Classifying spaces of degenerating mixed Hodge structures, II: Spaces of $\mathrm{SL}(2)$-orbits

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To the memory of Professor Masayoshi Nagata

Abstract We construct an enlargement of the classifying space of mixed Hodge structures with polarized graded quotients by adding mixed Hodge theoretic version of SL(2)-orbits. This space has a real analytic structure and a log structure with sign. The SL(2)-orbit theorem in several variables for mixed Hodge structures can be understood naturally with this space.

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L'impossible voyage aux points à l'infini
N'a pas fait battre en vain le coeur du géomètre
— translated by Luc Illusie

0. Introduction

This is part II of our series of articles in which we study degeneration of mixed Hodge structures.

0.1.

We first review the case of pure Hodge structures. Let D be the classifying space of polarized Hodge structures of given weight and given Hodge numbers, defined

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by Griffiths [G]. Let $F_t \in D$ be a variation of polarized Hodge structure with complex analytic parameter $t = (t_1, \ldots, t_n)$, $t_1 \cdots t_n \neq 0$, which degenerates when $t \to 0 = (0, \ldots, 0)$. It is often asked how F_t and invariants of F_t, like Hodge metric of F_t, and so on, behave when $t \to 0$. Usually, F_t diverges in D and invariants of F_t also diverge.

There are two famous theorems concerning the degeneration of F_t, which are roughly reviewed in Section 0.3:

(1) the nilpotent orbit theorem (see [Sc]),
(2) the SL(2)-orbit theorem (see [Sc], [CKS]).

In [KU2] and [KU3] (an announcement is given in [KU1]), we constructed enlargements $D_{\mathrm{SL}(2)}$ and D_Σ of D, respectively. Roughly speaking, these theorems (1) and (2) are interpreted as in (1)′ and (2)′ below, respectively (see [KU3]).

(1)′ $(F_t \mod \Gamma) \in \Gamma\backslash D$ converges in $\Gamma\backslash D_\Sigma$, and asymptotic behaviors of invariants of F_t are described by coordinate functions around the limit point on $\Gamma\backslash D_\Sigma$.

(2)′ $F_t \in D$ converges in $D_{\mathrm{SL}(2)}$, and asymptotic behaviors of invariants of F_t are described by coordinate functions around the limit point on $D_{\mathrm{SL}(2)}$ (see Section 0.2).

Here in (1)′, Γ is the monodromy group of F_t which acts on D, and Σ is a certain cone decomposition which is chosen suitably for F_t. The space $\Gamma\backslash D_\Sigma$ is a kind of toroidal partial compactification of the quotient space $\Gamma\backslash D$ and has a kind of complex analytic structure. The space $D_{\mathrm{SL}(2)}$ has a kind of real analytic structure. For the study of asymptotic behaviors of real analytic objects such as Hodge metrics, $D_{\mathrm{SL}(2)}$ is a nice space in which to work.

0.2.

Now let D be the classifying space of mixed Hodge structures whose graded quotients for the weight filtrations are polarized, as defined in [U1]. The purpose of this article is to construct an enlargement $D_{\mathrm{SL}(2)}$ of D, which is a mixed Hodge theoretic version of $D_{\mathrm{SL}(2)}$ in [KU2]. A mixed Hodge theoretic version of the SL(2)-orbit theorem of [CKS] was obtained in [KNU1], and it is also interpreted in the form (2)′ above by using the present $D_{\mathrm{SL}(2)}$ (see Section 4.1 of this article).

In Part I ([KNU2]) of this series of articles, we constructed the Borel-Serre space D_{BS} which contains D as a dense open subset and which is a real analytic manifold with corners like the original Borel-Serre space in [BS]. These spaces $D_{\mathrm{SL}(2)}$ and D_{BS} belong to the following fundamental diagram of eight enlargements of D whose constructions will be given in forthcoming parts of this series of articles. This fundamental diagram for the pure case (see Section 0.1) was constructed in [KU3]:

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ D_{\Sigma,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\ \downarrow & & \downarrow & & & & \\ D_{\Sigma} & \leftarrow & D^{\sharp}_{\Sigma} & & & & \end{array}$$

In the next parts of this series, we will construct the rest of the spaces in this diagram. Among them, D_Σ is the space of nilpotent orbits. Degenerations of mixed Hodge structures of geometric origin also satisfy a nilpotent orbit theorem (see [SZ], [K], [Sa], [P1], etc.; a review is given in [KNU1, Section 12.10]). In the next articles in this series, we plan to interpret this in the style $(1)'$ above by using D_Σ in this diagram.

0.3.

We explain the contents of Sections 0.1 and 0.2 more precisely (but still roughly).

The nilpotent orbit theorem (in the pure case, see Section 0.1, and in the mixed case, see Section 0.2 also) says roughly that when $t=(t_1,\dots,t_n)\to 0$, we have

$$(F_t \bmod \Gamma) \sim \left(\exp\Bigl(\sum_{j=1}^{n} z_j N_j\Bigr) F \bmod \Gamma\right)$$

for some fixed Hodge filtration F ($\sim$ expresses "very near," but the precise meaning of it is not explained here), where z_j is a branch of $(2\pi i)^{-1}\log(t_j)$ and N_j is the logarithm of the local monodromy of F_t around the divisor $t_j=0$. In [KU3] for the pure case and in the next articles in this series for the mixed case, this is interpreted as the convergence

$$(F_t \bmod \Gamma) \to \bigl((\sigma, Z) \bmod \Gamma\bigr) \in \Gamma\backslash D_\Sigma,$$

where σ is the cone $\sum_{j=1}^n \mathbf{R}_{\geq 0} N_j$ and Z is the orbit $\exp\bigl(\sum_{j=1}^n \mathbf{C}N_j\bigr)F$. (As in the pure case, as a set, D_Σ is a set of such pairs (σ, Z).)

The SL(2)-orbit theorem in the pure case of Section 0.1, obtained in [CKS], says roughly that when $t\to 0$, $t_j\in\mathbf{R}_{>0}$, and $y_j/y_{j+1}\to\infty$, where $y_j=-(2\pi)^{-1}\times\log(t_j)$ for $1\leq j\leq n$ ($y_{n+1}=1$), we have

$$F_t \sim \rho\left(\begin{pmatrix}\sqrt{y_1} & 0\\ 0 & 1/\sqrt{y_1}\end{pmatrix},\dots,\begin{pmatrix}\sqrt{y_n} & 0\\ 0 & 1/\sqrt{y_n}\end{pmatrix}\right)\varphi(\mathbf{i}),$$

($\sim$ expresses "very near" again) where ρ is a homomorphism of algebraic groups $\mathrm{SL}(2,\mathbf{R})^n\to\mathrm{Aut}(D)$, φ is a complex analytic map $\mathfrak{h}^n\to D$ from the product $\mathfrak{h}^n$ of copies of the upper half-plane $\mathfrak{h}$, satisfying $\varphi(gz)=\rho(g)\varphi(z)$ for any $g\in\mathrm{SL}(2,\mathbf{R})^n$ and $z\in\mathfrak{h}^n$, and where $\mathbf{i}=(i,\dots,i)\in\mathfrak{h}^n$. In [KU3], this is interpreted as the convergence

$$F_t\to\mathrm{class}(\rho,\varphi)\in D_{\mathrm{SL}(2)}.$$

The SL(2)-orbit theorem in the mixed case of Section 0.2 obtained in [KNU1] says roughly that when $t\to 0$, $t_j\in\mathbf{R}_{>0}$, and $y_j/y_{j+1}\to\infty$, where $y_j=-(2\pi)^{-1}\times\log(t_j)$ for $1\leq j\leq n$ ($y_{n+1}=1$), we have

$$F_t \sim \operatorname{lift}\left(\bigoplus_{w\in\mathbf{Z}} y_1^{-w/2}\rho_w\left(\begin{pmatrix}\sqrt{y_1} & 0\\ 0 & 1/\sqrt{y_1}\end{pmatrix},\dots,\begin{pmatrix}\sqrt{y_n} & 0\\ 0 & 1/\sqrt{y_n}\end{pmatrix}\right)\right)\mathbf{r},$$

where (ρ_w,φ_w) $(w\in\mathbf{Z})$ is the SL(2)-orbit of pure weight w associated to the filtration on gr^W_w induced from F_t, $\mathbf{r}$ is a certain point of D which induces $\varphi_w(\mathbf{i})$ on each gr^W_w, and "lift" is the lifting to $\mathrm{Aut}(D)$ by the canonical splitting of the weight filtration associated to $\mathbf{r}$ (see Section 1.2). For details, see [KNU1] and also Section 2.4 of this article. By using the space $D_{\mathrm{SL}(2)}$ of this article, this is interpreted as the convergence

$$F_t \to \operatorname{class}\big((\rho_w,\varphi_w)_{w\in\mathbf{Z}},\mathbf{r}\big)\in D_{\mathrm{SL}(2)}.$$

Since $D_{\mathrm{SL}(2)}$ has a real analytic structure, we can discuss the differential of the extended period map $t\mapsto F_t$ at $t=0$. We hope that such a delicate structure of $D_{\mathrm{SL}(2)}$ is useful for the study of degeneration.

0.4.

Precisely, there are two natural spaces $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ which can sit in the place of $D_{\mathrm{SL}(2)}$ in the fundamental diagram. They coincide in the pure case and coincide always as sets but do not coincide in general. What we wrote in Section 0.3 is valid for both. They both have good properties, so that we do not choose one of them as a standard one (see Section 3.2.1 for more surveys).

0.5.

The organization of this article is as follows. In Section 1, we give preliminaries about basic facts on mixed Hodge structures. In Section 2, we define the space $D_{\mathrm{SL}(2)}$ as a set. In Section 3, we endow this set with topologies and with real analytic structures. (These spaces $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ are not necessarily real analytic spaces, but they have the sheaves of real analytic functions which we call the real analytic structures.) We study properties of these spaces. In Section 4, we consider how the degenerations of mixed Hodge structures are related to these spaces.

NOTATION

Fix a quadruple

$$\Phi_0=\big(H_0,W,(\langle\,,\,\rangle_w)_{w\in\mathbf{Z}},(h^{p,q})_{p,q\in\mathbf{Z}}\big),$$

where

- H_0 is a finitely generated free $\mathbf{Z}$-module;
- W is an increasing filtration on $H_{0,\mathbf{R}}:=\mathbf{R}\otimes_{\mathbf{Z}}H_0$ defined over $\mathbf{Q}$;
- $\langle\,,\,\rangle_w$ is a nondegenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w\times\mathrm{gr}^W_w\to\mathbf{R}$ defined over $\mathbf{Q}$ for each $w\in\mathbf{Z}$ which is symmetric if w is even and antisymmetric if w is odd; and
- $h^{p,q}$ is a nonnegative integer given for $p,q\in\mathbf{Z}$ such that $h^{p,q}=h^{q,p}$, $\mathrm{rank}_{\mathbf{Z}}(H_0)=\sum_{p,q}h^{p,q}$, and $\dim_{\mathbf{R}}(\mathrm{gr}^W_w)=\sum_{p+q=w}h^{p,q}$ for all w.

Let $\check{D}$ be the set of all decreasing filtrations F on $H_{0,\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Z}} H_0$ satisfying the following two conditions:

(1) $\dim(F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q})) = h^{p,q}$ for any $p, q \in \mathbf{Z}$;
(2) $\langle\, ,\rangle_w$ kills $F^p(\mathrm{gr}^W_w) \times F^q(\mathrm{gr}^W_w)$ for any $p, q, w \in \mathbf{Z}$ such that $p + q > w$.

Here $F(\mathrm{gr}^W_w)$ denotes the filtration on $\mathrm{gr}^W_{w,\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}^W_w$ induced by F.

Let D be the set of all decreasing filtrations $F \in \check{D}$ which also satisfy the following condition:

(3) $i^{p-q}\langle x, \bar{x}\rangle_w > 0$ for any nonzero $x \in F^p(\mathrm{gr}^W_w) \cap \overline{F^q(\mathrm{gr}^W_w)}$ and any $p, q, w \in \mathbf{Z}$ with $p + q = w$.

Then D is an open subset of $\check{D}$ and, for each $F \in D$ and $w \in \mathbf{Z}$, $F(\mathrm{gr}^W_w)$ is a Hodge structure on $(H_0 \cap W_w)/(H_0 \cap W_{w-1})$ of weight w with Hodge number $(h^{p,q})_{p+q=w}$ which is polarized by $\langle\, ,\rangle_w$. The space D is the classifying space of mixed Hodge structures of type Φ_0 introduced in [U1], which is a natural generalization to the mixed case of the Griffiths domain in [G]. These two are related by taking graded quotients by W as follows:

- $D(\mathrm{gr}^W_w)$: the D for $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\, ,\rangle_w, (h^{p,q})_{p+q=w})$ for each $w \in \mathbf{Z}$;
- $D(\mathrm{gr}^W) = \prod_{w\in\mathbf{Z}} D(\mathrm{gr}^W_w)$;
- $D \to D(\mathrm{gr}^W)$, $F \mapsto F(\mathrm{gr}^W) := (F(\mathrm{gr}^W_w))_{w\in\mathbf{Z}}$, the canonical surjection.

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$,

- G_A: the group of all A-automorphisms g of $H_{0,A} := A \otimes_{\mathbf{Z}} H_0$ compatible with W such that $\mathrm{gr}^W_w(g) : \mathrm{gr}^W_w \to \mathrm{gr}^W_w$ are compatible with $\langle\, ,\rangle_w$ for all w;
- $G_{A,u} := \{g \in G_A \mid \mathrm{gr}^W_w(g) = 1 \text{ for all } w \in \mathbf{Z}\}$, the *unipotent radical* of G_A;
- $G_A(\mathrm{gr}^W_w)$: the G_A of $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\, ,\rangle_w)$ for each $w \in \mathbf{Z}$;
- $G_A(\mathrm{gr}^W) := \prod_w G_A(\mathrm{gr}^W_w)$.

Then $G_A/G_{A,u} = G_A(\mathrm{gr}^W)$, and G_A is a semidirect product of $G_{A,u}$ and $G_A(\mathrm{gr}^W)$.

The natural action of $G_{\mathbf{C}}$ on $\check{D}$ is transitive, and $\check{D}$ is a complex homogeneous space under the action of $G_{\mathbf{C}}$. Hence $\check{D}$ is a complex analytic manifold. An open subset D of $\check{D}$ is also a complex analytic manifold. However, the action of $G_{\mathbf{R}}$ on D is not transitive in general (see the equivalent conditions (4), (5) below). The subgroup $G_{\mathbf{R}}G_{\mathbf{C},u}$ of $G_{\mathbf{C}}$ acts always transitively on D, and the action of $G_{\mathbf{C},u}$ on each fiber of $D \to D(\mathrm{gr}^W)$ is transitive.

- $\mathrm{spl}(W)$: the set of all isomorphisms $s : \mathrm{gr}^W = \bigoplus_w \mathrm{gr}^W_w \xrightarrow{\sim} H_{0,\mathbf{R}}$ of $\mathbf{R}$-vector spaces such that for any $w \in \mathbf{Z}$ and $v \in \mathrm{gr}^W_w$, $s(v) \in W_w$ and $v = (s(v) \bmod W_{w-1})$.
- We have the action $G_{\mathbf{R},u} \times \mathrm{spl}(W) \to \mathrm{spl}(W)$, $(g, s) \mapsto gs$.

For a fixed $s \in \mathrm{spl}(W)$, we have a bijection $G_{\mathbf{R},u} \xrightarrow{\sim} \mathrm{spl}(W), g \mapsto gs$. Via this bijection, we endow $\mathrm{spl}(W)$ with a structure of a real analytic manifold.

• $D_{\rm spl} := \{s(F) \mid s \in {\rm spl}(W), F \in D({\rm gr}^W)\} \subset D$, the subset of **R**-*split* elements.

Here $s(F)^p := s\big(\bigoplus_w F^p_{(w)}\big)$ for $F = (F_{(w)})_w \in D({\rm gr}^W)$.

• $D_{\rm nspl} := D \smallsetminus D_{\rm spl}$.

Then, $D_{\rm spl}$ is a closed real analytic submanifold of D, and we have a real analytic isomorphism ${\rm spl}(W) \times D({\rm gr}^W) \xrightarrow{\sim} D_{\rm spl}$, $(s, F) \mapsto s(F)$.

The following two conditions are equivalent (see [KNU2], Proposition 8.7):

(4) D is $G_{\mathbf{R}}$-homogeneous;
(5) $D = D_{\rm spl}$.

For example, if there is $w \in \mathbf{Z}$ such that $W_w = H_{0,\mathbf{R}}$ and $W_{w-2} = 0$, then the above equivalent conditions are satisfied. But in general these conditions are not satisfied (see Examples I, III, IV in Section 1.1).

For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$,

$\mathfrak{g}_A := {\rm Lie}(G_A)$ which is identified with $\{X \in {\rm End}_A(H_{0,A}) \mid X(W_w) \subset W_w$ for all w; $\langle {\rm gr}^W_w(X)(x), y\rangle_w + \langle x, {\rm gr}^W_w(X)(y)\rangle_w = 0$ for all $w, x, y\}$;
$\mathfrak{g}_{A,u} := {\rm Lie}(G_{A,u}) = \{X \in \mathfrak{g}_A \mid {\rm gr}^W_w(X) = 0$ for all $w\}$;
$\mathfrak{g}_A({\rm gr}^W_w)$: the $\mathfrak{g}_A$ of $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\,,\,\rangle_w)$ for each $w \in \mathbf{Z}$;
$\mathfrak{g}_A({\rm gr}^W) := \bigoplus_{w\in\mathbf{Z}} \mathfrak{g}_A({\rm gr}^W_w)$.

Then $\mathfrak{g}_A/\mathfrak{g}_{A,u} = \mathfrak{g}_A({\rm gr}^W)$.

1. Basic facts

We examine some examples, review some basic facts, and fix further notation which is used in this article.

1.1. Examples

1.1.1.

We give six simple examples (see Examples 0–V) of D for which the set $\{w \in \mathbf{Z} \mid {\rm gr}^W_w \neq 0\}$ is $\{-1\}$, $\{0,-2\}$, $\{0,-1\}$, $\{0,-3\}$, $\{0,-1,-2\}$, $\{0,1\}$, respectively. Among these, Examples I, II, and III are already presented in [KNU2, Sections 1.10–1.12] to illustrate the results in that article on each step. All these examples are retreated also to illustrate the results in this article on each step.

EXAMPLE 0

(This example belongs to the pure case, although Examples I–V below do not.) Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$. Let W be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_{-2} = 0 \subset W_{-1} = H_{0,\mathbf{R}}.$$

Let $\langle e_2, e_1\rangle_{-1} = 1$. Let $h^{-1,0} = h^{0,-1} = 1$, and let $h^{p,q} = 0$ for all the other (p, q).

For $\tau \in \mathbf{C}$, let $F(\tau)$ be the decreasing filtration on $H_{0,\mathbf{C}}$ defined by

$$F(\tau)^1 = 0 \subset F(\tau)^0 = \mathbf{C}(\tau e_1 + e_2) \subset F(\tau)^{-1} = H_{0,\mathbf{C}}.$$

Then we have an isomorphism of complex analytic manifolds

$$D \simeq \mathfrak{h},$$

where $\mathfrak{h}$ is the upper half-plane $\{x+iy \mid x,y \in \mathbf{R}, y>0\}$, in which $\tau \in \mathfrak{h}$ corresponds to $F(\tau) \in D$. This isomorphism naturally extends to $\check{D} \simeq \mathbf{P}^1(\mathbf{C})$.

EXAMPLE I

Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, and let W be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j=1$ (resp., $j=2$), let e'_j be the image of e_j in gr^W_{-2} (resp., gr^W_0). Let $\langle e'_2, e'_2\rangle_0 = 1$, $\langle e'_1, e'_1\rangle_{-2} = 1$, and let $h^{0,0} = h^{-1,-1} = 1$, $h^{p,q} = 0$ for all the other (p,q).

We have an isomorphism of complex analytic manifolds

$$D \simeq \mathbf{C}.$$

For $z \in \mathbf{C}$, the corresponding $F(z) \in D$ is defined as

$$F(z)^1 = 0 \subset F(z)^0 = \mathbf{C}(ze_1 + e_2) \subset F(z)^{-1} = H_{0,\mathbf{C}}.$$

The group $G_{\mathbf{Z},u}$ is isomorphic to $\mathbf{Z}$ and is generated by $\gamma \in G_{\mathbf{Z}}$, which is defined as

$$\gamma(e_1) = e_1, \qquad \gamma(e_2) = e_1 + e_2.$$

We have

$$G_{\mathbf{Z},u}\backslash D \simeq \mathbf{C}^\times,$$

where $(F(z) \bmod G_{\mathbf{Z},u})$ corresponds to $\exp(2\pi iz) \in \mathbf{C}^\times$.

The space $G_{\mathbf{Z},u}\backslash D$ is the classifying space of extensions of mixed Hodge structures of the form $0 \to \mathbf{Z}(1) \to * \to \mathbf{Z} \to 0$.

In this case, $D(\mathrm{gr}^W)$ is a one-point set.

EXAMPLE II

Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, and let

$$W_{-2} = 0 \subset W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1,2$ (resp., 3), let e'_j be the image of e_j in gr^W_{-1} (resp., gr^W_0). Let $\langle e'_3, e'_3\rangle_0 = 1$, let $\langle e'_2, e'_1\rangle_{-1} = 1$, and let $h^{0,0} = h^{0,-1} = h^{-1,0} = 1$, $h^{p,q} = 0$ for all the other (p,q).

Then we have isomorphisms of complex analytic manifolds

$$D \simeq \mathfrak{h} \times \mathbf{C}, \quad D(\mathrm{gr}^W) \simeq \mathfrak{h}.$$

Here $(\tau, z) \in \mathfrak{h} \times \mathbf{C}$ corresponds to $F = F(\tau, z) \in D$ given by

$$F^1 = 0 \subset F^0 = \mathbf{C}(\tau e_1 + e_2) + \mathbf{C}(ze_1 + e_3) \subset F^{-1} = H_{0,\mathbf{C}}.$$

The induced isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-1}) \simeq \mathfrak{h}$ is identified with the isomorphism $D \simeq \mathfrak{h}$ in Example 0.

The group $G_{\mathbf{Z},u}$ is isomorphic to $\mathbf{Z}^2$, where $(a,b) \in \mathbf{Z}^2$ corresponds to the element of $G_{\mathbf{Z}}$ which sends e_j to e_j for $j = 1,2$ and sends e_3 to $ae_1 + be_2 + e_3$. The quotient space $G_{\mathbf{Z},u}\backslash D$ is the universal elliptic curve over the upper half-plane $\mathfrak{h}$. For $\tau \in \mathfrak{h}$, the fiber of $G_{\mathbf{Z},u}\backslash D \to D(\mathrm{gr}^W) = \mathfrak{h}$ over τ is identified with the elliptic curve $E_\tau := \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. The Hodge structure on $H_0 \cap W_{-1}$ corresponding to τ is isomorphic to $H^1(E_\tau)(1)$. Here $H^1(E_\tau)$ denotes the Hodge structure $H^1(E_\tau, \mathbf{Z})$ of weight 1 endowed with the Hodge filtration and (1) denotes the Tate twist. The fiber of $G_{\mathbf{Z},u}\backslash D \to \mathfrak{h}$ over τ is the classifying space of extensions of mixed Hodge structures of the form

$$0 \to H^1(E_\tau)(1) \to * \to \mathbf{Z} \to 0.$$

EXAMPLE III

Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, and let

$$W_{-4} = 0 \subset W_{-3} = W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1,2$ (resp., 3), let e'_j be the image of e_j in gr^W_{-3} (resp., gr^W_0). Let $\langle e'_3, e'_3\rangle_0 = 1$, $\langle e'_2, e'_1\rangle_{-3} = 1$, and let $h^{0,0} = h^{-1,-2} = h^{-2,-1} = 1$, $h^{p,q} = 0$ for all the other (p,q).

Then we have isomorphisms of complex analytic manifolds

$$D \simeq \mathfrak{h} \times \mathbf{C}^2, \quad D(\mathrm{gr}^W) \simeq \mathfrak{h}.$$

Here $(\tau, z_1, z_2) \in \mathfrak{h} \times \mathbf{C}^2$ corresponds to $F = F(\tau, z_1, z_2) \in D$ given by

$$F^1 = 0 \subset F^0 = \mathbf{C}(z_1e_1 + z_2e_2 + e_3) \subset F^{-1} = F^0 + \mathbf{C}(\tau e_1 + e_2) \subset F^{-2} = H_{0,\mathbf{C}}.$$

The induced isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-3}) \simeq \mathfrak{h}$ is identified with the isomorphism $D \simeq \mathfrak{h}$ in Example 0 ($F \in D(\mathrm{gr}^W)$ corresponds to the twist $F(-1)$ of F, which belongs to the D in Example 0).

The group $G_{\mathbf{Z},u}$ is the same as in Example II. The Hodge structure on $H_0 \cap W_{-3}$ corresponding to $\tau \in \mathfrak{h} \simeq D(\mathrm{gr}^W_{-3})$ is isomorphic to $H^1(E_\tau)(2)$. The fiber of $G_{\mathbf{Z},u}\backslash D \to D(\mathrm{gr}^W) \simeq \mathfrak{h}$ over $\tau \in \mathfrak{h}$ is the classifying space of extensions of mixed Hodge structures of the form

$$0 \to H^1(E_\tau)(2) \to * \to \mathbf{Z} \to 0.$$

EXAMPLE IV

Let $H_0 = \mathbf{Z}^4 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$, and let

$$W_{-3} = 0 \subset W_{-2} = \mathbf{R}e_1 \subset W_{-1} = W_{-2} + \mathbf{R}e_2 + \mathbf{R}e_3 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1$ (resp., 2,3, resp., 4), let e'_j be the image of e_j in gr^W_{-2} (resp., gr^W_{-1}, resp., gr^W_0). Let $\langle e'_4, e'_4\rangle_0 = 1$, $\langle e'_1, e'_1\rangle_{-2} = 1$, let $\langle e'_3, e'_2\rangle_{-1} = 1$, and let $h^{0,0} = h^{0,-1} = h^{-1,0} = h^{-1,-1} = 1$, $h^{p,q} = 0$ for all the other (p,q).

Then we have isomorphisms of complex analytic manifolds

$$D = \mathfrak{h} \times \mathbf{C}^3, \quad D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-1}) = \mathfrak{h}.$$

Here $(\tau, z_1, z_2, z_3) \in \mathfrak{h} \times \mathbf{C}^3$ corresponds to $F = F(\tau, z_1, z_2, z_3) \in D$ given by $F^{-1} = H_{0,\mathbf{C}}$, $F^1 = 0$, and

$$F^0 = \mathbf{C}(z_1 e_1 + \tau e_2 + e_3) + \mathbf{C}(z_2 e_1 + z_3 e_2 + e_4).$$

The induced isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-1}) \simeq \mathfrak{h}$ is identified with the isomorphism $D \simeq \mathfrak{h}$ in Example 0.

There is a bijection $G_{\mathbf{Z},u} \simeq \mathbf{Z}^5$ (but not a group isomorphism), where $(a_j)_{1 \le j \le 5} \in \mathbf{Z}^5$ corresponds to the element of $G_{\mathbf{Z},u}$ which sends e_1 to e_1, e_2 to $a_1 e_1 + e_2$, e_3 to $a_2 e_1 + e_3$, and e_4 to $a_3 e_1 + a_4 e_2 + a_5 e_3 + e_4$.

EXAMPLE V

Let $H_0 = \mathbf{Z}^5 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4 + \mathbf{Z}e_5$, and let

$$W_{-1} = 0 \subset W_0 = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3 \subset W_1 = H_{0,\mathbf{R}}.$$

For $j = 1, 2, 3$ (resp., $4, 5$), let e'_j be the image of e_j in gr^W_0 (resp., gr^W_1). Let $\langle e'_5, e'_4 \rangle_1 = 1$, $\langle e'_1, e'_3 \rangle_0 = 2$, $\langle e'_2, e'_2 \rangle_0 = -1$, and $\langle e'_j, e'_k \rangle_0 = 0$ $(j + k \neq 4,\ 1 \le j,\ k \le 3)$, and let $h^{1,-1} = h^{0,0} = h^{-1,1} = h^{1,0} = h^{0,1} = 1$ and $h^{p,q} = 0$ for all the other (p, q).

Let $\mathfrak{h}^{\pm} = \{x + iy \mid x, y \in \mathbf{R}, y \neq 0\} = \mathfrak{h} \sqcup (-\mathfrak{h})$. Then we have isomorphisms of complex analytic manifolds

$$D \simeq \mathfrak{h}^{\pm} \times \mathfrak{h} \times \mathbf{C}^3, \quad D(\mathrm{gr}^W_0) \simeq \mathfrak{h}^{\pm},\ D(\mathrm{gr}^W_1) \simeq \mathfrak{h}.$$

Here $(\tau_0, \tau_1, z_1, z_2, z_3) \in \mathfrak{h}^{\pm} \times \mathfrak{h} \times \mathbf{C}^3$ corresponds to $F = F(\tau_0, \tau_1, z_1, z_2, z_3) \in D$ given by $F^2 = 0$, $F^{-1} = H_{0,\mathbf{C}}$, and

$$F^1 = \mathbf{C}(\tau_0^2 e_1 + 2\tau_0 e_2 + e_3) + \mathbf{C}(z_1 e_1 + z_2 e_2 + \tau_1 e_4 + e_5),$$
$$F^0 = F^1 + \mathbf{C}(\tau_0 e_1 + e_2) + \mathbf{C}(z_3 e_1 + e_4).$$

Let $F(\tau)$ be the filtration in Example 0 corresponding to $\tau \in \mathfrak{h}$. The induced isomorphism $D(\mathrm{gr}^W_1) \simeq \mathfrak{h}$ sends $\tau \in \mathfrak{h}$ to the Tate twist $F(\tau)(-1)$ of $F(\tau)$. The induced isomorphism $D(\mathrm{gr}^W_0) \simeq \mathfrak{h}^{\pm}$ sends $\tau \in \mathfrak{h}^{\pm}$ to $\mathrm{Sym}^2(F(\tau))(-1) \in D(\mathrm{gr}^W_0)$ (see Section 1.1.2).

The group $G_{\mathbf{Z},u}$ is isomorphic to $\mathbf{Z}^6$, where $(a_j)_{1 \le j \le 6} \in \mathbf{Z}^6$ corresponds to the element of $G_{\mathbf{Z}}$ which sends e_j to e_j for $j = 1, 2, 3$, e_4 to $a_1 e_1 + a_2 e_2 + a_3 e_3 + e_4$, and e_5 to $a_4 e_1 + a_5 e_2 + a_6 e_3 + e_5$.

1.1.2.

REMARK

For the computations of Example V in Section 1.1.1 and in Sections 3.6 and 4.2.4 later, we describe here the classifying space D_2 of polarized Hodge structures of weight 2 underlain by the second symmetric power of the Tate twist (by -1) of $(H_0, \langle\ ,\ \rangle_{-1})$ in Example 0.

The domain $D(\mathrm{gr}^W_0)$ in Example V of Section 1.1.1 is identified with D_2 via the Tate twist.

Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}f_1 + \mathbf{Z}f_2$, let $W_0 = 0 \subset W_1 = H_{0,\mathbf{R}}$, and let $\langle f_2, f_1\rangle_1 = 1$. Then $\mathrm{Sym}^2(H_0) = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, where $e_1 := f_1^2,\ e_2 := f_1f_2,\ e_3 := f_2^2$, and the induced polarization on $\mathrm{Sym}^2(H_0)$, which is defined by

$$\langle x_1x_2, y_1y_2\rangle_2 = \langle x_1, y_1\rangle_1\langle x_2, y_2\rangle_1 + \langle x_1, y_2\rangle_1\langle x_2, y_1\rangle_1 \quad (x_j, y_j \in H_0, j = 1, 2),$$

is given by

$$\langle e_1, e_3\rangle_2 = \langle e_3, e_1\rangle_2 = 2, \qquad \langle e_2, e_2\rangle_2 = -1, \qquad \langle e_j, e_k\rangle_2 = 0$$

otherwise.

For $v = \omega_1e_1 + \omega_2e_2 + \omega_3e_3 \in \mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3$ to be Hodge type $(2, 0)$, the Riemann-Hodge bilinear relations are

$$\langle v, v\rangle_2 = 4\omega_1\omega_3 - \omega_2^2 = 0,$$
$$\langle Cv, \bar{v}\rangle_2 = i^2\langle v, \bar{v}\rangle_2 = -4\,\mathrm{Re}(\omega_1\bar{\omega}_3) + |\omega_2|^2 > 0,$$

where C is the Weil operator. Hence the classifying space D_2 and its compact dual $\check{D}_2$ of the Hodge structures of weight 2, with Hodge type $h^{2,0} = h^{1,1} = h^{0,2} = 1$ and $h^{p,q} = 0$ otherwise, and with the polarization $\langle\,,\,\rangle_2$, is as follows:

$$\check{D}_2 = \big\{\mathbf{C}(\omega_1e_1 + \omega_2e_2 + \omega_3e_3) \subset \mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3 \mid 4\omega_1\omega_3 - \omega_2^2 = 0\big\} \simeq \mathbf{P}^1(\mathbf{C}),$$

$$(1) \qquad D_2 = \big\{\mathbf{C}(\omega_1e_1 + \omega_2e_2 + \omega_3e_3) \in \check{D} \mid -4\,\mathrm{Re}(\omega_1\bar{\omega}_3) + |\omega_2|^2 > 0\big\} \simeq \mathfrak{h}^{\pm}.$$

The isomorphism is given by $\omega_1e_1 + \omega_2e_2 + \omega_3e_3 = \omega^2e_1 + 2\omega e_2 + e_3 \leftrightarrow \omega$.

Assigning $g \in \mathrm{SL}(2, \mathbf{R})$ to $\mathrm{sym}^2(g) \in \mathrm{Aut}(H_{0,\mathbf{R}}, \langle\,,\,\rangle_2)$, we have an exact sequence

$$(2) \qquad 1 \to \{\pm 1\} \to \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Aut}(H_{0,\mathbf{R}}, \langle\,,\,\rangle_2) \to \{\pm 1\} \to 1.$$

The isomorphism (1) is compatible with (2).

1.2. Canonical splittings of weight filtrations for mixed Hodge structures

Let W and D be as in the notation at the end of the introduction. In this section, we review the canonical splitting $s = \mathrm{spl}_W(F) \in \mathrm{spl}(W)$ of the weight filtration W associated to $F \in D$, defined by the theory of Cattani, Kaplan, and Schmid [CKS]. This canonical splitting s appeared naturally in the SL(2)-orbit theorem for mixed Hodge structures proved in our previous article [KNU1]. The definition of s was reviewed in detail in [KNU1, Section 1], although the formulation there is different from the one in this section. The canonical splitting plays important roles in the present series of our articles.

1.2.1.
Let $F = (F_{(w)})_w \in D(\mathrm{gr}^W)$. Regard F as the filtration $\bigoplus_w F_{(w)}$ on $\mathrm{gr}^W_{\mathbf{C}} = \bigoplus_w \mathrm{gr}^W_{w,\mathbf{C}}$, and let $H_F^{p,q} = H^{p,q}_{F_{(p+q)}} \subset \mathrm{gr}^W_{p+q,\mathbf{C}}$. Let

$$L^{-1,-1}_{\mathbf{R}}(F) = \Big\{\delta \in \mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W) \;\Big|\; \delta(H_F^{p,q}) \subset \bigoplus_{p'<p,q'<q} H_F^{p',q'} \text{ for all } p, q \in \mathbf{Z}\Big\}.$$

All elements of $L_{\mathbf{R}}^{-1,-1}(F)$ are nilpotent. Let

$$\mathcal{L} = \mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)_{\leq -2}$$

be the set of all $\mathbf{R}$-linear maps $\delta : \mathrm{gr}^W \to \mathrm{gr}^W$ such that $\delta(\mathrm{gr}^W_w) \subset \bigoplus_{w' \leq w-2} \mathrm{gr}^W_{w'}$ for any $w \in \mathbf{Z}$. Denote

$$\mathcal{L}(F) = L_{\mathbf{R}}^{-1,-1}(F) \subset \mathcal{L}.$$

$\mathcal{L}(F)$ is sometimes denoted simply by L.

In this Section 1.2, we review the isomorphism of real analytic manifolds

$$D \simeq \{(s, F, \delta) \in \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times \mathcal{L} \mid \delta \in \mathcal{L}(F)\}$$

obtained in the work [CKS] (see Section 1.2.5). For $F' \in D$, the corresponding (s, F, δ) consists of $F = F'(\mathrm{gr}^W)$, $\delta = \delta(F') \in \mathcal{L}(F)$ defined in Section 1.2.2, and the canonical splitting $s = \mathrm{spl}_W(F')$ of W associated to F' explained in Section 1.2.3.

1.2.2.
For $F' \in D$, there is a unique pair $(s', \delta) \in \mathrm{spl}(W) \times \mathcal{L}(F'(\mathrm{gr}^W))$ such that

$$F' = s'\big(\exp(i\delta)F'(\mathrm{gr}^W)\big)$$

(see [CKS]). This is the definition of $\delta = \delta(F')$ associated to F'.

1.2.3.
Let $F' \in D$, and let $s' \in \mathrm{spl}(W)$ and $\delta \in \mathcal{L}(F'(\mathrm{gr}^W))$ be as in Section 1.2.2. Then the canonical splitting $s = \mathrm{spl}_W(F')$ of W associated to F' is defined by

$$s = s' \exp(\zeta),$$

where $\zeta = \zeta(F'(\mathrm{gr}^W), \delta)$ is a certain element of $\mathcal{L}(F'(\mathrm{gr}^W))$ determined by $F'(\mathrm{gr}^W)$ and δ in the following way.

Let $\delta_{p,q}$ $(p, q \in \mathbf{Z})$ be the (p,q)-Hodge component of δ with respect to $F'(\mathrm{gr}^W)$ defined by

$$\delta = \sum_{p,q} \delta_{p,q} \quad \big(\delta_{p,q} \in \mathcal{L}_{\mathbf{C}}(F'(\mathrm{gr}^W)) = \mathbf{C} \otimes_{\mathbf{R}} \mathcal{L}(F'(\mathrm{gr}^W))\big),$$

$$\delta_{p,q}(H^{k,l}_{F'(\mathrm{gr}^W)}) \subset H^{k+p,l+q}_{F'(\mathrm{gr}^W)} \quad \text{for all } k, l \in \mathbf{Z}.$$

Then the (p,q)-Hodge component $\zeta_{p,q}$ of $\zeta = \zeta(F'(\mathrm{gr}^W), \delta)$ with respect to $F'(\mathrm{gr}^W)$ is given as a certain universal Lie polynomial of $\delta_{p',q'}$ $(p', q' \in \mathbf{Z}, p' \leq -1, q' \leq -1)$ (see [CKS] and [KNU1, Section 1]). For example,

$$\zeta_{-1,-1} = 0,$$

$$\zeta_{-1,-2} = -\frac{i}{2}\delta_{-1,-2},$$

$$\zeta_{-2,-1} = \frac{i}{2}\delta_{-2,-1}.$$

1.2.4.
For $F \in D(\mathrm{gr}^W)$ and $\delta \in \mathcal{L}(F)$, we define a filtration $\theta(F,\delta)$ on $\mathrm{gr}^W_{\mathbf{C}}$ by

$$\theta(F,\delta) = \exp(-\zeta)\exp(i\delta)F,$$

where $\zeta = \zeta(F,\delta)$ is the element of $\mathcal{L}(F)$ associated to the pair (F,δ) as in Section 1.2.3.

PROPOSITION 1.2.5
We have an isomorphism of real analytic manifolds

$$D \simeq \{(s,F,\delta) \in \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times \mathcal{L} \mid \delta \in \mathcal{L}(F)\},$$
$$F' \mapsto (\mathrm{spl}_W(F'), F'(\mathrm{gr}^W), \delta(F')),$$

whose inverse is given by $(s,F,\delta) \mapsto s(\theta(F,\delta))$.

1.2.6.
For $g = (g_w)_w \in G_{\mathbf{R}}(\mathrm{gr}^W) = \prod_w G_{\mathbf{R}}(\mathrm{gr}^W_w)$, we have

$$g\theta(F,\delta) = \theta(gF, \mathrm{Ad}(g)\delta),$$

where $\mathrm{Ad}(g)\delta = g\delta g^{-1}$.

1.2.7.
For $F \in D(\mathrm{gr}^W)$, $\delta \in \mathcal{L}(F)$, and $s \in \mathrm{spl}(W)$, the element $s(\theta(F,\delta))$ of D belongs to D_{spl} if and only if $\delta = 0$.

1.2.8.
REMARK
The results in Section 1.2 are valid for W defined over $\mathbf{R}$, that is, without assuming that W is being defined over $\mathbf{Q}$.

1.2.9.
We consider Examples I–V in Section 1.1.1. For these examples, $\mathcal{L}(F) = L_{\mathbf{R}}^{-1,-1}(F) \subset \mathcal{L}$ in Section 1.2.1 is independent of the choice of $F \in D(\mathrm{gr}^W)$, and we denote it simply by L. By Proposition 1.2.5, we have a real analytic presentation of D,

$$D \simeq \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times L. \qquad (1)$$

The relation with the complex analytic presentation of D given in Section 1.1.1 is as follows. We use the notation in Section 1.1.1.

EXAMPLE I
We have $\mathrm{spl}(W) \simeq \mathbf{R}$ by assigning $s \in \mathbf{R}$ to the splitting of W defined by $e'_2 \mapsto se_1 + e_2$, $D(\mathrm{gr}^W)$ is one point, and $L \simeq \mathbf{R}$, $\delta \leftrightarrow d$, by $\delta(e'_2) = de'_1$ (see Section 1.2.3).

The relation with the complex analytic presentation $D \simeq \mathbf{C}$ in Example I in Section 1.1.1 and the real analytic presentation (1) of D is as follows. The composition

$$\mathbf{C} \simeq D \simeq \mathrm{spl}(W) \times L \simeq \mathbf{R} \times \mathbf{R}$$

is given by

$$z \leftrightarrow (s, d), \quad z = s + id.$$

Conversely, we have

$$s = \mathrm{Re}(z), \qquad d = \mathrm{Im}(z).$$

This is because the ζ associated to $\delta \in L$ is equal to $\zeta_{-1,-1} = 0$ (see Section 1.2.3).

EXAMPLE II

We have $\mathrm{spl}(W) \simeq \mathbf{R}^2$, $s \leftrightarrow (s_1, s_2)$, by $s(e'_3) = s_1e_1 + s_2e_2 + e_3$ and $s(e'_j) = e_j$ ($j = 1, 2$), and we have $L = 0$.

The relation with the complex analytic presentation $D \simeq \mathfrak{h} \times \mathbf{C}$ in Example II in Section 1.1.1 and the real analytic presentation (1) of D is as follows. The composition

$$\mathfrak{h} \times \mathbf{C} \simeq D \simeq \mathrm{spl}(W) \times D(\mathrm{gr}^W) \simeq \mathbf{R}^2 \times \mathfrak{h}$$

is given by

$$(\tau, z) \leftrightarrow \big((s_1, s_2), \tau\big) \quad \text{with } z = s_1 - s_2\tau.$$

Conversely, we have

$$s_1 = \mathrm{Re}(z) - \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\mathrm{Re}(\tau), \qquad s_2 = -\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}.$$

EXAMPLE III

We have $\mathrm{spl}(W) \simeq \mathbf{R}^2$, $s \leftrightarrow (s_1, s_2)$, by $s(e'_3) = s_1e_1 + s_2e_2 + e_3$ and $s(e'_j) = e_j$ ($j = 1, 2$), and we have $L \simeq \mathbf{R}^2$, $\delta \leftrightarrow (d_1, d_2)$, by $\delta(e'_3) = d_1e'_1 + d_2e'_2$.

