ and $\beta_2 \neq 0$ (cf. [Mor1, Theorem 23]),
- cE/2 if $e = 3$ and $\beta_2 = 0$ (cf. [Mor1, Theorem 25]).

Every other singular point, if any, is easily seen to be an ordinary double point, in particular, a type (III) point:

(i) Case $\beta_1 \neq 0$. In this case we can assume $\beta_1 = -1$ by change of coordinate $y_1 \mapsto -y_1/\beta_1$, and we are in case (i) of Theorem 1.7. In this case, $[0:1]$ is the only singular point and it can be of type $\frac{1}{2}(1,1,1)$, cAx/2, cD/2 or cE/2 as above.

(ii) Case $\beta_1 = 0$. In this case we are in case (ii) of Theorem 1.7. The type of singularity of (X, C) in our example is

- $1/2(1,1,1) +$ (III) $+$ (III) if $\beta_2 = 0$ and $e = 1$,
- cAx/2 $+$ (III) if $\beta_2 \neq 0$ and $e = 2$,
- cAx/2 if $\beta_2 = 0$ and $e = 2$,
- cD/2 $+$ (III) if $\beta_2 \neq 0$ and $e = 3$, and
- cE/2 if $\beta = 0$ and $e = 3$.

In particular, we have shown that all types of terminal index two singularities can appear on $\mathbb{Q}$-conic bundles as in Theorem 1.7.

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