The relation with the complex analytic presentation $D \simeq \mathfrak{h} \times \mathbf{C}^2$ in Example III in Section 1.1.1 and the real analytic presentation (1) of D is as follows. The composition

$$\mathfrak{h} \times \mathbf{C}^2 \simeq D \simeq \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times L \simeq \mathbf{R}^2 \times \mathfrak{h} \times \mathbf{R}^2$$

is given by

$$(\tau, z_1, z_2) \leftrightarrow \big((s_1, s_2), \tau, (d_1, d_2)\big), \tag{2}$$

where

$$z_1 = s_1 + \Big(\frac{\mathrm{Re}(\tau)}{2\,\mathrm{Im}(\tau)} + i\Big)d_1 - \frac{\mathrm{Re}(\tau)^2 + \mathrm{Im}(\tau)^2}{2\,\mathrm{Im}(\tau)}d_2, \tag{3}$$

$$z_2 = s_2 + \frac{1}{2\,\mathrm{Im}(\tau)}d_1 + \Big(-\frac{\mathrm{Re}(\tau)}{2\,\mathrm{Im}(\tau)} + i\Big)d_2.$$

Conversely, we have

$$
\begin{aligned}
d_1 &= \operatorname{Im}(z_1), \qquad d_2 = \operatorname{Im}(z_2),\\
s_1 &= \operatorname{Re}(z_1) - \frac{\operatorname{Re}(\tau)}{2\operatorname{Im}(\tau)}\operatorname{Im}(z_1) + \frac{\operatorname{Re}(\tau)^2+\operatorname{Im}(\tau)^2}{2\operatorname{Im}(\tau)}\operatorname{Im}(z_2), \qquad (4)\\
s_2 &= \operatorname{Re}(z_2) - \frac{1}{2\operatorname{Im}(\tau)}\operatorname{Im}(z_1) + \frac{\operatorname{Re}(\tau)}{2\operatorname{Im}(\tau)}\operatorname{Im}(z_2).
\end{aligned}
$$

We explain that the correspondence (2) is described as in (3) and (4). Write $\tau = x+iy$ with $x, y \in \mathbf{R}$, $y>0$. We have in $\operatorname{gr}^W_{-3,\mathbf{C}}$ the Hodge decomposition $\delta(e'_3) = d_1e'_1 + d_2e'_2 = A+B$, where

$$A = \frac{d_1 - d_2\bar\tau}{2yi}(\tau e'_1 + e'_2), \qquad B = \frac{-d_1 + d_2\tau}{2yi}(\bar\tau e'_1 + e'_2)$$

with respect to the element $F \in D(\operatorname{gr}^W_{-3}) = D(\operatorname{gr}^W)$ corresponding to $\tau \in \mathfrak{h}$. This shows that the (p,q)-Hodge component $\delta_{p,q}$ of δ is given as follows. We have $\delta_{p,q} = 0$ for $(p,q) \neq (-1,-2), (-2,-1)$, and $\delta_{-1,-2}$ sends e'_3 to A, and $\delta_{-2,-1}$ sends e'_3 to B. Since $\zeta(F,\delta) = -(i/2)\delta_{-1,-2} + (i/2)\delta_{-2,-1}$ (see Section 1.2.3), this shows that $\zeta(F,\delta)$ sends e'_3 to

$$v := \frac{-d_1x + d_2(x^2+y^2)}{2y}e'_1 + \frac{-d_1 + d_2x}{2y}e'_2.$$

Hence $\theta(F,\delta) = \exp(-\zeta(F,\delta))\exp(i\delta)F$ is the decreasing filtration of $\operatorname{gr}^W_{\mathbf{C}}$ characterized by the following properties: $\theta(F,\delta)^1 = 0$, $\theta(F,\delta)^{-2} = \operatorname{gr}^W_{\mathbf{C}}$, $\theta(F,\delta)^0$ is generated over $\mathbf{C}$ by $-v + id_1e'_1 + id_2e'_2 + e'_3$, and $\theta(F,\delta)^{-1}$ is generated over $\mathbf{C}$ by $\theta(F,\delta)^0$ and $\tau e'_1 + e'_2$. The above (3) follows from this, and (4) follows from (3).

EXAMPLE IV

We have $\operatorname{spl}(W) \simeq \mathbf{R}^5$, $s \leftrightarrow (s_j)_{1\le j\le 5}$, by $s(e'_1) = e_1$, $s(e'_2) = s_1e_1 + e_2$, $s(e'_3) = s_2e_1 + e_3$, and $s(e'_4) = s_3e_1 + s_4e_2 + s_5e_3 + e_4$, and we have $L \simeq \mathbf{R}$, $\delta \leftrightarrow d$, by $\delta(e'_4) = de'_1$.

The relation with the complex analytic presentation $D \simeq \mathfrak{h} \times \mathbf{C}^3$ in Example IV in Section 1.1.1 and the real analytic presentation (1) of D is as follows. The composition

$$\mathfrak{h} \times \mathbf{C}^3 \simeq D \simeq \operatorname{spl}(W) \times D(\operatorname{gr}^W) \times L \simeq \mathbf{R}^5 \times \mathfrak{h} \times \mathbf{R}$$

is given by

$$(\tau, z_1, z_2, z_3) \leftrightarrow \big((s_1,\dots,s_5), \tau, d\big),$$

where

$$z_1 = s_1\tau + s_2, \qquad z_2 = s_3 - s_5(s_1\tau + s_2) + id, \qquad z_3 = s_4 - s_5\tau.$$

Conversely, we have

$$s_1 = \frac{\operatorname{Im}(z_1)}{\operatorname{Im}(\tau)}, \qquad s_2 = \operatorname{Re}(z_1) - \frac{\operatorname{Im}(z_1)}{\operatorname{Im}(\tau)}\operatorname{Re}(\tau),$$

$$s_3 = \mathrm{Re}(z_2) - \frac{\mathrm{Im}(z_3)}{\mathrm{Im}(\tau)}\mathrm{Re}(z_1), \qquad s_4 = \mathrm{Re}(z_3) - \frac{\mathrm{Im}(z_3)}{\mathrm{Im}(\tau)}\mathrm{Re}(\tau),$$
$$s_5 = -\frac{\mathrm{Im}(z_3)}{\mathrm{Im}(\tau)}, \qquad d = \mathrm{Im}(z_2) - \frac{\mathrm{Im}(z_1)\,\mathrm{Im}(z_3)}{\mathrm{Im}(\tau)}.$$

This follows from $\zeta = \zeta_{-1,-1} = 0$ (see Section 1.2.3).

EXAMPLE V

We have $\mathrm{spl}(W) \simeq \mathbf{R}^6$, $s \leftrightarrow (s_j)_{1\le j\le 6}$, by $s(e'_4) = s_1e_1 + s_2e_2 + s_3e_3 + e_4$, $s(e'_5) = s_4e_1 + s_5e_2 + s_6e_3 + e_5$, and $s(e'_j) = e_j$ $(j = 1, 2, 3)$, and $L = 0$.

The relation with the complex analytic presentation $D \simeq \mathfrak{h}^{\pm} \times \mathfrak{h} \times \mathbf{C}^3$ in Example V in Section 1.1.1 and the real analytic presentation (1) of D is as follows. The composition

$$\mathfrak{h}^{\pm} \times \mathfrak{h} \times \mathbf{C}^3 \simeq D \simeq \mathrm{spl}(W) \times D(\mathrm{gr}^W) \simeq \mathbf{R}^6 \times \mathfrak{h}^{\pm} \times \mathfrak{h}$$

is given by

$$(\tau_0, \tau_1, z_1, z_2, z_3) \leftrightarrow \big((s_1, \dots, s_6), \tau_0, \tau_1\big),$$

where

$$z_1 = s_1\tau_1 - s_3\tau_0^2\tau_1 + s_4 - s_6\tau_0^2, \qquad z_2 = s_2\tau_1 - 2s_3\tau_0\tau_1 + s_5 - 2s_6\tau_0,$$
$$z_3 = s_1 - s_2\tau_0 + s_3\tau_0^2.$$

From this we can obtain presentations of s_j $(1 \le j \le 6)$ in terms of τ_0, τ_1, z_1, z_2, z_3, but we do not write them down here.

2. The set $D_{\mathrm{SL}(2)}$

2.1. $\mathrm{SL}(2)$-orbits in pure case

We review SL(2)-orbits in the case of pure weight. We also prove some new results here.

Let $w \in \mathbf{Z}$, and assume $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$.

2.1.1.

Let $n \ge 0$, and consider a pair (ρ, φ) consisting of a homomorphism

$$\rho:\ \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$$

of algebraic groups which is defined over $\mathbf{R}$ and a holomorphic map $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$ satisfying the following condition:

$$\varphi(gz) = \rho(g)\varphi(z) \quad \text{for any } g \in \mathrm{SL}(2, \mathbf{C})^n, z \in \mathbf{P}^1(\mathbf{C})^n.$$

2.1.2.

As in [KU3, Section 5] (see also [KU2, Section 3]), we call (ρ, φ) as in Section 2.1.1 an SL(2)-*orbit in n variables* if it further satisfies the following two conditions (1) and (2):

(1) $$\varphi(\mathfrak{h}^n) \subset D.$$

(2) $\rho_*\big(\mathrm{fil}^p_z(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})\big) \subset \mathrm{fil}^p_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}})$ for any $z\in\mathbf{P}^1(\mathbf{C})^n$ and any $p\in\mathbf{Z}$.

Here in (1), $\mathfrak{h}=\{x+iy \mid x,y\in\mathbf{R}, y>0\}\subset\mathbf{P}^1(\mathbf{C})$ as in Section 1.1. In (2), ρ_* denotes the Lie algebra homomorphism $\mathfrak{sl}(2,\mathbf{C})^{\oplus n}\to\mathfrak{g}_{\mathbf{C}}$ induced by ρ,

$$\mathrm{fil}^p_z\big(\mathfrak{sl}(2,\mathbf{C})^{\oplus n}\big) = \Big\{X\in\mathfrak{sl}(2,\mathbf{C})^{\oplus n} \;\Big|\; X\Big(\bigoplus_{j=1}^n F^r_{z_j}(\mathbf{C}^2)\Big)\subset\bigoplus_{j=1}^n F^{r+p}_{z_j}(\mathbf{C}^2)\quad(\forall r\in\mathbf{Z})\Big\},$$

where for $a\in\mathbf{P}^1(\mathbf{C})$, $F^r_a(\mathbf{C}^2)=\mathbf{C}^2$ if $r\le 0$, $F^1_a(\mathbf{C}^2)=\mathbf{C}\binom{a}{1}$ if $a\in\mathbf{C}$, $F^1_\infty(\mathbf{C}^2)=\mathbf{C}\binom{1}{0}$, $F^r_a(\mathbf{C}^2)=0$ for $r\ge 2$, and

$$\mathrm{fil}^p_F(\mathfrak{g}_{\mathbf{C}})=\{X\in\mathfrak{g}_{\mathbf{C}} \mid XF^r\subset F^{r+p} \text{ for all } r\in\mathbf{Z}\}\quad \text{for} F\in\check{D}.$$

PROPOSITION 2.1.3

Let (ρ,φ) be as in Section 2.1.1.

(i) *Condition (1) in Section 2.1.2 is satisfied if there exists $z\in\mathfrak{h}^n$ such that $\varphi(z)\in D$.*

(ii) *Condition (2) in Section 2.1.2 is satisfied if there exists $z\in\mathbf{P}^1(\mathbf{C})^n$ such that $\rho_*\big(\mathrm{fil}^p_z(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})\big)\subset\mathrm{fil}^p_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}})$ for all $p\in\mathbf{Z}$.*

Proof

We prove (i). Any element z' of $\mathfrak{h}^n$ is written in the form gz with $g\in\mathrm{SL}(2,\mathbf{R})^n$. Hence $\varphi(z')=\rho(g)\varphi(z)\in D$.

We prove (ii). Any element z' of $\mathbf{P}^1(\mathbf{C})^n$ is written in the form gz with $g\in\mathrm{SL}(2,\mathbf{C})^n$. Hence

$$\begin{aligned}\rho_*\big(\mathrm{fil}^p_{z'}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})\big) &= \rho_*\big(\mathrm{Ad}(g)\,\mathrm{fil}^p_z(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})\big)\\ &= \mathrm{Ad}\big(\rho(g)\big)\rho_*\big(\mathrm{fil}^p_z(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})\big)\\ &\subset \mathrm{Ad}\big(\rho(g)\big)\,\mathrm{fil}^p_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}})=\mathrm{fil}^p_{\varphi(z')}(\mathfrak{g}_{\mathbf{C}}).\end{aligned}$$

□

2.1.4.

We fix notation. Assume that we are given (ρ,φ) as in Section 2.1.1.

Let

$$N_j, Y_j, N_j^+\in\mathfrak{g}_{\mathbf{R}}\quad(1\le j\le n),$$

$$N_j=\rho_*\begin{pmatrix}0&1\\0&0\end{pmatrix}_j,\qquad Y_j=\rho_*\begin{pmatrix}-1&0\\0&1\end{pmatrix}_j,\qquad N_j^+=\rho_*\begin{pmatrix}0&0\\1&0\end{pmatrix}_j,$$

where $(\,)_j$ means the embedding $\mathfrak{sl}(2)\to\mathfrak{sl}(2)^{\oplus n}$ into the jth factor.

PROPOSITION 2.1.5

Let (ρ,φ) be as in Section 2.1.1. Fix $F\in\varphi(\mathbf{C}^n)$. Then condition (2) in Section 2.1.2 is satisfied if and only if

(2′) $\quad N_jF^p\subset F^{p-1}$ *for any $1\le j\le n$ and any $p\in\mathbf{Z}$.*

Proof
Since $F=\varphi((z_j)_j)=\exp\bigl(\sum_{j=1}^n z_jN_j\bigr)\varphi(\mathbf{0})$ for some $(z_j)_j\in\mathbf{C}^n$, where $\mathbf{0}=0^n\in\mathbf{P}^1(\mathbf{C})^n$, condition (2′) for $F\in\varphi(\mathbf{C}^n)$ is equivalent to condition (2′) for $F=\varphi(\mathbf{0})$. Note that $\mathrm{fil}^p_{\mathbf{0}}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})=0$ if $p\ge 2$, that $\mathrm{fil}^1_{\mathbf{0}}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})$ is generated as a $\mathbf{C}$-vector space by the matrices $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)_j$ $(1\le j\le n)$, that $\mathrm{fil}^0_{\mathbf{0}}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})$ is generated as a $\mathbf{C}$-vector space by $\mathrm{fil}^1_{\mathbf{0}}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})$ and the matrices $\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)_j$ $(1\le j\le n)$, and that $\mathrm{fil}^p_{\mathbf{0}}(\mathfrak{sl}(2,\mathbf{C})^{\oplus n})=\mathfrak{sl}(2,\mathbf{C})^{\oplus n}$ if $p\le -1$. Hence, by Proposition 2.1.3(ii), condition (2) in Section 2.1.2 is equivalent to

$$N_j\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^{p-1},\qquad Y_j\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^p,\qquad N_j^+\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^{p+1}\quad\text{for any } j,\ p.$$

Hence, if condition (2) in Section 2.1.2 is satisfied, then (2′) is satisfied for $F=\varphi(\mathbf{0})$.

Assume that condition (2′) is satisfied for $F=\varphi(\mathbf{0})$. We show that condition (2) in Section 2.1.2 is satisfied. For any diagonal matrices $g_1,\dots,g_n$ in $\mathrm{SL}(2,\mathbf{C})$, we have $(g_1,\dots,g_n)\mathbf{0}=\mathbf{0}$ and hence $\rho(g_1,\dots,g_n)\varphi(\mathbf{0})=\varphi(\mathbf{0})$. From this, we have $Y_j\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^p$ for all j and all $p\in\mathbf{Z}$. It remains to prove $N_j^+\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^{p+1}$ for all j and all $p\in\mathbf{Z}$. The following argument is given in [U2, Section 2] in the case $n=1$. By the theory of representations of $\mathfrak{sl}(2,\mathbf{R})^{\oplus n}$ and by the property $Y_j\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^p$ for any j and any p, we have a direct sum decomposition as an $\mathbf{R}$-vector space

$$H_{0,\mathbf{R}}=\bigoplus_{(a,b)\in S}P_{a,b}$$

with $S=\{(a,b)\in\mathbf{Z}^n\times\mathbf{Z}^n\mid a\ge b\ge -a, a(j)\equiv b(j)\bmod 2\text{ for }1\le j\le n\}$, having the following properties (1)–(4). Here, for $a,b\in\mathbf{Z}^n$, $a\ge b$ means $a(j)\ge b(j)$ for all $1\le j\le n$. For $1\le j\le n$, let e_j be the element of $\mathbf{Z}^n$ defined by $e_j(k)=1$ if $k=j$ and $e_j(k)=0$ if $k\ne j$.

(1) On $P_{a,b}$, Y_j acts as the multiplication by $b(j)$.

(2) Let $(a,b)\in S$. If $b(j)\ne -a(j)$, $N_j(P_{a,b})\subset P_{a,b-2e_j}$, and the map $N_j: P_{a,b}\to P_{a,b-2e_j}$ is an isomorphism. If $b(j)=-a(j)$, N_j annihilates $P_{a,b}$.

(3) Let $(a,b)\in S$. If $b(j)\ne a(j)$, $N_j^+(P_{a,b})\subset P_{a,b+2e_j}$, and for some nonzero rational number c, the map $N_j^+: P_{a,b}\to P_{a,b+2e_j}$ is c times the inverse of the isomorphism $N_j: P_{a,b+2e_j}\xrightarrow{\sim}P_{a,b}$. If $b(j)=a(j)$, N_j^+ annihilates $P_{a,b}$.

(4) For any $p\in\mathbf{Z}$, $\varphi(\mathbf{0})^p=\bigoplus_{(a,b)\in S}\varphi(\mathbf{0})^p\cap P_{a,b,\mathbf{C}}$. For any $(a,b)\in S$, $P_{a,b}$ with the filtration $(\varphi(\mathbf{0})^p\cap P_{a,b,\mathbf{C}})_{p\in\mathbf{Z}}$ is an $\mathbf{R}$-Hodge structure of weight $w+\sum_{j=1}^n b(j)$.

For $(a,b)\in S$ such that $b(j)\ne a(j)$, by (2′) with $F=\varphi(\mathbf{0})$ and (4), the bijection $N_j: P_{a,b+2e_j}\xrightarrow{\sim}P_{a,b}$ in the above (2) sends the $(p+1,q+1)$-Hodge component of $P_{a,b+2e_j,\mathbf{C}}$ with $p+q=w+\sum_{j=1}^n b(j)$ bijectively onto the (p,q)-Hodge component of $P_{a,b,\mathbf{C}}$ for the Hodge structure in (4). Hence, by (3), N_j^+ sends the (p,q)-Hodge component of $P_{a,b,\mathbf{C}}$ with $p+q=w+\sum_{j=1}^n b(j)$ onto the $(p+1,q+1)$-Hodge component of $P_{a,b+2e_j,\mathbf{C}}$. This proves $N_j^+\varphi(\mathbf{0})^p\subset\varphi(\mathbf{0})^{p+1}$ for any p. □

2.1.6.
For $1 \le j \le n$, define the increasing filtration $W^{(j)}$ on $H_{0,\mathbf{R}}$ as follows. Note that

$$H_{0,\mathbf{R}} = \bigoplus_{m \in \mathbf{Z}^n} V_m,$$

where Y_j acts on V_m as the multiplication by $m(j)$. Let

$$\begin{aligned} W_k^{(j)} &= \bigoplus_{m \in \mathbf{Z}^n, m(1)+\cdots+m(j) \le k-w} V_m \\ &= (\text{the part of } H_{0,\mathbf{R}} \text{ on which eigenvalues of } Y_1+\cdots+Y_j \text{ are } \le k-w). \end{aligned}$$

Here w is the integer such that $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$ as at the beginning of this section.

Let $s^{(j)}$ be the splitting of $W^{(j)}$ given by the eigenspaces of $Y_1 + \cdots + Y_j$. That is, $s^{(j)}$ is the unique splitting of $W^{(j)}$ for which the image of $\mathrm{gr}_k^{W^{(j)}}$ under $s^{(j)}$ is the part of $H_{0,\mathbf{R}}$ on which $Y_1 + \cdots + Y_j$ acts as the multiplication by $k - w$ for any $k \in \mathbf{Z}$.

PROPOSITION 2.1.7
An SL(2)*-orbit in n variables is determined by $((W^{(j)})_{1 \le j \le n}, \varphi(\mathbf{i}))$.*

This is proved in [KU2, Lemma 3.10].

In Sections 2.1.8 and 2.1.10, we characterize the splitting $s^{(j)}$ of $W^{(j)}$ given in Section 2.1.6 in terms of the canonical splittings and the Borel-Serre splittings, respectively.

PROPOSITION 2.1.8
Let (ρ, φ) be an SL(2)*-orbit in n variables, and take j such that $1 \le j \le n$. Let $y_k \in \mathbf{R}_{\ge 0}$ $(1 \le k \le n)$, and assume $y_k > 0$ for $j < k \le n$. Then $(W^{(j)}, \varphi(iy_1, \ldots, iy_n))$ is a mixed Hodge structure, and $s^{(j)}$ coincides with the canonical splitting (see Section 1.2.3; cf. Section 1.2.8) associated to this mixed Hodge structure.*

Proof
Let $F = \varphi(iy_1, \ldots, iy_n)$, $F' = \varphi(0, \ldots, 0, iy_{j+1}, \ldots, iy_n)$. Then $F = \exp(iy_1N_1 + \cdots + iy_jN_j)F'$, $(W^{(j)}, F)$ is an $\mathbf{R}$-mixed Hodge structure, $(W^{(j)}, F')$ is an $\mathbf{R}$-split $\mathbf{R}$-mixed Hodge structure, and the canonical splitting of $W^{(j)}$ associated to F' is given by $Y_1 + \cdots + Y_j$. We have $\delta(F) = y_1N_1 + \cdots + y_jN_j$. Since this δ has only $(-1,-1)$-Hodge component, $\zeta = 0$ by Section 1.2.3, and hence $Y_1 + \cdots + Y_j$ is also the canonical splitting of $W^{(j)}$ associated to F. □

2.1.9.
Let W' be an increasing filtration on $H_{0,\mathbf{R}}$ such that there exists a group homomorphism $\alpha : \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ such that, for $k \in \mathbf{Z}$, $W'_k = \bigoplus_{m \le k-w} H(m)$, where $H(m) := \{x \in H_{0,\mathbf{R}} \mid \alpha(t)x = t^m x \ (t \in \mathbf{R}^\times)\}$.

We define the real analytic map

$$\mathrm{spl}^{\mathrm{BS}}_{W'} : D \to \mathrm{spl}(W')$$

as follows. Let $P = (G^{\circ}_{W'})_{\mathbf{R}}$ be the parabolic subgroup of $G_{\mathbf{R}}$ defined by W'. (Here G° is the connected component of G as an algebraic group containing 1.) Let P_u be the unipotent radical of P, and let S_P be the maximal $\mathbf{R}$-split torus of the center of P/P_u. Let $\mathbf{G}_{m,\mathbf{R}} \to S_P$, $t \mapsto (t^{k-w}$ on $\mathrm{gr}^{W'}_k)_k$, be the weight map induced by α. For $F \in D$, let K_F be the maximal compact subgroup of $G_{\mathbf{R}}$ consisting of the elements of $G_{\mathbf{R}}$ which preserve the Hodge metric $\langle C_F(\bullet), \bar{\bullet}\rangle_w$, where C_F is the Weil operator associated to F. Let $S_P \to P$ be the Borel-Serre lifting homomorphism at F, which assigns $a \in S_P$ to the element $a_F \in P$ uniquely determined by the following condition: The class of a_F in P/P_u coincides with a, and $\theta_{K_F}(a_F) = a_F^{-1}$, where θ_{K_F} is the Cartan involution at K_F which coincides with $\mathrm{ad}(C_F)$ in the present situation ([KU3, Section 5.1.3], [KNU1, Section 8.1]). Then the composite $\mathbf{G}_{m,\mathbf{R}} \to S_P \to P$ defines an action of $\mathbf{G}_{m,\mathbf{R}}$ on $H_{0,\mathbf{R}}$, and we call the corresponding splitting of W' the *Borel-Serre splitting at* F and denote it by $\mathrm{spl}^{\mathrm{BS}}_{W'}(F)$.

It is easy to see that the map $\mathrm{spl}^{\mathrm{BS}}_{W'} : D \to \mathrm{spl}(W'), F \mapsto \mathrm{spl}^{\mathrm{BS}}_{W'}(F)$, is real analytic.

PROPOSITION 2.1.10

Let (ρ, φ) *be an* SL(2)*-orbit in* n *variables, let* $y_j > 0$ $(1 \le j \le n)$*, and let* $p = \varphi(iy_1, \ldots, iy_n) \in D$*. Then*

$$s^{(j)} = \mathrm{spl}^{\mathrm{BS}}_{W^{(j)}}(p) \quad (1 \le j \le n).$$

See [KU2, Lemma 3.9] for the proof.

2.1.11.

Let E be a finite-dimensional vector space over a field, and let W' be an increasing filtration on E such that $W'_w = E$ for $w \gg 0$ and $W'_w = 0$ for $w \ll 0$.

Recall (see [D, Section 1.6]) that for a nilpotent endomorphism N of (E, W'), an increasing filtration M on E is called a *relative monodromy filtration* of N with respect to W' if the following two conditions are satisfied.

(1) $N(M_k) \subset M_{k-2}$ for any $k \in \mathbf{Z}$.

(2) N^k induces an isomorphism $\mathrm{gr}^M_{w+k}\,\mathrm{gr}^{W'}_w \xrightarrow{\sim} \mathrm{gr}^M_{w-k}\,\mathrm{gr}^{W'}_w$ for any $w \in \mathbf{Z}$ and any $k \ge 0$.

If a relative monodromy filtration exists, it is unique and is denoted by $M(N, W')$. In the case where W' is pure, that is, $W'_w = E$ and $W'_{w-1} = 0$ for some w, then $M(N, W')$ exists.

Let (ρ, φ) be as in Section 2.1.1. For the family of filtrations in Section 2.1.6, we see that, for $0 \le j \le k \le n$, $W^{(k)}$ is the relative monodromy filtration of $\sum_{j<l\le k} N_l$ with respect to $W^{(j)}$ $(W^{(0)} := W)$.

For an increasing filtration W' on E such that $W'_w = E$ for $w \gg 0$ and $W'_w = 0$ for $w \ll 0$, define the *mean value of the weights* $\mu(W') \in \mathbf{Q}$ *of* W' and the *variance of the weights* $\sigma^2(W') \in \mathbf{Q}$ *of* W' by

$$\mu(W') = \sum_{w \in \mathbf{Z}} \dim(\mathrm{gr}_w^{W'})w/\dim(E),$$

$$\sigma^2(W') = \sum_{w \in \mathbf{Z}} \dim(\mathrm{gr}_w^{W'})(w - \mu(W'))^2/\dim(E).$$

PROPOSITION 2.1.12

Let N be a nilpotent endomorphism of (E, W') as in Section 2.1.11. Assume that the relative monodromy filtration $M = M(N, W')$ exists. Then the following hold:

(i) $\mu(M) = \mu(W')$,
(ii) $\sigma^2(M) > \sigma^2(W')$ *unless* $M = W'$.

Proof

For each k, we have the equality

$$\dim(\mathrm{gr}_k^M) = \sum_w \dim(\mathrm{gr}_w^{W'} \mathrm{gr}_k^M) = \sum_w \dim(\mathrm{gr}_k^M \mathrm{gr}_w^{W'}). \tag{1}$$

Taking $\sum_k (\cdots)k/\dim(E)$ of (1), and using Section 2.1.11(2), we obtain (i). Let $\mu = \mu(M) = \mu(W')$. By taking $\sum_k (\cdots)(k-\mu)^2/\dim(E)$ of (1), (ii) is reduced to the inequality $\sum_k d_k(k-\mu)^2 > (\sum_k d_k)(w-\mu)^2$ unless $d_k = 0$ for any $k \neq w$, where $d_k = \dim(\mathrm{gr}_k^M \mathrm{gr}_w^{W'})$ for each w. This inequality is obtained again by using Section 2.1.11(2). □

PROPOSITION 2.1.13

Let (ρ, φ) be an SL(2)*-orbit in n variables, and let $W^{(j)}$ $(1 \le j \le n)$ be as in Section 2.1.6. Let $W^{(0)} = W$.*

(i) *Let $1 \le j \le n$. Then $W^{(j-1)} = W^{(j)}$ if and only if the jth component* $\mathrm{SL}(2, \mathbf{C}) \to G_{\mathbf{C}}$ *of ρ is the trivial homomorphism.*

(ii) *For $0 \le j \le n$, let $\sigma^2(j) = \sigma^2(W^{(j)})$ be as in Section 2.1.11 for the increasing filtration $W^{(j)}$ on the $\mathbf{R}$-vector space $H_{0,\mathbf{R}}$. Then $\sigma^2(j) \le \sigma^2(j')$ if $0 \le j \le j' \le n$.*

(iii) *Let $0 \le j \le n$, $0 \le j' \le n$. Then $W^{(j)} = W^{(j')}$ if and only if $\sigma^2(j) = \sigma^2(j')$.*

Statement (i) was proved in [KU2, Section 3]. Statements (ii) and (iii) follow from Proposition 2.1.12.

2.1.14.
Let (ρ, φ) be an SL(2)-orbit in n variables in pure case. Put $W^{(0)} = W$. We define *rank of* (ρ, φ) as the number of the elements of the set $\{j \mid 1 \le j \le n, W^{(j)} \neq W^{(j-1)}\}$.

2.1.15.
EXAMPLE 0
Recall that in this case, D is identified with the upper half-plane $\mathfrak{h}$. Let ρ be the standard isomorphism $\mathrm{SL}(2, \mathbf{C}) \to G_{\mathbf{C}}$, and let $\varphi: \mathbf{P}^1(\mathbf{C}) \to \check{D}$ be the natural isomorphism in Section 1.1.1. Then (ρ, φ) is an SL(2)-orbit in one variable of rank 1.

2.2. Nilpotent orbits and SL(2)-orbits in pure case

We consider the pure case. Let $w \in \mathbf{Z}$, and assume $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$.

2.2.1.
Let $F \in \check{D}$, and let $N_1, \ldots, N_n$ be elements of $\mathfrak{g}_{\mathbf{R}}$ such that $N_j N_k = N_k N_j$ for any j, k and such that N_j is nilpotent as a linear map $H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$ for any j.

We say that the map

$$\mathbf{C}^n \to \check{D}, \qquad (z_1, \ldots, z_n) \mapsto \exp\Big(\sum_{j=1}^{n} z_j N_j\Big) F$$

is a *nilpotent orbit* if the following conditions (1) and (2) are satisfied:

(1) $\exp(\sum_{j=1}^{n} z_j N_j) F \in D$ if $\mathrm{Im}(z_j) \gg 0$ for all j,
(2) $N_j F^p \subset F^{p-1}$ for any j and any p.

In this case, we say also that $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit.*

2.2.2.
Assume that $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit. By [CK], for $y_j \in \mathbf{R}_{\ge 0}$, the filtration $M(y_1 N_1 + \cdots + y_n N_n, W)$ (see Section 2.1.11) depends only on the set $\{j \mid y_j \neq 0\}$. For $1 \le j \le n$, let $W^{(j)} = M(N_1 + \cdots + N_j, W)$.

2.2.3.
Assume that $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit. Then by Cattani, Kaplan, and Schmid [CKS], an SL(2)-orbit (ρ, φ) is canonically associated to $(N_1, \ldots, N_n, F)$. (The homomorphism ρ is given in [CKS, Theorem 4.20], and φ is defined by $\varphi(g\mathbf{0}) = \rho(g)\hat{F}$ $(g \in \mathrm{SL}(2, \mathbf{C})^n)$, where $\mathbf{0} = 0^n \in \mathbf{P}^1(\mathbf{C})^n$.) By [KNU1], this SL(2)-orbit is characterized in the style of the following theorem.

THEOREM 2.2.4
Assume that $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit.*

(i) *Let* $1 \le j \le n$. *Then, when* $y_k \in \mathbf{R}_{>0}$ *and* $y_k/y_{k+1} \to \infty$ *(*$1 \le k \le n$*,* y_{n+1} *means* 1*), the Borel-Serre splitting* $\mathrm{spl}^{\mathrm{BS}}_{W^{(j)}}(\exp(\sum_{k=1}^{n} iy_k N_k)F)$ *converges in* $\mathrm{spl}(W^{(j)})$ *(see [KNU1, Section 8.7]).*

Let $s^{(j)} \in \mathrm{spl}(W^{(j)})$ *be the limit.*

(ii) *There is a homomorphism* $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ *of algebraic groups over* $\mathbf{R}$ *characterized by the following property. For any* $1 \le j \le n$ *and any* $k \in \mathbf{Z}$*, we have*

$$s^{(j)}(\mathrm{gr}_k^{W^{(j)}}) = \{ v \in H_{0,\mathbf{R}} \mid \tau_j(t)v = t^k v \text{ for any } t \in \mathbf{R}^\times \},$$

where $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ *is the* j*th component of* τ.

(iii) *There exists a unique* $\mathrm{SL}(2)$*-orbit* (ρ, φ) *in* n *variables characterized by the following properties (1) and (2).*

(1) *The associated weight filtrations* $W^{(j)}$ *(*$1 \le j \le n$*) are the same as* $W^{(j)}$ *in Section 2.2.2.*

(2) *The point* $\varphi(\mathbf{i})$ *is the limit in* D *of*

$$\tau\Big(\sqrt{\frac{y_2}{y_1}}, \dots, \sqrt{\frac{y_{n+1}}{y_n}}\Big)^{-1} \exp\Big(\sum_{j=1}^{n} iy_j N_j\Big)F \quad (y_j > 0, y_j/y_{j+1} \to \infty \ (1 \le j \le n))$$

(y_{n+1} *means* 1*), where* τ *is as in (ii).*

(iv) *The associated torus action* $\tilde{\rho}$ *(see [KU2, Section 3.1(4)]) of* (ρ, φ) *and the homomorphism* τ *in (ii) are related as follows:*

$$\tau(t_1, \dots, t_n) = \Big(\prod_{j=1}^{n} t_j\Big)^{w} \tilde{\rho}(t_1, \dots, t_n).$$

2.2.5.

EXAMPLE 0

Let (N, F) be as follows: $N(e_2) = e_1$, $N(e_1) = 0$, $F = F(z)$ with $z \in i \cdot \mathbf{R}$ in the notation of Section 1.1.1. Then (N, F) generates a nilpotent orbit, and the associated SL(2)-orbit is the one in Section 2.1.15.

In fact, $\exp(iyN)F = F(z + iy)$, and $\tau(t)$ in Theorem 2.2.4(ii) sends e_1 to $t^{-2}e_1$ and e_2 to e_2. Hence $\tau(1/\sqrt{y})^{-1} \exp(iyN)F = F((z+iy)/y) \to F(i)$ as $y \to \infty$.

2.2.6.

Assume that (N, F) generates a nilpotent orbit in the pure case in Section 2.2.1 for $n = 1$. Let $W^{(1)} = M(N, W)$ be as in Section 2.2.2. Then $(W^{(1)}, F)$ is a mixed Hodge structure, and the splitting $s^{(1)}$ of $W^{(1)}$ given by the SL(2)-orbit (see Section 2.1.6) associated to (N, F) coincides with the canonical splitting of $W^{(1)}$ associated to F (see Section 1.2).

2.2.7.

More generally, for any mixed Hodge structure, its canonical splitting (see Section 1.2) is obtained as in Section 2.2.6 by embedding the mixed Hodge structure into a pure nilpotent orbit.

In fact, let (M,F) be a mixed Hodge structure on an $\mathbf{R}$-vector space V. Let k be an integer such that all the weights of (M,F) are not greater than k. It is shown in [KNU1] that there exist an $\mathbf{R}$-vector space V', an $\mathbf{R}$-linear injective map $q: V \to V'$, a nilpotent endomorphism N of V', and a decreasing filtration F' on $V'_{\mathbf{C}}$ such that the pair (N,F') generates a nilpotent orbit on V' in the pure case of weight k in Section 2.2.1 for $n=1$, which satisfy the following conditions.

Let W' be the trivial weight filtration on V' of weight k, and let $W^{(1)} = M(N,W')$ be as in Section 2.2.2. Then, $0 \to (V,M,F) \xrightarrow{q} (V',W^{(1)},F') \xrightarrow{N} (V', W^{(1)}[-2],F'(-1))$ is an exact sequence of mixed Hodge structures, where $[-2]$ is the shift by -2 and (-1) is the Tate twist by -1, and the restriction of the splitting $s^{(1)}$ of $W^{(1)}$, given by the SL(2)-orbit associated to (N,F') on V', to $\mathrm{Ker}(N : \mathrm{gr}^{W^{(1)}} \to \mathrm{gr}^{W^{(1)}[-2]}) \simeq \mathrm{gr}^M$ coincides with the canonical splitting of M on V associated to F.

For the proof, see [KNU1, Section 3].

2.3. SL(2)-orbits in mixed case

Now we consider the mixed version of Section 2.1. Let W be as in the notation.

2.3.1.
For $n \geq 0$, let $\mathcal{D}'_{\mathrm{SL}(2),n}$ be the set of pairs $((\rho_w,\varphi_w)_{w\in\mathbf{Z}},\mathbf{r})$, where (ρ_w,φ_w) is an SL(2)-orbit in n variables for gr^W_w for each $w \in \mathbf{Z}$ and $\mathbf{r}$ is an element of D such that $\mathbf{r}(\mathrm{gr}^W_w) = \varphi_w(\mathbf{i})$ for each $w \in \mathbf{Z}$. Here $\mathbf{i} = (i,\dots,i) \in \mathbf{C}^n \subset \mathbf{P}^1(\mathbf{C})^n$.

2.3.2.
Let $\mathcal{D}_{\mathrm{SL}(2),n}$ be the set of all triples $((\rho_w,\varphi_w)_{w\in\mathbf{Z}},\mathbf{r},J)$, where $((\rho_w,\varphi_w)_{w\in\mathbf{Z}},\mathbf{r}) \in \mathcal{D}'_{\mathrm{SL}(2),n}$ and J is a subset of $\{1,\dots,n\}$ satisfying the following conditions (1) and (2). Let

$$J' = \{j \mid 1 \leq j \leq n, \text{there is } w \in \mathbf{Z} \text{ such that the } j\text{th component } \mathrm{SL}(2) \to G_{\mathbf{R}}(\mathrm{gr}^W_w) \text{ of } \rho_w \text{ is a nontrivial homomorphism}\}.$$

(1) If $\mathbf{r} \in D_{\mathrm{spl}}$, $J = J'$.
(2) If $\mathbf{r} \in D_{\mathrm{nspl}}$, either $J = J'$ or $J = J' \cup \{k\}$ for some $k < \min J'$.

Let

$$\mathcal{D}_{\mathrm{SL}(2)} = \bigsqcup_{n\geq 0} \mathcal{D}_{\mathrm{SL}(2),n}.$$

We call an element of $\mathcal{D}_{\mathrm{SL}(2),n}$ an SL(2)-*orbit in* n *variables* and call an element of $\mathcal{D}_{\mathrm{SL}(2)}$ an SL(2)-*orbit*. Note that, in the pure case, J is determined uniquely by $(\rho_w)_w$ since $D = D_{\mathrm{spl}}$.

We call the cardinality of the set J the *rank* of the SL(2)-orbit.

2.3.3.
Let $\mathcal{D}_{\mathrm{SL}(2),n,\sharp} \subset \mathcal{D}_{\mathrm{SL}(2),n}$ be the set of all SL(2)-orbits in n variables of rank n.

For an element $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ of $\mathcal{D}_{\mathrm{SL}(2),n,\sharp}$, $J = \{1, \ldots, n\}$. Hence, by forgetting J, the set $\mathcal{D}_{\mathrm{SL}(2),n,\sharp}$ is identified with the subset of $\mathcal{D}'_{\mathrm{SL}(2),n}$ (see Section 2.3.1) consisting of all elements $((\rho_w, \varphi_w)_w, \mathbf{r})$ satisfying the following conditions (1) and (2).

(1) If $2 \le j \le n$, there exists $w \in \mathbf{Z}$ such that the jth component of ρ_w is a nontrivial homomorphism.

(2) If $\mathbf{r} \in D_{\mathrm{spl}}$ and $n \ge 1$, there exists $w \in \mathbf{Z}$ such that the first component of ρ_w is a nontrivial homomorphism.

As is seen later in Section 2.5, for the construction of the space $D_{\mathrm{SL}(2)}$, it is sufficient to consider SL(2)-orbits in n variables of rank r with $r = n$. We call this type of SL(2)-orbit a *nondegenerate* SL(2)*-orbit of rank* n or, for simplicity, an SL(2)*-orbit of rank* n, and we regard it as an element of $\mathcal{D}'_{\mathrm{SL}(2),n}$.

On the other hand, the generality of the definition in Section 2.3.2 with the auxiliary data J is natural in Section 2.4 when we consider the relations with nilpotent orbits.

2.3.4.
If $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ is an SL(2)-orbit in n variables of rank r, we have the *associated* SL(2)*-orbit* $((\rho'_w, \varphi'_w)_w, \mathbf{r})$ *in* r *variables of rank* r, defined as follows. Let $J = \{a(1), \ldots, a(r)\}$ with $a(1) < \cdots < a(r)$. Then

$$\rho'_w(g_{a(1)}, \ldots, g_{a(r)}) := \rho_w(g_1, \ldots, g_n), \qquad \varphi'_w(z_{a(1)}, \ldots, z_{a(r)}) := \varphi_w(z_1, \ldots, z_n).$$

Note that, for any $w \in \mathbf{Z}$, ρ_w factors through the projection $\mathrm{SL}(2)^n \to \mathrm{SL}(2)^J$ to the J-component, and φ_w factors through the projection $\mathbf{P}^1(\mathbf{C})^n \to \mathbf{P}^1(\mathbf{C})^J$ to the J-component, and hence $(\rho_w, \varphi_w)_w$ is essentially the same as $(\rho'_w, \varphi'_w)_w$.

2.3.5. Associated torus action
Assume that we are given an SL(2)-orbit in n variables $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$.

We define the associated homomorphism of algebraic groups over $\mathbf{R}$,

$$\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W),$$

as follows. Let $s_{\mathbf{r}} : \mathrm{gr}^W \xrightarrow{\sim} H_{0,\mathbf{R}}$ be the canonical splitting of W associated to $\mathbf{r}$ (see Section 1.2). Then

$$\tau(t_1, \ldots, t_n) = s_{\mathbf{r}} \circ \left(\bigoplus_{w \in \mathbf{Z}} \Big(\prod_{j=1}^{n} t_j \Big)^w \rho_w(g_1, \ldots, g_n) \text{ on } \mathrm{gr}^W_w \right) \circ s_{\mathbf{r}}^{-1}$$

$$\text{with} \quad g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix}.$$

For $1 \le j \le n$, let $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the jth component of τ.

REMARK

The induced action of $\tau(t)$ $(t\in\mathbf{R}^n_{>0})$ on D is described as follows. For $s(\theta(F,\delta))\in D$ with $s\in\mathrm{spl}(W)$, $F\in D(\mathrm{gr}^W)$, $\delta\in\mathcal{L}(F)$ (see Proposition 1.2.5), we have

$$\tau(t)s\bigl(\theta(F,\delta)\bigr)=s'\bigl(\theta(F',\delta')\bigr)$$

with $s'=\tau(t)s\,\mathrm{gr}^W(\tau(t))^{-1}$, $F'=\mathrm{gr}^W(\tau(t))F$, $\delta'=\mathrm{Ad}\bigl(\mathrm{gr}^W(\tau(t))\bigr)\delta$.

2.3.6. Associated family of weight filtrations

In the situation of Section 2.3.5, for $1\le j\le n$, we define the associated jth weight filtration $W^{(j)}$ on $H_{0,\mathbf{R}}$ as follows. For $k\in\mathbf{Z}$, $W^{(j)}_k$ is the direct sum of $\{x\in H_{0,\mathbf{R}}\mid \tau_j(t)x=t^\ell x\ (\forall t\in\mathbf{R}^\times)\}$ over all $\ell\le k$.

By definition, we have $W^{(j)}_k=\sum_{w\in\mathbf{Z}}s_{\mathbf{r}}(W^{(j)}_k(\mathrm{gr}^W_w))$, and $W^{(j)}_k(\mathrm{gr}^W_w)$ coincides with the kth filter of the jth weight filtration on gr^W_w associated to the SL(2)-orbit (ρ_w,φ_w) in n variables.

PROPOSITION 2.3.7

(i) *An* SL(2)*-orbit in n variables $((\rho_w,\varphi_w)_w,\mathbf{r},J)$ is uniquely determined by $\bigl((W^{(j)}(\mathrm{gr}^W))_{1\le j\le n},\mathbf{r},J\bigr)$.*

(ii) *An* SL(2)*-orbit in n variables $((\rho_w,\varphi_w)_w,\mathbf{r},J)$ is uniquely determined by $(\tau,\mathbf{r},J)$.*

Proof

(i) In the pure case, this is Proposition 2.1.7. The general case is clearly reduced to the pure case.

(ii) This follows from (i) since the family of weight filtrations $(W^{(j)}(\mathrm{gr}^W))_{1\le j\le n}$ is determined by τ. □

PROPOSITION 2.3.8

Let $((\rho_w,\varphi_w)_w,\mathbf{r},J)$ be an SL(2)*-orbit in n variables, and let $W^{(j)}$ $(1\le j\le n)$ be as in Section 2.3.6. Let $W^{(0)}=W$.*

(i) *Let $1\le j\le n$. Then $W^{(j)}=W^{(j-1)}$ if and only if for any $w\in\mathbf{Z}$, the jth factor $\mathrm{SL}(2,\mathbf{C})\to G_{\mathbf{C}}(\mathrm{gr}^W_w)$ of ρ_w is the trivial homomorphism.*

(ii) *For $0\le j\le n$, let $\sigma^2(j)=\sum_{w\in\mathbf{Z}}\sigma^2(W^{(j)}(\mathrm{gr}^W_w))$, where $\sigma^2(W^{(j)}(\mathrm{gr}^W_w))$ is the variance (see Section 2.1.11) of the increasing filtration $W^{(j)}(\mathrm{gr}^W_w)$ on the $\mathbf{R}$-vector space gr^W_w. Then, $\sigma^2(j)\le\sigma^2(j')$ if $0\le j\le j'\le n$.*

(iii) *Let $0\le j\le n$, $0\le j'\le n$. Then, $W^{(j)}=W^{(j')}$ if and only if $\sigma^2(j)=\sigma^2(j')$.*

Proof

This is also reduced to the pure case, Proposition 2.1.13. □

2.3.9.
We describe the kind of SL(2)-orbits of positive rank which exist in Examples I–V. We consider only an SL(2)-orbit in r variables of rank r; hence, $J=\{1,\dots,r\}$ in the following (see Section 2.3.3).

EXAMPLE I
Any SL(2)-orbit of rank >0 is of rank 1. An SL(2)-orbit in one variable of rank 1 is $((\rho_w,\varphi_w)_w,\mathbf{r})$, where ρ_w is the trivial homomorphism from SL(2) to $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ and φ_w is the unique map from $\mathbf{P}^1(\mathbf{C})$ onto the one-point set $D(\mathrm{gr}^W_w)$, and $\mathbf{r}$ is any element of $D_{\mathrm{nspl}}=\mathbf{C}\smallsetminus\mathbf{R}$. We have $W^{(1)}=W$. Later we refer to the case $\mathbf{r}=F(i)\in D$ (i.e., $\mathbf{r}=i\in\mathbf{C}=D$) as Section 2.3.9, Example I.

EXAMPLE II
Any SL(2)-orbit of rank >0 is of rank 1. An SL(2)-orbit in one variable of rank 1 is $((\rho_w,\varphi_w)_w,\mathbf{r})$, where (ρ_w,φ_w) is of rank 0 for $w\neq-1$, and (ρ_{-1},φ_{-1}) is of rank 1. An example of such an SL(2)-orbit is that (ρ_{-1},φ_{-1}) is the SL(2)-orbit in Section 2.1.15, and $\mathbf{r}=F(i,z)$ in the notation of Section 1.1.1, Example II. For this SL(2)-orbit, $W^{(1)}$ is given by

$$W^{(1)}_{-3}=0\subset W^{(1)}_{-2}=W^{(1)}_{-1}=\mathbf{R}e_1\subset W^{(1)}_0=H_{0,\mathbf{R}}.$$

EXAMPLE III
There are three cases for SL(2)-orbits in r variables of rank $r>0$. For any of them, (ρ_w,φ_w) is of rank 0 unless $w=-3$.

Case 1: $r=1$ and (ρ_{-3},φ_{-3}) is of rank 1. An example of such an SL(2)-orbit is given as follows: (ρ_{-3},φ_{-3}) is $(\rho,\varphi(1))$ of Section 2.1.15 (we identify $\check{D}(\mathrm{gr}^W_{-3})$ with $\mathbf{P}^1(\mathbf{C})$ via the Tate twist), and $\mathbf{r}=F(i,z_1,i)$ for $z_1\in\mathbf{C}$ (see Section 1.1.1). For this SL(2)-orbit,

$$W^{(1)}_{-5}=0\subset W^{(1)}_{-4}=W^{(1)}_{-3}=\mathbf{R}e_1\subset W^{(1)}_{-2}=W^{(1)}_{-1}=W^{(1)}_{-3}+\mathbf{R}e_2\subset W^{(1)}_0=H_{0,\mathbf{R}}.$$

Case 2: $r=1$ and (ρ_{-3},φ_{-3}) is of rank 0. An example of such an SL(2)-orbit is given as follows: ρ_{-3} is the trivial homomorphism onto $\{1\}$, φ_{-3} is the constant map with value $i\in\mathfrak{h}=D(\mathrm{gr}^W_{-3})$, and $\mathbf{r}=F(i,z_1,z_2)$ with $(z_1,z_2)\in\mathbf{C}^2\smallsetminus\mathbf{R}^2$. For this SL(2)-orbit, $W^{(1)}=W$.

Case 3: $r=2$ and (ρ_{-3},φ_{-3}) is of rank 1. The homomorphism $\rho_{-3}:\ \mathrm{SL}(2,\mathbf{C})^2\to G_{\mathbf{C}}(\mathrm{gr}^W_{-3})=\mathrm{SL}(2,\mathbf{C})$ factors through the second projection onto $\mathrm{SL}(2,\mathbf{C})$, and $\varphi_{-3}:\mathbf{P}^1(\mathbf{C})^2\to\check{D}(\mathrm{gr}^W_{-3})=\mathbf{P}^1(\mathbf{C})$ factors through the second projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such an SL(2)-orbit is given as follows: $\rho_{-3}(g_1,g_2)=g_2$, $\varphi_{-3}(p_1,p_2)=p_2$, and $\mathbf{r}=F(i,z_1,z_2)$, where $(z_1,z_2)\in\mathbf{C}^2\smallsetminus\mathbf{R}^2$. For this SL(2)-orbit, $W^{(1)}=W$ and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 1.

EXAMPLE IV

There are three cases for SL(2)-orbits in r variables of rank $r>0$. For any of them, (ρ_w,φ_w) is of rank 0 unless $w=-1$.

Case 1: $r=1$ and (ρ_{-1},φ_{-1}) is of rank 1. An example of such an SL(2)-orbit is given as follows: (ρ_{-1},φ_{-1}) is the standard one (which is identified with (ρ,φ) in Section 2.1.15 by the identification of e_2', e_3' here with e_1,e_2 there), and $\mathbf{r}=F(i,z_1,z_2,z_3)$ for $z_1,z_2,z_3\in\mathbf{C}$ (see Section 1.1.1). For this SL(2)-orbit,

$$W^{(1)}_{-3}=0\subset W^{(1)}_{-2}=W^{(1)}_{-1}=\mathbf{R}e_1+\mathbf{R}e_2\subset W^{(1)}_0=H_{0,\mathbf{R}}.$$

Case 2: $r=1$ and (ρ_{-1},φ_{-1}) is of rank 0. An example of such an SL(2)-orbit is given as follows: ρ_{-1} is the trivial homomorphism onto $\{1\}$, φ_{-1} is the constant map with value $i\in\mathfrak{h}=D(\mathrm{gr}^W_{-1})$, and $\mathbf{r}=F(i,z_1,z_2,z_3)$ with $\mathrm{Im}(z_2)\neq\mathrm{Im}(z_1)\,\mathrm{Im}(z_3)$ (the last condition says that $F(i,z_1,z_2,z_3)\in D_{\mathrm{nspl}}$). For this SL(2)-orbit, $W^{(1)}=W$.

Case 3: $r=2$ and (ρ_{-1},φ_{-1}) is of rank 1. The homomorphism $\rho_{-1}:\ \mathrm{SL}(2,\mathbf{C})^2\to G_{\mathbf{C}}(\mathrm{gr}^W_{-1})$ factors through the second projection onto $\mathrm{SL}(2,\mathbf{C})$, and $\varphi_{-1}:\mathbf{P}^1(\mathbf{C})^2\to\check{D}(\mathrm{gr}^W_{-1})=\mathbf{P}^1(\mathbf{C})$ factors through the second projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such an SL(2)-orbit is given as follows: $\rho_{-1}(g_1,g_2)=g_2$, $\varphi_{-1}(p_1,p_2)=p_2$, and $\mathbf{r}=F(i,z_1,z_2,z_3)$ with $\mathrm{Im}(z_2)\neq\mathrm{Im}(z_1)\,\mathrm{Im}(z_3)$. For this SL(2)-orbit, $W^{(1)}=W$ and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 1.

EXAMPLE V

There are five cases for SL(2)-orbits in r variables of rank $r>0$. For any of them, (ρ_w,φ_w) is of rank 0 if $w\notin\{0,1\}$.

Case 1 (resp., Case 2): $r=1$ and (ρ_0,φ_0) is of rank 1 (resp., 0), and (ρ_1,φ_1) is of rank 0 (resp., 1). An example of such an SL(2)-orbit is given as follows: (ρ_0,φ_0) (resp., (ρ_1,φ_1)) is $(\mathrm{Sym}^2(\rho),\mathrm{Sym}^2(\varphi)(-1))$ (resp., $(\rho,\varphi(-1))$) for the standard (ρ,φ) in Section 2.1.15 via a suitable identification, where (-1) means the Tate twist, and $\mathbf{r}=F(i,i,z_1,z_2,z_3)$ for $z_1,z_2,z_3\in\mathbf{C}$. For this SL(2)-orbit,

$$\begin{aligned}W^{(1)}_{-3}=0\subset W^{(1)}_{-2}=W^{(1)}_{-1}=\mathbf{R}e_1\subset W^{(1)}_0=W^{(1)}_{-1}+\mathbf{R}e_2\\ \subset W^{(1)}_1=W^{(1)}_0+\mathbf{R}e_4+\mathbf{R}e_5\subset W^{(1)}_2=H_{0,\mathbf{R}}\end{aligned}$$

(resp., $W^{(1)}_{-1}=0\subset W^{(1)}_0=W^{(1)}_1=\mathbf{R}e_1+\mathbf{R}e_2+\mathbf{R}e_3+\mathbf{R}e_4\subset W^{(1)}_2=H_{0,\mathbf{R}}$).

Case 3: $r=1$, and both (ρ_0,φ_0) and (ρ_1,φ_1) are of rank 1. An example of such an SL(2)-orbit is given as follows: $\rho_0=\mathrm{Sym}^2(\rho)$, $\varphi_0=\mathrm{Sym}^2(\varphi)(-1)$, $\rho_1=\rho$, $\varphi_1=\varphi(-1)$ for the standard (ρ,φ) in Section 2.1.15 via a suitable identification, and $\mathbf{r}=F(i,i,z_1,z_2,z_3)$ for $z_1,z_2,z_3\in\mathbf{C}$. For this SL(2)-orbit,

$$\begin{aligned}W^{(1)}_{-3}=0\subset W^{(1)}_{-2}=W^{(1)}_{-1}=\mathbf{R}e_1\subset W^{(1)}_0=W^{(1)}_1=W^{(1)}_{-1}+\mathbf{R}e_2+\mathbf{R}e_4\\ \subset W^{(1)}_2=H_{0,\mathbf{R}}.\end{aligned}$$

Case 4 (resp., Case 5): $r = 2$, both (ρ_0, φ_0) and (ρ_1, φ_1) are of rank 1, $\rho_0 : \mathrm{SL}(2,\mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}_0^W)$ factors through the first (resp., second) projection onto $\mathrm{SL}(2,\mathbf{C})$, $\varphi_0 : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}_0^W)$ factors through the first (resp., second) projection onto $\mathbf{P}^1(\mathbf{C})$, $\rho_1 :$ $\mathrm{SL}(2,\mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}_1^W)$ factors through the second (resp., first) projection onto $\mathrm{SL}(2,\mathbf{C})$, and $\varphi_1 : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}_1^W)$ factors through the second (resp., first) projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such an SL(2)-orbit is given as follows. For $j = 1$ (resp., 2), $\rho_0(g_1, g_2) = \mathrm{Sym}^2(g_j)$, $\varphi_0(p_1, p_2) = p_j \in \mathbf{P}^1(\mathbf{C}) \simeq \check{D}(\mathrm{gr}_0^W)$ (cf. Section 1.1.2), $\rho_1(g_1, g_2) = g_{3-j}$, $\varphi_1(p_1, p_2) = p_{3-j}(-1) \in \mathbf{P}^1(\mathbf{C}) \simeq \check{D}(\mathrm{gr}_1^W)$, and $\mathbf{r} = F(i, i, z_1, z_2, z_3)$ with $z_1, z_2, z_3 \in \mathbf{C}$. For this SL(2)-orbit, $W^{(1)}$ is the $W^{(1)}$ in the example in Case 1 (resp., Case 2) and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 3.

2.4. Nilpotent orbits and SL(2)-orbits in mixed case

2.4.1.
Let $N_j \in \mathfrak{g}_{\mathbf{R}}$ $(1 \le j \le n)$, and let $F \in \check{D}$. We say that $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit* if the following conditions (1)–(4) are satisfied.

(1) The $\mathbf{R}$-linear maps $N_j : H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$ are nilpotent for all j, and $N_j N_k = N_k N_j$ for all j, k.

(2) If $y_j \gg 0$ $(1 \le j \le n)$, then $\exp(\sum_{j=1}^n i y_j N_j) F \in D$.

(3) We have $N_j F^p \subset F^{p-1}$ for all j and p (Griffiths transversality).

(4) Let J be any subset of $\{1, \ldots, n\}$. Then for $y_j \in \mathbf{R}_{>0}$ $(j \in J)$, the relative monodromy filtration $M(\sum_{j \in J} y_j N_j, W)$ (see Section 2.1.11) exists. Furthermore, this filtration is independent of the choice of $y_j \in \mathbf{R}_{>0}$.

In the pure case, by Section 2.2.2, $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit in this sense if and only if it does in the sense of Section 2.2.1.

Let $\mathcal{D}_{\mathrm{nilp},n}$ be the set of all $(N_1, \ldots, N_n, F)$ which generate nilpotent orbits.

For $(N_1, \ldots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$, we call the map

$$(z_1, \ldots, z_n) \mapsto \exp\Big(\sum_{j=1}^n z_j N_j\Big) F$$

a *nilpotent orbit in n variables*.

In the terminology of Kashiwara [K], $\mathcal{D}_{\mathrm{nilp},n}$ is the set of all $(N_1, \ldots, N_n, F)$ such that $(H_{0,\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \ldots, N_n)$, with $\bar{F}$ the complex conjugate of F, is an infinitesimal mixed Hodge module.

We prove Theorem 2.4.2, Proposition 2.4.3, and Theorem 2.4.5. Theorem 2.4.2(i) was already proved in Theorem 0.5 of our previous article [KNU1].

THEOREM 2.4.2
Let $(N_1, \ldots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$. For each $w \in \mathbf{Z}$, let (ρ_w, φ_w) be the SL(2)*-orbit in n variables for gr_w^W associated to $(\mathrm{gr}_w^W(N_1), \ldots, \mathrm{gr}_w^W(N_n), F(\mathrm{gr}_w^W))$, which generates a nilpotent orbit for gr_w^W (see Section 2.2.3). Let $k = \min(\{j \mid 1 \le j \le n, N_j \ne 0\} \cup \{n+1\})$.*

(i) *If* $y_j \in \mathbf{R}_{>0}$ *and* $y_j/y_{j+1} \to \infty$ *(*$1 \le j \le n$*,* y_{n+1} *means* 1*), the canonical splitting* $\mathrm{spl}_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ *of* W *associated to* $\exp(\sum_{j=1}^n iy_jN_j)F$ *(see Section 1.2.3) converges in* $\mathrm{spl}(W)$*.*

Let $s \in \mathrm{spl}(W)$ *be the limit.*

(ii) *Let* $\tau : \mathbf{G}_{m,\mathbf{R}}^n \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ *be the homomorphism of algebraic groups defined by*

$$\tau(t_1,\dots,t_n) = s \circ \Big(\bigoplus_{w\in\mathbf{Z}}\Big(\Big(\prod_{j=1}^n t_j\Big)^w \rho_w(g_1,\dots,g_n) \ \text{on}\ \mathrm{gr}_w^W\Big)\Big) \circ s^{-1},$$

where g_j *is as in Section 2.3.5. Then, as* $y_j > 0$*,* $y_1 = \cdots = y_k$*,* $y_j/y_{j+1} \to \infty$ *(*$k \le j \le n$*,* y_{n+1} *means* 1*),*

$$\tau\Big(\sqrt{\frac{y_2}{y_1}},\dots,\sqrt{\frac{y_{n+1}}{y_n}}\Big)^{-1} \exp\Big(\sum_{j=1}^n iy_jN_j\Big)F$$

converges in D*.*

Let $\mathbf{r}_1 \in D$ *be the limit.*

(iii) *Let*

$$J' = \{j \mid 1 \le j \le n,\ \text{the } j\text{th component of } \rho_w \text{ is nontrivial for some } w \in \mathbf{Z}\}.$$

Let $J = J' = \emptyset$ *if* $k = n+1$*, and let* $J = J' \cup \{k\}$ *otherwise. Then*

$$\big((\rho_w,\varphi_w)_w, \mathbf{r}_1, J\big) \in \mathcal{D}_{\mathrm{SL}(2),n}.$$

(iv) *The family of weight filtrations (see Section 2.3.6) and the torus action (see Section 2.3.5) associated to* $((\rho_w,\varphi_w)_w, \mathbf{r}_1, J)$ *coincide with* $(M(N_1 + \cdots + N_j, W))_{1\le j\le n}$ *and* τ *in (ii), respectively.*

We prove this theorem later in Section 2.4.8.

By this theorem, we have a map

$$\psi : \mathcal{D}_{\mathrm{nilp},n} \to \mathcal{D}_{\mathrm{SL}(2),n}, \qquad (N_1,\dots,N_n,F) \mapsto ((\rho_w,\varphi_w)_w, \mathbf{r}_1, J),$$

(for the notation, see Sections 2.4.1, 2.3.2). For $p \in \mathcal{D}_{\mathrm{nilp},n}$, we call $\psi(p) \in \mathcal{D}_{\mathrm{SL}(2),n}$ the SL(2)-*orbit associated to* p. Note that this definition is slightly different from that in [KNU1, Section 0.2]. Note also that though in the definition of a nilpotent orbit in Section 2.4.1, the order of $N_1,\dots,N_n$ in $(N_1,\dots,N_n,F)$ is not important, when we consider the SL(2)-orbit associated to $(N_1,\dots,N_n,F)$, the order of $N_1,\dots,N_n$ becomes essential.

Even when $k=1$, the $\mathbf{r}_1$ in Theorem 2.4.2(ii) is not $\mathbf{r}$ but $\exp(\varepsilon_0)\mathbf{r}$ in the main theorem [KNU1, Theorem 0.5], although the s in Theorem 2.4.2(i) coincides with $\mathrm{spl}_W(\mathbf{r}_1)$ (see Section 1.2.3).

PROPOSITION 2.4.3

Let $(N_1,\dots,N_n,F) \in \mathcal{D}_{\mathrm{nilp},n}$*, and let* $W^{(j)} = M(N_1 + \cdots + N_j, W)$ *for* $1 \le j \le n$ *(cf. Section 2.2.2 in the pure case). Let* $k = \min(\{j \mid 1 \le j \le n, N_j \ne 0\} \cup \{n+1\})$*. Then the following two conditions (1) and (2) are equivalent.*

(1) *For any* $k \le j \le n$, $\big(W^{(j)}, \exp\big(\sum_{l=j+1}^n iN_l\big)F\big)$ *is an* $\mathbf{R}$*-split mixed Hodge structure.*

(2) *For any* $k \le j \le n$ *and for any* $y_l \in \mathbf{R}_{>0}$ $(j < l \le n)$, $\big(W^{(j)}, \exp\big(\sum_{l=j+1}^n iy_lN_l\big)F\big)$ *is an* $\mathbf{R}$*-split mixed Hodge structure.*

We prove this proposition later in Section 2.4.9.

2.4.4.
Let $\mathcal{D}_{\mathrm{nilp,SL(2)},n} \subset \mathcal{D}_{\mathrm{nilp},n}$ be the set of all $(N_1, \dots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$ which satisfy the equivalent conditions in Proposition 2.4.3.

For example, $\mathcal{D}_{\mathrm{nilp,SL(2)},1}$ is the set of all $(N, F) \in \mathcal{D}_{\mathrm{nilp},1}$ such that $N = 0$ or $(M(N,W), F)$ is an $\mathbf{R}$-split mixed Hodge structure.

THEOREM 2.4.5
For $p = (N_1, \dots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$, *let* $k = \min(\{j \mid 1 \le j \le n, N_j \ne 0\} \cup \{n+1\})$, *and let* $\phi(p) = (N_1, \dots, N_k, N_{k+1}^{\Delta}, \dots, N_n^{\Delta}, F')$, *where* $F' = F$ *if* $k = n+1$ *and* $F' = \hat{F}_{(n)}$ *otherwise* ($N_j^{\Delta} \in \mathfrak{g}_{\mathbf{R}}$ $(k < j \le n)$ *and* $\hat{F}_{(n)} \in \check{D}$ *are as in [KNU1, Sections 10.1–10.2]; we review these objects in Section 2.4.6, Proposition 2.4.7 below).*

(i) *For* $p \in \mathcal{D}_{\mathrm{nilp},n}$, *we have* $\phi(p) \in \mathcal{D}_{\mathrm{nilp},n}$ *and* $\phi(\phi(p)) = \phi(p)$.

(ii) *We have* $\mathcal{D}_{\mathrm{nilp,SL(2)},n} = \{p \in \mathcal{D}_{\mathrm{nilp},n} \mid \phi(p) = p\}$.

(iii) *The map* $\psi : \mathcal{D}_{\mathrm{nilp,SL(2)},n} \to \mathcal{D}_{\mathrm{SL(2)},n}$ *is injective. This map is described via Proposition 2.3.7 as follows. For* $p = (N_1, \dots, N_n, F) \in \mathcal{D}_{\mathrm{nilp,SL(2)},n}$, *the family of weight filtrations associated to* $\psi(p)$ *is given as in Theorem 2.4.2(iv),* $\mathbf{r}_1 = \exp(iN_1 + \cdots + iN_n)F$, *and* $J = \{j \mid 1 \le j \le n, N_j \ne 0\}$. *If* $J = \{a(1), \dots, a(r)\}$ $(a(1) < \cdots < a(r))$ *and if* p' *denotes* $(N_{a(1)}, \dots, N_{a(r)}, F)$, $\psi(p')$ *coincides with the* $\mathrm{SL}(2)$*-orbit in* r *variables of rank* r *associated to* $\psi(p)$ *(see Section 2.3.4).*

(iv) *In the pure case, the map* $\psi : \mathcal{D}_{\mathrm{nilp,SL(2)},n} \to \mathcal{D}_{\mathrm{SL(2)},n}$ *is bijective. The converse map is given by* $(\rho, \varphi) \mapsto (N_1, \dots, N_n, \varphi(\mathbf{0}))$, *where* N_j *is the operator associated to* ρ *in Section 2.1.4.*

(v) *The map* $\psi : \mathcal{D}_{\mathrm{nilp},n} \to \mathcal{D}_{\mathrm{SL(2)},n}$ *factors as* $\mathcal{D}_{\mathrm{nilp},n} \xrightarrow{\phi} \mathcal{D}_{\mathrm{nilp,SL(2)},n} \xhookrightarrow{\psi} \mathcal{D}_{\mathrm{SL(2)},n}$.

(vi) *Assume* $p = (N_1, \dots, N_n, F) \in \mathcal{D}_{\mathrm{nilp,SL(2)},n}$. *Let*

$$\psi(p) = \big((\rho_w, \varphi_w)_w, \mathbf{r}_1, J\big)$$

(see Theorem 2.4.2), and let $(W^{(j)})_{1 \le j \le n}$ *be the family of weight filtrations associated to* $\psi(p)$. *Then* $(W^{(j)}, \mathbf{r}_1)$ *is a mixed Hodge structure for* $1 \le j \le n$, *and* p *is recovered from* $\psi(p)$ *by the following (1) and (2).*

(1) *Let* $k = \min(J \cup \{n+1\})$. *For* $1 \le j < k$, $N_j = 0$. *For* $k \le j \le n$, $\sum_{l=k}^j N_l = s^{(j)}\delta(W^{(j)}, \mathbf{r}_1)(s^{(j)})^{-1}$, *where* $s^{(j)}$ *is the* $s_{\mathbf{r}_1}$*-lift (cf. Section 2.4.6; see Section 2.3.5 for* $s_{\mathbf{r}_1}$*) of* $(s^{(j)}$ *of* $(\rho_w, \varphi_w))_w$.

(2) *If* $k = n+1$, $F = \mathbf{r}_1$. *Otherwise,* $(W^{(n)}, F)$ *is the* $\mathbf{R}$*-split mixed Hodge structure associated to the mixed Hodge structure* $(W^{(n)}, \mathbf{r}_1)$.

We prove this theorem later in Section 2.4.10.

The injection $\psi : \mathcal{D}_{\mathrm{nilp},\mathrm{SL}(2),n} \to \mathcal{D}_{\mathrm{SL}(2),n}$ need not be surjective although it is bijective in the pure case (see Theorem 2.4.5(iv); see also Section 2.4.11, Example III).

Some readers may prefer to define an SL(2)-orbit as an element of $\bigsqcup_n \mathcal{D}_{\mathrm{nilp},\mathrm{SL}(2),n}$, not using $\mathcal{D}_{\mathrm{SL}(2),n}$. The reason we use the set $\mathcal{D}_{\mathrm{SL}(2),n}$ is that the space $D_{\mathrm{SL}(2)}$ of the classes of SL(2)-orbits defined by using $\mathcal{D}_{\mathrm{SL}(2),n}$ has nice properties (e.g., Theorem 3.5.15).

We now give preparations for the proofs of Theorem 2.4.2, Proposition 2.4.3, and Theorem 2.4.5.

2.4.6.

Let $(N_1,\ldots,N_n,F) \in \mathcal{D}_{\mathrm{nilp},n}$. In the following, we review an alternative construction of s,τ, and $\mathbf{r}_1$ by a finite number of algebraic steps, not by a limit. In particular, we review the definition of $\hat{F}_{(n)}$.

For $0 \le j \le n$, we denote $M(\sum_{k=1}^{j} N_k, W)$ by $W^{(j)}$. In particular, $W^{(0)} = W$.

For $0 \le j \le n$, we define an $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$ and the associated splitting $s^{(j)}$ of $W^{(j)}$ inductively starting from $j=n$ and ending at $j=0$ (see [KNU1, Section 10.1]; in the pure case, see [CKS]). Note that, in the definition of mixed Hodge structure, we do not assume that the weight filtration is rational (cf. Section 1.2.8). First, $(W^{(n)},F)$ is a mixed Hodge structure, as is proved by Deligne (see [K, Proposition 5.2.1]). Let $(W^{(n)},\hat{F}_{(n)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(n)},F)$. Then $(W^{(n-1)},\exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. Let $(W^{(n-1)},\hat{F}_{(n-1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to $(W^{(n-1)},\exp(iN_n)\hat{F}_{(n)})$. Then $(W^{(n-2)},\exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure. This process continues. In this way we define $\hat{F}_{(j)}$ inductively as the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(j)},\exp(iN_{j+1})\hat{F}_{(j+1)})$ and define $s^{(j)}$ to be the splitting of $W^{(j)}$ associated to $\hat{F}_{(j)}$. The splitting s in Theorem 2.4.2(i) is nothing but $s^{(0)}$ (see [KNU1, Section 10.1.2]).

Thus we have $s^{(j)} = \mathrm{spl}_{W^{(j)}}(\exp(iN_{j+1})\hat{F}_{(j+1)})$, $\hat{F}_{(j)} = s^{(j)}\big((\exp(iN_{j+1})\cdot \hat{F}_{(j+1)})(\mathrm{gr}^{W^{(j)}})\big)$ $(N_{n+1} := 0,\ \hat{F}_{(n+1)} := F)$. We also have $\mathbf{r}_1 = \exp(iN_k)\hat{F}_{(k)}$, where $k = \min(\{j \mid 1 \le j \le n, N_j \ne 0\} \cup \{n+1\})$ (cf. Section 2.4.8).

These $s^{(j)}$ $(0 \le j \le n)$ are compatible in the sense that we have a direct sum decomposition

$$H_{0,\mathbf{R}} = \bigoplus_{\theta \in \mathbf{Z}^{n+1}} H_{0,\mathbf{R}}^{[\theta]}, \quad \text{where } H_{0,\mathbf{R}}^{[\theta]} = \bigcap_{j=0}^{n} s^{(j)}(\mathrm{gr}_{\theta(j)}^{W^{(j)}}).$$

This compatibility is expressed also in the following way. Let

$$\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W) \quad (0 \le j \le n)$$

be the homomorphism of algebraic groups over $\mathbf{R}$ characterized as follows. For $a \in \mathbf{R}^\times$ and $w \in \mathbf{Z}$, $\tau_j(a)$ acts on $s^{(j)}(\mathrm{gr}_w^{W^{(j)}})$ as the multiplication by a^w. Then the compatibility of $s^{(j)}$ $(0 \le j \le n)$ in question is expressed as the fact that $\tau_j(a)\tau_k(b) = \tau_k(b)\tau_j(a)$ for any j, k and any $a, b \in \mathbf{R}_{>0}$. Let

$$\tau(a) = \prod_{j=1}^{n} \tau_j(a_j) \quad \text{for } a = (a_j)_j \in \mathbf{R}^n_{>0}.$$

This τ coincides with the τ in Theorem 2.4.2(ii). Note also that $s^{(j)}$ is the s-lift of $(s^{(j)}$ of $(\rho_w, \varphi_w))_w$; that is, it coincides with the composite $\mathrm{gr}^{W^{(j)}} \cong \bigoplus_w \mathrm{gr}^{W^{(j)}}(\mathrm{gr}^W_w) \to \bigoplus_w \mathrm{gr}^W_w \xrightarrow{s} H_{0,\mathbf{R}}$, where the first arrow is the sum of the splittings $s^{(j)}$ on gr^W_w with respect to (ρ_w, φ_w). In [KNU1, Section 10.3], we denoted $\tau_j(\sqrt{a})^{-1}$ for $a \in \mathbf{R}_{>0}$ by $t^{(j)}(a)$ and $\tau((\sqrt{a_{j+1}/a_j})_j)$ for $a = (a_j)_j \in \mathbf{R}^n_{>0}$ $(a_{n+1} := 1)$ by $t(a)$.

Any $h \in \mathfrak{g}_{\mathbf{R}}$ is decomposed uniquely in the form

$$h = \sum_{\theta \in \mathbf{Z}^{n+1}} h^{[\theta]}, \qquad h^{[\theta]} \in \mathfrak{g}_{\mathbf{R}}, \qquad h^{[\theta]}(H^{[\theta']}_{0,\mathbf{R}}) \subset H^{[\theta+\theta']}_{0,\mathbf{R}} \quad (\forall \theta' \in \mathbf{Z}^{n+1}).$$

PROPOSITION 2.4.7

Let the notation be as above.

(i) *Let $1 \le j \le n$, and let $\theta = (\theta(k))_{0 \le k \le n} \in \mathbf{Z}^{n+1}$ $(\theta(k) \in \mathbf{Z})$. Then $N^{[\theta]}_j = 0$ unless $\theta(k) = -2$ for $j \le k \le n$.*

(ii) *Let $1 \le j \le n$, and define $\hat{N}_j$ (resp., N^{Δ}_j, resp., $\hat{N}'_j$) to be the sum of $N^{[\theta]}_j$, where θ ranges over all elements of $\mathbf{Z}^{n+1}$ such that $\theta(k) = 0$ for $0 \le k \le j-1$ (resp., for $1 \le k \le j-1$, resp., for $k = j-1$). Then*

$$\hat{N}_j = \hat{N}'_j.$$

Consequently,

$$N^{\Delta}_j = \hat{N}_j \quad \textit{for } 2 \le j \le n, \qquad N^{\Delta}_1 = N_1.$$

(iii) *We have $N_j\hat{N}_k = \hat{N}_k N_j$ if $1 \le j < k \le n$.*

(iv) *We have $\hat{N}_j\hat{N}_k = \hat{N}_k\hat{N}_j$ and $N^{\Delta}_j N^{\Delta}_k = N^{\Delta}_k N^{\Delta}_j$ for all j, k.*

(v) *Assume $1 \le j \le k \le n$. Then $(W^{(k)}, \hat{F}_{(j)})$ is a mixed Hodge structure. The $\mathbf{R}$-split mixed Hodge structure associated to $(W^{(k)}, \hat{F}_{(j)})$ is $(W^{(k)}, \hat{F}_{(k)})$, $(s^{(k)})^{-1}N_\ell s^{(k)}$ and $(s^{(k)})^{-1}\hat{N}_\ell s^{(k)}$ belong to $L^{-1,-1}_{\mathbf{R}}(W^{(k)}, \hat{F}_{(k)}(\mathrm{gr}^{W^{(k)}}))$ (which is a subset of $\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^{W^{(k)}})$ defined similarly to $L^{-1,-1}_{\mathbf{R}}(F)$ in Section 1.2.1) for all $\ell \le k$, and $\delta(W^{(k)}, \hat{F}_{(j)}) = (s^{(k)})^{-1}(\sum_{j<\ell\le k} \hat{N}_\ell)s^{(k)}$.*

REMARK 1

Thus $(N^{\Delta}_1, \ldots, N^{\Delta}_n)$ is nothing but $(N_1, \hat{N}_2, \ldots, \hat{N}_n)$. In [KNU1], Proposition 2.4.7(ii) above was not recognized, so we did not unify the notation N^{Δ}_j and $\hat{N}_j$.

REMARK 2

In the case $j \ge k$, $N_j\hat{N}_k = \hat{N}_k N_j$ in Proposition 2.4.7(iii) need not be true. For example, in Section 1.1.1, Example III, if we take N in Section 2.4.11, Example III below as N_j for $1 \le j \le n$ and take F in Section 2.4.11, Example III, then $\hat{N}_1$ sends e_1 and e_3 to zero and e_2 to e_1, so $N_j\hat{N}_1 = 0$, but $\hat{N}_1 N_j$ is not zero for any j. (On the other hand, in this example, $\hat{N}_j = 0$ for $j \ge 2$, and hence $N_1\hat{N}_j = \hat{N}_j N_1$ is trivially true for $j \ge 2$.)

Proof of Proposition 2.4.7
The assertion (i) is explained in [KNU1, Section 10.3]. We give the proofs of the remaining statements.

Let $1 \le j \le n$. By [KNU1, Section 10.1.4], $\hat{F}_{(j)} = s(\varphi(\{0\}^j \times \{i\}^{n-j}))$. Here s is the splitting of W associated to $\mathbf{r}_1$. From this, we have the following.

(1) The filtration $\hat{F}_{(j)}$ coincides with $s^{(k)}\left(\bigoplus_w \hat{F}_{(j)}(\mathrm{gr}_w^{W^{(k)}})\right)$ if $0 \le k \le j$.

By (1) and by $(s^{(j)})^{-1}N_k s^{(j)} \in L_{\mathbf{R}}^{-1,-1}(W^{(j)}, \hat{F}_{(j)}(\mathrm{gr}^{W^{(j)}}))$ for $1 \le k \le j$, we have

(2) The endomorphism $(s^{(j)})^{-1}\hat{N}_k s^{(j)}$ and $(s^{(j)})^{-1}\hat{N}'_k s^{(j)}$ belong to $L_{\mathbf{R}}^{-1,-1}(W^{(j)}, \hat{F}_{(j)}(\mathrm{gr}^{W^{(j)}}))$ for $1 \le k \le j$.

We prove (ii). By (1), we see that

$$\hat{F}_{(j-1)} = \exp(i\hat{N}'_j)\hat{F}_{(j)},$$

and since $(W^{(j)}, \hat{F}_{(j)})$ is an $\mathbf{R}$-split mixed Hodge structure, we have by (2),

$$\delta(W^{(j)}, \hat{F}_{(j-1)}) = (s^{(j)})^{-1}\hat{N}'_j s^{(j)}. \tag{3}$$

Note that $\zeta = 0$ since δ has only $(-1,-1)$-Hodge component (see Section 1.2.3).

Next, by [KNU1, Proposition 10.4(1)],

$$\hat{F}_{(j-1)} = \exp(i\hat{N}_j)\hat{F}_{(j)}.$$

Hence by (1) and (2), we have

$$\delta(W^{(j)}, \hat{F}_{(j-1)}) = (s^{(j)})^{-1}\hat{N}_j s^{(j)}. \tag{4}$$

Comparing (3) and (4), we conclude that $\hat{N}_j = \hat{N}'_j$.

We prove (iii). Since $N_j^{[\theta]} = 0$ unless $\theta(k-1) = -2$ by (i), and since $\hat{N}_k = \hat{N}'_k$ by (ii), $N_j\hat{N}_k$ (resp., $\hat{N}_k N_j$) is the sum of $(N_j N_k)^{[\theta]}$ (resp., $(N_k N_j)^{[\theta]}$), where θ ranges over all elements of $\mathbf{Z}^{n+1}$ such that $\theta(k-1) = -2$. But $N_j N_k = N_k N_j$; (iii) follows.

We prove (iv). We may assume $j < k$. Then, by (ii), $\hat{N}_j\hat{N}_k$ (resp., $\hat{N}_k\hat{N}_j$) is the sum of $(N_j\hat{N}_k)^{[\theta]}$ (resp., $(\hat{N}_k N_j)^{[\theta]}$), where θ ranges over all elements of $\mathbf{Z}^{n+1}$ such that $\theta(j-1) = 0$. But $N_j\hat{N}_k = \hat{N}_k N_j$ by (iii). The first assertion of (iv) follows, and hence the second follows.

The rest is (v). Again by [KNU1, Proposition 10.4(1)], we have $\hat{F}_{(j)} = \exp(\sum_{j<\ell\le k} i\hat{N}_\ell)\hat{F}_{(k)}$. This implies (v) by the same argument as in the proof of (ii). □

2.4.8. Proof of Theorem 2.4.2
The assertion (i) is contained in [KNU1, Theorem 0.5].

We prove (ii). It is clear in the case when $k=n+1$. When $k\le n$, the proof of [KNU1, Proposition 10.4(2)] shows (ii), and furthermore, $\mathbf{r}_1=\exp(iN_k+iN_{k+1}^{\Delta}+\cdots+iN_n^{\Delta})\hat{F}_{(n)}$.

We prove (iii). We may assume $k\le n$. By the pure case, $\mathbf{r}_1(\mathrm{gr}_w^W)=\varphi_w(\mathbf{i})$. By the calculation of $\mathbf{r}_1$ in the above proof of (ii) together with [KNU1, Proposition 10.4(1)] and Proposition 2.4.7(ii), we have the following.

CLAIM
$\mathbf{r}_1$ *in Theorem 2.4.2(ii) coincides with* $\exp(iN_k)\hat{F}_{(k)}$.

If $\mathrm{gr}^W(N_k)\neq 0$, then by the pure case, $k\in J'$, and hence there is no problem (see Section 2.3.2). Assume $\mathrm{gr}^W(N_k)=0$. Then $W^{(k)}=W$, and hence $(W,\hat{F}_{(k)})$ is an $\mathbf{R}$-split mixed Hodge structure. Since N_k sends the (p,q)-Hodge component of $(W,\hat{F}_{(k)})$ to the $(p-1,q-1)$-Hodge component, we have $\delta(W,\exp(iN_k)\hat{F}_{(k)})=s^{-1}N_ks$. This shows that if $N_k\neq 0$, then $\mathbf{r}_1=\exp(iN_k)\hat{F}_{(k)}$ (see the claim above) belongs to D_{nspl} (see Section 1.2.7). Hence $((\rho_w,\varphi_w)_w,\mathbf{r}_1,J)\in\mathcal{D}_{\mathrm{SL}(2),n}$ (see Section 2.3.2).

We prove Theorem 2.4.2(iv). Since $s^{(0)}$ in Section 2.4.6 coincides with $\mathrm{spl}_W(\mathbf{r}_1)$ and also with the s in Theorem 2.4.2(i) (by the claim), it is reduced to the pure case that τ in Section 2.4.6 coincides with the torus action associated to $((\rho_w,\varphi_w)_w,\mathbf{r}_1,J)$ in Section 2.3.5 and also with the τ in Theorem 2.4.2(ii). This also shows the statement for the associated weight filtrations. □

2.4.9. Proof of Proposition 2.4.3
We may assume $k\le n$. It is enough to show that (1) implies (2). Assume (1). By Section 2.4.6, we have $\hat{F}_{(j)}=\exp(\sum_{l=j+1}^n iN_l)F$ for $k\le j\le n$. This gives $\hat{F}^{(n)}=F$ and also $\delta(W^{(n)},\hat{F}_{(j)})=(s^{(n)})^{-1}(\sum_{l=j+1}^n N_l)s^{(n)}$ for $k\le j\le n$. Comparing this with $\delta(W^{(n)},\hat{F}_{(j)})=(s^{(n)})^{-1}(\sum_{l=j+1}^n \hat{N}_l)s^{(n)}$ $(k\le j\le n)$ obtained in Proposition 2.4.7(v), we have $\hat{N}_j=N_j$ for $k<j\le n$. This implies (2). □

2.4.10. Proof of Theorem 2.4.5
We prove (i). We may assume $k\le n$. We show $\phi(p)\in\mathcal{D}_{\mathrm{nilp},n}$ by checking conditions (1)–(4) in Section 2.4.1. Condition (1) is satisfied by Proposition 2.4.7(ii)–(iv). Section 2.4.1(2) is seen by reduction to the pure case. Section 2.4.1(3) (Griffiths transversality) for N_j follows from [KNU1, Proposition 5.7] and Section 2.4.1(3) for $\hat{N}_j$ is deduced from it and from (1) in the proof of Proposition 2.4.7. We show Section 2.4.1(4) (concerning relative monodromy filtration). By Kashiwara [K, Theorem 4.4.1] and by Proposition 2.4.7(ii), it is sufficient to show that the relative monodromy filtration exists for $\hat{N}_j$ $(k\le j\le n)$ and for N_k. For N_k, this is included in the assumption. For $\hat{N}_j$ $(k\le j\le n)$, this is easy since $\hat{N}_j$ is of weight zero with respect to $s^{(0)}$. Once $\phi(p)\in\mathcal{D}_{\mathrm{nilp},n}$ is verified, it is easy to see that $\phi\circ\phi=\phi$.

The assertion (ii) is essentially proved in Section 2.4.9. The assertion (iii) is proved later. The assertion (iv) is known as the pure case (see [KU2, Section 6]). The assertion (v) is easy.

We prove (vi). For $k \le j \le n$, we have $\mathbf{r}_1 = \exp(\sum_{l=k}^{j} iN_l)\hat{F}_{(j)}$ in the notation of Section 2.4.6. In particular, we have $\mathbf{r}_1 = \exp(\sum_{l=k}^{n} iN_l)F$. The assertion (vi) is deduced from these relations.

We prove (iii). The injectivity follows from (vi).

To prove $J = \{j \mid 1 \le j \le n, N_j \ne 0\}$, we first show the following.

CLAIM

For $k < j \le n$, $W^{(j)} \ne W^{(j-1)}$ if and only if $N_j \ne 0$.

Proof

Since N_j is of weight zero with respect to $s^{(j-1)}$, the N_j is zero if and only if $\mathrm{gr}_w^{W^{(j-1)}}(N_j)$ is zero for any w. The latter condition is equivalent to $W^{(j)} = W^{(j-1)}$. □

By this claim, we have the description of J. The remaining parts of (iii) are easy. This finishes the proof of Theorem 2.4.5. □

2.4.11.

For some examples in Examples I–III, we describe here the map $\psi : \mathcal{D}_{\mathrm{nilp},1} \to \mathcal{D}_{\mathrm{SL}(2),1}$, $(N,F) \mapsto ((\rho_w, \varphi_w)_w, \mathbf{r}_1, J)$, in Theorem 2.4.2, the torus actions $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}(H_{0,\mathbf{R}}, W)$ $(j = 0, 1)$, and nilpotent endomorphisms $\hat{N}$ and N^{Δ} (see Proposition 2.4.7).

EXAMPLE I

Let $N(e_1) = 0$, $N(e_2) = e_1$, and let $F = F(i)$. Then (N,F) generates a nilpotent orbit. The canonical splitting of W associated to $\exp(iy_1N)F$ $(y_1 > 0)$ sends e_2' to e_2. From this we have $\tau(t)e_1 = t^{-2}e_1$, $\tau(t)e_2 = e_2$. For $t = 1/\sqrt{y_1}$, we have $\lim_{t\to 0} \tau(t)^{-1}\exp(iy_1N)F = F(i)$. Hence the image $((\rho_w, \varphi_w)_w, \mathbf{r}_1, J)$ of (N,F) under ψ consists of Section 2.3.9, Example I with $J = \{1\}$.

We have $W^{(1)} = W$, and $\tau_1 = \tau_0 = \tau$. Hence $\hat{N} = 0$, $N^{\Delta} = N$.

EXAMPLE II

We consider the following example (N,F) which generates a nilpotent orbit. Let $N(e_2) = e_1$, $N(e_j) = 0$ $(j = 1, 3)$, and let $F = F(i, ia)$ with $a \in \mathbf{R}$.

By Section 1.2.9, $(s_1, s_2) \in \mathbf{R}^2$ corresponding to the canonical splitting $\mathrm{spl}_W(\exp(iy_1N)F)$ is $s_1 = 0$, $s_2 = -a/(1+y_1)$. When $y_1 \to \infty$, (s_1, s_2) converges to $(0,0)$ in $\mathrm{spl}(W) = \mathbf{R}^2$. From this, we have $\tau(t)e_1 = t^{-2}e_1$, $\tau(t)e_j = e_j$ $(j = 2, 3)$. For $t = 1/\sqrt{y_1}$, we have $\lim_{t\to 0}\tau(t)^{-1}\exp(iy_1N)F = \mathbf{r}_1 \in D$, where $\mathbf{r}_1^1 := 0$, $\mathbf{r}_1^0 := \mathbf{C}(ie_1 + e_2) + \mathbf{C}e_3$, $\mathbf{r}_1^{-1} := H_{0,\mathbf{C}}$. Hence the image $((\rho_w, \varphi_w)_w, \mathbf{r}_1, J)$ of (N,F) under ψ consists of the example in Section 2.3.9, Example II with $z = ia$ and $J = \{1\}$.

The torus actions τ_0, τ_1 which induce the splittings of the filtrations W, $W^{(1)}$ in Section 1.1.1, Example II, and Section 2.3.9, Example II, respectively, are as follows: $\tau_0(t)e_j = t^{-1}e_j$ $(j = 1,2)$, $\tau_0(t)e_3 = e_3$; $\tau_1 = \tau$ above. Hence $\hat{N} = N^{\Delta} = N$.

EXAMPLE III

We consider the following example (N,F) which generates a nilpotent orbit. Let $N(e_3) = e_2$, $N(e_2) = e_1$, $N(e_1) = 0$, and let $F = F(i,z,i)$ with $z \in \mathbf{C}$.

By Section 1.2.9, $(s_1,s_2) \in \mathbf{R}^2$ corresponding to the canonical splitting $\mathrm{spl}_W(\exp(iy_1N)F)$ is $s_1 = \mathrm{Re}(z) + (1/2)$, $s_2 = -\mathrm{Im}(z)/2(1+y_1)$. When $y_1 \to \infty$, (s_1,s_2) converges to $(\mathrm{Re}(z) + (1/2), 0)$ in $\mathrm{spl}(W) = \mathbf{R}^2$. From this, we have $\tau(t)e_1 = t^{-4}e_1$, $\tau(t)e_2 = t^{-2}e_2$, $\tau(t)e_3 = e_3 + (1-t^{-4})(\mathrm{Re}(z) + (1/2))e_1$. For $t = 1/\sqrt{y_1}$, we have $\lim_{t\to 0}\tau(t)^{-1}\exp(iy_1N)F = \mathbf{r}_1 \in D$, where $\mathbf{r}_1^1 := 0$, $\mathbf{r}_1^0 := \mathbf{C}(\mathrm{Re}(z)e_1 + ie_2 + e_3)$, $\mathbf{r}_1^{-1} := \mathbf{r}_1^0 + \mathbf{C}(ie_1 + e_2)$, $\mathbf{r}_1^{-2} := H_{0,\mathbf{C}}$. Hence the image $((\rho_w,\varphi_w)_w, \mathbf{r}_1, J)$ of (N,F) under ψ consists of the example in Section 2.3.9, Example III, Case 1 with $z_1 = \mathrm{Re}(z)$ and $J = \{1\}$.

There is no nilpotent orbit whose associated SL(2)-orbit is in Case 2 or 3 in Section 2.3.9, Example III (cf. the comment after Theorem 2.4.5). (In Examples I, II, IV, and V, all SL(2)-orbits come from nilpotent orbits.)

In the following, assume $\mathrm{Re}(z) = -1/2$ for simplicity. The torus actions τ_0, τ_1 which induce the splittings of the filtrations W, $W^{(1)}$, in Section 1.1.1, Example III and in Case 1 of Section 2.3.9, Example III, respectively, are as follows: $\tau_0(t)e_j = t^{-3}e_j$ $(j = 1,2)$, $\tau_0(t)e_3 = e_3$; $\tau_1 = \tau$ above. Hence $\hat{N} = N^{[(0,-2)]}$ is given by $\hat{N}(e_2) = e_1$, $\hat{N}(e_j) = 0$ $(j = 1,3)$; $N^{\Delta} = N$.

2.5. Definition of the set $D_{\mathrm{SL}(2)}$

2.5.1.

Two nondegenerate SL(2)-orbits $p = ((\rho_w,\varphi_w)_w, \mathbf{r})$ and $p' = ((\rho'_w,\varphi'_w)_w, \mathbf{r}')$ in n variables of rank n (see Section 2.3.3) are said to be *equivalent* if there is a $t \in \mathbf{R}^n_{>0}$ such that

$$\rho'_w = \mathrm{Int}\big(\mathrm{gr}^W_w(\tau(t))\big) \circ \rho_w, \qquad \varphi'_w = \mathrm{gr}^W_w\big(\tau(t)\big) \circ \varphi_w \quad (\forall w \in \mathbf{Z}), \qquad \mathbf{r}' = \tau(t)\mathbf{r}.$$

Here $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the torus action associated to $((\rho_w,\varphi_w)_w, \mathbf{r})$ defined in Section 2.3.5.

Note that this is actually an equivalence relation. We explain this. For $t = (t_j)_j \in \mathbf{R}^n_{>0}$, we write $\tilde{\rho}_w(t) = \rho_w(g_1,\ldots,g_n)$ in Section 2.3.5. Since $\mathrm{gr}^W_w(\tau(t)) = (\prod_{j=1}^n t_j)^w \tilde{\rho}_w(t)$ for $t \in \mathbf{R}^n_{>0}$ (see Section 2.3.5), we have $\tilde{\rho}'_w = \tilde{\rho}_w$ as homomorphisms $\mathbf{G}^n_{m,\mathbf{R}} \to G_{\mathbf{R}}(\mathrm{gr}^W_w)$ for any w. On the other hand, the splittings of W associated to $\mathbf{r}$ and to $\mathbf{r}' = \tau(t)\mathbf{r}$ coincide by the remark in Section 2.3.5. From these it follows that τ of p and τ of p' coincide. The axioms of equivalence relations can be now easily checked.

An SL(2)-orbit $((\rho_w,\varphi_w)_w, \mathbf{r}, J)$ in n variables of rank r and an SL(2)-orbit $((\rho'_w,\varphi'_w)_w, \mathbf{r}', J')$ in n' variables of rank r' are said to be *equivalent* if $r = r'$

and their associated SL(2)-orbits in r variables of rank r (see Section 2.3.4) are equivalent.

The class determines and is determined by the associated set of weight filtrations, the associated torus action, and the associated torus orbit; that is, we have the following.

PROPOSITION 2.5.2
Let $p = ((\rho_w, \varphi_w)_w, \mathbf{r})$ *be a nondegenerate* SL(2)*-orbit of rank* n*.*

(i) *The* $W^{(j)}$ *of* p*, the* τ *and the* τ_j *of* p $(1 \le j \le n)$*, the canonical splitting of* W *associated to* $\mathbf{r}$ *(see Section 1.2.3), and* $Z = \tau(\mathbf{R}^n_{>0})\mathbf{r}$ *depend only on the equivalence class of* p*. Here* τ *is the homomorphism in Section 2.3.5 associated to* p*.* Z *is called the* torus orbit associated to p*.*

(ii) *The equivalence class of* p *is determined by* $\big((W^{(j)}(\mathrm{gr}^W))_{1\le j\le n}, Z\big)$*, where* Z *is as above.*

(iii) *The equivalence class of* p *is determined by* (τ, Z)*, where* τ *and* Z *are as above.*

Proof
We prove (i). The statement for $W^{(j)}$ follows from $\tau(t)W^{(j)} = W^{(j)}$ $(t \in (\mathbf{R}^\times)^n)$, the statements for τ and for the splitting were proved in Section 2.5.1, and the rest is clear.

The statements (ii) and (iii) follow from (i) and from Proposition 2.3.7. □

2.5.3.
Let $D_{\mathrm{SL}(2)}$ be the set of all equivalence classes of SL(2)-orbits satisfying the following condition (C).

Take an SL(2)-orbit $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ in n variables which is a representative of the class in question.

(C) For each $w \in \mathbf{Z}$ and for each $1 \le j \le n$, the weight filtration $W^{(j)}(\mathrm{gr}^W_w)$ is rational.

(This condition is independent of the choice of the representative by Proposition 2.5.2(i).)

As a set, we have

$$D_{\mathrm{SL}(2)} = \bigsqcup_{n\ge 0} D_{\mathrm{SL}(2),n},$$

where $D_{\mathrm{SL}(2),n}$ is the set of equivalence classes of SL(2)-orbits of rank n (see Section 2.3.3) with rational associated weight filtrations. We identify $D_{\mathrm{SL}(2),0}$ with D in the evident way.

Let $D_{\mathrm{SL}(2),\mathrm{spl}}$ be the subset of $D_{\mathrm{SL}(2)}$ consisting of the classes of $((\rho_w, \varphi_w)_w, \mathbf{r})$ with $\mathbf{r} \in D_{\mathrm{spl}}$ (see the notation in Section 0). (The last condition is independent of the choice of the representative.) Let $D_{\mathrm{SL}(2),\mathrm{nspl}} = D_{\mathrm{SL}(2)} \smallsetminus D_{\mathrm{SL}(2),\mathrm{spl}}$.

2.5.4.
We have a canonical projection

$$D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = \prod_{w \in \mathbf{Z}} D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W),$$

$$\mathrm{class}\big((\rho_w, \varphi_w)_w, \mathbf{r}\big) \mapsto \big(\mathrm{class}(\rho_w, \varphi_w)\big)_w.$$

Here $D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)$ is the $D_{\mathrm{SL}(2)}$ for $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\, ,\, \rangle_w)$. Note that in the pure case, the definition of $D_{\mathrm{SL}(2)}$ coincides with that of [KU2].

2.5.5.
As in the notation in Section 0, let $\mathrm{spl}(W)$ be the set of all splittings of W. We have a canonical map

$$D_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$$

as $\mathrm{class}((\rho_w, \varphi_w)_w, \mathbf{r}) \mapsto s$, where s denotes the canonical splitting of W associated to $\mathbf{r}$ (see Proposition 2.5.2(i)).

2.5.6.
For $p \in D_{\mathrm{SL}(2)}$, we denote by τ_p and Z_p the corresponding τ and Z, respectively (see Proposition 2.5.2(iii)).

2.5.7.
Later, in Section 3.2, we define two topologies on the set $D_{\mathrm{SL}(2)}$. Basic properties of these topologies are the following (see Section 3.2, Theorem 4.1.1).

(i) If $p \in D_{\mathrm{SL}(2)}$ is the class of $(\tau_p, \mathbf{r})$, then we have, in $D_{\mathrm{SL}(2)}$,

$$\tau_p(t)\mathbf{r} \to p \qquad \text{when } t \in \mathbf{R}^n_{>0} \text{ tends to } \mathbf{0}.$$

Here n is the rank of p and $\mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^n_{\geq 0}$.

(ii) If $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit and if the monodromy filtration of $\mathrm{gr}_w^W(N_1) + \cdots + \mathrm{gr}_w^W(N_j)$ is rational for any $w \in \mathbf{Z}$ and any $1 \leq j \leq n$, then we have, in $D_{\mathrm{SL}(2)}$,

$$\exp\Big(\sum_{j=1}^{n} iy_j N_j\Big) F \to p$$

when $y_j > 0, y_j/y_{j+1} \to \infty$ ($1 \leq j \leq n$, y_{n+1} denotes 1), where p denotes the class of the SL(2)-orbit associated to $(N_1, \ldots, N_n, F)$ by Theorem 2.4.2.

This (ii) is the basic principle that lies in our construction of the topologies on $D_{\mathrm{SL}(2)}$. Our SL(2)-orbit theorem [KNU1, Theorem 0.5] says roughly that, when $y_j/y_{j+1} \to \infty$ $(1 \leq j \leq n,\ y_{n+1} = 1)$, $\exp(\sum_{j=1}^n iy_j N_j)F$ is near to $\tau_p(\sqrt{y_2/y_1}, \ldots, \sqrt{y_{n+1}/y_n})\mathbf{r}$, where $\mathbf{r} \in Z_p$. Hence (i) is natural in view of (ii).

3. Real analytic structures of $D_{\mathrm{SL}(2)}$

3.1. Spaces with real analytic structures and log structures with sign

We discuss a category $\mathcal{B}_{\mathbf{R}}$ of spaces with real analytic structures and its logarithmic version, a category $\mathcal{B}_{\mathbf{R}}(\log)$. In Section 3.1.11, Proposition 3.1.12 and Section 3.1.13, we consider log modifications in $\mathcal{B}_{\mathbf{R}}(\log)$ associated to cone decompositions.

3.1.1. The categories $\mathcal{B}_{\mathbf{R}}$, $\mathcal{B}'_{\mathbf{R}}$, and $\mathcal{C}_{\mathbf{R}}$

We define three full subcategories

$$\mathcal{B}_{\mathbf{R}} \subset \mathcal{B}'_{\mathbf{R}} \subset \mathcal{C}_{\mathbf{R}}$$

of the category of local ringed spaces over $\mathbf{R}$.

We first define $\mathcal{B}'_{\mathbf{R}}$. An object of $\mathcal{B}'_{\mathbf{R}}$ is a local ringed space $(S, \mathcal{O}_S)$ over $\mathbf{R}$ such that the following holds locally on S. There are $n \geq 0$ and a morphism $\iota : S \to \mathbf{R}^n$ of local ringed spaces over $\mathbf{R}$ from S to the real analytic manifold $\mathbf{R}^n$ such that ι is injective, the topology of S coincides with the one induced from the topology of $\mathbf{R}^n$ via ι, and the canonical map $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_S$ is surjective. Here $\mathcal{O}_{\mathbf{R}^n}$ denotes the sheaf of $\mathbf{R}$-valued real analytic functions on $\mathbf{R}^n$, and $\iota^{-1}(\,)$ denotes the inverse image of a sheaf. Morphisms of $\mathcal{B}'_{\mathbf{R}}$ are those of local ringed spaces over $\mathbf{R}$.

Let $\mathcal{B}_{\mathbf{R}}$ be the full subcategory of $\mathcal{B}'_{\mathbf{R}}$ consisting of all objects for which, locally on S, we can take $\iota : S \to \mathbf{R}^n$ as above such that the kernel of the surjection $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_S$ is a finitely generated ideal.

Of course, a real analytic manifold is an object of $\mathcal{B}_{\mathbf{R}}$. An example of an object of $\mathcal{B}_{\mathbf{R}}$ which often appears in this article is $\mathbf{R}^n_{\geq 0}$ with the inverse image of the sheaf of real analytic functions on $\mathbf{R}^n$.

For an object $(S, \mathcal{O}_S)$ of $\mathcal{B}'_{\mathbf{R}}$, we often call $\mathcal{O}_S$ the sheaf of real analytic functions of S, although $(S, \mathcal{O}_S)$ need not be a real analytic space.

We define another category $\mathcal{C}_{\mathbf{R}}$ as follows. An object of $\mathcal{C}_{\mathbf{R}}$ is a local ringed space $(S, \mathcal{O}_S)$ over $\mathbf{R}$ such that for any open set U of S and for any $n \geq 0$, the canonical map $\mathrm{Mor}(U, \mathbf{R}^n) \to \mathcal{O}_S(U)^n, \varphi \mapsto (\varphi_j)_{1 \leq j \leq n}$, is bijective, where $\mathbf{R}^n$ is regarded as a real analytic manifold as usual, $\mathrm{Mor}(U, \mathbf{R}^n)$ is the set of all morphisms in the category of local ringed spaces over $\mathbf{R}$, and φ_j denotes the pullback of the jth coordinate function of $\mathbf{R}^n$ via φ. Morphisms of $\mathcal{C}_{\mathbf{R}}$ are those of local ringed spaces over $\mathbf{R}$.

It is easily seen that real analytic manifolds, C^∞-manifolds (with the sheaves of C^∞-functions), and any topological spaces with the sheaves of real-valued continuous functions belong to $\mathcal{C}_{\mathbf{R}}$.

LEMMA 3.1.2

We have

$$\mathcal{B}'_{\mathbf{R}} \subset \mathcal{C}_{\mathbf{R}}.$$

Proof
Let S be an object of $\mathcal{B}'_{\mathbf{R}}$. Let $\mathrm{Mor}_S(-,\mathbf{R}^n)$ be the sheaf on S of morphisms into $\mathbf{R}^n$. We prove that the map $\mathrm{Mor}_S(-,\mathbf{R}^n) \to \mathcal{O}_S^n$ is an isomorphism. We first prove the surjectivity. A local section of $\mathcal{O}_S^n$ comes, locally on S, from an element of $\mathcal{O}_{\mathbf{R}^m}(V)^n$ for some open set V of $\mathbf{R}^m$ and for some morphism $S \to V$. Since $\mathcal{O}_{\mathbf{R}^m}(V)^n = \mathrm{Mor}(V,\mathbf{R}^n)$, a local section of $\mathcal{O}_S^n$ comes from $\mathrm{Mor}_S(-,\mathbf{R}^n)$ locally on S. It remains to prove the injectivity of $\mathrm{Mor}_S(-,\mathbf{R}^n) \to \mathcal{O}_S^n$. We prove the following.

CLAIM
For any $s \in S$, the local ring $\mathcal{O}_{S,s}$ is Noetherian.

This is reduced to the fact that the local rings of the real analytic manifold $\mathbf{R}^n$ are Noetherian. These local rings are the rings of convergent Taylor series. Hence they are Noetherian.

Now we return to the proof of Lemma 3.1.2. Assume that two morphisms $f, g : S \to \mathbf{R}^n$ induce the same element $(\varphi_j)_j$ of $\mathcal{O}_S(S)^n$. The underlying map $S \to \mathbf{R}^n$ of sets induced by f and g are given by $s \mapsto (\varphi_j(s))_j$, and hence they coincide. To prove $f = g$, it is sufficient to prove that for any $s \in S$ with image $s' = f(s) = g(s) \in \mathbf{R}^n$ and for any element h of $\mathcal{O}_{\mathbf{R}^n,s'}$, the pullbacks $f^*(h), g^*(h) \in \mathcal{O}_{S,s}$ coincide. Let m be the maximal ideal of $\mathcal{O}_{S,s}$, and let m' be the maximal ideal of $\mathcal{O}_{\mathbf{R}^n,s'}$. Let $r \geq 1$. Then $h \bmod (m')^r$ is expressed as a polynomial over $\mathbf{R}$ in the coordinate functions $t_j\mathbf{R}^n$. Hence $f^*(h) \equiv g^*(h) \bmod m^r$. Since $\mathcal{O}_{S,s}$ is Noetherian, the canonical map $\mathcal{O}_{S,s} \to \varprojlim_r \mathcal{O}_{S,s}/m^r$ is injective. Hence $f^*(h) = g^*(h)$ in $\mathcal{O}_{S,s}$. □

PROPOSITION 3.1.3
The category $\mathcal{B}'_{\mathbf{R}}$ has fiber products, and $\mathcal{B}_{\mathbf{R}}$ is stable under taking fiber products. The underlying topological space of a fiber product in $\mathcal{B}'_{\mathbf{R}}$ is the fiber product of the underlying topological spaces. The fiber product in $\mathcal{B}'_{\mathbf{R}}$ is also a fiber product in $\mathcal{C}_{\mathbf{R}}$.

Proof
Let $S' \to S$ and $S'' \to S$ be morphisms in $\mathcal{B}'_{\mathbf{R}}$.

Working locally on S, S', and S'', we may assume that there are injective morphisms $\iota : S \to \mathbf{R}^n$, $\iota' : S' \to \mathbf{R}^{n'}$, $\iota'' : S'' \to \mathbf{R}^{n''}$ such that the topologies of S, S', S'' are induced from those of $\mathbf{R}^n$, $\mathbf{R}^{n'}$, and $\mathbf{R}^{n''}$, respectively, and such that the homomorphisms $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_S$, $(\iota')^{-1}(\mathcal{O}_{\mathbf{R}^{n'}}) \to \mathcal{O}_{S'}$, and $(\iota'')^{-1}(\mathcal{O}_{\mathbf{R}^{n''}}) \to \mathcal{O}_{S''}$ are surjective. Let I' and I'' be the kernels of the last two homomorphisms, respectively. Let t_j $(1 \leq j \leq n)$ be the jth coordinate function of $\mathbf{R}^n$. Working locally on S', we may assume that for an open neighborhood U' of S' in $\mathbf{R}^{n'}$, there are elements $s'_j \in \mathcal{O}(U')$ $(1 \leq j \leq n)$ such that the restriction of s'_j to S' coincides with the pullback of t_j for each j. Similarly, working locally on S'', we may assume that for an open neighborhood U'' of S'' in $\mathbf{R}^{n''}$, there are elements $s''_j \in \mathcal{O}(U'')$ $(1 \le j \le n)$ such that the restriction of s''_j to S'' coincides with the pullback of t_j for each j. Let $F := S' \times_S S'' \subset V := U' \times U'' \subset \mathbf{R}^{n'+n''}$. Endow F with the topology as the fiber product, and endow it with the inverse image of

$$\mathcal{O}_V/J \quad \text{with } J = \big(I'\mathcal{O}_V + I''\mathcal{O}_V + (s'_1 - s''_1)\mathcal{O}_V + \cdots + (s'_n - s''_n)\mathcal{O}_V\big).$$

Here $I'\mathcal{O}_V + I''\mathcal{O}_V$ denotes the ideal of $\mathcal{O}_V$ generated by the inverse images of I' and I''. When we regard the diagram $S' \to S \leftarrow S''$ as the one in $\mathcal{C}_{\mathbf{R}}$ by Lemma 3.1.2, we can show that F is the fiber product of it in $\mathcal{C}_{\mathbf{R}}$, and hence F is the fiber product also in $\mathcal{B}'_{\mathbf{R}}$. If S, S', S'' belong to $\mathcal{B}_{\mathbf{R}}$, we can assume that I' and I'' are finitely generated. Then the ideal J is finitely generated. □

We now begin to discuss log structures.

LEMMA 3.1.4

Let $(S, \mathcal{O}_S)$ be an object of $\mathcal{C}_{\mathbf{R}}$. Let $\mathcal{O}^{\times}_{S,>0}$ be the subsheaf of $\mathcal{O}^{\times}_S$ consisting of all local sections whose values are strictly greater than zero. Then $\{\pm 1\} \xrightarrow{\sim} \mathcal{O}^{\times}_S/\mathcal{O}^{\times}_{S,>0}$. Furthermore, $\mathcal{O}^{\times}_{S,>0}$ coincides with the image of $\mathcal{O}^{\times}_S \to \mathcal{O}^{\times}_S, f \mapsto f^2$.

Proof

The isomorphisms $\mathbf{R}_{>0} \times \{\pm 1\} \xrightarrow{\sim} \mathbf{R}^{\times}$ and $\mathbf{R}_{>0} \xrightarrow{\sim} \mathbf{R}_{>0}, x \mapsto x^2$, of real analytic manifolds induce isomorphisms of sheaves

$$\mathcal{O}^{\times}_{S,>0} \times \{\pm 1\} \cong \mathrm{Mor}_S(-, \mathbf{R}_{>0} \times \{\pm 1\}) \xrightarrow{\sim} \mathrm{Mor}_S(-, \mathbf{R}^{\times}) \cong \mathcal{O}^{\times}_S,$$

$$\mathcal{O}^{\times}_{S,>0} \xrightarrow{\sim} \mathcal{O}^{\times}_{S,>0}, f \mapsto f^2,$$

respectively. This proves Lemma 3.1.4. □

DEFINITION 3.1.5

For an object S of $\mathcal{C}_{\mathbf{R}}$, a *log structure with sign* on S is an integral log structure M_S on S in the sense of Fontaine and Illusie (see [KU3, Section 2.1]) endowed with a subgroup sheaf $M^{\mathrm{gp}}_{S,>0}$ of M^{gp}_S satisfying the following three conditions. Here $M^{\mathrm{gp}}_S \supset M_S$ denotes the sheaf of commutative groups $\{ab^{-1} \mid a, b \in M_S\}$ associated to the sheaf M_S of commutative monoids.

(1) We have $M^{\mathrm{gp}}_{S,>0} \supset \mathcal{O}^{\times}_{S,>0}$.

(2) We have $\mathcal{O}^{\times}_S/\mathcal{O}^{\times}_{S,>0} \xrightarrow{\sim} M^{\mathrm{gp}}_S/M^{\mathrm{gp}}_{S,>0}$.

(3) Let $M_{S,>0} := M_S \cap M^{\mathrm{gp}}_{S,>0} \subset M^{\mathrm{gp}}_S$. Then the image of $M_{S,>0}$ in $\mathcal{O}_S$ under the structural map $M_S \to \mathcal{O}_S$ of the log structure has values in $\mathbf{R}_{\ge 0} \subset \mathbf{R}$ at any points of S.

(We note that $(M_{S,>0})^{\mathrm{gp}} = M^{\mathrm{gp}}_{S,>0}$, and thus $M^{\mathrm{gp}}_{S,>0}$ is recovered from $M_{S,>0}$.)

Let $\mathcal{B}_{\mathbf{R}}(\log)$ (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$, resp., $\mathcal{C}_{\mathbf{R}}(\log)$) be the category of objects of $\mathcal{B}_{\mathbf{R}}$ (resp., $\mathcal{B}'_{\mathbf{R}}$, resp., $\mathcal{C}_{\mathbf{R}}$) endowed with an fs log structure (see [KU3, Section 2.1]) with sign.

If S is an object of $\mathcal{C}_{\mathbf{R}}(\log)$ such that the structural map $M_S \to \mathcal{O}_S$ is injective and also the canonical map from $\mathcal{O}_S$ to the sheaf of real-valued functions on S is injective, then for an object S' of $\mathcal{C}_{\mathbf{R}}(\log)$, a morphism $f : S \to S'$ in $\mathcal{C}_{\mathbf{R}}(\log)$ is determined by its underlying map $\bar{f}$ of sets. For such S and an object S' of $\mathcal{C}_{\mathbf{R}}(\log)$, and for a map $g : S \to S'$ of sets, we sometimes say that g is a morphism of $\mathcal{C}_{\mathbf{R}}(\log)$ if $g = \bar{f}$ for some morphism $f : S \to S'$ of $\mathcal{C}_{\mathbf{R}}(\log)$.

We introduce some terminologies.

A *trivial log structure with sign* is the log structure $M_S = \mathcal{O}_S^{\times}$ with $M_{S,>0}^{\mathrm{gp}} = \mathcal{O}_{S,>0}^{\times}$.

The *inverse image* of a log structure with sign is the following. For a morphism $S' \to S$ in $\mathcal{C}_{\mathbf{R}}$ and for a log structure M_S with sign on S, the inverse image $M_{S'}$ of M_S on S', which is a log structure with sign on S', is defined as follows. As a log structure, $M_{S'}$ is the inverse image of M_S (see [KU3, Section 2.1.3]). $M_{S',>0}^{\mathrm{gp}}$ is the subgroup sheaf of $M_{S'}^{\mathrm{gp}}$ generated by $\mathcal{O}_{S',>0}^{\times}$ and the inverse image of $M_{S,>0}^{\mathrm{gp}}$.

A *chart* of an fs log structure with sign is the following. Let S be an object of $\mathcal{C}_{\mathbf{R}}(\log)$. A chart of M_S with sign is a pair of an fs monoid $\mathcal{S}$ and a homomorphism $h : \mathcal{S} \to M_{S,>0}$ such that $h : \mathcal{S} \to M_S$ is a chart of the fs log structure M_S (see [KU3, Section 2.1.5]) and such that $M_{S,>0}$ is generated by $\mathcal{O}_{S,>0}^{\times}$ and $h(\mathcal{S})$ as a sheaf of monoids. A chart of M_S with sign exists locally on S. This is shown by the fact that $M_{S,>0}/\mathcal{O}_{S,>0}^{\times} \to M_S/\mathcal{O}_S^{\times}$ is an isomorphism.

3.1.6. Real toric varieties, real analytic manifolds with corners

As standard examples of objects of $\mathcal{B}_{\mathbf{R}}(\log)$, we have real toric varieties and also real analytic manifolds with corners.

Let $\mathcal{S}$ be an fs monoid. We regard $S = \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ as follows and call it a real toric variety associated to $\mathcal{S}$: $\mathcal{O}_S$ is the sheaf of real-valued functions on S which belong to $\mathcal{O}_X|_S$. Here $X = \mathrm{Hom}(\mathcal{S}, \mathbf{C}^{\mathrm{mult}}) = \mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$, and $\mathcal{O}_X$ denotes the sheaf of complex analytic functions on X; M_S is the log structure associated to $\mathcal{S} \to \mathcal{O}_S$; $M_{S,>0}^{\mathrm{gp}}$ is generated by $\mathcal{S}^{\mathrm{gp}}$ and $\mathcal{O}_{S,>0}^{\times}$.

For any object T of $\mathcal{C}_{\mathbf{R}}(\log)$, we have

$$\mathrm{Mor}\big(T, \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}})\big) = \mathrm{Hom}(\mathcal{S}, M_{T,>0}).$$

In the case $\mathcal{S} = \mathbf{N}^n$, we have $S = \mathbf{R}_{\geq 0}^n$. We usually regard $\mathbf{R}_{\geq 0}^n$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ in this way.

A real analytic manifold with corners S is a local ringed space over $\mathbf{R}$ which has an open covering $(U_\lambda)_\lambda$ such that for each λ, U_λ is isomorphic to an open set of the object $\mathbf{R}_{\geq 0}^{n(\lambda)}$ of $\mathcal{B}_{\mathbf{R}}(\log)$ for some $n(\lambda) \geq 0$. The inverse images on U_λ of the fs log structures with sign of $\mathbf{R}_{\geq 0}^{n(\lambda)}$ glue together to an fs log structure with sign on S canonically. Thus a real analytic manifold with corners is regarded canonically as an object of $\mathcal{B}_{\mathbf{R}}(\log)$.

PROPOSITION 3.1.7

The category $\mathcal{B}'_{\mathbf{R}}(\log)$ *has fiber products, and* $\mathcal{B}_{\mathbf{R}}(\log)$ *is stable under taking fiber products. A fiber product in* $\mathcal{B}_{\mathbf{R}}(\log)$ *is a fiber product in* $\mathcal{C}_{\mathbf{R}}(\log)$*. The underlying object of* $\mathcal{B}'_{\mathbf{R}}$ *(resp., the underlying topological space) of a fiber product* $S' \times_S S''$ *in* $\mathcal{B}'_{\mathbf{R}}(\log)$ *coincides with the fiber product in* $\mathcal{B}'_{\mathbf{R}}$ *(resp., fiber product as topological spaces) if one of the following conditions (1) and (2) is satisfied.*

(1) *The log structure of* S *is trivial.*

(2) *The log structure of* S' *coincides with the inverse image of the log structure of* S*.*

This is a real analytic version of the complex analytic theory about the category $\mathcal{B}(\log)$ in [KU3, Section 2.1.10]. The proof is given by the same arguments there.

We next consider toric geometry in $\mathcal{B}_{\mathbf{R}}(\log)$ and log modifications in $\mathcal{B}_{\mathbf{R}}(\log)$ and in $\mathcal{B}'_{\mathbf{R}}(\log)$. These are real analytic versions of those in $\mathcal{B}(\log)$ (see [KU3, Section 3.6]).

3.1.8.

Let N be a finitely generated free abelian group whose group law is denoted additively. A *rational fan* in $N_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Z}} N$ is a nonempty set Σ of sharp rational finitely generated cones in $N_{\mathbf{R}}$ satisfying the following conditions (1) and (2).

(1) If $\sigma \in \Sigma$, any face of σ belongs to Σ.

(2) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of σ.

Here a *finitely generated cone* in $N_{\mathbf{R}}$ is a subset of $N_{\mathbf{R}}$ of the form $\{\sum_{j=1}^{n} a_j N_j \mid a_j \in \mathbf{R}_{\geq 0}\}$ with $N_1, \ldots, N_n \in N_{\mathbf{R}}$.

A finitely generated cone in $N_{\mathbf{R}}$ is said to be *rational* if we can take $N_1, \ldots,$ $N_n \in N_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} N$ in the above.

A finitely generated cone σ in $N_{\mathbf{R}}$ is said to be *sharp* if $\sigma \cap (-\sigma) = \{0\}$.

For a finitely generated cone σ in $N_{\mathbf{R}}$, a *face* of σ is a nonempty subset τ of σ satisfying the following conditions (3) and (4).

(3) If $x, y \in \tau$ and $a, b \in \mathbf{R}_{\geq 0}$, then $ax + by \in \tau$.

(4) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

A face of a finitely generated cone σ in $N_{\mathbf{R}}$ is a finitely generated cone in $N_{\mathbf{R}}$. It is rational if σ is rational.

3.1.9.

Let N be as in Section 3.1.8, and let Σ be a rational fan in $N_{\mathbf{R}}$. Recalling the definition of the (complex analytic) toric variety toric(Σ) corresponding to Σ (see [O, Section 1.2]; see also [KU3, Section 3.3]), we define a subset $|\text{toric}|(\Sigma)$ of toric(Σ) and a structure of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ on $|\text{toric}|(\Sigma)$.

Let $M = \text{Hom}(N, \mathbf{Z})$, and denote the group law of M multiplicatively.

For $\sigma \in \Sigma$, let

$$\mathcal{S}(\sigma) = \{\chi \in M \mid \chi : N_{\mathbf{R}} \to \mathbf{R} \text{ sends } \sigma \text{ to } \mathbf{R}_{\geq 0}\}.$$

Then

$$\sigma = \{x \in N_{\mathbf{R}} \mid \chi : N_{\mathbf{R}} \to \mathbf{R} \text{ sends } x \text{ into } \mathbf{R}_{\geq 0} \text{ for any } \chi \in \mathcal{S}(\sigma)\}.$$

We have $\mathcal{S}(\sigma)^{\mathrm{gp}} = M$, where $\mathcal{S}(\sigma)^{\mathrm{gp}} = \{ab^{-1} \mid a, b \in \mathcal{S}(\sigma)\}$.

For $\sigma \in \Sigma$, let $\mathrm{toric}(\sigma) = \mathrm{Spec}(\mathbf{C}[\mathcal{S}(\sigma)])_{\mathrm{an}} = \mathrm{Hom}(\mathcal{S}(\sigma), \mathbf{C}^{\mathrm{mult}})$, where $\mathbf{C}^{\mathrm{mult}}$ denotes $\mathbf{C}$ regarded as a multiplicative monoid. Then we have an open covering

$$\mathrm{toric}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathrm{toric}(\sigma).$$

Let

$$|\mathrm{toric}|(\Sigma) = \bigcup_{\sigma \in \Sigma} |\mathrm{toric}|(\sigma) \subset \mathrm{toric}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathrm{toric}(\sigma)$$
$$\text{with } |\mathrm{toric}|(\sigma) := \mathrm{Hom}\big(\mathcal{S}(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0}\big).$$

Then $|\mathrm{toric}|(\Sigma)$ has the unique structure of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ whose restriction to each open subsets $|\mathrm{toric}|(\sigma)$ coincides with the one given in Section 3.1.6.

Note that $|\mathrm{toric}|(\Sigma) \supset \mathrm{Hom}(M, \mathbf{R}_{>0}) = N \otimes \mathbf{R}_{>0}$, which is the restriction of $\mathrm{toric}(\Sigma) \supset \mathrm{Hom}(M, \mathbf{C}^\times) = N \otimes \mathbf{C}^\times$. As a subset of $\mathrm{toric}(\Sigma)$, $|\mathrm{toric}|(\Sigma)$ coincides with the closure of $N \otimes \mathbf{R}_{>0}$ in $\mathrm{toric}(\Sigma)$.

There is a canonical bijection between $\mathrm{toric}(\Sigma)$ (resp., $|\mathrm{toric}|(\Sigma)$) and the set of all pairs (σ, h), where $\sigma \in \Sigma$ and h is a homomorphism $\mathcal{S}(\sigma)^\times \to \mathbf{C}^\times$ (resp., $\mathcal{S}(\sigma)^\times \to \mathbf{R}_{>0}$). Here $\mathcal{S}(\sigma)^\times$ denotes the group of invertible elements of $\mathcal{S}(\sigma)$. Indeed, for such a pair (σ, h), the corresponding element of $\mathrm{toric}(\sigma) = \mathrm{Hom}(\mathcal{S}(\sigma), \mathbf{C}^{\mathrm{mult}})$ (resp., $|\mathrm{toric}|(\sigma) = \mathrm{Hom}(\mathcal{S}(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0})$) is defined to be the homomorphism sending $x \in \mathcal{S}(\sigma)$ to $h(x)$ if $x \in \mathcal{S}(\sigma)^\times$ and to zero if $x \notin \mathcal{S}(\sigma)^\times$.

3.1.10.
Let Σ and Σ' be rational fans in $N_{\mathbf{R}}$, and assume that the following condition (1) is satisfied.

(1) For each $\tau \in \Sigma'$, there is $\sigma \in \Sigma$ such that $\tau \subset \sigma$.

Then we have a morphism $\mathrm{toric}(\Sigma') \to \mathrm{toric}(\Sigma)$ of complex analytic spaces (resp., a morphism $|\mathrm{toric}|(\Sigma') \to |\mathrm{toric}|(\Sigma)$ in $\mathcal{B}_{\mathbf{R}}(\log)$) which induces the morphisms $\mathrm{toric}(\tau) \to \mathrm{toric}(\sigma)$ (resp., $|\mathrm{toric}|(\tau) \to |\mathrm{toric}|(\sigma)$) ($\tau \in \Sigma'$ $\sigma \in \Sigma$, $\tau \subset \sigma$) induced by the inclusion maps $\tau \subset \sigma$.

Under condition (1), let $\Sigma' \to \Sigma$ be the map which sends $\tau \in \Sigma'$ to the smallest $\sigma \in \Sigma$ with $\tau \subset \sigma$. Then the map $\mathrm{toric}(\Sigma') \to \mathrm{toric}(\Sigma)$ (resp., $|\mathrm{toric}|(\Sigma') \to |\mathrm{toric}|(\Sigma)$) sends the point of $\mathrm{toric}(\Sigma')$ (resp., $|\mathrm{toric}|(\Sigma')$) corresponding to the pair (τ, h') ($\tau \in \Sigma'$, h' is a homomorphism $\mathcal{S}(\tau)^\times \to \mathbf{C}^\times$ (resp., $\mathcal{S}(\tau)^\times \to \mathbf{R}_{>0}$) to the point of $\mathrm{toric}(\Sigma)$ (resp., $|\mathrm{toric}|(\Sigma)$) corresponding to the pair (σ, h), where σ is the image of τ under the map $\Sigma' \to \Sigma$, and h is the composite of $\mathcal{S}(\sigma)^\times \to \mathcal{S}(\tau)^\times$ with h'.

3.1.11.
Let Σ be a finite rational fan in $N_{\mathbf{R}}$.

A *finite rational subdivision* of Σ is a finite rational fan Σ' in $N_{\mathbf{R}}$ satisfying condition 3.1.10(1) and also the following condition (1):

(1) $$\bigcup_{\tau\in\Sigma'}\tau=\bigcup_{\sigma\in\Sigma}\sigma.$$

For a finite rational subdivision Σ' of Σ, the maps $\mathrm{toric}(\Sigma')\to\mathrm{toric}(\Sigma)$ and $|\mathrm{toric}|(\Sigma')\to|\mathrm{toric}|(\Sigma)$ are proper.

PROPOSITION 3.1.12

Let S be an object of $\mathcal{B}_{\mathbf{R}}(\log)$ (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$). Let $\mathcal{S}$ be an fs monoid, and let $\mathcal{S}\to M_S/\mathcal{O}_S^\times$ be a homomorphism which lifts locally on S to a chart $\mathcal{S}\to M_{S,>0}$ of fs log structure with sign (see Definition 3.1.5). Let Σ be a finite rational subdivision of the cone $\mathrm{Hom}(\mathcal{S},\mathbf{R}^{\mathrm{add}}_{\geq 0})$. Then we have an object $S(\Sigma)$ of $\mathcal{B}_{\mathbf{R}}(\log)$ (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$) having the following universal property.

(1) *If T is an object of $\mathcal{C}_{\mathbf{R}}(\log)$ over S, then there is at most one morphism $T\to S(\Sigma)$ over S. We have a criterion for the existence of such a morphism: such a morphism exists if and only if, for any $t\in T$ and for any homomorphism $h:(M_T/\mathcal{O}_T^\times)_t\to\mathbf{N}$, there exists $\sigma\in\Sigma$ such that the composite $\mathcal{S}\to(M_S/\mathcal{O}_S^\times)_s\to(M_T/\mathcal{O}_T^\times)_t\to\mathbf{N}$ (s is the image of t in S) belongs to σ.*

The map $S(\Sigma)\to S$ is proper and surjective.

Proof
This $S(\Sigma)$ is obtained as follows. By taking $N=\mathrm{Hom}(\mathcal{S}^{\mathrm{gp}},\mathbf{Z})$ and $M=\mathcal{S}^{\mathrm{gp}}$, define $|\mathrm{toric}|(\Sigma)$ as in Section 3.1.9. Locally on S, take a lift $\mathcal{S}\to M_{S,>0}$ of $\mathcal{S}\to M_S/\mathcal{O}_S^\times$, and consider the corresponding morphism $S\to\mathrm{Hom}(\mathcal{S},\mathbf{R}^{\mathrm{mult}}_{\geq 0})$ (see Section 3.1.6). Then $S(\Sigma)$ is obtained as the fiber product (see Proposition 3.1.7) of $S\to\mathrm{Hom}(\mathcal{S},\mathbf{R}^{\mathrm{mult}}_{\geq 0})\leftarrow|\mathrm{toric}|(\Sigma)$. The universal property is proved similarly to the complex analytic case (see [KU3, Proposition 3.6.1, Section 3.6.11]). □

The object $S(\Sigma)$ is called the *log modification* of S associated to the subdivision Σ of the cone $\mathrm{Hom}(\mathcal{S},\mathbf{R}^{\mathrm{add}}_{\geq 0})$. It is the real analytic version of the complex analytic log modification in the category $\mathcal{B}(\log)$ in [KU3, Definition 3.6.12].

3.1.13.
We use the notation in Proposition 3.1.12. As a set, the log modification $S(\Sigma)$ is identified with the set of all triples (s,σ,h), where $s\in S$, $\sigma\in\Sigma$, and if $P(\sigma)$ denotes the image of $\mathcal{S}(\sigma)$ (see Section 3.1.9 for $N=\mathrm{Hom}(\mathcal{S}^{\mathrm{gp}},\mathbf{Z})$ and $M=\mathcal{S}^{\mathrm{gp}}$) in $(M_S/\mathcal{O}_S^\times)_s^{\mathrm{gp}}$ and $P'(\sigma)$ denotes the inverse image of $P(\sigma)$ in $M^{\mathrm{gp}}_{S,>0,s}$, then h is a homomorphism $P'(\sigma)^\times\to\mathbf{R}_{>0}$, satisfying the following conditions (1) and (2).

(1) We have $P(\sigma)^\times\cap(M_S/\mathcal{O}_S^\times)_s=\{1\}$.

(2) The restriction of h to $\mathcal{O}^{\times}_{S,>0,s}$ $(\subset P'(\sigma)^{\times})$ is the evaluation map at s.

This is the real analytic version of the complex analytic theory (see [KU3, Lemma 3.6.15]).

3.2. Real analytic structures of $D_{\mathrm{SL}(2)}$

3.2.1.
We define two structures on the set $D_{\mathrm{SL}(2)}$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$. We denote $D_{\mathrm{SL}(2)}$ with these structures by $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$. There is a morphism $D^{I}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ whose underlying map is the identity map of $D_{\mathrm{SL}(2)}$. The log structure with sign of $D^{I}_{\mathrm{SL}(2)}$ coincides with the inverse image (see Definition 3.1.5) of that of $D^{II}_{\mathrm{SL}(2)}$.

In the pure case, these two structures coincide, and the topology of $D_{\mathrm{SL}(2)}$ given by these structures coincides with the one defined in [KU2].

$D^{II}_{\mathrm{SL}(2)}$ is proper over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ (see Theorem 3.5.16). This shows that our definition of $D_{\mathrm{SL}(2)}$ in the mixed case provides sufficiently many points at infinity. This properness is a good property of $D^{II}_{\mathrm{SL}(2)}$ which $D^{I}_{\mathrm{SL}(2)}$ need not have. On the other hand, $D^{I}_{\mathrm{SL}(2)}$ is nice for norm estimates (see Proposition 4.2.2), but $D^{II}_{\mathrm{SL}(2)}$ need not be.

The sheaf of rings on $D^{I}_{\mathrm{SL}(2)}$ is called the *sheaf of real analytic functions* (or the *real analytic structure*) *on* $D_{\mathrm{SL}(2)}$ *in the first sense*, and that on $D^{II}_{\mathrm{SL}(2)}$ is called the *sheaf of real analytic functions* (or the *real analytic structure*) *on* $D_{\mathrm{SL}(2)}$ *in the second sense*. The topology of $D^{I}_{\mathrm{SL}(2)}$ is called the *stronger topology of* $D_{\mathrm{SL}(2)}$, and that of $D^{II}_{\mathrm{SL}(2)}$ is called the *weaker topology of* $D_{\mathrm{SL}(2)}$. These two topologies often differ.

In Section 3.2, we characterize the structures of $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ as objects of $\mathcal{B}_{\mathbf{R}}(\log)$ by certain nice properties of them (see Theorem 3.2.10). The existences of such structures are proved in Sections 3.3 and 3.4.

3.2.2.
We define sets $\mathcal{W}$, $\overline{\mathcal{W}}$, a subset $D^{I}_{\mathrm{SL}(2)}(\Psi)$ of $D_{\mathrm{SL}(2)}$ for $\Psi \in \mathcal{W}$, and a subset $D^{II}_{\mathrm{SL}(2)}(\Phi)$ of $D_{\mathrm{SL}(2)}$ for $\Phi \in \overline{\mathcal{W}}$, as follows.

For $p \in D_{\mathrm{SL}(2)}$, let $\mathcal{W}(p)$ be the set of weight filtrations associated to p.

By an *admissible set of weight filtrations on* $H_{0,\mathbf{R}}$ we mean a finite set Ψ of increasing filtrations on $H_{0,\mathbf{R}}$ such that $\Psi = \mathcal{W}(p)$ for some element p of $D_{\mathrm{SL}(2)}$. We denote by $\mathcal{W}$ the set of all admissible sets of weight filtrations on $H_{0,\mathbf{R}}$.

For $\Psi \in \mathcal{W}$, we define a subset $D^{I}_{\mathrm{SL}(2)}(\Psi)$ of $D_{\mathrm{SL}(2)}$ by

$$D^{I}_{\mathrm{SL}(2)}(\Psi) = \{p \in D_{\mathrm{SL}(2)} \mid \mathcal{W}(p) \subset \Psi\}.$$

Note that $D_{\mathrm{SL}(2)}$ is covered by the subsets $D^{I}_{\mathrm{SL}(2)}(\Psi)$ for $\Psi \in \mathcal{W}$. Furthermore, $D_{\mathrm{SL}(2)}$ is covered by the subsets $D^{I}_{\mathrm{SL}(2)}(\Psi)$ for $\Psi \in \mathcal{W}$ with $W \notin \Psi$ and the subsets $D^{I}_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}} := D^{I}_{\mathrm{SL}(2)}(\Psi) \cap D_{\mathrm{SL}(2),\mathrm{nspl}}$ for $\Psi \in \mathcal{W}$ with $W \in \Psi$. As is stated in Theorem 3.2.10, these are open coverings of $D_{\mathrm{SL}(2)}$ for the topology of $D^I_{\mathrm{SL}(2)}$.

For $p \in D_{\mathrm{SL}(2)}$, let

$$\overline{\mathcal{W}}(p) = \{W'(\mathrm{gr}^W) \mid W' \in \mathcal{W}(p), W' \neq W\},$$

where $W'(\mathrm{gr}^W)$ is the filtration on $\mathrm{gr}^W = \bigoplus_w \mathrm{gr}^W_w$ induced by W'; that is, $W'(\mathrm{gr}^W)_k := \bigoplus_w W'_k(\mathrm{gr}^W_w) \subset \bigoplus_w \mathrm{gr}^W_w$.

By an *admissible set of weight filtrations on* gr^W we mean a finite set Φ of increasing filtrations on gr^W such that $\Phi = \overline{\mathcal{W}}(p)$ for some element p of $D_{\mathrm{SL}(2)}$. We denote by $\overline{\mathcal{W}}$ the set of all admissible sets of weight filtrations on gr^W.

For $\Phi \in \overline{\mathcal{W}}$, we define a subset $D^{II}_{\mathrm{SL}(2)}(\Phi)$ of $D_{\mathrm{SL}(2)}$ by

$$D^{II}_{\mathrm{SL}(2)}(\Phi) = \{p \in D_{\mathrm{SL}(2)} \mid \overline{\mathcal{W}}(p) \subset \Phi\}.$$

As a set, $D_{\mathrm{SL}(2)}$ is covered by $D^{II}_{\mathrm{SL}(2)}(\Phi)$ $(\Phi \in \overline{\mathcal{W}})$. As is stated in Theorem 3.2.10, this is an open covering for the topology of $D^{II}_{\mathrm{SL}(2)}$.

We have a canonical map

$$\mathcal{W} \to \overline{\mathcal{W}}$$

which sends $\Psi \in \mathcal{W}$ to $\bar{\Psi} := \{W'(\mathrm{gr}^W) \mid W' \in \Psi, W' \neq W\} \in \overline{\mathcal{W}}$. For $\Psi \in \mathcal{W}$, we have $D^I_{\mathrm{SL}(2)}(\Psi) \subset D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$.

3.2.3.

Let $\Psi \in \mathcal{W}$. A homomorphism $\alpha : \mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ of algebraic groups over $\mathbf{R}$ is called a *splitting of* Ψ if it satisfies the following conditions (1) and (2).

(1) The corresponding direct sum decomposition

$$H_{0,\mathbf{R}} = \bigoplus_{\mu \in X} S_\mu \quad (X := \mathbf{Z}^{\Psi})$$

into eigen $\mathbf{R}$-subspaces S_μ satisfies

$$W'_{w'} = \sum_{\mu \in X, \mu(W') \leq w'} S_\mu$$

for all $W' \in \Psi$ and for all $w' \in \mathbf{Z}$.

(2) For all $w \in \mathbf{Z}$ and all $t \in \mathbf{G}^{\Psi}_{m,\mathbf{R}}$, $\iota(t)^{-w}\alpha_w(t)$ is contained in $G_{\mathbf{R}}(\mathrm{gr}^W_w)$, where $\alpha_w : \mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ is the homomorphism induced by α, and ι is the composite of the multiplication $\mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathbf{G}_{m,\mathbf{R}}$ and the canonical map $\mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$, $a \mapsto$ (multiplication by a).

A splitting of Ψ exists: If Ψ is associated to $p \in D_{\mathrm{SL}(2)}$, the torus action τ_p associated to p (see Sections 2.5.6, 2.3.5) is a splitting of Ψ. Here and hereafter, we identify $\{1, \ldots, n\}$ (n is the rank of p) with Ψ via the bijection $j \mapsto W^{(j)}$, which is independent of the choice of p by Proposition 2.3.8.

Let $\Phi \in \overline{\mathcal{W}}$. A homomorphism $\alpha : \mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ of algebraic groups over $\mathbf{R}$ is called a *splitting of* Φ if it satisfies the following conditions $(\bar{1})$ and $(\bar{2})$.

$(\bar{1})$ The corresponding direct sum decomposition

$$\mathrm{gr}^W = \bigoplus_{\mu \in X} S_\mu \quad (X := \mathbf{Z}^\Phi)$$

into eigen $\mathbf{R}$-subspaces S_μ satisfies

$$W'_{w'} = \sum_{\mu \in X, \mu(W') \le w'} S_\mu$$

for all $W' \in \Phi$ and for all $w' \in \mathbf{Z}$.

$(\bar{2})$ For all $w \in \mathbf{Z}$ and all $t \in \mathbf{G}^\Phi_{m,\mathbf{R}}$, $\iota(t)^{-w}\alpha_w(t)$ is contained in $G_\mathbf{R}(\mathrm{gr}^W_w)$, where $\alpha_w : \mathbf{G}^\Phi_{m,\mathbf{R}} \to \mathrm{Aut}_\mathbf{R}(\mathrm{gr}^W_w)$ is the w-component of α, and ι is the composite of the multiplication $\mathbf{G}^\Phi_{m,\mathbf{R}} \to \mathbf{G}_{m,\mathbf{R}}$ and the canonical map $\mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_\mathbf{R}(\mathrm{gr}^W_w)$.

A splitting of Φ exists. For $p \in D_{\mathrm{SL}(2)}$, let $\bar{\tau}_p$ be $\mathrm{gr}^W(\tau_p)$ in the case $W \notin \mathcal{W}(p)$, and in the case $W \in \mathcal{W}(p)$, let $\bar{\tau}_p$ be the restriction of $\mathrm{gr}^W(\tau_p)$ to $\mathbf{G}^{\overline{\mathcal{W}}(p)}_{m,\mathbf{R}}$ which we identify with the part of $\mathbf{G}^{\mathcal{W}(p)}_{m,\mathbf{R}}$ with the W-component removed. Then if $\Phi = \overline{\mathcal{W}}(p)$, $\bar{\tau}_p$ is a splitting of Φ.

REMARK

Under condition $(\bar{1})$, condition $(\bar{2})$ is equivalent to the following condition: for all $w \in \mathbf{Z}$, the direct sum decomposition

$$\mathrm{gr}^W_w = \bigoplus_{\mu \in X} S_{w,\mu}$$

corresponding to α_w satisfies

$$\langle S_{w,\mu}, S_{w,\mu'} \rangle = 0$$

unless $\mu + \mu' = (2w, \ldots, 2w)$.

3.2.4.

Let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp., $W \in \Psi$). If a real analytic map $\beta : D \to \mathbf{R}^\Psi_{>0}$ (resp., $D_{\mathrm{nspl}} \to \mathbf{R}^\Psi_{>0}$) satisfies the following (1) for any splitting α of Ψ, then we call β a *distance to Ψ-boundary*:

$$\beta\bigl(\alpha(t)p\bigr) = t\beta(p) \quad (t \in \mathbf{R}^\Psi_{>0},\ p \in D\ (\text{resp.}, D_{\mathrm{nspl}})). \tag{1}$$

Let $\Phi \in \overline{\mathcal{W}}$. If a real analytic map $\beta : D(\mathrm{gr}^W) \to \mathbf{R}^\Phi_{>0}$ satisfies the following $(\bar{1})$ for any splitting α of Φ, then we call β a *distance to Φ-boundary*:

$$\beta\bigl(\alpha(t)p\bigr) = t\beta(p) \quad (t \in \mathbf{R}^\Phi_{>0},\ p \in D(\mathrm{gr}^W)). \tag{$\bar{1}$}$$

The proofs of Propositions 3.2.5–3.2.7 and 3.2.9 are given in Section 3.3.

PROPOSITION 3.2.5

(i) *Let $\Psi \in \mathcal{W}$. Then a distance to Ψ-boundary exists.*

(ii) *Let $\Phi \in \overline{\mathcal{W}}$. Then a distance to Φ-boundary exists.*

PROPOSITION 3.2.6

(i) *Let $\Psi \in \mathcal{W}$, let α be a splitting of Ψ, and let β be a distance to Ψ-boundary. Assume $W \notin \Psi$ (resp., $W \in \Psi$), and consider the map*

$$\nu_{\alpha,\beta} : D \ (\textit{resp.},\ D_{\mathrm{nspl}}) \to \mathbf{R}^{\Psi}_{>0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}\big(W'(\mathrm{gr}^W)\big),$$

$$p \mapsto \big(\beta(p), \alpha\beta(p)^{-1}p, \mathrm{spl}_W(p), (\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W)))_{W' \in \Psi}\big).$$

Here $\mathrm{spl}_W(p)$ is the canonical splitting of W associated to p in Section 1.2, and $\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W))$ is the Borel-Serre splitting of $W'(\mathrm{gr}^W)$ associated to $p(\mathrm{gr}^W)$ in Section 2.1.9. Let $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$), let J be the set of weight filtrations associated to p (see Section 2.3.6), let $\tau_p : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the associated torus action (see Sections 2.5.6, 2.3.5), and let $\mathbf{r} \in D$ be a point on the torus orbit (see Proposition 2.5.2) associated to p. Then, when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$, $\nu_{\alpha,\beta}(\tau_p(t)\mathbf{r})$ converges in $\mathbf{R}^{\Psi}_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}(W'(\mathrm{gr}^W))$. This limit depends only on p and is independent of the choice of $\mathbf{r}$.

(ii) *Let $\Phi \in \overline{\mathcal{W}}$, let α be a splitting of Φ, and let β be a distance to Φ-boundary. Consider the map*

$$\nu_{\alpha,\beta} : D \to \mathbf{R}^{\Phi}_{>0} \times D(\mathrm{gr}^W) \times \mathcal{L} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W'),$$

$$p \mapsto \big(\beta(p(\mathrm{gr}^W)), \alpha\beta(p(\mathrm{gr}^W))^{-1}p(\mathrm{gr}^W), \mathrm{Ad}(\alpha\beta(p(\mathrm{gr}^W)))^{-1}\delta(p),$$
$$\mathrm{spl}_W(p), (\mathrm{spl}^{\mathrm{BS}}_{W'}(p(\mathrm{gr}^W)))_{W' \in \Phi}\big).$$

Here $\mathcal{L}$ is in Section 1.2.1 and $\delta(p)$ denotes δ of p. Let $p \in D^{II}_{\mathrm{SL}(2)}(\Phi)$, let J be the set of weight filtrations associated to p, let $\tau_p : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the associated torus action, and let $\mathbf{r} \in D$ be a point on the torus orbit associated to p. Then, when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$, $\nu_{\alpha,\beta}(\tau_p(t)\mathbf{r})$ converges in $\mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W')$. This limit depends only on p and is independent of the choice of $\mathbf{r}$.

We recall the compactified vector space $\bar{V}$ associated to a weightened finite-dimensional $\mathbf{R}$-vector space $V = \bigoplus_{w \in \mathbf{Z}} V_w$ such that $V_w = 0$ unless $w \leq -1$. It is a compact real analytic manifold with boundary. For $t \in \mathbf{R}_{>0}$ and $v = \sum_{w \in \mathbf{Z}} v_w \neq 0$ ($v_w \in V_w$), let $t \circ v = \sum_w t^w v_w$. Then as a set, $\bar{V}$ is the disjoint union of V and the points $0 \circ v$ ($v \in V \smallsetminus \{0\}$), where $0 \circ v$ is the limit point in $\bar{V}$ of $t \circ v$ with $t \in \mathbf{R}_{>0}$, $t \to 0$. We have $0 \circ v = 0 \circ v'$ if and only if $v' = t \circ v$ for some $t \in \mathbf{R}_{>0}$.

Since $\bar{V}$ is a real analytic manifold with boundary (a special case of a real analytic manifold with corners), $\bar{V}$ is regarded as an object of $\mathcal{B}(\log)$ (see Section 3.1.6).

Since $\mathcal{L}$ is a finite-dimensional weightened $\mathbf{R}$-vector space of weights ≤ -2, we have the associated compactified vector space $\bar{\mathcal{L}} \supset \mathcal{L}$.

In Proposition 3.2.6, in both (i) and (ii), we denote the limit of $\nu_{\alpha,\beta}(\tau_p(t)\mathbf{r})$ by $\nu_{\alpha,\beta}(p)$.

As we see in Section 3.3.10, in Proposition 3.2.6(ii), the $\bar{\mathcal{L}}$-component of $\nu_{\alpha,\beta}(p)$ belongs to $\mathcal{L}$ (resp., $\bar{\mathcal{L}}\smallsetminus\mathcal{L}$) if and only if $W \notin \mathcal{W}(p)$ (resp., $W \in \mathcal{W}(p)$).

PROPOSITION 3.2.7

(i) *Let $\Psi \in \mathcal{W}$, let α be a splitting of Ψ, and let β be a distance to Ψ-boundary. Then, in the case $W \notin \Psi$ (resp., $W \in \Psi$), the map*

$$\begin{aligned}\nu_{\alpha,\beta} : D^I_{\mathrm{SL}(2)}(\Psi) \ \big(\textit{resp.}, D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}\big) \\ \to \mathbf{R}^{\Psi}_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}\big(W'(\mathrm{gr}^W)\big)\end{aligned}$$

is injective.

(ii) *Let $\Phi \in \overline{\mathcal{W}}$, let α be a splitting of Φ, and let β be a distance to Φ-boundary. Then the map*

$$\nu_{\alpha,\beta} : D^{II}_{\mathrm{SL}(2)}(\Phi) \to \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W')$$

is injective.

3.2.8.

Here, for $\Psi \in \mathcal{W}$, we define a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ on the set $D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) in the case $W \notin \Psi$ (resp., $W \in \Psi$), depending on choices of a splitting α of Ψ and a distance to Ψ-boundary β. Also, for $\Phi \in \overline{\mathcal{W}}$, we define a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ on the set $D^{II}_{\mathrm{SL}(2)}(\Phi)$ depending on choices of a splitting α of Φ and a distance to Φ-boundary β.

Let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp., $W \in \Psi$). Let $A = D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $A = D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$), let $B = \mathbf{R}^{\Psi}_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}(W'(\mathrm{gr}^W))$, and regard B as an object of $\mathcal{B}_{\mathbf{R}}(\log)$. Define the topology of A to be the one as a subspace of B in which A is embedded by $\nu_{\alpha,\beta}$ in Proposition 3.2.7(i). We define the sheaf of real analytic functions on A as follows. For an open set U of A and a function $f : U \to \mathbf{R}$, we say that f is a real analytic function if and only if, for each $p \in U$, there are an open neighborhood U' of p in U, an open neighborhood U'' of U' in B, and a real analytic function $g : U'' \to \mathbf{R}$ such that the restrictions to U' of f and g coincide. Then A belongs to $\mathcal{B}'_{\mathbf{R}}$. Define the log structure with sign on A as the inverse image (see Definition 3.1.5) of the log structure with sign of B.

Let $\Phi \in \overline{\mathcal{W}}$. Let $A = D^{II}_{\mathrm{SL}(2)}(\Phi)$, let $B = \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W')$, and regard B as an object of $\mathcal{B}_{\mathbf{R}}(\log)$. Define the topology of A to be the one as a subspace of B in which A is embedded by $\nu_{\alpha,\beta}$ in Proposition 3.2.7(ii). We define the sheaf of real analytic functions on A as follows. For an open set U of A and a function $f : U \to \mathbf{R}$, we say that f is a real analytic function if and only if, for each $p \in U$, there are an open neighborhood U' of p in U, an open neighborhood U'' of U' in B, and a real analytic function $g : U'' \to \mathbf{R}$ such that the restrictions to U' of f and g coincide. Then A belongs to $\mathcal{B}'_{\mathbf{R}}$. Define the log structure with sign on A as the inverse image of the log structure with sign of B.

PROPOSITION 3.2.9

(i) *Let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp., $W \in \Psi$). Then the structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ on $D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) in Section 3.2.8 is independent of the choices of α and β.*

(ii) *Let $\Phi \in \overline{\mathcal{W}}$. Then the structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ on $D^{II}_{\mathrm{SL}(2)}(\Phi)$ in Section 3.2.8 is independent of the choices of α and β.*

The following theorem is proved in Section 3.4.

THEOREM 3.2.10

(i) *There exists a unique structure $D^I_{\mathrm{SL}(2)}$ of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ on the set $D_{\mathrm{SL}(2)}$ having the following property: For any $\Psi \in \mathcal{W}$, $D^I_{\mathrm{SL}(2)}(\Psi)$ and $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$ are open in $D^I_{\mathrm{SL}(2)}$, and if $W \notin \Psi$ (resp., $W \in \Psi$), the induced structure on $D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) coincides with the structure in Proposition 3.2.9 as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$.*

(ii) *There exists a unique structure $D^{II}_{\mathrm{SL}(2)}$ of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ on the set $D_{\mathrm{SL}(2)}$ having the following property: for any $\Phi \in \overline{\mathcal{W}}$, $D^{II}_{\mathrm{SL}(2)}(\Phi)$ is open in $D^{II}_{\mathrm{SL}(2)}$, and the induced structure on $D_{\mathrm{SL}(2)}(\Phi)$ coincides with the structure in Proposition 3.2.9 as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$.*

(iii) *The topology of $D^{II}_{\mathrm{SL}(2)}$ is coarser than or equal to that of $D^I_{\mathrm{SL}(2)}$, and the sheaf of real analytic functions on $D^{II}_{\mathrm{SL}(2)}$ is contained in the sheaf of real analytic functions on $D^I_{\mathrm{SL}(2)}$. Thus we have a morphism $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ of local ringed spaces over $\mathbf{R}$. The log structure with sign on $D^I_{\mathrm{SL}(2)}$ coincides with the inverse image of that of $D^{II}_{\mathrm{SL}(2)}$. Thus we have a morphism $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ in $\mathcal{B}_{\mathbf{R}}(\log)$ whose underlying map of sets is the identity map of $D_{\mathrm{SL}(2)}$. In the pure case (i.e., in the case where $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$ for some $w \in \mathbf{Z}$), the last morphism is an isomorphism, and the topology of $D_{\mathrm{SL}(2)}$ given by these structures coincides with the one defined in [KU2].*

3.2.11.

In Proposition 3.2.12 below, we give characterizations of the topologies of $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$. Recall (see [Bn, chapitre 1, section 8, no. 4]) that a topological space X is said to be *regular* if it is Hausdorff and if for any point x of X and any neighborhood U of x there is a closed neighborhood of x contained in U.

Recall (see [Bn, chapitre 1, section 8, no. 5]) that the topology of a regular topological space X is determined by the restrictions of neighborhoods of each point to a dense subset X' of X. Precisely speaking, if T_1 and T_2 are topologies on a set X and if X' is a subset of X, then T_1 and T_2 coincide if the following conditions (1) and (2) are satisfied.

(1) The space X is regular for T_1 and also for T_2, and the subset X' is dense in X for T_1 and also for T_2.

(2) Let $x \in X$, and for $j = 1, 2$, let S_j be the set $\{X' \cap U \mid U$ is a neighborhood of x in X for $T_j\}$ of subsets of X'. Then $S_1 = S_2$.

This condition (2) is equivalent to the following condition (2′).

(2′) For any $x \in X$ and for any directed family $(x_\lambda)_\lambda$ of elements of X', $(x_\lambda)_\lambda$ converges to x for T_1 if and only if it converges to x for T_2.

The topologies of $D^I_{\mathrm{SL}(2)}$ and that of $D^{II}_{\mathrm{SL}(2)}$ have the following characterizations.

PROPOSITION 3.2.12

(i) *The topology of* $D^I_{\mathrm{SL}(2)}$ *is the unique topology which satisfies the following conditions (1) and (2).*

(1) *For any admissible set* Ψ *of weight filtrations on* $H_{0,\mathbf{R}}$, $D^I_{\mathrm{SL}(2)}(\Psi)$ *(see Section 3.2.2) is open and regular, and* D *is dense in it.*

(2) *For any* $p \in D_{\mathrm{SL}(2)}$ *and for any family* $(p_\lambda)_{\lambda\in\Lambda}$ *of points of* D *with a directed ordered set* Λ, (p_λ) *converges to* p *in* $D^I_{\mathrm{SL}(2)}$ *if and only if the following (a), (b), and (c.I) are satisfied. Let* n *be the rank of* p *(see Sections 2.5.1, 2.3.2–2.3.3), let* $((\rho_w, \varphi_w)_w, \mathbf{r})$ *be an* $\mathrm{SL}(2)$*-orbit in* n *variables which represents* p*, let* $\Psi = \mathcal{W}(p)$*, and let* $\tau : \mathbf{G}_m^\Psi \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ *be the homomorphism of algebraic groups associated to* p *(see Section 2.3.5).*

(a) *The canonical splitting of* W *associated to* p_λ *converges to the canonical splitting of* W *associated to* $\mathbf{r}$.

(b) *For each* $1 \le j \le n$ *and* $w \in \mathbf{Z}$*, the Borel-Serre splitting* $\mathrm{spl}^{\mathrm{BS}}_{W^{(j)}(\mathrm{gr}^W_w)}(p_\lambda(\mathrm{gr}^W_w))$ *of* $W^{(j)}(\mathrm{gr}^W_w)$ *at* $p_\lambda(\mathrm{gr}^W_w)$ *(see Section 2.1.9) converges to the Borel-Serre splitting of* $W^{(j)}(\mathrm{gr}^W_w)$ *at* $\mathbf{r}(\mathrm{gr}^W_w)$.

(c.I) *There is a family* $(t_\lambda)_{\lambda\in\Lambda}$ *of elements of* $\mathbf{R}^n_{>0}$ *such that* $t_\lambda \to \mathbf{0}$ *in* $\mathbf{R}^n_{\ge 0}$ *and such that* $\tau(t_\lambda)^{-1}p_\lambda \to \mathbf{r}$.

(ii) *The topology of* $D^{II}_{\mathrm{SL}(2)}$ *is the unique topology which satisfies the following conditions (1) and (2).*

(1) *For any admissible set* Φ *of weight filtrations on* gr^W, $D^{II}_{\mathrm{SL}(2)}(\Phi)$ *(see Section 3.2.2) is open and regular, and* D *is dense in it.*

(2) *For any* $p \in D_{\mathrm{SL}(2)}$ *and for any family* $(p_\lambda)_{\lambda\in\Lambda}$ *of points of* D *with a directed ordered set* Λ, (p_λ) *converges to* p *in* $D^{II}_{\mathrm{SL}(2)}$ *if and only if (a) and (b) in (i) and the following (c.II) are satisfied. Let* n, $((\rho_w, \varphi_w)_w, \mathbf{r})$, Ψ*, and* τ *be as in (2) of (i). Let* $\Phi = \overline{\mathcal{W}}(p) = \bar{\Psi}$.

(c.II) *There is a family* $(t_\lambda)_{\lambda\in\Lambda}$ *of elements of* $\mathbf{R}^\Phi_{>0} \subset \mathbf{R}^\Psi_{>0}$ *such that* $t_\lambda \to \mathbf{0}$ *in* $\mathbf{R}^\Phi_{\ge 0} \subset \mathbf{R}^\Psi_{\ge 0}$ *and such that* $(\tau(t_\lambda)^{-1}p_\lambda)(\mathrm{gr}^W) \to \mathbf{r}(\mathrm{gr}^W)$ *and* $\delta(\tau(t_\lambda)^{-1}p_\lambda) \to \delta(\mathbf{r})$.

The proof of Proposition 3.2.12 is given in Section 3.4.

3.2.13.

EXAMPLE 0

Consider the pure case Example 0 in Section 1.1.1. Let $\Psi = \{W'\}$, where $W'_{-3} = 0 \subset W'_{-2} = W'_{-1} = \mathbf{R}e_1 \subset W'_0 = H_{0,\mathbf{R}}$. Then we have a splitting α of Ψ defined by $\alpha(t)e_1 = t^{-2}e_1$, $\alpha(t)e_2 = e_2$, and we have a distance β to Ψ-boundary defined by $\beta(x+iy) = y^{-1/2}$ $(x+iy \in \mathfrak{h} = D, x, y \in \mathbf{R}, y > 0)$. Then the map

$$\nu_{\alpha,\beta} : D \to \mathbf{R}_{>0} \times D \times \mathrm{spl}(W'), \qquad p \mapsto \big(\beta(p), \alpha\beta(p)^{-1}p, \mathrm{spl}^{\mathrm{BS}}_{W'}(p)\big)$$

is described as

$$x + iy \mapsto \Big(y^{-1/2}, \frac{x}{y} + i, x\Big) \quad (x, y \in \mathbf{R}, y > 0),$$

where we identify $\mathrm{spl}(W')$ with $\mathbf{R}$ in the standard way. We can identify $D^I_{\mathrm{SL}(2)}(\Psi)$ with $\{x + iy \mid x, y \in \mathbf{R}, 0 < y \le \infty\}$ (see Section 3.6.1). The extended map $\nu_{\alpha,\beta} : D^I_{\mathrm{SL}(2)}(\Psi) \to \mathbf{R}_{\ge 0} \times D \times \mathrm{spl}(W')$ sends $x + i\infty$ to $(0, i, x)$.

3.3. Proofs of Propositions 3.2.5–3.2.7 and 3.2.9

3.3.1.

Let $\overline{\mathcal{W}}$ be as in Section 3.2.2. For each $w \in \mathbf{Z}$, let $\mathcal{W}(\mathrm{gr}^W_w)$ be the set of all admissible sets of weight filtrations on gr^W_w. We have a canonical map

$$\overline{\mathcal{W}} \to \mathcal{W}(\mathrm{gr}^W_w), \qquad \Phi \mapsto \big\{W'(\mathrm{gr}^W_w) \bigm| W' \in \Phi, W'(\mathrm{gr}^W_w) \ne W(\mathrm{gr}^W_w)\big\}.$$

This map sends $\overline{\mathcal{W}}(p)$ for $p \in D_{\mathrm{SL}(2)}$ to $\mathcal{W}(p(\mathrm{gr}^W_w))$.

For $\Phi \in \overline{\mathcal{W}}$ and $w \in \mathbf{Z}$, let $\Phi(w) \in \mathcal{W}(\mathrm{gr}^W_w)$ be the image of Φ under the above map.

We sometimes denote elements of Φ and elements of $\Phi(w)$ by the small letters j, k, and so on.

Note that Φ is a totally ordered set by Proposition 2.3.8 (for $j, k \in \Phi$, $j \le k$ means $\sigma^2(j) \le \sigma^2(k)$), and $\{W(\mathrm{gr}^W_w)\} \cup \Phi(w)$ is also a totally ordered set by Proposition 2.1.13 with respect to σ^2. (Note that $W(\mathrm{gr}^W_w) \le j$ for any $j \in \Phi(w)$.) The canonical map $\Phi \to \{W(\mathrm{gr}^W_w)\} \cup \Phi(w), W' \mapsto W'(\mathrm{gr}^W_w)$, preserves the ordering.

LEMMA 3.3.2

We use the notation in Section 3.3.1.

(i) *For $\Phi \in \overline{\mathcal{W}}$ and $w \in \mathbf{Z}$, the map $\Phi \to \prod_{w \in \mathbf{Z}}(\{W(\mathrm{gr}^W_w)\} \cup \Phi(w))$, $W' \mapsto (W'(\mathrm{gr}^W_w))_{w \in \mathbf{Z}}$, is injective.*

By this injection, we identify Φ and its image and denote the latter also by Φ.

(ii) *We have the bijection from $\overline{\mathcal{W}}$ onto the set of pairs $(\Phi', (\Phi'(w))_{w \in \mathbf{Z}})$, where $\Phi'(w)$ is an element of $\mathcal{W}(\mathrm{gr}^W_w)$ for each $w \in \mathbf{Z}$ and Φ' is a subset of $\prod_{w \in \mathbf{Z}}(\{W(\mathrm{gr}^W_w)\} \cup \Phi'(w))$ satisfying conditions (1)–(3) below. The bijection sends $\Phi \in \overline{\mathcal{W}}$ to $(\Phi, (\Phi(w))_{w \in \mathbf{Z}})$.*

(1) *For each $w \in \mathbf{Z}$, the image of the projection $\Phi' \to \{W(\mathrm{gr}^W_w)\} \cup \Phi'(w)$, which we denote by $j \mapsto j(w)$, contains $\Phi'(w)$.*

(2) *For each* $j \in \Phi'$, *there is* $w \in \mathbf{Z}$ *such that* $j(w) \in \{W(\mathrm{gr}^W_w)\} \cup \Phi'(w)$ *belongs to* $\Phi'(w)$.

(3) *For any* $j, k \in \Phi'$, *one of the following (a), (b) holds.*

(a) $j(w) \le k(w)$ *for all* $w \in \mathbf{Z}$.

(b) $j(w) \ge k(w)$ *for all* $w \in \mathbf{Z}$.

Proof

The assertion (i) is clear.

We prove (ii). The injectivity of the map $\Phi \mapsto (\Phi, (\Phi(w))_w)$ follows from (i). We prove the surjectivity. Let $(\Phi', (\Phi'(w))_w)$ be a pair satisfying (1)–(3). For $w \in \mathbf{Z}$, let $n(w)$ be the cardinality of $\Phi'(w)$, let (ρ'_w, φ'_w) be an SL(2)-orbit on gr^W_w in $n(w)$ variables of rank $n(w)$ whose associated set of weight filtrations is $\Phi'(w)$, and let $\mathbf{r}(w) \in D(\mathrm{gr}^W_w)$ be a point on the torus orbit associated to (ρ'_w, φ'_w). Take a point $\mathbf{r}$ of $D_{\mathrm{SL}(2)}$ such that $\mathbf{r}(\mathrm{gr}^W) = (\mathbf{r}(w))_w$. Let n be the cardinality of Φ', write $\Phi' = \{\phi_1, \dots, \phi_n\}$ ($\phi_1(w) \le \cdots \le \phi_n(w)$ for all $w \in \mathbf{Z}$), write $\Phi'(w) = \{\phi_{w,1}, \dots, \phi_{w,n(w)}\}$ ($\phi_{w,1} < \cdots < \phi_{w,n(w)}$), and let $e_w : \{1, \dots, n(w)\} \to \{1, \dots, n\}$ be the injection defined by $e_w(k) = \min\{j \mid \phi_j(w) = \phi_{w,k}\}$. Let $p \in D_{\mathrm{SL}(2)}$ be the class of the SL(2)-orbit $((\rho_w, \varphi_w)_w, \mathbf{r})$ in n variables of rank n, where

$$\rho_w(g_1, \dots, g_n) = \rho'_w(g_{e_w(1)}, \dots, g_{e_w(n(w))}),$$

$$\varphi_w(z_1, \dots, z_n) = \varphi'_w(z_{e_w(1)}, \dots, z_{e_w(n(w))}).$$

Then the pair $(\Phi', (\Phi'(w))_w)$ is the image of $\mathcal{W}(p) \in \overline{\mathcal{W}}$. □

LEMMA 3.3.3

Let $\Phi \in \overline{\mathcal{W}}$, *and let* $(\Phi(w))_w$ *be the image of* Φ *in* $\prod_w \mathcal{W}(\mathrm{gr}^W_w)$. *Then there is a bijection between the set of all splittings of* Φ *and the set of all families* $(\alpha_w)_{w \in \mathbf{Z}}$, *where* α_w *is a splitting of* $\Phi(w)$ *for each* w. *This bijection sends a splitting* α *of* Φ *to the following family* $(\alpha_w)_w$. *For* $w \in \mathbf{Z}$, *let* $e_w : \Phi(w) \to \Phi$ *be the map defined by* $e_w(k) = \min\{j \in \Phi \mid j(\mathrm{gr}^W_w) = k\}$. *Then* α_w *is the composite* $\mathbf{G}^{\Phi(w)}_{m,\mathbf{R}} \to \mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_w)$, *where the first arrow is induced from* e_w *and the second arrow is given by* α.

Proof

From a family $(\alpha_w)_w$ of splittings α_w of $\Phi(w)$, the corresponding splitting α of Φ is recovered as follows. For $w \in \mathbf{Z}$, let $R(w) = \{W(\mathrm{gr}^W_w)\} \cup \Phi(w)$. Let $\mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathbf{G}^{R(w)}_{m,\mathbf{R}} = \mathbf{G}_{m,\mathbf{R}} \times \mathbf{G}^{\Phi(w)}_{m,\mathbf{R}}$ be the homomorphism induced by the map $\Phi \to R(w)$, $W' \mapsto W'(\mathrm{gr}^W_w)$. Then the action of $\mathbf{G}^{\Phi}_m$ on gr^W_w by α is defined to be the composite $\mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathbf{G}_{m,\mathbf{R}} \times \mathbf{G}^{\Phi(w)}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_w)$, where the last arrow is $(t, t') \mapsto t^w \alpha_w(t')$. □

LEMMA 3.3.4

Let $\Phi \in \overline{\mathcal{W}}$. *For each* $w \in \mathbf{Z}$, *let* $\beta_w : D(\mathrm{gr}^W_w) \to \mathbf{R}^{\Phi(w)}_{>0}$ *be a distance to* $\Phi(w)$-*boundary. Let* $h : \mathbf{Z}^{\Phi} \to \prod_{w \in \mathbf{Z}} \mathbf{Z}^{\Phi(w)}$ *be an injective homomorphism induced by the map* $\Phi \to \prod_{w \in \mathbf{Z}}(\{W(\mathrm{gr}^W_w)\} \cup \Phi(w))$. *Then there is a homomorphism*

$h' : \prod_{w\in\mathbf{Z}} \mathbf{Z}^{\Phi(w)} \to \mathbf{Z}^{\Phi}$ such that the composite $\mathbf{Z}^{\Phi} \xrightarrow{h} \prod_{w\in\mathbf{Z}} \mathbf{Z}^{\Phi(w)} \xrightarrow{h'} \mathbf{Z}^{\Phi}$ is the identity map, and, for such an h', the composite $D(\mathrm{gr}^W) \to \prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{\Phi(w)} \to \mathbf{R}_{>0}^{\Phi}$, where the first arrow is $(\beta_w)_w$ and the second arrow is induced by h', is a distance to Φ-boundary.

Proof
Since the cokernel of h is torsion free, there is such an h'. The rest follows from Lemma 3.3.3. □

LEMMA 3.3.5
Let $\Psi \in \mathcal{W}$, and let $\Phi \in \overline{\mathcal{W}}$ be the image of Ψ under the canonical map $\mathcal{W} \to \overline{\mathcal{W}}$ (see Section 3.2.2). Let $\beta : D(\mathrm{gr}^W) \to \mathbf{R}_{\geq 0}^{\Phi}$ be a distance to Φ-boundary.

(i) *Assume $W \notin \Psi$. Then the map*

$$D \to D(\mathrm{gr}^W) \xrightarrow{\beta} \mathbf{R}_{>0}^{\Phi} \simeq \mathbf{R}_{>0}^{\Psi}, \quad x \mapsto \beta\big(x(\mathrm{gr}^W)\big),$$

is a distance to Ψ-boundary, where the last isomorphism is induced from the canonical bijection $\Psi \to \Phi, W' \mapsto W'(\mathrm{gr}^W)$.

(ii) *Assume $W \in \Psi$. Let $\gamma : D_{\mathrm{nspl}} \to \mathbf{R}_{>0}$ be a real analytic map such that $\gamma(\alpha(t)x) = t_W\gamma(x)$ for any $t \in \mathbf{R}_{>0}^{\Psi}$ and $x \in D_{\mathrm{nspl}}$, where t_W denotes the W-component of t. Then the map*

$$D_{\mathrm{nspl}} \to \mathbf{R}_{>0} \times \mathbf{R}_{>0}^{\Phi} \simeq \mathbf{R}_{>0}^{\Psi}, \qquad x \mapsto \big(\gamma(x), \beta(x(\mathrm{gr}^W))\big)$$

is a distance to Ψ-boundary.

This is proved easily.

3.3.6.
We prove Proposition 3.2.5 (the existence of β)

Proof
Assume first that we are in the pure case. In this case, the existence of β is proved in [KU2, Proposition 4.12].

In fact, there is a mistake in [KU2], for [KU2, Proposition 4.12] does not hold for a general compatible family of $\mathbf{Q}$-rational increasing filtrations in the sense of [KU2]. The proof for Proposition 4.12 there assumed the injectivity of the splitting (denoted ν there), but, for a general compatible family, a splitting is not necessarily injective. On the other hand, for an admissible set of weight filtrations, any splitting is injective, and for such a family, the proof there is correct, and hence the conclusion of [KU2], Proposition 4.12 holds.

The existence of a distance to Φ-boundary β for $\Phi \in \overline{\mathcal{W}}$ follows from the pure case by Lemma 3.3.4.

We prove the existence of a distance to Ψ-boundary β for $\Psi \in \mathcal{W}$. Let Φ be the image $\bar{\Psi}$ of Ψ in $\overline{\mathcal{W}}$ as in Section 3.2.2.

If $W \notin \Psi$, the existence of β follows from Lemma 3.3.5(i). Assume $W \in \Psi$. It is sufficient to construct a real analytic map $\gamma : D_{\mathrm{nspl}} \to \mathbf{R}_{>0}$ having the property stated in Lemma 3.3.5(ii). Fix $\bar{\mathbf{r}} = (\bar{\mathbf{r}}_w)_w \in D(\mathrm{gr}^W)$ and, for each $w \le -1$, fix a $K'_{\bar{\mathbf{r}}_w}$-invariant positive definite symmetric $\mathbf{R}$-bilinear form $(\ ,\ )_w$ on the component L_w of $L := \mathcal{L}(\bar{\mathbf{r}})$ (see Section 1.2.1) of weight w. Here $K'_{\bar{\mathbf{r}}_w}$ is the isotropy subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ at $\bar{\mathbf{r}}_w$, which is compact so that there is such a form. Let $f : L - \{0\} \to \mathbf{R}_{>0}, f(v) := (\sum_{w \le -1} (v_w, v_w)_w^{-1/w})^{-1/2}$, where v_w denotes the component of v of weight w. For $F \in D(\mathrm{gr}^W)$, if g is an element of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $F = g\bar{\mathbf{r}}$, then we have an isomorphism $\mathrm{Ad}(g)^{-1} : \mathcal{L}(F) \xrightarrow{\sim} L$. The map $f_F : \mathcal{L}(F) - \{0\} \to \mathbf{R}_{>0}, v \mapsto f(\mathrm{Ad}(g)^{-1}v)$, is independent of the choice of g. This is because $(g')^{-1}g \in \prod_w K'_{\bar{\mathbf{r}}_w}$ if $g, g' \in G_{\mathbf{R}}(\mathrm{gr}^W)$ and $g\bar{\mathbf{r}} = g'\bar{\mathbf{r}}$. Define $\gamma' : D_{\mathrm{nspl}} \to \mathbf{R}_{>0}$ by $\gamma'\big(s(\theta(F,\delta))\big) = f_F(\delta)$. Let α be any splitting of Ψ. Then $\gamma'(\alpha(t)x) = (\prod_{W' \in \Psi} t_{W'})\gamma'(x)$ for $t \in \mathbf{R}^{\Psi}_{>0}$ and $x \in D_{\mathrm{nspl}}$, where $t_{W'} \in \mathbf{R}_{>0}$ denotes the W'-component of t. For $x \in D_{\mathrm{nspl}}$, define $\gamma(x) = \gamma'(x) \cdot \prod_{W' \in \Phi} \beta(x(\mathrm{gr}^W))^{-1}_{W'}$, where $\beta(x(\mathrm{gr}^W))_{W'}$ denotes the W'-component of $\beta(x(\mathrm{gr}^W))$. Then γ has the property stated in Lemma 3.3.5(ii). □

3.3.7.

We start to prove Proposition 3.2.6. The last assertions of (i) and (ii) are clear once the preceding convergences are shown. We then prove the convergences in Sections 3.3.7–3.3.12.

Here we prove the following part of Proposition 3.2.6(i).

Let $\Psi \in \mathcal{W}$, and assume $W \notin \Psi$ (resp., $W \in \Psi$), let β be a distance to Ψ-boundary, let $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$), and let $\mathbf{r} \in D$ be a point on the torus orbit associated to p. Let J be the set of weight filtrations associated to p. Then $\beta(\tau_p(t)\mathbf{r})$ $(t \in \mathbf{R}^J_{>0})$ converges in $\mathbf{R}^{\Psi}_{\ge 0}$ when t tends to 0^J.

Proof

Take a splitting α of Ψ, and let $\alpha_J : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}(H_{0,\mathbf{R}})$ be the restriction of α to the J-component $\mathbf{G}^J_{m,\mathbf{R}}$ of $\mathbf{G}^{\Psi}_{m,\mathbf{R}}$. Let $H_{0,\mathbf{R}} = \bigoplus_{m \in \mathbf{Z}^J} S(J,m)$ be the decomposition associated to α_J. Since both τ_p and α_J split J, there is a unique element u of $G_{\mathbf{R}}$ such that $\tau_p = \mathrm{Int}(u)(\alpha_J)$ and such that $(1-u)S(J,m) \subset \bigoplus_{m' < m} S(J,m')$ for any $m \in \mathbf{Z}^J$. We have

$$\beta\big(\tau_p(t)\mathbf{r}\big) = \beta\big(u\alpha_J(t)u^{-1}\mathbf{r}\big) = \beta\big(\alpha_J(t)u_t u^{-1}\mathbf{r}\big) = \iota_J(t)\beta(u_t u^{-1}\mathbf{r}),$$

where $u_t = \mathrm{Int}(\alpha_J(t))^{-1}(u)$, and $\iota_J : \mathbf{R}^J_{>0} \to \mathbf{R}^{\Psi}_{>0}$ is the canonical injective homomorphism from the J-component. When $t \to 0^J$, u_t converges to 1, as is easily seen. Hence $\beta(\tau_p(t)\mathbf{r})$ converges to $0^J\beta(u^{-1}\mathbf{r})$ in $\mathbf{R}^{\Psi}_{\ge 0}$, where 0^J denotes the element of $\mathbf{R}^{\Psi}_{\ge 0}$ whose jth component for $j \in \Psi$ is 0 if $j \in J$ and is 1 if $j \notin J$. □

REMARK

In [KU2, Proposition 4.12], the corresponding statement in the pure case was treated, but on the second line after the proof of it, the factor corresponding to 0^J here is missing.

3.3.8.
We prove the following part of Proposition 3.2.6(ii).

Let $\Phi \in \overline{\mathcal{W}}$, let β be a distance to Φ-boundary, let $p \in D^{II}_{\mathrm{SL}(2)}(\Phi)$, and let $\mathbf{r}$ be a point on the torus orbit associated to p. Let J be the set of weight filtrations associated to p. Then $\beta(\tau_p(t)\mathbf{r}(\mathrm{gr}^W))$ $(t \in \mathbf{R}^J_{>0})$ converges in $\mathbf{R}^\Phi_{\geq 0}$ when t tends to 0^J.

Proof
Let $\bar{J} \in \overline{\mathcal{W}}$ be the image of J, and let $\bar{\tau}_p$ be as in Section 3.2.3. Take a splitting α of Φ, let $\alpha_{\bar{J}} : \mathbf{G}^{\bar{J}}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W)$ be the restriction of α to the $\bar{J}$-component $\mathbf{G}^{\bar{J}}_{m,\mathbf{R}}$ of $\mathbf{G}^{\Phi}_{m,\mathbf{R}}$, and let $\mathrm{gr}^W = \bigoplus_{m \in \mathbf{Z}^{\bar{J}}} \bar{S}(\bar{J}, m)$ be the decomposition associated to $\alpha_{\bar{J}}$. Since both $\bar{\tau}_p$ and $\alpha_{\bar{J}}$ split $\bar{J}$, there is a unique element u of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $\bar{\tau}_p = \mathrm{Int}(u)(\alpha_{\bar{J}})$ and such that $(1-u)\bar{S}(\bar{J}, m) \subset \bigoplus_{m' < m} \bar{S}(\bar{J}, m')$ for any $m \in \mathbf{Z}^{\bar{J}}$. We have

$$\begin{aligned}\beta\big(\tau_p(t)\mathbf{r}(\mathrm{gr}^W)\big) &= \beta\big(u\alpha_{\bar{J}}(t_{\bar{J}})u^{-1}\mathbf{r}(\mathrm{gr}^W)\big)\\ &= \beta\big(\alpha_{\bar{J}}(t_{\bar{J}})u_t u^{-1}\mathbf{r}(\mathrm{gr}^W)\big) = \iota_{\bar{J}}(t_{\bar{J}})\beta\big(u_t u^{-1}\mathbf{r}(\mathrm{gr}^W)\big),\end{aligned}$$

where $u_t = \mathrm{Int}(\alpha_{\bar{J}}(t_{\bar{J}}))^{-1}(u)$, $\iota_{\bar{J}} : \mathbf{R}^{\bar{J}}_{>0} \to \mathbf{R}^{\Phi}_{>0}$ is the canonical injective homomorphism from the $\bar{J}$-component, and $t_{\bar{J}}$ is the $\bar{J}$-component of t. Here we identify $\bar{J}$ with J (resp., $J \smallsetminus \{W\}$) if $W \notin J$ (resp., $W \in J$). When $t \to 0^J$, u_t converges to 1 as is easily seen. Hence $\beta(\tau_p(t)\mathbf{r}(\mathrm{gr}^W))$ converges to $0^{\bar{J}}\beta(u^{-1}\mathbf{r}(\mathrm{gr}^W))$, where $0^{\bar{J}}$ denotes the element of $\mathbf{R}^{\Phi}_{\geq 0}$ whose jth component for $j \in \Phi$ is 0 if $j \in \bar{J}$ and is 1 if $j \notin \bar{J}$. □

3.3.9.
We prove the following part of Proposition 3.2.6(i).

Let the notation be as in Section 3.3.7, let α be a splitting of Ψ, and let $\mu : D \to D$ be the map $x \mapsto \alpha\beta(x)^{-1}x$. Then $\mu(\tau_p(t)\mathbf{r})$ converges in D when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$.

Proof
We have

$$\mu\big(\tau_p(t)\mathbf{r}\big) = \mu\big(u\alpha_J(t)u^{-1}\mathbf{r}\big) = \mu\big(\alpha_J(t)u_t u^{-1}\mathbf{r}\big) = \mu(u_t u^{-1}\mathbf{r}) \to \mu(u^{-1}\mathbf{r})$$

when $t \to 0^J$. □

3.3.10.
We prove the following part of Proposition 3.2.6(ii).

Let the notation be as in Section 3.3.8, let α be a splitting of Φ, and let $\mu = (\mu_1, \mu_2) : D \to D(\mathrm{gr}^W) \times \mathcal{L}$ be the map $x \mapsto \big(\alpha\beta(x(\mathrm{gr}^W))^{-1}x(\mathrm{gr}^W), \mathrm{Ad}(\alpha\beta(x(\mathrm{gr}^W)))^{-1}\delta(x)\big)$. Then, $\mu(\tau_p(t)\mathbf{r})$ converges in $D(\mathrm{gr}^W) \times \bar{\mathcal{L}}$ when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$.

Proof
This μ_1 factors through the projection $D \to D(\mathrm{gr}^W)$ and

$$\mu_1\big(\tau_p(t)\mathbf{r}\big) = \mu_1\big(u\alpha_{\bar J}(t_{\bar J})u^{-1}\mathbf{r}(\mathrm{gr}^W)\big) = \mu_1\big(\alpha_{\bar J}(t_{\bar J})u_t u^{-1}\mathbf{r}(\mathrm{gr}^W)\big)$$
$$= \mu_1\big(u_t u^{-1}\mathbf{r}(\mathrm{gr}^W)\big) \to \mu\big(u^{-1}\mathbf{r}(\mathrm{gr}^W)\big)$$

when $t \to 0^J$. Assume $W \notin J$, and identify J and $\bar J$ via the canonical bijection. Then

$$\begin{aligned}\mu_2\big(\tau_p(t)\mathbf{r}\big) &= \big(\mathrm{Ad}\,\alpha\beta(\bar\tau_p(t)\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\mathrm{Ad}\big(\bar\tau_p(t)\big)\delta(\mathbf{r})\\ &= \mathrm{Ad}\big(\alpha\beta(u_t u^{-1}\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\mathrm{Ad}(u_t u^{-1})\delta(\mathbf{r})\\ &\to \mathrm{Ad}\big(\alpha\beta(u^{-1}\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\mathrm{Ad}(u^{-1})\delta(\mathbf{r})\end{aligned}$$

when $t \to 0^J$. Next, assume $W \in J$ and identify $J \smallsetminus \{W\}$ with $\bar J$ via the canonical bijection. For $t \in \mathbf{R}^J_{>0}$, write $t = (t', t_{\bar J})$, where $t' \in \mathbf{R}_{>0}$ denotes the W-component of t and $t_{\bar J}$ denotes the $\bar J$-component of t. Then

$$\begin{aligned}\mu_2\big(\tau_p(t)\mathbf{r}\big) &= \mathrm{Ad}\big(\alpha\beta(\bar\tau_p(t)\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\big(t' \circ \mathrm{Ad}(\bar\tau_p(t))\delta(\mathbf{r})\big)\\ &= t' \circ \mathrm{Ad}\big(\alpha\beta(u_t u^{-1}\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\mathrm{Ad}(u_t u^{-1})\delta(\mathbf{r})\\ &\to 0 \circ \mathrm{Ad}\big(\alpha\beta(u^{-1}\mathbf{r}(\mathrm{gr}^W))\big)^{-1}\mathrm{Ad}(u^{-1})\delta(\mathbf{r})\end{aligned}$$

when $t \to 0^J$. Here, for $t \in \mathbf{R}_{>0}$ and $\delta = \sum_{w \le -2}\delta_w \in \mathcal{L}$, we write $t \circ \delta = \sum t^w \delta_w$, and $0 \circ \delta = \lim_{t\to 0} t \circ \delta$ in $\bar{\mathcal{L}}$. □

3.3.11.
Since the convergences of the canonical splittings are trivial (cf. Sections 2.4.6, 2.5.5), to prove Proposition 3.2.6, the rest is the convergences of Borel-Serre splittings. To see the latter, we may and do assume that we are in the pure case.

Let Ψ be an admissible set of weight filtrations, and let $W' \in \Psi$. Fix an SL(2)-orbit q whose associated set of weight filtrations is Ψ. Let $X = \mathbf{Z}^\Psi$, and let $\mathfrak{g}_\mathbf{R} = \bigoplus_{m \in X}\mathfrak{g}_{\mathbf{R},m}$ be the direct sum decomposition, where $t \in (\mathbf{R}^\times)^\Psi$ acts via τ_q on $\mathfrak{g}_{\mathbf{R},m}$ as the multiplication by t^m.

In this paragraph, we prove the following.

Let $\mathbf{r}$ be a point on the torus orbit associated to q. Let J be a subset of Ψ, and let τ_J be the restriction of τ_q to the J-component $\mathbf{G}^J_{m,\mathbf{R}}$ of $\mathbf{G}^\Psi_{m,\mathbf{R}}$. Let $h \in \mathfrak{g}_\mathbf{R}$ be an element whose m-component is zero ($m \in X$) unless $m(j) < 0$ for all $j \in J$. Then there are an open neighborhood U of 0^J in $\mathbf{R}^J_{\ge 0}$ and real analytic maps $f_1 : U \to G_{W',\mathbf{R}}$ and $f_2 : U \to K_\mathbf{r}$ such that $\mathrm{Int}(\tau_J(t))^{-1}(\exp(h)) = f_1(t)f_2(t)$ for any $t \in U \cap \mathbf{R}^J_{>0}$, and, furthermore, $\mathrm{Int}(\tau_J(t))(f_1(t))$ extends to a real analytic map on U.

To prove this, first, we take an $\mathbf{R}$-subspace V of $\mathfrak{g}_\mathbf{R}$ satisfying the following (1)–(3).

(1) We have $\mathfrak{g}_\mathbf{R} = V \oplus \mathrm{Lie}(K_\mathbf{r})$.
(2) The vector space V is the sum of $V_{\pm m} := V \cap (\mathfrak{g}_{\mathbf{R},m} + \mathfrak{g}_{\mathbf{R},-m})$ for $m \in X$.

(3) We have $\mathrm{Lie}(G_{W',u,\mathbf{R}}) \subset V \subset \mathrm{Lie}(G_{W',\mathbf{R}})$.

Then there exist an open neighborhood O of zero in $\mathfrak{g}_{\mathbf{R}}$ and a real analytic function $a=(a_1,a_2): O \to V \oplus \mathrm{Lie}(K_{\mathbf{r}})$ having the following properties (4)–(7).

(4) For any $x \in O$, $\exp(x) = \exp(a_1(x))\exp(a_2(x))$.
(5) We have $a(0) = (0,0)$.
(6) The map $\exp : O \to G_{\mathbf{R}}$ is an injective open map.
(7) For $k=1,2$, a_k has the form of absolutely convergent series $a_k = \sum_{r=0}^{\infty} a_{k,r}$, where $a_{k,r}$ is the part of degree r in the Taylor expansion of a_k at zero, such that $a_{k,r}(x) = l_{k,r}(x \otimes \cdots \otimes x)$ for some linear map $l_{k,r} : \mathfrak{g}_{\mathbf{R}}^{\otimes r} \to \mathfrak{g}_{\mathbf{R}}$ having the following property: If $m_1,\dots,m_r \in X$ and $x_j \in \mathfrak{g}_{\mathbf{R},m_j}$ for $1 \le j \le r$, then $l_{k,r}(x_1 \otimes \cdots \otimes x_r) \in \sum_m \mathfrak{g}_{\mathbf{R},m}$, where m ranges over all elements of X satisfying $|m| \le |m_1| + \cdots + |m_r|$. Here $|\quad| : \mathbf{Z}^{\Psi} \to \mathbf{N}^{\Psi}$ is the map sending $(m(j))_j$ to $(|m(j)|)_j$.

This is proved similarly as [KU3, Lemma 10.3.4]. Or, if we choose V such that $V = \mathrm{Lie}(\tilde{\rho}(\mathbf{R}^n_{>0})) \oplus L$ for some L as in [KU3, Section 10.1.2] (such a choice is always possible), this is seen by [KU3, Lemma 10.3.4] just by taking $a_1(x) = H(f_1(x), f_2(x))$, $a_2(x) = f_3(x)$, where $H(x,y) = x + y + (1/2)[x,y] + \cdots$ is a Hausdorff series.

Now consider the decomposition $h = \sum_{m \in X} h_m$ ($h_m \in \mathfrak{g}_{\mathbf{R},m}$). By assumption, $h_m = 0$ unless $m(j) < 0$ for any $j \in J$. Then $\mathrm{Ad}(\tau_J(t))^{-1}(h) = \sum_{m \in X} t^{-m_J} h_m$ ($t \in \mathbf{R}^J_{>0}$) extends to a real analytic map $g : \mathbf{R}^J_{\ge 0} \to \mathfrak{g}_{\mathbf{R}}$ sending 0^J to zero, where $m_J \in \mathbf{Z}^J$ is the J-component of m. Let $U = g^{-1}(O)$, $f_j = \exp \circ a_j \circ g$ ($j=1,2$). It is enough to show that $\mathrm{Ad}(\tau_J(t))\big(a_1(g(t))\big)$ extends to a real analytic map around zero. This is a consequence of the property (7) of a_1. In fact, in the notation in (7), $a_1(g(t)) = a_1\big(\sum t^{-m_J} h_m\big)$ is the infinite formal sum of $t^{-((m_1)_J + \cdots + (m_r)_J)} l_{1,r}(h_{m_1} \otimes \cdots \otimes h_{m_r})$ ($m_j \in X, h_{m_j} \in \mathfrak{g}_{\mathbf{R},m_j}$ ($1 \le j \le r$)). Since the weights m of $l_{1,r}(h_{m_1} \otimes \cdots \otimes h_{m_r})$ satisfy $|m| \le |m_1| + \cdots + |m_r|$, we conclude that $\mathrm{Ad}(\tau_J(t))\big(a_1(g(t))\big)$ extends to a real analytic map over 0^J, as desired.

3.3.12.
We continue to assume that we are in the pure situation.

Let Ψ be an admissible set of weight filtrations, and let $W' \in \Psi$. We prove the following, which completes the proof of Proposition 3.2.6.

Let $p \in D_{\mathrm{SL}(2)}(\Psi)$, and let $\mathbf{r}$ be a point on the torus orbit associated to p. Let J be the set of weight filtrations associated to p. Then $\mathrm{spl}^{\mathrm{BS}}_{W'}(\tau_p(t)\mathbf{r})$ ($t \in \mathbf{R}^J_{>0}$) converges in $\mathrm{spl}(W')$ when t tends to 0^J.

REMARK 1
The proof is easy when $W' \in J$ (Borel-Serre splitting is then constant on the torus orbit) but is not when $W' \notin J$ (see Remark 3 after the proof).

Proof
Since Ψ is admissible, Ψ is the set of weight filtrations associated to some $q \in D_{\mathrm{SL}(2)}$. Let $\mathbf{r}_q$ be a point on the torus orbit associated to q. Then, by [KU3, Section 6.4.4, Claim 1], there exist $v \in G_{J,\mathbf{R}}$ and $k \in K_{\mathbf{r}_q}$ such that $\tau_p = \mathrm{Int}(v)(\tau_{q,J})$ and $\mathbf{r} = vk\mathbf{r}_q$. Here $G_{J,\mathbf{R}} = \{g \in G_\mathbf{R} \mid gW'' = W''$ for any $W'' \in J\}$, and $\tau_{q,J}$ denotes the restriction of τ_q to the J-component $\mathbf{G}^J_{m,\mathbf{R}}$ of $\mathbf{G}^\Psi_{m,\mathbf{R}}$.

Let $G_\mathbf{R}(J)$ be the $\mathbf{R}$-algebraic subgroup of $G_{J,\mathbf{R}}$ consisting of all elements of $G_\mathbf{R}$ which commute with any element of $\tau_{q,J}(\mathbf{G}^J_{m,\mathbf{R}})$. Then we have the projection $G_{J,\mathbf{R}} \to G_\mathbf{R}(J), a \mapsto a(J)$, where $a(J)$ on $S(J,m)$ $(m \in \mathbf{Z}^J)$ (see Section 3.3.7) is defined to be the $(S(J,m) \to S(J,m))$-component of $a : S(J,m) \to \bigoplus_{m' \le m} S(J,m')$. The composite $G_\mathbf{R}(J) \to G_{J,\mathbf{R}} \to G_\mathbf{R}(J)$ is the identity map. Since $G_\mathbf{R}(J)$ is reductive, any element of $G_\mathbf{R}(J)$ is expressed in the form bc, where $b \in G_\mathbf{R}(J) \cap G_{W',\mathbf{R}}$ and $c \in G_\mathbf{R}(J) \cap K_{\mathbf{r}_q}$. Write the image of v in $G_\mathbf{R}(J)$ as bc by using such b and c. Then $v = bv_uc$ with $v_u \in G_{J,\mathbf{R}}$ satisfying $(v_u - 1)S(J,m) \subset \bigoplus_{m' < m} S(J,m)$ for any $m \in \mathbf{Z}^J$. We have $\mathrm{Int}(\tau_{q,J}(t))^{-1}(v_u) \to 1$ when $t \to 0^J$ in $\mathbf{R}^J_{\ge 0}$. Hence, by Section 3.3.11, there are an open neighborhood U of 0^J in $\mathbf{R}^J_{\ge 0}$ and real analytic maps $b_u : U \to G_{W',\mathbf{R}}$ and $c_u : U \to K_{\mathbf{r}_q}$ such that $\mathrm{Int}(\tau_{q,J}(t))^{-1}(v_u) = \mathrm{Int}(\tau_{q,J}(t))^{-1}(b_u(t))c_u(t)$ for any $t \in U \cap \mathbf{R}^J_{>0}$. We have, for $t \in U \cap \mathbf{R}^J_{>0}$,

$$\tau_p(t)\mathbf{r} = v\tau_{q,J}(t)k\mathbf{r}_q = bb_u(t)\tau_{q,J}(t)c_u(t)ck\mathbf{r}_q,$$

and hence

$$\begin{aligned}\mathrm{spl}^{\mathrm{BS}}_{W'}\big(\tau_p(t)\mathbf{r}\big) &= \mathrm{Int}\big(bb_u(t)\big)\,\mathrm{Int}\big(\tau_{q,J}(t)\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(c_u(t)ck\mathbf{r}_q)\big)\\ &= \mathrm{Int}\big(bb_u(t)\big)\,\mathrm{Int}\big(\tau_{q,J}(t)\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(\mathbf{r}_q)\big) = \mathrm{Int}\big(bb_u(t)\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(\mathbf{r}_q)\big)\\ &\to \mathrm{Int}\big(bb_u(0^J)\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(\mathbf{r}_q)\big). \qquad \square\end{aligned}$$

REMARK 2
In the above proof, $\mathrm{spl}^{\mathrm{BS}}_{W'}(\mathbf{r}_q)$ coincides with the splitting of W' associated to q.

REMARK 3
In the case $W' \in J$, $\mathrm{spl}^{\mathrm{BS}}_{W'}(\tau_p(t)\mathbf{r})$ constantly coincides with $\mathrm{Int}(v)\,\mathrm{spl}^{\mathrm{BS}}_{W'}(\mathbf{r}_q)$ with v as in the above proof.

3.3.13. Proof of Proposition 3.2.7 (injectivity of $\nu_{\alpha,\beta}$)
Recall that a point of $D_{\mathrm{SL}(2)}$ is determined by the associated weight filtrations and the associated torus orbit (see Proposition 2.5.2(ii)).

First, let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp., $W \in \Psi$). We prove that the map

$$\begin{aligned}\nu_{\alpha,\beta} : D^I_{\mathrm{SL}(2)}(\Psi)\ &\big(\text{resp.}, D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}\big)\\ &\to \mathbf{R}^\Psi_{\ge 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}\big(W'(\mathrm{gr}^W)\big)\end{aligned}$$

is injective. Denote $\nu_{\alpha,\beta}(p)$ by $\big(\beta(p), \mu(p), \mathrm{spl}_W(p), (\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W)))_{W'\in\Psi}\big)$. (Note that the symbol μ was introduced in Section 3.3.9.)

Let $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ (resp., $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$). Then the set $J \subset \Psi$ of weight filtrations associated to p is recovered from $\beta(p)$ as

$$J = \{j \in \Psi \mid \beta(p)_j = 0\}.$$

Let α_J be the restriction of α to the J-component $\mathbf{G}^J_{m,\mathbf{R}}$ of $\mathbf{G}^\Psi_{m,\mathbf{R}}$. Since both $\mathrm{gr}^W(\tau_p)$ and $\mathrm{gr}^W(\alpha_J)$ split $W'(\mathrm{gr}^W)$ for all $W' \in J$, there is a unique element u of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $\mathrm{gr}^W(\tau_p) = \mathrm{Int}(u)(\mathrm{gr}^W(\alpha_J))$ and such that $(1-u)\bar{S}(\bar{J}, m) \subset \bigoplus_{m'<m} \bar{S}(\bar{J}, m')$ for any $m \in \mathbf{Z}^{\bar{J}}$ (cf. Section 3.3.8). This u is characterized by the following property (1).

(1) For any $W' \in J$, $u^{-1}\,\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W))$ coincides with the splitting of $W'(\mathrm{gr}^W)$ defined by the W'-component of α.

The torus orbit associated to p is recovered as

$$\big\{\mathrm{spl}_W(p)\theta\big(u\,\mathrm{gr}^W(\alpha(t))(\mu(p)(\mathrm{gr}^W)), \mathrm{Ad}(u\alpha(t))(\delta(\mu(p)))\big) \bigm| t \in \mathbf{R}^\Psi_{>0}, \beta(p) = 0^J t\big\}.$$

Next, let $\Phi \in \overline{\mathcal{W}}$. We prove that the map

$$\nu_{\alpha,\beta} : D^{II}_{\mathrm{SL}(2)}(\Phi) \to \mathbf{R}^\Phi_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W'\in\Phi} \mathrm{spl}(W')$$

is injective. Denote $\nu_{\alpha,\beta}(p)$ by

$$\big(\beta(p(\mathrm{gr}^W)), \mu(p(\mathrm{gr}^W)), \mathrm{spl}_W(p), (\mathrm{spl}^{\mathrm{BS}}_{W'}(p(\mathrm{gr}^W)))_{W'\in\Phi}\big).$$

Let $p \in D^{II}_{\mathrm{SL}(2)}(\Phi)$. Let J be the set of weight filtrations associated to p. Let $\bar{J} = \{W'(\mathrm{gr}^W) \mid W' \in J, W' \neq W\} \subset \Phi$. Then $\bar{J}$ is recovered from $\beta(p(\mathrm{gr}^W))$ as

$$\bar{J} = \{j \in \Phi \mid \beta\big(p(\mathrm{gr}^W)\big)_j = 0\}.$$

Let $\mu(p(\mathrm{gr}^W)) = (x, y)$ with $x \in D(\mathrm{gr}^W)$ and $y \in \bar{\mathcal{L}}$ (see Section 3.3.10). If $y \in \mathcal{L}$, J is the lifting of $\bar{J}$ on $H_{0,\mathbf{R}}$ by $\mathrm{spl}_W(p)$. If $y \in \bar{\mathcal{L}} \smallsetminus \mathcal{L}$, J is the union of $\{W\}$ and the lifting of $\bar{J}$ on $H_{0,\mathbf{R}}$ by $\mathrm{spl}_W(p)$.

Let $\alpha_{\bar{J}}$ be the restriction of α to the $\bar{J}$-component $\mathbf{G}^{\bar{J}}_{m,\mathbf{R}}$ of $\mathbf{G}^\Phi_{m,\mathbf{R}}$. Since both $\bar{\tau}_p$ and $\alpha_{\bar{J}}$ split all $W' \in \bar{J}$, there is a unique element u of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $\mathrm{gr}^W(\bar{\tau}_p) = \mathrm{Int}(u)(\alpha_{\bar{J}})$ and such that $(1-u)\bar{S}(\bar{J}, m) \subset \bigoplus_{m'<m} \bar{S}(\bar{J}, m')$ for any $m \in \mathbf{Z}^{\bar{J}}$. This u is characterized by the following property (1).

(1) For any $W' \in \bar{J}$, $u^{-1}\,\mathrm{spl}^{\mathrm{BS}}_{W'}(p(\mathrm{gr}^W))$ coincides with the splitting of W' defined by the W'-component of α.

If $W \notin J$ (note that $y \in \mathcal{L}$ in this case), the torus orbit associated to p is recovered as

$$\big\{\mathrm{spl}_W(p)\theta\big(u\alpha(t)(x), \mathrm{Ad}(u\alpha(t))(y)\big) \bigm| t \in \mathbf{R}^\Phi_{>0}, \beta(p) = 0^J t\big\}.$$

If $W \in J$, y has the shape $0 \circ z$ with $z \in \mathcal{L} \smallsetminus \{0\}$ (see Section 3.3.10), and the torus orbit associated to p is recovered as

$$\big\{\mathrm{spl}_W(p)\theta\big(u\alpha(t)(x), t' \circ \mathrm{Ad}(u\alpha(t))(z)\big) \bigm| t \in \mathbf{R}^\Phi_{>0}, \beta(p) = 0^J t, t' \in \mathbf{R}_{>0}\big\}.$$

Proposition 3.2.7 is proved. □

3.3.14. Proof of Proposition 3.2.9
The proofs of (i) and (ii) are similar. We give here the proof of (ii).

To prove that another choice (α', β') gives the same structure as (α, β), we may assume either $\alpha = \alpha'$ or $\beta = \beta'$.

Assume first $\alpha = \alpha'$. Then we have a commutative diagram in which the right vertical arrow is a morphism of $\mathcal{B}_{\mathbf{R}}(\log)$:

$$\begin{array}{ccccc} D^{II}_{\mathrm{SL}(2)}(\Phi) & \xrightarrow{\text{by } \nu_{\alpha,\beta}} & \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} & & (t, y, \delta) \\ \| & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \xrightarrow{\text{by } \nu_{\alpha,\beta'}} & \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} & & \big(t\beta'(y), \alpha\beta'(y)^{-1}y, \mathrm{Ad}(\alpha\beta'(y))^{-1}\delta\big) \end{array}$$

Assume $\beta = \beta'$. Then $\alpha' = \mathrm{Int}(u)\alpha$ for some $u \in G_{\mathbf{R}}$ such that $(u-1)W'_w \subset W'_{w-1}$ for any $W' \in \Phi$ and for any $w \in \mathbf{Z}$. For $t \in \mathbf{R}^{\Phi}_{>0}$, let $u_t = \alpha(t)^{-1}u\alpha(t)$. Then as is easily seen, the map $\mathbf{R}^{\Phi}_{>0} \to G_{\mathbf{R}}, t \mapsto u_t$, extends to a real analytic map $\mathbf{R}^{\Phi}_{\geq 0} \to G_{\mathbf{R}}$, which we still denote by $t \mapsto u_t$. We have a commutative diagram in which the right vertical arrow is a morphism in $\mathcal{B}_{\mathbf{R}}(\log)$:

$$\begin{array}{ccccc} D^{II}_{\mathrm{SL}(2)}(\Phi) & \xrightarrow{\text{by } \nu_{\alpha,\beta}} & \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} & & (t, y, \delta) \\ \| & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \xrightarrow{\text{by } \nu_{\alpha',\beta}} & \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} & & \big(t, uu_t^{-1}y, \mathrm{Ad}(uu_t^{-1})\delta\big) \end{array}$$

These commutative diagrams prove Proposition 3.2.9(ii). □

3.4. Local properties of $D_{\mathrm{SL}(2)}$

In this subsection, we prove Theorem 3.2.10 and Proposition 3.2.12, give local descriptions of $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ (Theorems 3.4.4, 3.4.6), and prove a criterion (Proposition 3.4.29) for the coincidence of $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$.

3.4.1.
Let $p \in D_{\mathrm{SL}(2)}$, let $\Phi = \overline{\mathcal{W}}(p)$ (see Section 3.2.2), let $\mathbf{r}$ be a point on the torus orbit associated to p, and let $\bar{\mathbf{r}} = \mathbf{r}(\mathrm{gr}^W)$. Fix $\mathbf{R}$-subspaces

$$R \subset \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W), \qquad S \subset \mathrm{Lie}(K_{\bar{\mathbf{r}}})$$

satisfying the following conditions (a), (b), and (c). Here $K_{\bar{\mathbf{r}}} = \prod_w K_{\bar{\mathbf{r}}_w}$ with $K_{\bar{\mathbf{r}}_w}$ the maximal compact subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ corresponding to $\bar{\mathbf{r}}_w$ (see [KU3, Section 5.1.2]), where we write $\bar{\mathbf{r}} = (\bar{\mathbf{r}}_w)_w$ as in Section 3.3.6. Note that $K_{\bar{\mathbf{r}}_w} \supset K'_{\bar{\mathbf{r}}_w}$ for all w.

(a) We have $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = R \oplus \mathrm{Lie}(\tilde{\rho}(\mathbf{R}^{\Phi}_{>0})) \oplus \mathrm{Lie}(K_{\bar{\mathbf{r}}})$.

Here $\tilde{\rho}$ is the homomorphism $\mathbf{G}^{\Phi}_{m,\mathbf{R}} \to G_{\mathbf{R}}(\mathrm{gr}^W)$ defined by

$$\tilde{\rho}(t_1, \dots, t_n) = \bigoplus_{w \in \mathbf{Z}} \big(\rho_w(g_1, \dots, g_n) \text{ on } \mathrm{gr}^W_w\big) \quad \text{with } g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix},$$

where n is the number of the elements of Φ and $((\rho_w, \varphi_w)_w, \mathbf{r})$ is the SL(2)-orbit in n variables of rank n with class p (cf. Section 2.3.5).

(b) We have $\mathrm{Lie}(K_{\bar{\mathbf{r}}}) = S \oplus \mathrm{Lie}(K'_{\bar{\mathbf{r}}})$, where $K'_{\bar{\mathbf{r}}} = \prod_w K'_{\bar{\mathbf{r}}_w}$ (cf. Section 3.3.6).

We introduce the notation to state condition (c). Let

$$\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = \bigoplus_{m \in \mathbf{Z}^\Phi} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m$$

be the direct decomposition associated to the adjoint action of $\mathbf{G}_{m,\mathbf{R}}^\Phi$ via $\tilde{\rho}$. Note that this action coincides with the adjoint action of $\mathbf{G}_{m,\mathbf{R}}^\Phi$ via $\bar{\tau}_p$ (see Section 3.2.3). Thus

$$\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m := \big\{x \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \;\big|\; \mathrm{Ad}\big(\bar{\tau}_p(t)\big)x = t^m x \text{ for all } t \in (\mathbf{R}^\times)^\Phi\big\}.$$

Condition (c) is the following.

(c) We have $R = \sum_{m \in \mathbf{Z}^\Phi} R \cap (\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m + \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{-m})$.

Such R and S exist. The proof of the existence for the pure case is in [KU3, Section 10.1.2], and the general case is similar to it. We remark that when we are given a parabolic subgroup P of $G_{\mathbf{R}}(\mathrm{gr}^W)$, we can take $R \subset \mathrm{Lie}(P)$.

3.4.2.

Let the notation be as in Section 3.4.1. We define objects $Y^{II}(p, \mathbf{r}, S)$ and $Y^{II}(p, \mathbf{r}, R, S)$ of $\mathcal{B}_{\mathbf{R}}(\log)$.

Let $L = \mathcal{L}(\bar{\mathbf{r}})$ (see Section 1.2.1).

We define sets $Z(p)$ and $Z(p, R)$. Let

$$Z(p) \subset \mathbf{R}^\Phi_{\geq 0} \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$$

be the set of all (t, f, g, h) satisfying the following conditions (1) and (2). Let $J = J(t) := \{j \in \Phi \mid t_j = 0\}$.

(1) For $m \in \mathbf{Z}^\Phi$, $g_m = 0$ unless $m(j) = 0$ for all $j \in J$, $f_m = 0$ unless $m(j) \leq 0$ for all $j \in J$, and $h_m = 0$ unless $m(j) \geq 0$ for all $j \in J$.

Here $(\,)_m$ for $m \in \mathbf{Z}^\Phi$ denotes the m-component for the adjoint action of $\mathbf{G}^\Phi_{m,\mathbf{R}}$ under $\bar{\tau}_p$.

(2) Let t' be any element of $\mathbf{R}^\Phi_{>0}$ such that $t'_j = t_j$ for any $j \in \Phi \smallsetminus J$. If $m \in \mathbf{Z}^\Phi$ and $m(j) = 0$ for any $j \in J$, then $g_m = \mathrm{Ad}(\bar{\tau}_p(t'))^{-1}(f_m)$ and $g_m = \mathrm{Ad}(\bar{\tau}_p(t'))(h_m)$.

Let

$$Z(p, R) \subset Z(p)$$

be the subset consisting of all elements (t, f, g, h) satisfying the following condition (3).

(3) We have $g \in R$ and $f_m + h_{-m} \in R$ for all $m \in \mathbf{Z}^\Phi$.

Let

$$Y^{II}(p, \mathbf{r}, S) \subset Z(p) \times S \times \bar{L} \times \mathfrak{g}_{\mathbf{R},u}$$
$$\big(\text{resp., } Y^{II}(p, \mathbf{r}, R, S) \subset Z(p, R) \times S \times \bar{L} \times \mathfrak{g}_{\mathbf{R},u}\big)$$

be the set consisting of all elements (t,f,g,h,k,δ,u) $((t,f,g,h)\in Z(p)$ (resp., $Z(p,R)$), $k\in S, \delta\in\bar{L}, u\in\mathfrak{g}_{\mathbf{R},u}$) satisfying the following condition (4).

(4) We have $\exp(k)\bar{\mathbf{r}}\in(K_{\bar{\mathbf{r}}}\cap G_{\mathbf{R}}(\mathrm{gr}^W)_J)\cdot\bar{\mathbf{r}}$ with $J=J(t)$, where $G_{\mathbf{R}}(\mathrm{gr}^W)_J=\{g\in G_{\mathbf{R}}(\mathrm{gr}^W)\mid gW'=W'$ for any $W'\in J\}$.

We endow $Y^{II}(p,\mathbf{r},S)$ (resp., $Y^{II}(p,\mathbf{r},R,S)$) with the following structure as an object of $\mathcal{B}_{\mathbf{R}}(\log)$.

Let $E=\mathbf{R}^{\Phi}_{\geq 0}\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S\times\bar{L}\times\mathfrak{g}_{\mathbf{R},u}$. Let $A=Y^{II}(p,\mathbf{r},S)$ (resp., $A=Y^{II}(p,\mathbf{r},R,S)$).

We endow A with the topology as a subspace of E.

We define the sheaf of real analytic functions on A as follows. For an open set U of A and for a map $f:U\to\mathbf{R}$, we say that f is real analytic if and only if, for any $p\in U$, there are an open neighborhood U' of p in U, an open neighborhood U'' of U' in E, and a real analytic function g on U'', such that the restrictions to U' of f and g coincide.

We show that with this sheaf of rings over $\mathbf{R}$, A is an object of $\mathcal{B}_{\mathbf{R}}$. Let $\mathcal{O}_E$ be the sheaf of real analytic functions on E. Let I be the ideal of $\mathcal{O}_E$ generated by the following sections $a_{m,l}$ and $b_{m,l}$ given for elements m of $\mathbf{Z}^{\Phi}$ and for $\mathbf{R}$-linear maps $l:\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\to\mathbf{R}$:

$$a_{m,l}(t,f,g,h,k,\delta,u)=\Big(\prod_{j\in\Phi,m(j)\leq 0}t_j^{-m(j)}\Big)l(f_m)-\Big(\prod_{j\in\Phi,m(j)\geq 0}t_j^{m(j)}\Big)l(g_m),$$

$$b_{m,l}(t,f,g,h,k,\delta,u)=\Big(\prod_{j\in\Phi,m(j)\leq 0}t_j^{-m(j)}\Big)l(g_m)-\Big(\prod_{j\in\Phi,m(j)\geq 0}t_j^{m(j)}\Big)l(h_m).$$

Here $(\ )_m$ denotes the mth component with respect to the adjoint action of $\mathbf{G}^{\Phi}_{m,\mathbf{R}}$ by $\bar{\tau}_p$, $\prod_{j\in\Phi,m(j)\leq 0}$ means the product over all $j\in\Phi$ such that $m(j)\leq 0$, and $\prod_{j\in\Phi,m(j)\geq 0}$ is defined in a similar way. Then I is a finitely generated ideal. Indeed, if $l_1,\dots,l_r$ form a basis of the dual $\mathbf{R}$-vector space of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$, a_{m,l_j} and b_{m,l_j} $(1\leq j\leq r)$ such that $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m\neq 0$ (there are only finitely many such m) generate I. Furthermore, the inverse image of $\mathcal{O}_E/I$ on $Y^{II}(p,\mathbf{r},S)$ coincides with the sheaf of real analytic functions on $Y^{II}(p,\mathbf{r},S)$. Hence $Y^{II}(p,\mathbf{r},S)$ is an object of $\mathcal{B}_{\mathbf{R}}$. Let I' be the ideal of $\mathcal{O}_E$ generated by I and by the following sections c_l and $d_{m,l}$ given for elements m of $\mathbf{Z}^{\Phi}$ and $\mathbf{R}$-linear maps $l:\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\to\mathbf{R}$ which kill R:

$$c_l(t,f,g,h,k,\delta,u)=l(g),$$

$$d_{m,l}(t,f,g,h,k,\delta,u)=l(f_m+h_{-m}).$$

As is easily seen, I' is a finitely generated ideal. Furthermore, the inverse image of $\mathcal{O}_E/I'$ on $Y^{II}(p,\mathbf{r},R,S)$ coincides with the sheaf of real analytic functions on $Y^{II}(p,\mathbf{r},R,S)$. Hence $Y^{II}(p,\mathbf{r},R,S)$ is also an object of $\mathcal{B}_{\mathbf{R}}$.

We define the log structures with sign of $Y^{II}(p,\mathbf{r},S)$ and of $Y^{II}(p,\mathbf{r},R,S)$ to be the inverse images of the log structure with sign of $\mathbf{R}^{\Phi}_{\geq 0}$. This endows $Y^{II}(p,\mathbf{r},S)$ and $Y^{II}(p,\mathbf{r},R,S)$ with structures of objects of $\mathcal{B}_{\mathbf{R}}(\log)$.

3.4.3.
Define an open subset $Y_0^{II}(p,\mathbf{r},S)$ of $Y^{II}(p,\mathbf{r},S)$ by

$$Y_0^{II}(p,\mathbf{r},S)=\{(t,f,g,h,k,\delta,u)\in Y^{II}(p,\mathbf{r},S)\mid t\in\mathbf{R}^{\Phi}_{>0},\delta\in L\}.$$

We define an open subset $Y_0^{II}(p,\mathbf{r},R,S)$ of $Y^{II}(p,\mathbf{r},R,S)$ by

$$Y_0^{II}(p,\mathbf{r},R,S)=Y^{II}(p,\mathbf{r},R,S)\cap Y_0^{II}(p,\mathbf{r},S).$$

We have isomorphisms of real analytic manifolds

$$Y_0^{II}(p,\mathbf{r},S)\xrightarrow{\sim}\mathbf{R}^{\Phi}_{>0}\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S\times L\times\mathfrak{g}_{\mathbf{R},u},$$
$$Y_0^{II}(p,\mathbf{r},R,S)\xrightarrow{\sim}\mathbf{R}^{\Phi}_{>0}\times R\times S\times L\times\mathfrak{g}_{\mathbf{R},u},$$

given by

$$(t,f,g,h,k,\delta,u)\mapsto(t,g,k,\delta,u),$$

whose inverse maps are given by

$$f=\mathrm{Ad}\big(\bar\tau_p(t)\big)(g),\qquad h=\mathrm{Ad}\big(\bar\tau_p(t)\big)^{-1}(g).$$

We have a morphism of real analytic manifolds

$$\eta^{II}_{p,\mathbf{r},S}:Y_0^{II}(p,\mathbf{r},S)\to D,\qquad(t,f,g,h,k,\delta,u)\mapsto\exp(u)s_{\mathbf{r}}\theta\big(d\bar{\mathbf{r}},\mathrm{Ad}(d)\delta\big)$$

with $s_{\mathbf{r}}=\mathrm{spl}_W(\mathbf{r})$, $d=\bar\tau_p(t)\exp(g)\exp(k)=\exp(f)\bar\tau_p(t)\exp(k)$.

Let

$$\eta^{II}_{p,\mathbf{r},R,S}:Y_0^{II}(p,\mathbf{r},R,S)\to D$$

be the induced morphism.

THEOREM 3.4.4

Let the notation be as above. If U is a sufficiently small open neighborhood of $0:=(0,0,0,0)$ in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$ and if $Y^{II}(p,\mathbf{r},S,U)$ (resp., $Y^{II}(p,\mathbf{r},R,S,U)$) denotes the open set of $Y^{II}(p,\mathbf{r},S)$ (resp., $Y^{II}(p,\mathbf{r},R,S)$) consisting of all elements (t,f,g,h,k,δ,u) such that $(f,g,h,k)\in U$, we have the following.

(i) *There is a unique morphism $Y^{II}(p,\mathbf{r},S,U)\to D^{II}_{\mathrm{SL}(2)}(\Phi)$ in the category $\mathcal{B}'_{\mathbf{R}}(\log)$ whose restriction to $Y_0^{II}(p,\mathbf{r},S,U)=Y_0^{II}(p,\mathbf{r},S)\cap Y^{II}(p,\mathbf{r},S,U)$ coincides with the restriction of $\eta^{II}_{p,\mathbf{r},S}$ (Section 3.4.3).*

(ii) *The restriction of the morphism in (i) induces an open immersion $Y^{II}(p,\mathbf{r},R,S,U)\to D^{II}_{\mathrm{SL}(2)}(\Phi)$ in the category $\mathcal{B}'_{\mathbf{R}}(\log)$ which sends $(0^{\Phi},0,0,0,0,\delta(\mathbf{r}),0)\in Y^{II}(p,\mathbf{r},R,S,U)$ to p.*

The proof of this theorem is given later in Sections 3.4.18–3.4.19.

REMARK

From the proof of Theorem 3.4.4 given below, we see that if q is the image of

$(t,f,g,h,k,\delta,u)\in Y^{II}(p,\mathbf{r},S,U)$ in $D_{\mathrm{SL}(2)}(\Phi)$, then $q\in D_{\mathrm{SL}(2),\mathrm{spl}}$ if and only if $\delta=0$, and $W\in\mathcal{W}(q)$ if and only if $\delta\in\bar{L}\smallsetminus L$.

3.4.5.
Next, we consider $D^I_{\mathrm{SL}(2)}$.

Let $\Psi=\mathcal{W}(p)$. Let $\Phi,\mathbf{r},R,S$ be as before in Section 3.4.1.

We define an object $Y^I(p,\mathbf{r},R,S)$ of $\mathcal{B}_{\mathbf{R}}(\log)$ first in the case $W\notin\mathcal{W}(p)$. Let

$$(*)\qquad Y^I(p,\mathbf{r},R,S)\subset Y^{II}(p,\mathbf{r},R,S)\times\mathfrak{g}_{\mathbf{R},u}$$

be the set consisting of all elements (t,f,g,h,k,δ,u,v) $((t,f,g,h,k,\delta,u)\in Y^{II}(p,\mathbf{r},R,S),\ v\in\mathfrak{g}_{\mathbf{R},u})$ satisfying the following conditions (5)–(7). Via the bijection $\Psi\to\Phi$, we regard τ_p as a homomorphism $\mathbf{G}^{\Phi}_{m,\mathbf{R}}\to\mathrm{Aut}(H_{0,\mathbf{R}},W)$. Let $\mathfrak{g}_{\mathbf{R},u}=\bigoplus_{m\in\mathbf{Z}^{\Phi}}\mathfrak{g}_{\mathbf{R},u,m}$ be the corresponding direct sum decomposition. Denote by u_m the m-component of $u\in\mathfrak{g}_{\mathbf{R},u}$.

(5) For $m\in\mathbf{Z}^{\Phi}$, $u_m=0$ unless $m(j)\le 0$ for all $j\in J=J(t)$, and $v_m=0$ unless $m(j)=0$ for all $j\in J$.

(6) Let t' be any element of $\mathbf{R}^{\Phi}_{>0}$ such that $t'_j=t_j$ for any $j\in\Phi\smallsetminus J$. If $m\in\mathbf{Z}^{\Phi}$ and $m(j)=0$ for any $j\in J$, then $v_m=\mathrm{Ad}(\tau_p(t'))^{-1}(u_m)$.

(7) We have $\delta\in L$ in $\bar{L}$.

We endow $Y^I(p,\mathbf{r},R,S)$ with a structure of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ via the injection $Y^I(p,\mathbf{r},R,S)\hookrightarrow E\times\mathfrak{g}_{\mathbf{R},u}$, just as we endowed $Y^{II}(p,\mathbf{r},R,S)$ with it via the injection $Y^{II}(p,\mathbf{r},R,S)\hookrightarrow E$ in Section 3.4.2.

Next, in the case $W\in\mathcal{W}(p)$, we define an object $Y^I(p,\mathbf{r},R,S)$ of $\mathcal{B}_{\mathbf{R}}(\log)$ by fixing a closed real analytic subspace $L^{(1)}$ of $L\smallsetminus\{0\}$ such that $\mathbf{R}_{>0}\times L^{(1)}\to L\smallsetminus\{0\},(a,x)\mapsto a\circ x$, is an isomorphism of real analytic manifolds. Via the evident bijection between Ψ and the disjoint union of $\{W\}$ and Φ, we regard τ_p as a homomorphism $\mathbf{G}_{m,\mathbf{R}}\times\mathbf{G}^{\Phi}_{m,\mathbf{R}}\to\mathrm{Aut}(H_{0,\mathbf{R}},W)$. Let

$$(*)\qquad Y^I(p,\mathbf{r},R,S)\subset\mathbf{R}_{\ge0}\times Y^{II}(p,\mathbf{r},R,S)\times\mathfrak{g}_{\mathbf{R},u}$$

be the set consisting of all elements $(t_0,t,f,g,h,k,\delta,u,v)$ $(t_0\in\mathbf{R}_{\ge0},\ (t,f,g,h,k,\delta,u)\in Y^{II}(p,\mathbf{r},R,S),\ v\in\mathfrak{g}_{\mathbf{R},u})$ satisfying the following conditions (5′)–(7′).

(5′) Condition (5) holds, and furthermore, in the case $t_0=0$, we have $\exp(v)s_{\mathbf{r}}=s_{\mathbf{r}}$.

(6′) Let t' be any element of $\mathbf{R}^{\Phi}_{>0}$ such that $t'_j=t_j$ for any $j\in\Phi\smallsetminus J$. Let $m\in\mathbf{Z}^{\Phi}$, and assume $m(j)=0$ for any $j\in J$. If $t_0\ne0$, then $v_m=\mathrm{Ad}(\tau_p(t_0,t'))^{-1}(u_m)$. If $t_0=0$, then $v_m=\mathrm{Ad}(\tau_p(1,t'))^{-1}(u_m)$.

(7′) We have $\delta\in L^{(1)}$.

We endow $Y^I(p,\mathbf{r},R,S)$ with a structure of an object in $\mathcal{B}_{\mathbf{R}}(\log)$ via the injection $Y^I(p,\mathbf{r},R,S)\hookrightarrow\mathbf{R}_{\ge0}\times B\times\mathfrak{g}_{\mathbf{R},u}$.

We define a canonical morphism $Y^I(p,\mathbf{r},R,S)\to Y^{II}(p,\mathbf{r},R,S)$. In the case $W\notin\mathcal{W}(p)$, it is just the canonical projection. In the case $W\in\mathcal{W}(p)$, it is the morphism $(t_0, t', f, g, h, k, \delta, u, v) \mapsto (t', f, g, h, k, t_0 \circ \delta, u)$. In both cases, this morphism is injective.

Define an open subset $Y_0^I(p, \mathbf{r}, R, S)$ of $Y^I(p, \mathbf{r}, R, S)$ by the inverse image of $Y_0^{II}(p, \mathbf{r}, R, S)$ (see Section 3.4.3). Then we have an isomorphism of real analytic manifolds $Y_0^I(p, \mathbf{r}, R, S) \xrightarrow{\sim} Y_0^{II}(p, \mathbf{r}, R, S)$.

Combining this with $\eta^{II}_{p,\mathbf{r},R,S}$ (see Section 3.4.3), we have a morphism of real analytic manifolds

$$\eta^I_{p,\mathbf{r},R,S} : Y_0^I(p, \mathbf{r}, R, S) \to D.$$

THEOREM 3.4.6

Let the notation be as above. Assume $W \notin \Psi$ (resp., $W \in \Psi$). Then if U is a sufficiently small open neighborhood of $0 := (0,0,0,0)$ in $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times R \times \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times S$ and if $Y^I(p, \mathbf{r}, R, S, U)$ denotes the open set of $Y^I(p, \mathbf{r}, R, S)$ defined as the inverse image of U by the canonical map $Y^I(p, \mathbf{r}, R, S) \to \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times R \times \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times S$, then there is an open immersion $Y^I(p, \mathbf{r}, R, S, U) \to D^I_{\mathrm{SL}(2)}(\Psi)$ in the category $\mathcal{B}'_\mathbf{R}(\log)$ which sends $(0^\Phi, 0, 0, 0, 0, \delta(\mathbf{r}), 0, 0)$ (resp., $(0^\Psi, 0, 0, 0, 0, \delta(\mathbf{r})^{(1)}, 0, 0)$, where $\delta(\mathbf{r})^{(1)} \in L^{(1)}$ (see Section 3.4.5) such that $\delta(\mathbf{r}) = 0 \circ \delta(\mathbf{r})^{(1)}$) to p and whose restriction to $Y^I(p, \mathbf{r}, R, S, U) \cap Y_0^I(p, \mathbf{r}, R, S)$ coincides with the restriction of $\eta^I_{p,\mathbf{r},R,S}$ (see Section 3.4.5).

The proof is given in Section 3.4.20.

3.4.7.

Before we start to prove Theorems 3.4.4 and 3.4.6, we make some preparations.

Let the notation be as in Section 3.4.1. Then there exist an open neighborhood O of zero in $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)$ and a real analytic function $c = (c_1, c_2, c_3) : O \to \mathbf{R}^\Phi_{>0} \times R \times S$ having the following properties (1)–(4).

(1) For any $x \in O$, $\exp(x)\bar{\mathbf{r}} = \bar{\tau}_p(c_1(x)) \exp(c_2(x)) \exp(c_3(x))\bar{\mathbf{r}}$.

(2) We have $c(0) = (1, 0, 0)$.

(3) The map $\exp : O \to G_\mathbf{R}(\mathrm{gr}^W)$ is an injective open map.

(4) For $k = 2, 3$, c_k has the form of absolutely convergent series $c_k = \sum_{r=0}^\infty c_{k,r}$, where $c_{k,r}$ is the part of degree r in the Taylor expansion of c_k at zero, such that $c_{k,r}(x) = l_{k,r}(x \otimes \cdots \otimes x)$ for some linear map $l_{k,r} : \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)^{\otimes r} \to \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)$ having the following property: if $m_1, \dots, m_r \in \mathbf{Z}^\Phi$ and $x_j \in \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)_{m_j}$ for $1 \le j \le r$, then $l_{k,r}(x_1 \otimes \cdots \otimes x_r) \in \sum_m \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)_m$, where m ranges over all elements of $\mathbf{Z}^\Phi$ of the form $\sum_{1 \le j \le r} e_j m_j$ with $e_j \in \{1, -1\}$ for each j.

This is proved similarly as [KU3, Lemma 10.3.4] (cf. also Section 3.3.11). It is clear that there is a real analytic c satisfying (1)–(3) unique up to restrictions of domains of definitions. The property (4) of Taylor expansion can be checked formally as follows.

Consider the following formal calculation:

$$\exp(x) = \exp(t^{(1)} + b^{(1)} + k^{(1)}) = \exp(t^{(1)}) \exp(b^{(1)} + x^{(1)}) \exp(k^{(1)})$$

$$= \exp(t^{(1)})\exp(b^{(1)}+t^{(2)}+b^{(2)}+k^{(2)})\exp(k^{(1)})$$
$$= \exp(t^{(1)})\exp(t^{(2)})\exp(b^{(1)}+b^{(2)}+x^{(2)})\exp(k^{(2)})\exp(k^{(1)}) = \cdots .$$

Here $x \in O$, $t^{(j)} \in \mathrm{Lie}(\tilde{\rho}(\mathbf{R}^{\Phi}_{>0}))$ with $\tilde{\rho}$ being as in Section 3.4.1 (note that the actions of $\tilde{\rho}(t)$ and $\bar{\tau}_p(t)$ for $t \in \mathbf{R}^{\Phi}_{>0}$ on $D(\mathrm{gr}^W)$ coincide), and $b^{(j)} \in R$, $k^{(j)} \in S$, $x^{(j)} \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ for any j. Then we have $\tilde{\rho}(c_1(x)) = \exp(t^{(1)})\exp(t^{(2)})\cdots$, $c_2(x) = b^{(1)}+b^{(2)}+\cdots$, and $\exp(c_3(x)) = \cdots\exp(k^{(2)})\exp(k^{(1)})$ formally. From these, we can prove property (4) formally.

3.4.8.
We prove Theorem 3.4.4 up to Section 3.4.19. After that, we prove Theorem 3.4.6. Let p, Φ, and $\mathbf{r}$ be as in Section 3.4.1. In the notation in Section 3.4.7, let $U = \exp(O)\bar{\mathbf{r}}$ which is an open neighborhood of $\bar{\mathbf{r}}$ in $D(\mathrm{gr}^W)$. By Section 3.4.7, there is a real analytic map

$$a = (a_1, a_2, a_3) : U \to \mathbf{R}^{\Phi}_{>0} \times R \times S$$

such that for any $y \in U$, we have $y = \bar{\tau}_p(a_1(y))\exp(a_2(y))\exp(a_3(y))\bar{\mathbf{r}}$. (Just put $a_j(\exp(x)\bar{\mathbf{r}}) = c_j(x)$ for $x \in O$.)

3.4.9.
Fix a distance β to Φ-boundary such that $\beta(\bar{\mathbf{r}}) = 1$. Here we denote $\beta(x) = \beta(x(\mathrm{gr}^W))$ $(x \in D)$ by abuse of notation. Let $\mu : D(\mathrm{gr}^W) \to D(\mathrm{gr}^W)$ be the real analytic map defined by $\mu(x) = \bar{\tau}_p(\beta(x))^{-1}x$. Denote the composite $D \to D(\mathrm{gr}^W) \xrightarrow{\mu} D(\mathrm{gr}^W)$ also by μ by abuse of notation. Let $D(U) \subset D$ be the inverse image of U by μ.

Let

$$b = b_{R,S} : D(U) \to Y_0^{II}(p, \mathbf{r}, R, S)$$

be the real analytic map $x \mapsto (t, f, g, h, k, \delta, u)$, where $t = \beta(x)a_1(\mu(x))$, $f = \mathrm{Ad}(\bar{\tau}_p(t))(a_2(\mu(x)))$, $g = a_2(\mu(x))$, $h = \mathrm{Ad}(\bar{\tau}_p(t))^{-1}(a_2(\mu(x)))$, $k = a_3(\mu(x))$, $\delta = \mathrm{Ad}(\bar{\tau}_p(t)\exp(g)\exp(k))^{-1}(\delta(x))$, and u is characterized by $\mathrm{spl}_W(x) = \exp(u)\cdot \mathrm{spl}_W(\mathbf{r})$.

Recall that, in Theorem 3.4.4, for an open neighborhood U' of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$, we denote by $Y^{II}(p, \mathbf{r}, S, U')$ the subset of $Y^{II}(p, \mathbf{r}, S)$ consisting of all elements $(t, f, g, h, k, \delta, u)$ such that $(f, g, h, k) \in U'$. We also defined $Y_0^{II}(p, \mathbf{r}, S, U')$ and $Y^{II}(p, \mathbf{r}, R, S, U')$ there. Now, we define $Y_0^{II}(p, \mathbf{r}, R, S, U') = Y^{II}(p, \mathbf{r}, R, S, U') \cap Y_0^{II}(p, \mathbf{r}, S)$.

The next two lemmas are easily seen.

LEMMA 3.4.10
The composite $D(U) \xrightarrow{b} Y_0^{II}(p, \mathbf{r}, R, S) \to D$ is the canonical inclusion.

LEMMA 3.4.11
If U' is sufficiently small, the image of $Y_0^{II}(p, \mathbf{r}, S, U')$ in D is contained in $D(U)$

and the map $Y_0^{II}(p,\mathbf{r},R,S,U') \to D(U) \to Y_0^{II}(p,\mathbf{r},R,S)$ *is the canonical inclusion.*

3.4.12.
We define

$$p(J,\mathbf{r},z,\delta,u) \in D^{II}_{\mathrm{SL}(2)}(\Phi)$$

as follows for a subset J of Φ, a point $\mathbf{r}$ on the torus orbit associated to p (see Proposition 2.5.2), an element z of $G_{\mathbf{R}}(\mathrm{gr}^W)$ which satisfies

(1) $z \in G_{\mathbf{R}}(\mathrm{gr}^W)_J$,

an element δ of $\bar{L}$, and an element u of $\mathfrak{g}_{\mathbf{R},u}$.

This $p(J,\mathbf{r},z,\delta,u)$ is the unique element of $D_{\mathrm{SL}(2)}$ which satisfies the following (2)–(5).

(2) The set of weight filtrations on gr^W associated to $p(J,\mathbf{r},z,\delta,u)$ is J.

(3) The torus action $\bar{\tau}$ associated to $p(J,\mathbf{r},z,\delta,u)$ is $\mathrm{Int}(z)(\bar{\tau}_{p,J}) : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W)$, where $\bar{\tau}_{p,J}$ denotes the restriction of $\bar{\tau}_p : \mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W)$ (see Sections 2.5.6, 2.3.5) to the J-component of $\mathbf{G}^{\Phi}_{m,\mathbf{R}}$.

(4) We have $\delta \in L$ in $\bar{L}$ if and only if W does not belong to the set of weight filtrations associated to $p(J,\mathbf{r},z,\delta,u)$.

(5) The torus orbit associated to $p(J,\mathbf{r},z,\delta,u)$ (see Proposition 2.5.2) contains $\exp(u)s_{\mathbf{r}}\theta\big(z(\mathbf{r}(\mathrm{gr}^W)),\mathrm{Ad}(z)(\delta)\big)$ if $\delta \in L$, and contains $\exp(u)s_{\mathbf{r}}\theta\big(z(\mathbf{r}(\mathrm{gr}^W)),\mathrm{Ad}(z)(\delta')\big)$ if $\delta \in \bar{L} \smallsetminus L$ and $\delta = 0 \circ \delta'$ with $\delta' \in L \smallsetminus \{0\}$.

This $p(J,\mathbf{r},z,\delta,u)$ is constructed as follows. Let n be the cardinality of $\Psi = \mathcal{W}(p)$, and identify Ψ with $\{1,\dots,n\}$ as a totally ordered set for the ordering in Proposition 2.3.8. In the case $W \notin \Psi$, consider the bijection $\Psi \to \Phi$. In the case $W \in \Psi$, consider the bijection $\Psi \smallsetminus \{W\} \to \Phi$. Via these bijections, embed $J \subset \Phi$ into Ψ. In the case $\delta \in L$ (resp., $\delta \in \bar{L} \smallsetminus L$), let $m = \sharp(J)$ (resp., $m = \sharp(J)+1$), and write $J = \{j_1,\dots,j_m\} \subset \Psi$ with $j_1 < \cdots < j_m$ (resp., $J = \{j_2,\dots,j_m\} \subset \Psi$ with $j_2 < \cdots < j_m$). Let $((\rho_w,\varphi_w)_w,\mathbf{r})$ be an SL(2)-orbit in n variables of rank n whose class in $D_{\mathrm{SL}(2)}$ is p. Then, in the case $\delta \in L$ (resp., $\delta \in \bar{L} \smallsetminus L$), the $p(J,\mathbf{r},z,\delta,u)$ is the class of the following SL(2)-orbit $((\rho',\varphi') = (\rho'_w,\varphi'_w)_w,\mathbf{r}')$ in m variables of rank m:

$$\rho'(g_1,\dots,g_m) := \mathrm{Int}(z)\big(\rho(g'_1,\dots,g'_n)\big),$$

$$\varphi'(z_1,\dots,z_m) := z\varphi(z'_1,\dots,z'_n),$$

$$\mathbf{r}' := \exp(u)s_{\mathbf{r}}\theta\big(z(\mathbf{r}(\mathrm{gr}^W)),\mathrm{Ad}(z)(\delta)\big)$$

(resp., $\mathbf{r}' := \exp(u)s_{\mathbf{r}}\theta\big(z(\mathbf{r}(\mathrm{gr}^W)),\mathrm{Ad}(z)(\delta')\big)$ with $\delta' \in L \smallsetminus \{0\}, \delta = 0 \circ \delta'$), where g'_j and z'_j $(1 \le j \le n)$ are as follows. If $j \le j_k$ for some k, define $g'_j := g_k$ and $z'_j := z_k$ for the smallest integer k with $j \le j_k$. Otherwise, $g'_j := 1$ and $z'_j := i$.

Let $Y_1 := Y_1^{II}(p,\mathbf{r},S)$ be the subset of $Y^{II}(p,\mathbf{r},S)$ consisting of all elements (t,f,g,h,k,δ,u) such that $h_m = 0$ unless $m(j) = 0$ for all $j \in J(t)$. We have $Y_1 \supset Y_0 := Y_0^{II}(p,\mathbf{r},S)$. We have the following.

(6) A point $(t,f,g,h,k,\delta,u)\in Y_1^{II}(p,\mathbf{r},S)$ is the limit of $y(t',\delta')\in Y_0^{II}(p,\mathbf{r},S)$ defined by $y(t',\delta')=\big(t',f,\mathrm{Ad}(\tau_p(t'))^{-1}(f),\mathrm{Ad}(\tau_p(t'))^{-2}(f),k,\delta',u\big)$, where $t'\in\mathbf{R}^{\Phi}_{>0}$, $\delta'\in L$, and t' tends to t and δ' tends to δ. Write $\exp(k)\cdot\bar{\mathbf{r}}=k'\cdot\bar{\mathbf{r}}$ with $k'\in K_{\bar{\mathbf{r}}}\cap G_{\mathbf{R}}(\mathrm{gr}^W)_J$. Note that k' commutes with $\bar\tau_p(t')$. The image of $y(t',\delta')$ in D is $\exp(u)s_{\mathbf{r}}\theta(z(\bar{\mathbf{r}}),\mathrm{Ad}(z)(\delta''))$, where $z=\exp(f)k'\bar\tau_p(t')$ and $\delta''=\mathrm{Ad}((k')^{-1}\exp(k))(\delta')$.

We extend the map $\eta^{II}_{p,\mathbf{r},S}:Y_0\to D$ in Section 3.4.3 to a map

$$\eta^{II}_{p,\mathbf{r},S}:Y_1\to D^{II}_{\mathrm{SL}(2)}(\Phi),$$
$$\eta^{II}_{p,\mathbf{r},S}(t,f,g,h,k,\delta,u)=p(J,\mathbf{r},z,\delta',u),$$

where J, z, and δ' are defined as follows. Let $J=\{j\in\Phi\mid t_j=0\}$. Let t' be an element of $\mathbf{R}^{\Phi}_{>0}$ such that $t'_j=t_j$ for any $j\in\Phi\smallsetminus J$, and let k' be an element of $K_{\bar{\mathbf{r}}}\cap G_{\mathbf{R}}(\mathrm{gr}^W)_J$ such that $\exp(k)\cdot\bar{\mathbf{r}}=k'\cdot\bar{\mathbf{r}}$. Let $z=\exp(f)k'\bar\tau_p(t')$ and $\delta'=\mathrm{Ad}((k')^{-1}\exp(k))\delta$.

We use the following fact (7) which is deduced from [KU3, Section 10.2.16].

(7) Let $\mu:D^{II}_{\mathrm{SL}(2)}(\Phi)\to D(\mathrm{gr}^W)$ be the extension of $\alpha\beta(x(\mathrm{gr}^W))^{-1}x(\mathrm{gr}^W)$ ($x\in D$) given in Proposition 3.2.6(ii). Then, if $p'\in D^{II}_{\mathrm{SL}(2)}(\Phi)$ and if $\mu(p')$ is sufficiently near to $\mu(p)$, p' is expressed as $p(J,\mathbf{r},z,\delta',u)$ as above.

LEMMA 3.4.13

There are an open neighborhood U' of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$ and a morphism $\xi:Y^{II}(p,\mathbf{r},S,U')\to Y^{II}(p,\mathbf{r},R,S)$ which satisfy the following conditions: $\eta^{II}_{p,\mathbf{r},S}$ sends $Y_0^{II}(p,\mathbf{r},S,U')$ into $D(U)$, and the restriction of ξ to $Y_0^{II}(p,\mathbf{r},S,U')$ coincides with the composite $Y_0^{II}(p,\mathbf{r},S,U')\xrightarrow{\eta^{II}_{p,\mathbf{r},S}}D(U)\xrightarrow{b}Y_0^{II}(p,\mathbf{r},R,S)$, where b is as in Section 3.4.9.

Proof
Let $x=\eta^{II}_{p,\mathbf{r},S}(t,f,g,h,k,\delta,u)$, and write $b(x)$ as $(t',f',g',h',k',\delta',u')$.

First, we show that each component $t',f',g',\dots$ extends real analytically over the boundary of $Y^{II}(p,\mathbf{r},S,U')$ for some U'. Since $\mu(x)=\bar\tau_p\big(\beta(\exp(g)\cdot\exp(k)\bar{\mathbf{r}})\big)^{-1}\exp(g)\exp(k)\bar{\mathbf{r}}$, this extends over the boundary. Hence so does $a_j\mu(x)$ for each $j=1,2,3$ (see Section 3.4.8). On the other hand, $\beta(x)=t\beta(\exp(g)\cdot\exp(k)\bar{\mathbf{r}})$, and this is also real analytic over the boundary because β is so. Thus t',g',k' extend. Further, $u'=u$ trivially extends. We have $\delta'=\mathrm{Ad}(\bar\tau_p(t')\exp(g')\cdot\exp(k'))^{-1}\,\mathrm{Ad}(\bar\tau_p(t)\exp(g)\exp(k))(\delta)$. Since g' and k' already extend and since $t't^{-1}=\beta(\exp(g)\exp(k)\bar{\mathbf{r}})a_1\mu(x)$ also extends, so does δ'.

The rest are f' and h', that is, to see that $\mathrm{Ad}(\bar\tau_p(t'))^{\pm1}a_2\mu(x)$ extend real analytically. We can replace t' in the last formula with t because $t'=t\beta(\exp(g)\cdot\exp(k)\bar{\mathbf{r}})a_1(\mu(x))$. Further, by Section 3.4.7 with the formal construction there, $a_2(\mu(x))=c_2(g)$. Hence, it is enough to show that $\mathrm{Ad}(\bar\tau_p(t))^{\pm1}c_2(g)$ extend.

Consider the decomposition $g=\sum_{m\in\mathbf{Z}^\Phi}g_m$ ($g_m\in\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m$). Then, by property (4) of c_2 in Section 3.4.7, $c_2(g)=c_2(\sum g_m)$ is the infinite formal sum of

$l_{2,r}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ ($m_j \in \mathbf{Z}^{\Phi}$, $g_{m_j} \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{m_j}$ $(1 \le j \le r)$). Now the weights m of $l_{2,r}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ satisfy $m = \sum e_j m_j$ with $e_j \in \{1, -1\}$. Decompose $l_{2,r}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ into $\sum_m l_{2,r,m}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ according to the weights, where m ranges over such $\sum e_j m_j$. We see that, for each m and $j \in \{1, -1\}$, $\bar{\tau}_p(t)^j l_{2,r,m}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ extends over the boundary. We explain the proof for $j = 1$. The other case is similar. In this case, we observe that $\bar{\tau}_p(t) l_{2,r,m}(g_{m_1} \otimes \cdots \otimes g_{m_r})$ is $(\prod (t^{m_j})^{e_j}) l_{2,r,m}(g_{m_1} \otimes \cdots \otimes g_{m_r}) = l_{2,r,m}((t^{m_1})^{e_1} g_{m_1} \otimes \cdots \otimes (t^{m_r})^{e_r} g_{m_r})$. Since $t^m g_m = f_m$ and $t^{-m} g_m = h_m$, the last function extends to a real analytic map over the boundary. Shrinking U' if necessary, we may assume that f and h are sufficiently near to zero, and the above infinite sum converges, as desired.

Next, we show that in the ambient product space containing $Y^{II}(p, \mathbf{r}, R, S)$, the image of each element $y = (t, f, g, h, k, \delta, u)$ of $Y^{II}(p, \mathbf{r}, S, U')$ by the extended coordinate functions in fact belongs to $Y^{II}(p, \mathbf{r}, R, S)$, which completes the proof. For $t' \in \mathbf{R}^{\Phi}_{>0}$ such that $t'_j = t_j$ for any $j \in \Phi \smallsetminus J$ with $J = J(t)$ and for $\delta' \in L$, let $y(t', \delta') = (t', f', g', h', k, \delta', u) \in Y^{II}_0(p, \mathbf{r}, S)$, where $f', g', h' \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ are defined as follows. Let $m \in \mathbf{Z}^{\Phi}$. Then $f'_m = (t')^{2m} h_m$, $g'_m = (t')^m h_m$, $h'_m = h_m$ if $m(j) \ge 0$ for any $j \in J$, $f'_m = f_m$, $g'_m = (t')^{-m} f_m$, $h'_m = (t')^{-2m} f_m$ if $m(j) \le 0$ for any $j \in J$ and $m(j) < 0$ for some $j \in J$, and $f'_m = g'_m = h'_m = 0$ otherwise. Here $(t')^m := \prod_{j \in \Phi} (t'_j)^{m(j)}$, and so on. Then $y(t', \delta') \to y$ in $Y^{II}(p, \mathbf{r}, S)$ when $t' \to t$ and $\delta' \to \delta$.

We have to prove that the limit $(t_0, f_0, g_0, \ldots)$ of the image $(t'', f'', g'', \ldots)$ of $y(t', \delta')$ in the ambient product space satisfies Section 3.4.2(1)–(4). First, it is easy to see $J := J(t_0) = J(t)$. Conditions (2) and (3) are deduced from the corresponding conditions on $(t'', f'', g'', \ldots)$. Condition (1) is also seen from condition (2) on $(t'', f'', g'', \ldots)$. For example, we show that $(f_0)_m = 0$ unless $m(j) \le 0$ for any $j \in J$. We have $f''_m = (t'')^m g''_m$ for any $m \in \mathbf{Z}^{\Phi}$. Since $t'' = t' \beta(\exp(g') \exp(k) \bar{\mathbf{r}}) a_1 \mu(y(t', \delta'))$, if there is some $j \in J$ such that $m(j) > 0$, the above equality implies $f''_m \to 0 \cdot (\lim g''_m) = 0$. Hence we have $(f_0)_m = 0$. Finally, (4) is seen as follows. Let k' be the element of $\mathrm{Lie}(K_{\bar{\mathbf{r}}})$ such that $\exp(g) = \exp(g_0) \exp(k')$ and $k'_m = 0$ unless $m(j) = 0$ for any $j \in J$. Then we have $\exp(k_0) = \exp(k') \exp(k)$. Hence k_0 satisfies (4). □

LEMMA 3.4.14

There are an open neighborhood U' of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$ and a morphism $Y^{II}(p, \mathbf{r}, S, U') \to B := \mathbf{R}^{\Phi}_{\ge 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W')$ whose restriction to $Y^{II}_0(p, \mathbf{r}, S, U')$ coincides with the composite $\nu_{\bar{\tau}_p, \beta} \circ \eta^{II}_{p, \mathbf{r}, S}$ (see Proposition 3.2.6, Section 3.4.3).

Proof

It is enough to show that the composite map from $Y^{II}_0(p, \mathbf{r}, S, U')$ extends componentwise over the boundary. The components except the last ones (Borel-Serre splittings) are easily treated. For example, the first two were already treated in the proof of Lemma 3.4.13. The extendability of Borel-Serre splittings is reduced to Lemma 3.4.13. In fact, let $W' \in \Phi$. Then, by Lemmas 3.4.10 and 3.4.13, it is sufficient to prove that the composite $Y_0^{II}(p,\mathbf{r},R,S,U') \to Y_0^{II}(p,\mathbf{r},S,U') \to \mathrm{spl}(W')$ extends to a real analytic map on $Y^{II}(p,\mathbf{r},R,S,U')$ under the assumption $R \subset \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$. Assuming this, we prove $f_m \in \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$ for any $(t,f,g,h,k,\delta,u) \in Y^{II}(p,\mathbf{r},R,S,U')$ and any $m \in \mathbf{Z}^\Phi$. This is clear if $m(W') \le 0$. If $m(W') \ge 0$, since $f_m + h_{-m} \in R \subset \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$ and $h_{-m} \in \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$, we have $f_m \in \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$. Thus $\exp(f)$ belongs to $G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$, so that the concerned component is $\mathrm{spl}^{\mathrm{BS}}_{W'}(\exp(f)\bar{\tau}_p(t)\exp(k)\bar{\mathbf{r}}) = \exp(f)\,\mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{\mathbf{r}})\,\mathrm{gr}^{W'}\exp(f)^{-1}$, which real analytically extends over the boundary. □

LEMMA 3.4.15

There exist open neighborhoods $U'' \subset U'$ of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$ such that, for any $y \in Y^{II}(p,\mathbf{r},S,U'')$, there exists $y_1 \in Y_1^{II}(p,\mathbf{r},S) \cap Y^{II}(p,\mathbf{r},S,U')$ such that (y_1,y) belongs to the closure of $Y_0^{II}(p,\mathbf{r},S) \times_D Y_0^{II}(p,\mathbf{r},S)$ in $Y_1^{II}(p,\mathbf{r},S) \times Y^{II}(p,\mathbf{r},S)$.

Proof

For any subset J of Φ, take $R = R_J$ as in Section 3.4.1 such that $\mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{J,u}) \subset R_J$. Here $G_{\mathbf{R}}(\mathrm{gr}^W)_{J,u}$ denotes the unipotent part of $G_{\mathbf{R}}(\mathrm{gr}^W)_J$. For this $R = R_J$, let U_J be the neighborhood U in Section 3.4.8, and let U' be a neighborhood of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$ such that $Y^{II}(p,\mathbf{r},S,U')$ is contained in $(\eta_{p,\mathbf{r},S})^{-1}\big(\bigcap_J D(U_J)\big)$.

Let $y = (t,f,g,h,k,\delta,u) \in Y^{II}(p,\mathbf{r},S,U')$. For $t' \in \mathbf{R}^\Phi_{>0}$ such that $t'_j = t_j$ for any $j \in \Phi \smallsetminus J$ with $J = J(t)$ and for $\delta' \in L$, consider $y(t',\delta')$ in the proof of Lemma 3.4.13.

Let $R = R_{J(t)}$. Then, for any (t',δ') which is sufficiently near to (t,δ), the point $y_1(t',\delta') := b_{R,S}\big(\eta^{II}_{p,\mathbf{r},S}(y(t',\delta'))\big)$ is well defined and $(y_1(t',\delta'), y(t',\delta')) \in Y_0^{II}(p,\mathbf{r},S) \times_D Y_0^{II}(p,\mathbf{r},S)$. Furthermore, $y_1(t',\delta')$ converges to an element y_1 of $Y^{II}(p,\mathbf{r},S)$ when $t' \to t$ and $\delta' \to \delta$ by Lemma 3.4.13. We show that the limit $y_1 = (t_0, f_0, g_0, h_0, \ldots)$ belongs to $Y_1^{II}(p,\mathbf{r},S)$; that is, $(h_0)_m = 0$ if $m(j) \ge 0$ for any $j \in J(t) = J(t_0)$ and if $m(j) > 0$ for some $j \in J(t_0)$. Fix such an m. Then we have $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{-m} \subset R_J$ and $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_m \cap R_J = \{0\}$. Hence the property $(f_0)_{-m} + (h_0)_m \in R_J$ implies $(h_0)_m = 0$.

Finally, for a sufficiently small $U'' \subset U'$, the above correspondence $y \mapsto y_1$ sends $Y^{II}(p,\mathbf{r},S,U'')$ into $Y^{II}(p,\mathbf{r},S,U')$. □

LEMMA 3.4.16

(i) *On the intersection of $Y_1 = Y_1^{II}(p,\mathbf{r},S)$ and $Y(U') := Y^{II}(p,\mathbf{r},S,U')$, the map $Y(U') \to B$ in Lemma 3.4.14 coincides with the restriction of the composite $Y_1 \to D^{II}_{\mathrm{SL}(2)}(\Phi) \to B$.*

(ii) *For a sufficiently small U', the image of $Y(U') \to B$ in Lemma 3.4.14 is contained in the image of $D^{II}_{\mathrm{SL}(2)}(\Phi)$.*

Proof

(i) This follows from Section 3.4.12(6). (ii) This follows from (i) and Lemma 3.4.15. □

LEMMA 3.4.17

Let U be a sufficiently small open neighborhood of $\bar{\mathbf{r}}$ in $D(\mathrm{gr}^W)$, and let $D^{II}_{\mathrm{SL}(2)}(U)$ be the inverse image of U under $D^{II}_{\mathrm{SL}(2)}(\Phi) \xrightarrow{\mu} D(\mathrm{gr}^W)$. Let $q \in D^{II}_{\mathrm{SL}(2)}(U)$, and let $\mathbf{r}_q$ be a point on the torus orbit associated to q. Then the limit $\lim_{t \to 0^{\mathcal{W}(q)}} b(\tau_q(t)\mathbf{r}_q)$ exists in $Y^{II}(p, \mathbf{r}, R, S)$ and is independent of the choice of $\mathbf{r}_q$.

Proof

We reduce this to Lemma 3.4.13. First, by Section 3.4.12(7), we may assume that q has the form $p(J, \mathbf{r}, z, \delta, u)$ such that $\mathbf{r}_q$ is the point in Section 3.4.12(5). Hence it is the image of some $y_1 = (s, f, g, h, k, \delta, u) \in Y_1$ by $\eta^{II}_{p,\mathbf{r},S}$ in Section 3.4.12. Then $\tau_q(t)\mathbf{r}_q$ is the image of $y_1(t) := \big(t', f, \mathrm{Ad}(\bar{\tau}_p(t'))^{-1} f, \mathrm{Ad}(\bar{\tau}_p(t'))^{-2} f, k, \delta'', u\big)$, where $t' \in \mathbf{R}^{\Phi}_{>0}$ such that $t'_j = t_j$ for any $j \in J$ and $t'_j = s_j$ for any $j \in \Phi \smallsetminus J$ and $\delta'' = \delta$ if $\delta \in L$ and $\delta'' = t_W \circ \delta'$ for $\delta' \in L$ in Section 3.4.12(5) if $\delta \in \bar{L} \smallsetminus L$. Since $y_1(t)$ converges to y_1, the sequence $b(\tau_q(t)\mathbf{r}_q)$ converges to the image of y_1 by ξ in Lemma 3.4.13. The last independency is clear. □

Denote this limit by $b(q)$. Thus b in Section 3.4.9 is extended to a map $D^{II}_{\mathrm{SL}(2)}(U) \to Y^{II}(p, \mathbf{r}, R, S)$.

3.4.18. Proof of Theorem 3.4.4

Theorem 3.4.4(i) follows from Lemma 3.4.16(ii). We prove Theorem 3.4.4(ii). We first describe the idea of the proof.

Locally on $Y(R, S) := Y^{II}(p, \mathbf{r}, R, S)$, we define an object X of $\mathcal{B}_{\mathbf{R}}(\log)$ which contains $Y(R, S)$ having the following properties.

(1) The morphism $Y(R, S) \to B$ (defined locally) extends to some explicit morphism $X \to B$ (locally). (It is explained in Section 3.4.19.)

(2) As an object of $\mathcal{B}_{\mathbf{R}}(\log)$, X is isomorphic to the product $\mathbf{R}^{\Phi}_{\geq 0} \times$ (a real analytic manifold) $\times \bar{L}$. Hence, for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is isomorphic to the ring of convergent power series in n variables over $\mathbf{R}$ for some n. Note that $Y(R, S)$ need not have this last property (because $Y(R, S)$ can have a singularity of the style $t_1^2 x = t_2 y$), and this is the reason why we use X here.

(3) The homomorphism $\mathcal{O}_X|_{Y(R,S)} \to \mathcal{O}_{Y(R,S)}$ is surjective. Here $\mathcal{O}_X|_{Y(R,S)}$ is the inverse image of $\mathcal{O}_X$ on $Y(R, S)$.

(4) The homomorphism $\mathcal{O}_B|_X \to \mathcal{O}_X$ is surjective. Here $\mathcal{O}_B|_X$ denotes the inverse image of $\mathcal{O}_B$ on X.

Although (3) is shown easily, (4) is not. But by the property of the local rings explained in (2), the property (4) is reduced to the surjectivity of $m_{B,y}/m^2_{B,y} \to m_{X,x}/m^2_{X,x}$, where $x \in X$ and y is the image of x in B. This is the injectivity of the map of tangent spaces $T_x(X) \to T_y(B)$, where $T_x(X)$ and $T_y(B)$ are $\mathbf{R}$-linear duals of $m_{X,x}/m_{X,x}^2$ and $m_{B,y}/m_{B,y}^2$, respectively, and this injectivity is explained in Section 3.4.19.

By (3) and (4), we have the surjectivity of $\mathcal{O}_B|_{Y(R,S)} \to \mathcal{O}_{Y(R,S)}$. Since $Y(R,S) \to B$ factors (locally) as $Y(R,S) \to A \to B$ by Lemma 3.4.16(ii), where $A := D^{II}_{\mathrm{SL}(2)}(\Phi)$, we see that the map $\mathcal{O}_A|_{Y(R,S)} \to \mathcal{O}_{Y(R,S)}$ is surjective.

Since the map $Y(R,S) \to A$ has the inverse map $A \to Y(R,S)$ (locally) by Lemma 3.4.17, $Y(R,S) \to A$ is bijective locally.

Since $\mathcal{O}_A|_{Y(R,S)} \to \mathcal{O}_{Y(R,S)}$ is injective (they are subsheaves of the sheaves of functions), we have $(Y(R,S), \mathcal{O}_{Y(R,S)}) \simeq (A, \mathcal{O}_A)$ locally. It is easy to see that this isomorphism preserves the log structures with sign.

3.4.19.

We give the definition of X and the proof of the property 3.4.18(4).

Actually X is constructed at each point of $Y(R,S)$. We give the construction at $\tilde{p} = (0^\Phi, 0, 0, 0, 0, \delta(\mathbf{r}), 0) \in Y(R,S)$ and the proof of the property 3.4.18(4) for the tangent space at $\tilde{p}$. The general case is similar.

We define the set X to be the subset of $E := \mathbf{R}^\Phi_{\geq 0} \times \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \times S \times \bar{L} \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements $(t, f, g, h, k, \delta, u)$ satisfying the following conditions (1)–(3).

(1) If $m \in \mathbf{Z}^\Phi$ and $m(j) \geq 0$ for any $j \in \Phi$, then $f_m = t^m g_m$ and $g_m = t^m h_m$. Here $t^m := \prod_{j\in\Phi} t_j^{m(j)}$.

(2) If $m \in \mathbf{Z}^\Phi$ and $m(j) \leq 0$ for any $j \in \Phi$, then $h_m = t^{-m} g_m$ and $g_m = t^{-m} f_m$.

(3) We have $g \in R$ and $f_m + h_{-m} \in R$ for all $m \in \mathbf{Z}^\Phi$.

Define the structure on X as an object of $\mathcal{B}_\mathbf{R}(\log)$ by using the embedding $X \subset E$ just as we defined the structure of $Y(R,S)$ as an object of $\mathcal{B}_\mathbf{R}(\log)$ by using the embedding $Y(R,S) \subset E$ in Section 3.4.2. Then it is clear that X is isomorphic to a product $\mathbf{R}^\Phi_{\geq 0} \times$ (a real analytic manifold) $\times \bar{L}$ as an object of $\mathcal{B}_\mathbf{R}(\log)$.

We give a morphism $X \to B$ which extends $Y(R,S) \to B$ and prove property 3.4.18(4) for it. We define the morphism componentwise. Let X_0 be the inverse image of $\mathbf{R}^\Phi_{>0} \times L$ by the natural map $X \to \mathbf{R}^\Phi_{\geq 0} \times \bar{L}$. First, we define $X_0 \to B' := \mathbf{R}^\Phi_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W)$ as the projection after $\nu_{\bar{\tau}_p,\beta} \circ \eta$, where η sends $(t, f, g, h, k, \delta, u)$ to $\exp(u) s_\mathbf{r} \theta(d\bar{\mathbf{r}}, \mathrm{Ad}(d)\delta)$ with $d = \bar{\tau}_p(t)\exp(g)\exp(k)$. Then this map $X_0 \to B'$ extends to $X \to B'$, as is seen easily in the same way as in Lemma 3.4.14. Next, for each $j = W' \in \Phi$, we give an extension to $\mathrm{spl}(W')$. Define $X_0 \to \mathrm{spl}(W')$ as follows. Consider the decomposition $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) = \mathfrak{g}_\leq \oplus \mathfrak{g}_{>0}$, where $\mathfrak{g}_\leq = \sum_{m(j)\leq 0} \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)_m$ and $\mathfrak{g}_> = \sum_{m(j)>0} \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)_m$. Then there are a neighborhood V_1 of zero in $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)$ and a real analytic map $(c_\leq, c_>) : V_1 \to \mathfrak{g}_\leq \times \mathfrak{g}_>$ such that for any $g \in V_1$, we have $\exp(g) = \exp(c_\leq(g))\exp(c_>(g))$. Further, let M be an $\mathbf{R}$-subspace of $\sum_{m(j)\geq 0} \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)_m$ containing $\mathfrak{g}_>$ such that $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) = M \oplus \mathrm{Lie}(K_{\bar{\mathbf{r}}})$. Then there are a neighborhood V_2 of zero in $\mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)$

and a real analytic map $(c_1', c_2') : V_2 \to M \times \mathrm{Lie}(K_{\bar{\mathbf{r}}})$ such that for any $g' \in V_2$, we have $\exp(g') = \exp(-Cc_1'(g'))\exp(c_2'(g'))$, where C is the Cartan involution at $\bar{\mathbf{r}}$. We define $X_0 \to \mathrm{spl}(W')$ (locally) as $\mathrm{spl}^{\mathrm{BS}}_{W'}\big(\exp(c_{\leq}(f))\exp(-C(c_1'(c_{>}(h))))\bar{\mathbf{r}}\big)$. This extends to $X \to \mathrm{spl}(W')$ and gives an extension of $Y(R,S) \to \mathrm{spl}(W')$ since $\mathrm{Int}(\bar{\tau}_p(t))\exp(g) = \exp(f)$, and so on, on $Y(R,S)$.

We prove the surjectivity of $\mathcal{O}_B|_X \to \mathcal{O}_X$. We write the proof of the surjectivity for the stalk at $\tilde{p}$. (The general case is similar.) It is sufficient to prove the injectivity of $T_{\tilde{p}}(X) \to T_q(B)$, where q denotes the image of $\tilde{p}$ in B.

The first tangent space is identified with the vector subspace V of $\mathbf{R}^{\Phi} \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S \times L \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements $(t, f, g, h, k, \delta, u)$ satisfying the following conditions (1) and (2).

(1) $f_m = g_m = 0$ if $m(j) \geq 0$ for any $j \in \Phi$, and $g_m = h_m = 0$ if $m(j) \leq 0$ for any $j \in \Phi$.

(2) We have $g \in R$ and $f_m + h_{-m} \in R$ for all $m \in \mathbf{Z}^{\Phi}$.

The injectivity of the map of tangent spaces in problem is reduced to the injectivity of the following map:

$$V \to \mathbf{R}^{\Phi} \times R \times S \times L \times \mathfrak{g}_{\mathbf{R},u} \times \Big(\prod_{j \in \Phi} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\Big),$$

$$(t, f, g, h, k, \delta, u) \mapsto \big(t, g, k, \delta, u, (v_j)_{j \in \Phi}\big),$$

$$\text{where} \quad v_j = \sum_{m(j) < 0} \big(f_m - C(h_{-m})\big).$$

Assume that the image of $(t, f, g, h, k, \delta, u) \in V$ under this map is zero. Then clearly we have $t = g = k = \delta = u = 0$. We have also the following.

(i) If $m(j) < 0$ for some $j \in \Phi$, then $f_m = h_{-m} = 0$.

Indeed, if $m(j) < 0$ for some $j \in \Phi$, then $f_m - C(h_{-m}) = 0$. Since $h_{-m} + C(h_{-m}) \in \mathrm{Lie}(K_{\bar{\mathbf{r}}})$, $f_m + h_{-m} \in R \cap \mathrm{Lie}(K_{\bar{\mathbf{r}}}) = 0$, and consequently we have (i).

This shows that if $m(j) < 0$ and $m(j') > 0$ for some $j, j' \in \Phi$, then $f_m = f_{-m} = h_m = h_{-m} = 0$. If $m(j) \leq 0$ for any $j \in \Phi$ and if $m(j) < 0$ for some $j \in \Phi$, then $f_m = h_{-m} = 0$ by (i) and $f_{-m} = h_m = 0$ by the definition of V. If $m(j) \geq 0$ for any $j \in \Phi$, we have similarly $f_m = h_m = f_{-m} = h_{-m} = 0$.

Theorem 3.4.4 is proved. □

3.4.20. Proof of Theorem 3.4.6

We deduce it from Theorem 3.4.4(ii) as follows.

Let $\Psi \in \mathcal{W}$, and let Φ be the image of Ψ in $\overline{\mathcal{W}}$ (see Section 3.2.2). Take a distance to Φ-boundary β.

Let E be the subset of $\mathbf{R}^{\Psi}_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements (t, u, v) satisfying conditions (5) and (6) (resp., (5′) and (6′)) in Section 3.4.5 in the case where $W \notin \Psi$ (resp., $W \in \Psi$). We regard E as an object of $\mathcal{B}_{\mathbf{R}}(\log)$, similarly to the case of $Y^{II}(p, \mathbf{r}, R, S)$ (see Section 3.4.2).

Assume first $W \notin \Psi$. Let $D^{II}_{\mathrm{SL}(2)}(\Phi)'$ be the open set of $D^{II}_{\mathrm{SL}(2)}(\Phi)$ consisting of all elements q such that $W \notin \mathcal{W}(q)$. (This condition is equivalent to the condition that the $\bar{\mathcal{L}}$-component of $\nu_{\bar{\tau}_p,\beta}(q)$ (see Proposition 3.2.6(ii)) be contained in $\mathcal{L}$.) Then $D^{I}_{\mathrm{SL}(2)}(\Psi)$ is the fiber product of

$$D^{II}_{\mathrm{SL}(2)}(\Phi)' \to \mathbf{R}^{\Phi}_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \leftarrow E$$

in $\mathcal{B}'_{\mathbf{R}}(\log)$, where the first arrow is given by $x \mapsto (\beta(x), u)$ with $\mathrm{spl}_W(x) = \exp(u)s_{\mathbf{r}}$, the second arrow sends (t,u,v) to (t,u), and the morphism $D^{I}_{\mathrm{SL}(2)}(\Psi) \to E$ is given by $x \mapsto (\beta(x), u, v)$ with $\mathrm{spl}_W(x) = \exp(u)s_{\mathbf{r}}$ and $\mathrm{spl}_W(y) = \exp(v)s_{\mathbf{r}}$ for the D-component y of $\nu_{\tau_p,\beta}$ (see Proposition 3.2.6(i)). Since $Y^I(p,\mathbf{r},R,S)$ is the fiber product of $Y^{II}(p,\mathbf{r},R,S) \to \mathbf{R}^{\Phi}_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \leftarrow E$, Theorem 3.4.6 is reduced to Theorem 3.4.4.

Next, assume $W \in \Psi$. Let $\beta_0 : \bar{\mathcal{L}} \smallsetminus \{0\} \to \mathbf{R}_{>0}$ be a real analytic function such that $\beta_0(a \circ \delta) = a\beta_0(\delta)$ for any $a \in \mathbf{R}_{>0}$ and $\delta \in \bar{\mathcal{L}} \smallsetminus \{0\}$. Denote the composite $D^{II}_{\mathrm{SL}(2)}(\Phi)_{\mathrm{nspl}} \to \bar{\mathcal{L}} \smallsetminus \{0\} \to \mathbf{R}_{\geq 0}$ also by β_0, where the first arrow is the $\bar{\mathcal{L}}$-component of $\nu_{\tau_p,\beta}$ (see Proposition 3.2.6(ii)). Then $(\beta_0, \beta) : D \to \mathbf{R}^{\Psi}_{>0} = \mathbf{R}_{>0} \times \mathbf{R}^{\Phi}_{>0}$ is a distance to Ψ-boundary. As an object of $\mathcal{B}'_{\mathbf{R}}(\log)$, $D^{I}_{\mathrm{SL}(2),\mathrm{nspl}}(\Psi)$ is the fiber product of

$$D^{II}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \to \mathbf{R}^{\Psi}_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \leftarrow E,$$

where the first arrow is given by $x \mapsto ((\beta_0,\beta)(x), u)$ with $\mathrm{spl}_W(x) = \exp(u)s_{\mathbf{r}}$. On the other hand, if we denote by $Y^*(p,\mathbf{r},R,S)_{\mathrm{nspl}}$ $(* = I, II)$ the open set of $Y^*(p,\mathbf{r},R,S)$ consisting of all elements satisfying $\delta \neq 0$, $Y^I(p,\mathbf{r},R,S)_{\mathrm{nspl}}$ is the fiber product of

$$Y^{II}(p,\mathbf{r},R,S)_{\mathrm{nspl}} \to \mathbf{R}^{\Psi}_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \leftarrow E$$

in $\mathcal{B}'_{\mathbf{R}}(\log)$, where the first arrow is given by $(t,f,g,h,k,\delta,u) \mapsto ((a,t),u)$ for $\delta = a \circ \delta^{(1)}$ with $\delta^{(1)} \in L^{(1)}$ (see Section 3.4.5). From these facts, Theorem 3.4.6 is reduced to Theorem 3.4.4 also in the case $W \in \Psi$.

Theorem 3.4.6 is proved. □

3.4.21. Proof of Theorem 3.2.10

We first prove Theorem 3.2.10(ii). Let $\Phi \in \overline{\mathcal{W}}$. We prove the following.

CLAIM 1

For $\Phi' \subset \Phi$, the inclusion map $D^{II}_{\mathrm{SL}(2)}(\Phi') \to D^{II}_{\mathrm{SL}(2)}(\Phi)$ is an open immersion in $\mathcal{B}'_{\mathbf{R}}(\log)$.

Let α be a splitting of Φ, and let β be a distance to Φ-boundary. Since $D^{II}_{\mathrm{SL}(2)}(\Phi')$ is the inverse image of $\{t \in \mathbf{R}^{\Phi}_{\geq 0} \mid t_j \neq 0 \text{ if } j \in \Phi \smallsetminus \Phi'\}$ under the map $\beta : D^{II}_{\mathrm{SL}(2)}(\Phi) \to \mathbf{R}^{\Phi}_{\geq 0}$, it is an open subset of $D^{II}_{\mathrm{SL}(2)}(\Phi)$. Let $\alpha' : \mathbf{G}^{\Phi'}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W)$ be the Φ'-component of α, and let $\beta' : D(\mathrm{gr}^W) \to \mathbf{R}^{\Phi'}_{>0}$ be the Φ'-component of $\beta : D(\mathrm{gr}^W) \to \mathbf{R}^{\Phi}_{>0}$. Then we have a commutative diagram

$$\begin{array}{ccc} D^{II}_{\mathrm{SL}(2)}(\Phi') & \to & \mathbf{R}^{\Phi'}_{\geq 0} \times D(\mathrm{gr}^W)' \times \bar{\mathcal{L}} \\ \cap & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \to & \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \end{array}$$

where $D(\mathrm{gr}^W)' = \{x \in D(\mathrm{gr}^W) \mid \beta'(x) = 1\}$, the upper horizontal arrow is induced by (α', β') as in Proposition 3.2.6, the lower horizontal arrow is induced by (α, β) as in Proposition 3.2.6, and the right vertical arrow sends $(t, x, \delta) \in \mathbf{R}^{\Phi'}_{\geq 0} \times D(\mathrm{gr}^W)' \times \bar{\mathcal{L}}$ to $((t, \beta(x)), \alpha\beta(x)^{-1}x, \mathrm{Ad}(\alpha\beta(x))^{-1}\delta)$. Here by the fact that $\beta(x)_j = 1$ for any $j \in \Phi'$, we regard $(t, \beta(x))$ as an element of $\mathbf{R}^{\Phi'}_{\geq 0} \times \mathbf{R}^{\Phi \smallsetminus \Phi'}_{>0} \subset \mathbf{R}^{\Phi}_{\geq 0}$. From this, we obtain the following.

CLAIM 2

Let $D^{II,\Phi}_{\mathrm{SL}(2)}(\Phi')$ be the set $D^{II}_{\mathrm{SL}(2)}(\Phi')$ endowed with the structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ as an open set of $D^{II}_{\mathrm{SL}(2)}(\Phi)$. Then the canonical inclusion map $D^{II,\Phi}_{\mathrm{SL}(2)}(\Phi') \to D^{II}_{\mathrm{SL}(2)}(\Phi')$ is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$. This morphism is an isomorphism if and only if, for any $W' \in \Phi$, the composite $D^{II}_{\mathrm{SL}(2)}(\Phi') \to D^{II}_{\mathrm{SL}(2)}(\Phi) \to \mathrm{spl}(W')$, where the last arrow is induced by $\mathrm{spl}^{\mathrm{BS}}_{W'}$, is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$.

By Claim 2 and Theorem 3.4.4, for the proof of Claim 1, it is sufficient to prove the following.

CLAIM 3

Let $p' \in D^{II}_{\mathrm{SL}(2)}(\Phi)$, and let $\Phi' = \overline{\mathcal{W}}(p') \subset \Phi$. Let $\mathbf{r}'$ be a point on the torus orbit associated to p'. Then, for a sufficiently small open neighborhood U of $(0^{\Phi'}, 0, 0, 0, 0, \delta(\mathbf{r}'), 0)$ in $Y^{II}(p', \mathbf{r}', S)$ (S is taken for $\mathbf{r}'$), the composite $U \to D^{II}_{\mathrm{SL}(2)}(\Phi') \to D^{II}_{\mathrm{SL}(2)}(\Phi) \to \mathrm{spl}(W')$ is a morphism of $\mathcal{B}'_{\mathbf{R}}(\log)$.

We prove Claim 3. Take $p \in D^{II}_{\mathrm{SL}(2)}(\Phi)$ such that $\Phi = \overline{\mathcal{W}}(p)$. Let $\alpha = \bar{\tau}_p$, and take a distance to Φ-boundary β such that $\beta(K_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}) = 1$. Note that such a β exists (cf. [KU2, Proposition 4.12]). For each $w \in \mathbf{Z}$, let $Q(w) \in \mathcal{W}(\mathrm{gr}^W_w)$ be the image of Φ, and let $Q = (Q(w))_w$. Let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) = \prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)(Q(w))$. Let $\bar{\mu} : D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \to D(\mathrm{gr}^W)$ be the extension of $D(\mathrm{gr}^W) \to D(\mathrm{gr}^W)$, $x \mapsto \alpha\beta(x)^{-1}x$, induced by Proposition 3.2.6(ii). Let $\alpha' = \bar{\tau}_{p'}$. We first prove the following.

CLAIM 4

There exists $y \in G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$ such that $\bar{\mu}(y^{-1}\bar{p}') \in K_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}$, where $\bar{p}' = p'(\mathrm{gr}^W)$.

In fact, by Claim 1 in [KU3, Section 6.4.4], there are $z \in G_{\mathbf{R}}(\mathrm{gr}^W)_{\Phi'}$ and $k \in K_{\bar{\mathbf{r}}}$ such that $\alpha' = \mathrm{Int}(z)(\alpha_{\Phi'})$ and $\bar{\mathbf{r}}' = zk\bar{\mathbf{r}}$, where $\alpha_{\Phi'}$ is the restriction of α to Φ'. Write $z = z_0 z_u$, where z_0 commutes with $\alpha_{\Phi'}(t)$ ($t \in (\mathbf{R}^\times)^{\Phi'}$) and $z_u \in G_{\mathbf{R}}(\mathrm{gr}^W)_{\Phi',u}$. We can write $z_0 = yk_0$, where y and k_0 commute with $\alpha_{\Phi'}(t)$ ($t \in (\mathbf{R}^\times)^{\Phi'}$), $y \in G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$, and $k_0 \in K_{\bar{\mathbf{r}}}$. We have $\bar{\mu}(y^{-1}\bar{p}') = k_0 k\bar{\mathbf{r}}$. In fact, since

$\bar{p}' = \lim \alpha'(t)\bar{\mathbf{r}}' = \lim z\alpha(t)k\bar{\mathbf{r}}$, $\bar{\mu}(y^{-1}\bar{p}')$ is the limit of $\bar{\mu}(y^{-1}z\alpha(t)k\bar{\mathbf{r}}) = \bar{\mu}(\alpha(t)\cdot y^{-1}z_tk\bar{\mathbf{r}}) = \bar{\mu}(y^{-1}z_tk\bar{\mathbf{r}})$, where $z_t = \bar{\tau}_p(t)^{-1}z\bar{\tau}_p(t)$, which converges to $\bar{\mu}(y^{-1}z_0k\bar{\mathbf{r}}) = \bar{\mu}(k_0k\bar{\mathbf{r}}) = k_0k\bar{\mathbf{r}} \in K_{\bar{\mathbf{r}}}\cdot\bar{\mathbf{r}}$.

Let y be as in Claim 4. Then, for $q \in D$ near p' in $D^{II}_{\mathrm{SL}(2)}(\Phi')$, $\mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{q}) = y\,\mathrm{spl}^{\mathrm{BS}}_{W'}(y^{-1}\bar{q})y(\mathrm{gr}^{W'})^{-1}$, where $\bar{q} = q(\mathrm{gr}^W)$. We denote the right-hand side of the last equation by $\mathrm{Int}(y)\,\mathrm{spl}^{\mathrm{BS}}_{W'}(y^{-1}\bar{q})$. From this, we may replace $\bar{p}'$ by $y^{-1}\bar{p}'$, and hence we may assume $\bar{\mu}(\bar{p}') \in K_{\bar{\mathbf{r}}}\cdot\bar{\mathbf{r}}$.

Take an $\mathbf{R}$-subspace V of $\mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}^W)_{W'})$ such that $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = V \oplus \mathrm{Lie}(K_{\bar{\mathbf{r}}})$. For $q \in D$ near p' in $D^{II}_{\mathrm{SL}(2)}(\Phi')$, write $\bar{\mu}(\bar{q}) \in \exp(v(\bar{q}))\cdot K_{\bar{\mathbf{r}}}\cdot\bar{\mathbf{r}}$ with $v(\bar{q}) \in V$, and write $f(\bar{q}) = \mathrm{Int}(\alpha\beta(\bar{q}))\big(\exp(v(\bar{q}))\big) \in G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$. Then, since $\bar{q} = \alpha\beta(\bar{q})\bar{\mu}(\bar{q})$, we have

$$\mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{q}) = \mathrm{Int}\big(f(\bar{q})\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(\alpha\beta(\bar{q})\bar{\mathbf{r}})\big) = \mathrm{Int}\big(f(\bar{q})\big)\big(\mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{\mathbf{r}})\big).$$

Here the last equality follows from $\mathrm{Int}(\alpha(t))\,\mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{\mathbf{r}}) = \mathrm{spl}^{\mathrm{BS}}_{W'}(\bar{\mathbf{r}})$ for any t. By Theorem 3.4.4 and the real analycity of a_1 in Section 3.3.11, $v(\bar{q})$ extends over the boundary, and hence so does $f(\bar{q})$; that is, for a sufficiently small open neighborhood U of $(0^{\Phi'},0,0,0,0,\delta(\mathbf{r}'),0)$ in $Y^{II}(p',\mathbf{r}',S)$, there is a morphism $U \to G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$ which is compatible with the map $Y^{II}_0(p',\mathbf{r}',S) \to G_{\mathbf{R}}(\mathrm{gr}^W)_{W'}$ induced by f. Hence $\mathrm{spl}^{\mathrm{BS}}_{W'}$ extends over the boundary. This completes the proof of Claim 3 and hence the proof of Claim 1.

By Claim 1, on $D_{\mathrm{SL}(2)}$, there is a unique structure as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ for which each $D^{II}_{\mathrm{SL}(2)}(\Phi)$ ($\Phi \in \overline{\mathcal{W}}$) is open and whose restriction to $D^{II}_{\mathrm{SL}(2)}(\Phi)$ coincides with the structure of $D^{II}_{\mathrm{SL}(2)}(\Phi)$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. By Theorem 3.4.4, this object $D^{II}_{\mathrm{SL}(2)}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$ belongs to $\mathcal{B}_{\mathbf{R}}(\log)$.

Next, Theorem 3.2.10(i) follows from Theorem 3.2.10(ii) and Theorem 3.4.6.

We prove Theorem 3.2.10(iii). It is clear that the identity map of $D_{\mathrm{SL}(2)}$ is a morphism $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ in $\mathcal{B}_{\mathbf{R}}(\log)$ and that the log structure with sign on $D^I_{\mathrm{SL}(2)}$ is the pullback of that of $D^{II}_{\mathrm{SL}(2)}$. It is also clear that, in the pure case, this morphism $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ is an isomorphism.

It remains to prove that in the pure case, the topology of $D_{\mathrm{SL}(2)}$ defined in [KU2] coincides with the topology defined in this article.

Assume that we are in the pure case.

The topology of $D_{\mathrm{SL}(2)}$ defined in [KU2] is characterized by the following properties (1) and (2) (see [KU3]).

(1) For any $\Psi \in \mathcal{W}$, $D^I_{\mathrm{SL}(2)}(\Psi)$ is open and is a regular space.

(2) Let $p \in D_{\mathrm{SL}(2)}$, let $\mathbf{r}$ be a point on the torus orbit associated to p, and let $\Psi = \mathcal{W}(p)$. Then, for a directed family $(p_\lambda)_\lambda$ of points of D, $(p_\lambda)_\lambda$ converges to p in $D_{\mathrm{SL}(2)}(\Psi)$ if and only if there exist $t_\lambda \in \mathbf{R}^{\Psi}_{>0}$, $g_\lambda \in \mathfrak{g}_{\mathbf{R}}$, $k_\lambda \in \mathrm{Lie}(K_{\mathbf{r}})$ such that $p_\lambda = \tau_p(t_\lambda)\exp(g_\lambda)\exp(k_\lambda)\mathbf{r}$, $t_\lambda \to 0^{\Psi}$ in $\mathbf{R}^{\Psi}_{\geq 0}$, $\mathrm{Ad}(\tau_p(t_\lambda))^j(g_\lambda) \to 0$ for $j = \pm 1, 0$, and $k_\lambda \to 0$.

It is sufficient to prove that the topology of $D^{II}_{\mathrm{SL}(2)}$ (i.e., the topology of $D^{I}_{\mathrm{SL}(2)}$) in this article satisfies this (1) and (2). Property (1) is clearly satisfied. We prove (2).

Assume $p_\lambda \to p$ for the topology of this article. By Theorem 3.4.4(ii), for some $\tilde{p}_\lambda = (t_\lambda, f_\lambda, g_\lambda, h_\lambda, k_\lambda) \in Y_0(p, \mathbf{r}, R, S) \subset \mathbf{R}^\Psi_{>0} \times \mathfrak{g}_\mathbf{R} \times \mathfrak{g}_\mathbf{R} \times \mathfrak{g}_\mathbf{R} \times \mathrm{Lie}(K_\mathbf{r})$ such that $p_\lambda = \tau_p(t_\lambda)\exp(g_\lambda)\exp(k_\lambda)\mathbf{r}$, we have $\tilde{p}_\lambda \to (0^\Psi, 0, 0, 0, 0)$ in $Y(p, \mathbf{r}, R, S)$. Since $f_\lambda = \mathrm{Ad}(\tau_p(t_\lambda))(g_\lambda)$ and $h_\lambda = \mathrm{Ad}(\tau_p(t_\lambda))^{-1}(g_\lambda)$, we have $t_\lambda \to 0^\Psi$, $\mathrm{Ad}(\tau_p(t_\lambda))^j(g_\lambda) \to 0$ for $j = \pm 1, 0$, and $k_\lambda \to 0$. Conversely, assume $p_\lambda = \tau_p(t_\lambda) \cdot \exp(g_\lambda)\exp(k_\lambda)\mathbf{r}$ for some $t_\lambda \in \mathbf{R}^\Psi_{>0}$, $g_\lambda \in \mathfrak{g}_\mathbf{R}$, $k_\lambda \in \mathrm{Lie}(K_\mathbf{r})$ such that $t_\lambda \to 0^\Psi$, $\mathrm{Ad}(\tau_p(t_\lambda))^j(g_\lambda) \to 0$ for $j = \pm 1, 0$, and $k_\lambda \to 0$. Then if we put $f_\lambda = \mathrm{Ad}(\tau_p(t_\lambda))(g_\lambda)$ and $h_\lambda = \mathrm{Ad}(\tau_p(t_\lambda))^{-1}(g_\lambda)$, $(t_\lambda, f_\lambda, g_\lambda, h_\lambda, k_\lambda)$ converges to $(0^\Psi, 0, 0, 0, 0)$ in $Y(p, \mathbf{r}, S)$. By Theorem 3.4.4(i), this shows that $\tau_p(t_\lambda)\exp(g_\lambda)\exp(k_\lambda)\mathbf{r}$ converges to p for the topology of this article.

Theorem 3.2.10 is proved. □

3.4.22.
In Propositions 3.4.23 and 3.4.27, we give local descriptions of $D^{II}_{\mathrm{SL}(2)}$ and $D^{I}_{\mathrm{SL}(2)}$ as topological spaces, respectively. Compared with the real analytic local descriptions in Theorems 3.4.4 and 3.4.6, we have simpler descriptions here.

We define a topological space $Z^{II}_{\mathrm{top}}(p, R)$ as the subspace of $\mathbf{R}^\Phi_{\geq 0} \times R$ consisting of all elements (t, a) satisfying the following condition (1).

(1) Let $m \in \mathbf{Z}^\Phi$. Then $a_m = 0$ unless either $m(j) \geq 0$ for all $j \in J$ or $m(j) \leq 0$ for all $j \in J$.

We define a topological space $Y^{II}_{\mathrm{top}}(p, \mathbf{r}, R, S)$ as the subspace of $Z^{II}_{\mathrm{top}}(p, R) \times S \times \bar{L} \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements (t, a, k, δ, u) ($(t, a) \in Z^{II}_{\mathrm{top}}(p, R)$, $k \in S$, $\delta \in \bar{L}$, $u \in \mathfrak{g}_{\mathbf{R},u}$) such that (t, k) satisfies condition (4) in Section 3.4.2. Let $Y^{II}_{0,\mathrm{top}}(p, \mathbf{r}, R, S)$ be the open set $\mathbf{R}^\Phi_{>0} \times R \times S \times L \times \mathfrak{g}_{\mathbf{R},u}$ of $Y^{II}_{\mathrm{top}}(p, \mathbf{r}, R, S)$, and let

$$\eta^{II}_{p,\mathbf{r},R,S,\mathrm{top}} : Y^{II}_{0,\mathrm{top}}(p, \mathbf{r}, R, S) \to D$$

be the continuous map

$$(t, a, k, \delta, u) \mapsto \exp(u) s_\mathbf{r} \theta\big(d\bar{\mathbf{r}}, \mathrm{Ad}(d)\delta\big)$$
$$\text{with } d = \bar{\tau}_p(t) \exp\Big(\sum_{m \in \mathbf{Z}^\Phi} g_m/(t^m + t^{-m})\Big) \exp(k).$$

Here $t^m = \prod_{j \in \Phi} t_j^{m(j)}$.

PROPOSITION 3.4.23
Let the notation be as in Theorem 3.4.4. Then there are an open neighborhood V of $(0^\Phi, 0, 0, \delta(\mathbf{r}), 0)$ in $Y^{II}_{\mathrm{top}}(p, \mathbf{r}, R, S)$ and an open immersion $V \to D^{II}_{\mathrm{SL}(2)}(\Phi)$ of topological spaces which sends $(0^\Phi, 0, 0, \delta(\mathbf{r}), 0)$ to p and whose restriction to $V \cap Y^{II}_{0,\mathrm{top}}(p, \mathbf{r}, R, S)$ coincides with the restriction of $\eta^{II}_{p,\mathbf{r},R,S,\mathrm{top}}$ (see Section 3.4.22).

3.4.24.
This Proposition 3.4.23 follows from Theorem 3.4.4, because we have a homeomorphism

$$Y^{II}(p,\mathbf{r},R,S) \cong Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S), \quad (t,f,g,h,k,\delta,u) \leftrightarrow (t,a,k,\delta,u)$$

$$\text{with } a = f+h,$$

$$f = \sum_m (1+t^{-2m})^{-1} a_m, \quad g = \sum_m (t^m + t^{-m})^{-1} a_m, \quad h = \sum_m (t^{2m}+1)^{-1} a_m,$$

where, in $\sum_m$, m ranges over all elements of $\mathbf{Z}^\Phi$ such that either $m(j) \geq 0$ for any $j \in J(t)$ or $m(j) \leq 0$ for any $j \in J(t)$. (Note that $(1+t^{-2m})^{-1}, (t^m + t^{-m})^{-1}, (t^{2m}+1)^{-1} \in \mathbf{R}$ are naturally defined for such m.)

3.4.25.
REMARK
In the pure case, at the beginning of [KU3, Section 10], it is suggested that the local homeomorphism with $Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S)$ in Proposition 3.4.23 may be used to define a real analytic structure of $D_{\mathrm{SL}(2)}$. If we do so, we regard $Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S)$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ by using the embedding $Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S) \hookrightarrow \mathbf{R}^\Phi_{\geq 0} \times R \times S \times \bar{L} \times \mathfrak{g}_{\mathbf{R},u}$ in the same way as we did so for $Y^{II}(p,\mathbf{r},R,S)$ by using the injection $Y^{II}(p,\mathbf{r},R,S) \hookrightarrow E$ (see Section 3.4.2). However, the definition of the real analytic structure of $D_{\mathrm{SL}(2)}$ in this article, which is given by the local homeomorphism with $Y^{II}(p,\mathbf{r},R,S)$, is slightly different from the suggested one in [KU3, Section 10]. The above map $Y^{II}(p,\mathbf{r},R,S) \to Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S)$ is real analytic and is a homeomorphism, but the inverse map need not be real analytic at $(0^\Phi,0,0,\delta(\mathbf{r}),0)$.

3.4.26.
We define the topological space $Y^I_{\mathrm{top}}(p,\mathbf{r},R,S)$ as follows.

In the case $W \notin \Psi$, let $Y^I_{\mathrm{top}}(p,\mathbf{r},R,S)$ be the subset of $Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S) \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements (t,a,k,δ,u,v) $((t,a,k,\delta,u) \in Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S),\ v \in \mathfrak{g}_{\mathbf{R},u})$ such that (t,δ,u,v) satisfies conditions (5)–(7) in Section 3.4.5.

Similarly, in the case $W \in \Psi$, let $Y^I_{\mathrm{top}}(p,\mathbf{r},R,S)$ be the subset of $\mathbf{R}_{\geq 0} \times Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S) \times \mathfrak{g}_{\mathbf{R},u}$ consisting of all elements (t_0,t,a,k,δ,u,v) $(t_0 \in \mathbf{R}_{\geq 0}$, $(t,a,k,\delta,u) \in Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S)$, $v \in \mathfrak{g}_{\mathbf{R},u})$ such that (t_0,t,δ,u,v) satisfies conditions (5′)–(7′) in Section 3.4.5.

We define a canonical map $Y^I_{\mathrm{top}}(p,\mathbf{r},R,S) \to Y^{II}_{\mathrm{top}}(p,\mathbf{r},R,S)$. If $W \notin \Psi$, it is the canonical projection. Otherwise, it is $(t_0,t',a,k,\delta,u,v) \mapsto (t',a,k,t_0 \circ \delta,u)$. Let $Y^I_{0,\mathrm{top}}(p,\mathbf{r},R,S)$ be the open set of $Y^I_{\mathrm{top}}(p,\mathbf{r},R,S)$ defined by the inverse image of $Y^{II}_{0,\mathrm{top}}(p,\mathbf{r},R,S)$ by this canonical map. Then $Y^I_{0,\mathrm{top}}(p,\mathbf{r},R,S) \to Y^{II}_{0,\mathrm{top}}(p,\mathbf{r},R,S)$ is a homeomorphism. Let $\eta^I_{p,\mathbf{r},R,S,\mathrm{top}} : Y^I_{0,\mathrm{top}}(p,\mathbf{r},R,S) \to D$ be the continuous map obtained from $\eta^{II}_{p,\mathbf{r},R,S,\mathrm{top}}$ and the last homeomorphism.

PROPOSITION 3.4.27

Let the notation be as in Theorem 3.4.6. Assume $W \notin \Psi$ (resp., $W \in \Psi$). Then there is an open neighborhood V of $v := (0^\Psi, 0, 0, \delta(\mathbf{r}), 0, 0)$ (resp., $(0^\Psi, 0, 0, \delta(\mathbf{r})^{(1)}, 0, 0)$, where $\delta(\mathbf{r})^{(1)} \in L^{(1)}$ (see Section 3.4.5) such that $\delta(\mathbf{r}) = 0 \circ \delta(\mathbf{r})^{(1)}$) in $Y^I_{\mathrm{top}}(p, \mathbf{r}, R, S)$ and an open immersion $V \to D^I_{\mathrm{SL}(2)}(\Psi)$ of topological spaces which sends v to p and whose restriction to $V \cap Y^I_{0,\mathrm{top}}(p, \mathbf{r}, R, S)$ coincides with the restriction of $\eta^I_{p,\mathbf{r},R,S,\mathrm{top}}$ (see Section 3.4.26).

This follows from Theorem 3.4.6, just as Proposition 3.4.23 follows from Theorem 3.4.4 in Section 3.4.24.

3.4.28. Proof of Proposition 3.2.12

We prove (i). It is sufficient to prove that the topology of $D^I_{\mathrm{SL}(2)}$ has property (2). Let $p \in D_{\mathrm{SL}(2)}$, and let Ψ be the set of weight filtrations associated to p. In the following, we assume $W \notin \Psi$. The case where $W \in \Psi$ is similar. Assume first that $(p_\lambda)_\lambda$ ($p_\lambda \in D$) converges to p. Then clearly (a) and (b) are satisfied. Take a distance to Ψ-boundary β such that $\beta(\mathbf{r}) = 1$, and let $\mu : D^I_{\mathrm{SL}(2)}(\Psi) \to D$ be the extension of $x \mapsto \tau_p\beta(x)^{-1}x$ given in Proposition 3.2.6(i). We show that (c.I) is satisfied for $t_\lambda := \beta(p_\lambda)$. We have $t_\lambda = \beta(p_\lambda) \to \beta(p) = 0^\Psi$ and $\tau_p(t_\lambda)^{-1}p_\lambda = \mu(p_\lambda) \to \mu(p) = \mathbf{r}$. Next, assume that (a), (b), and (c.I) are satisfied. Take $\alpha = \tau_p$, and take β such that $\beta(\mathbf{r}) = 1$. We prove $p_\lambda \to p$. It is sufficient to prove that $\nu_{\alpha,\beta}(p_\lambda)$ converges to $\nu_{\alpha,\beta}(p) = \big(0^\Psi, \mathbf{r}, \mathrm{spl}_W(\mathbf{r}), (\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(\mathbf{r}(\mathrm{gr}^W)))_{W' \in \Psi}\big)$ in $\mathbf{R}^\Psi_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}(W'(\mathrm{gr}^W))$. The $\mathrm{spl}(W)$-component and the $\mathrm{spl}(W'(\mathrm{gr}^W))$-component of $\nu_{\alpha,\beta}(p_\lambda)$ converge to $\mathrm{spl}_W(\mathbf{r})$ and to $\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(\mathbf{r}(\mathrm{gr}^W))$ by (a) and (b), respectively. Let $a_\lambda = t_\lambda^{-1}\beta(p_\lambda) \in \mathbf{R}^\Psi_{>0}$. By taking β of $\tau_p(t_\lambda)^{-1}p_\lambda \to \mathbf{r}$, we have $a_\lambda \to 1$. Since $t_\lambda \to 0^\Psi$, $\beta(p_\lambda) = t_\lambda a_\lambda$ converges to 0^Ψ. Finally, $\alpha\beta(p_\lambda)^{-1}p_\lambda = \tau_p(a_\lambda)^{-1}\tau_p(t_\lambda)^{-1}p_\lambda \to \mathbf{r}$.

The proof of (ii) is similar to that of (i).

Proposition 3.2.12 is proved. □

PROPOSITION 3.4.29

The following conditions (1)–(3) are equivalent.

(1) *The topology of $D^I_{\mathrm{SL}(2)}$ coincides with that of $D^{II}_{\mathrm{SL}(2)}$.*

(2) *$D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ coincide in $\mathcal{B}_{\mathbf{R}}(\log)$.*

(3) *For any $p \in D_{\mathrm{SL}(2)}$, for any $w, w' \in \mathbf{Z}$ such that $w > w'$, for any member W' of the set of weight filtrations associated to p, and for any $a, b \in \mathbf{Z}$ such that $\mathrm{gr}^{W'}_a(\mathrm{gr}^W_w) \neq 0$ and $\mathrm{gr}^{W'}_b(\mathrm{gr}^W_{w'}) \neq 0$, we have $a \geq b$.*

REMARKS

(i) Assume that the equivalent conditions of Proposition 3.4.29 are satisfied. Then, for any $\Psi \in \mathcal{W}$ and for $\bar{\Psi} = \{W'(\mathrm{gr}^W) \mid W' \in \Psi, W' \neq W\} \in \overline{\mathcal{W}}$, $D^I_{\mathrm{SL}(2)}(\Psi) = D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$ in $\mathcal{B}_{\mathbf{R}}(\log)$ if $W \in \Psi$, and $D^I_{\mathrm{SL}(2)}(\Psi)$ is an open subobject of $D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$ in general.

(ii) As is easily seen from Section 2.3.9, Examples I–IV in Section 1.1.1 satisfy the above condition (3), but Example V does not (see Section 3.6.2).

3.4.30. Proof of Proposition 3.4.29
We first prove that (1) implies (3). Assume that (3) is not satisfied. Then for some $p \in D_{\mathrm{SL}(2)}$, there exists $W' \in \mathcal{W}(p)$ having the following property. There are $w, w', a, b \in \mathbf{Z}$ such that $\mathrm{gr}_a^{W'}(\mathrm{gr}_w^W)$ and $\mathrm{gr}_b^{W'}(\mathrm{gr}_{w'}^W)$ are not zero, and $w > w'$ and $a < b$. There is a nonzero element u of $\mathfrak{g}_{\mathbf{R},u}$ such that the W'-component $\tau_{p,W'}$ satisfies $\mathrm{Ad}(\tau_{p,W'}(t))u = t^{b-a}u$ for all $t \in \mathbf{R}^\times$. Take any real number c such that $0 < c < b - a$. We have $W' \neq W$. For $t \geq 0$, let $\epsilon(t)$ be the element of $\mathbf{R}^\Psi_{\geq 0}$ whose W'-component is t and whose other components are all 1. Let Φ be the image of Ψ in $\overline{\mathcal{W}}$ (see Section 3.2.2). Take a point $\mathbf{r} \in D$ on the torus orbit associated to p, consider an element p' of $Y^I(p, \mathbf{r}, R, S)$ of the form $p' = (\epsilon(0), 0, 0, 0, 0, \delta, 0, 0) \in Y^I(p, \mathbf{r}, R, S)$, let $\bar{\epsilon}(t)$ be the image of $\epsilon(t)$ in $\mathbf{R}^\Phi_{\geq 0}$, and let $p'' = (\bar{\epsilon}(0), 0, 0, 0, 0, \delta, 0) \in Y^{II}(p, \mathbf{r}, R, S)$ be the image of p'. When $t \in \mathbf{R}_{>0}$ tends to 0, $(\bar{\epsilon}(t), 0, 0, 0, 0, \delta, t^c u) \in Y^{II}(p, \mathbf{r}, R, S)$ converges to p''. But this element of $Y^{II}(p, \mathbf{r}, R, S)$ is the image of $(\epsilon(t), 0, 0, 0, 0, \delta, t^c u, t^{c+a-b}u) \in Y^I(p, \mathbf{r}, R, S)$ which does not converge to p' when $t \to 0$ because $c + a - b < 0$. By Theorems 3.4.4 and 3.4.6, this proves that the topology of $D^I_{\mathrm{SL}(2)}$ and that of $D^{II}_{\mathrm{SL}(2)}$ are different.

It is clear that (2) implies (1).

It remains to prove that (3) implies (2). Assume that (3) is satisfied. As in Remark (i) after Proposition 3.4.29, $D^I_{\mathrm{SL}(2)}(\Psi)$ is an open set of $D^{II}_{\mathrm{SL}(2)}(\Phi)$.

By Theorems 3.4.4 and 3.4.6, it is sufficient to prove the following.

CLAIM
For a splitting α of Ψ, the map $\mathbf{R}^\Psi_{>0} \times \mathfrak{g}_{\mathbf{R},u} \to \mathfrak{g}_{\mathbf{R},u}, (t, u) \mapsto \mathrm{Ad}(\alpha(t))^{-1}(u)$ extends to a real analytic map $\mathbf{R}^\Psi_{\geq 0} \times \mathfrak{g}_{\mathbf{R},u} \to \mathfrak{g}_{\mathbf{R},u}$.

By (3), for the adjoint action of $\mathbf{G}^\Psi_{m,\mathbf{R}}$ by α, $\mathfrak{g}_{\mathbf{R},u}$ is the sum of the eigenspaces $(\mathfrak{g}_{\mathbf{R},u})_m$ for all $m \in \mathbf{Z}^\Psi$ such that $m \leq 0$. This proves the claim and completes the proof of Proposition 3.4.29. □

3.5. Global properties of $D_{\mathrm{SL}(2)}$

In Section 3.5, we prove that the projection $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is proper (see Theorem 3.5.16). We also prove results on the actions of a subgroup Γ of $G_\mathbf{Z}$ on $D^I_{\mathrm{SL}(2)}$ and on $D^{II}_{\mathrm{SL}(2)}$ (see Theorem 3.5.17).

Concerning the properness of $D^{II}_{\mathrm{SL}(2)}$ over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, we prove a more precise result. We define a log modification (see Proposition 3.1.12) $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, which is an object of $\mathcal{B}_\mathbf{R}(\log)$ and is proper over $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, such that the canonical projection $D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ factors as $D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. We prove that as an object of $\mathcal{B}_\mathbf{R}(\log)$, $D^{II}_{\mathrm{SL}(2)}$ is an $\bar{L}$-bundle over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ (see Theorem 3.5.15). Here $L = \mathcal{L}(F)$ for any fixed $F \in D(\mathrm{gr}^W)$, and $\bar{L}$ is the compactified vector space associated to L (see Proposition 3.2.6). This is an SL(2)-analogue of the fact (see [KNU2, Theorem 8.5]) that D_{BS} is an $\bar{L}$-bundle over $\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$. The properness of $D^{II}_{\mathrm{SL}(2)}$ over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ follows from this.

3.5.1.
We define the set $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$.

By an SL(2)-*orbit on* gr^W we mean a family $(\rho_w, \varphi_w)_{w\in\mathbf{Z}}$, where, for some $n \geq 0$, (ρ_w, φ_w) is an SL(2)-orbit for gr^W_w in n variables for any $w \in \mathbf{Z}$ satisfying the following condition (1).

(1) For $1 \leq j \leq n$, there is $w \in \mathbf{Z}$ such that the jth component of ρ_w is nontrivial.

This n is called the *rank of* $(\rho_w, \varphi_w)_w$.

We say two SL(2)-orbits $(\rho_w, \varphi_w)_w$ and $(\rho'_w, \varphi'_w)_w$ on gr^W are *equivalent* if their ranks coincide, say, n, and furthermore, there is $t = (t_1, \ldots, t_n) \in \mathbf{R}^n_{>0}$ such that

$$\rho'_w = \mathrm{Int}\big(\tilde{\rho}_w(t)\big)\rho_w, \qquad \varphi'_w = \tilde{\rho}_w(t)\varphi_w$$

for any $w \in \mathbf{Z}$, where $\tilde{\rho}_w(t)$ is as in Section 2.5.1.

Let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ be the set of all equivalence classes of SL(2)-orbits on gr^W.

Note that the SL(2)-orbits on gr^W just defined are in fact what should be called nondegenerate SL(2)-orbits on gr^W. We omitted this adjective in the above definition since we use only nondegenerate ones for the study of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$.

3.5.2.
The canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = \prod_{w\in\mathbf{Z}} D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ factors as

$$D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W),$$

where the second arrow is

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W), \quad \big(\text{class of } (\rho_w, \varphi_w)_w\big) \mapsto \big(\text{class of } (\rho_w, \varphi_w)\big)_w,$$

and the first arrow is defined as follows. Let $p \in D_{\mathrm{SL}(2)}$ be the class of an SL(2)-orbit $((\rho_w, \varphi_w)_w, \mathbf{r})$ in n variables of rank n, and let Ψ be the associated set of weight filtrations. Then the image $\tilde{p}$ of p in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ is the class of the following SL(2)-orbit $(\rho'_w, \varphi'_w)_w$ on gr^W. If $W \notin \Psi$, $(\rho'_w, \varphi'_w)_w = (\rho_w, \varphi_w)_w$, and hence $\tilde{p}$ is of rank n. If $W \in \Psi$, then $(\rho'_w, \varphi'_w)_w$ is an SL(2)-orbit on gr^W of rank $n-1$ defined by

$$\rho'_w(g_1, \ldots, g_{n-1}) = \rho_w(1, g_1, \ldots, g_{n-1}), \qquad \varphi'_w(z_1, \ldots, z_{n-1}) = \varphi_w(i, z_1, \ldots, z_{n-1}),$$

for $w \in \mathbf{Z}$.

The map $D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ is surjective.

The map $D_{\mathrm{SL}(2)} \to \overline{\mathcal{W}}, p \mapsto \overline{\mathcal{W}}(p)$ (see Section 3.2.2) factors through $D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$. For $q \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$, we denote by $\overline{\mathcal{W}}(q) \in \overline{\mathcal{W}}$ the element $\overline{\mathcal{W}}(p)$ for p an element of $D_{\mathrm{SL}(2)}$ with image q in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$, which is independent of the choice of p.

The map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is also surjective. This is shown as follows. For each $w \in \mathbf{Z}$, let (ρ_w, φ_w) be an SL(2)-orbit on gr^W_w in $n(w)$ variables of rank $n(w)$. Let $n = \max\{n(w) \mid w \in \mathbf{Z}\}$, and let (ρ'_w, φ'_w) be the SL(2)-orbit on gr^W_w in n variables defined by $\rho'_w(g_1, \ldots, g_n) = \rho_w(g_1, \ldots, g_{n(w)})$ and $\varphi'_w(z_1, \ldots, z_n) = \varphi_w(z_1, \ldots, z_{n(w)})$. Then (class of $(\rho_w, \varphi_w))_w \in \prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ is the image of the element (class of $(\rho'_w, \varphi'_w)_w$) in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ (cf. Section 3.5.1).

The map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ need not be injective (see Corollary 3.5.12, Example V in Section 3.5.13). There are two reasons for this. The first reason is as follows. For SL(2)-orbits $(\rho_w, \varphi_w)_w$ and $(\rho'_w, \varphi'_w)_w$ on gr^W, their images in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ coincide if and only if (ρ_w, φ_w) and (ρ'_w, φ'_w) are equivalent for all w, and the last equivalences are given by elements of $\mathbf{R}^{n(w)}_{>0}$ which can depend on $w \in \mathbf{Z}$ (here $n(w) = \mathrm{rank}(\rho_w, \varphi_w) = \mathrm{rank}(\rho'_w, \varphi'_w)$) not like the equivalence between $(\rho_w, \varphi_w)_w$ and $(\rho'_w, \varphi'_w)_w$ defined as in Section 3.5.1. The second reason is as follows. For $p \in D_{\mathrm{SL}(2)}$, the image of p in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ still remembers $\overline{\mathcal{W}}(p) \in \overline{\mathcal{W}}$, but the image of p in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ remembers only the image of this element of $\overline{\mathcal{W}}$ in $\prod_w \mathcal{W}(\mathrm{gr}^W_w)$ (see Section 3.3.1). As in Lemma 3.3.2, the map $\overline{\mathcal{W}} \to \prod_w \mathcal{W}(\mathrm{gr}^W_w)$ is described as $\big(\Phi, (\Phi(w))_w\big) \mapsto (\Phi(w))_w$ and is not necessarily injective.

3.5.3.
For $Q = (Q(w))_{w \in \mathbf{Z}} \in \prod_{w \in \mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$ (see Section 3.3.1), let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ be the open set of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ defined by

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) = \prod_{w \in \mathbf{Z}} D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)(Q(w)) \subset D_{\mathrm{SL}(2)}(\mathrm{gr}^W),$$

as in Section 3.4.21.

Define

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q) \subset D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$$

as the inverse image of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$. For $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$, p belongs to $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ if and only if $\Phi := \overline{\mathcal{W}}(p)$ satisfies $\Phi(w) \subset Q(w)$ for all $w \in \mathbf{Z}$.

3.5.4.
Let $Q = (Q(w))_w \in \prod_{w \in \mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, let $S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$, and let $\mathcal{S} = \bigoplus_{w \in \mathbf{Z}} \mathbf{N}^{Q(w)}$. Then we have a canonical surjective homomorphism $\mathcal{S} \to M_S/\mathcal{O}^\times_S$ characterized as follows. For any distance to $Q(w)$-boundary $\beta_w = (\beta_{w,j})_{j \in Q(w)} : D(\mathrm{gr}^W_w) \to \mathbf{R}^{Q(w)}_{>0}$ given for each $w \in \mathbf{Z}$, this homomorphism sends $m = \big((m(w,j))_{j \in Q(w)}\big)_w \in \mathcal{S}$ $(m(w,j) \in \mathbf{N})$ to $\big(\prod_{w \in \mathbf{Z}, j \in Q(w)} \beta^{m(w,j)}_{w,j}\big) \bmod \mathcal{O}^\times_S$. This homomorphism lifts locally on S to a chart $\mathcal{S} \to M_{S,>0}$.

In Sections 3.5.5 and 3.5.6 and Proposition 3.5.7, we define and study a finite rational subdivision Σ_Q of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0}) = \prod_{w \in \mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$, and in Theorem 3.5.9 we identify $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ with the associated log modification

$S(\Sigma_Q)$ (see Proposition 3.1.12) of S. We see in Section 3.5.10 that there is a unique structure on $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ for which each $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)\big(Q \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)\big)$ is open in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ and the induced structure on it coincides with the structure as the log modification.

3.5.5.
For $Q = (Q(w))_{w\in\mathbf{Z}} \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, we define a finite rational subdivision Σ_Q of the cone $\prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$ as follows.

First, we recall that, for a finite set Λ, the barycentric subdivision $\mathrm{Sd}(\Lambda)$ of the cone $\mathbf{R}^{\Lambda}_{\geq 0}$ is defined as follows (cf. [I, Section 2.8]). Let $J(\Lambda)$ be the set of all pairs (n, g), where n is a nonnegative integer and g is a function $\Lambda \to \{j \in \mathbf{Z} \mid 0 \leq j \leq n\}$ such that the image of g contains $\{j \in \mathbf{Z} \mid 1 \leq j \leq n\}$. For $(n, g) \in J(\Lambda)$, define the subcone $C(n, g)$ of $\mathbf{R}^{\Lambda}_{\geq 0}$ by

$$C(n,g) = \big\{(a_\lambda)_{\lambda\in\Lambda} \;\big|\; a_\lambda \leq a_\mu \text{ if } g(\lambda) \leq g(\mu), a_\lambda = 0 \text{ if } g(\lambda) = 0\big\}.$$

Then the set of cones $\mathrm{Sd}(\Lambda) := \{C(n,g) \mid (n,g) \in J(\Lambda)\}$ is a finite rational subdivision of $\mathbf{R}^{\Lambda}_{\geq 0}$ and is called the *barycentric subdivision of* $\mathbf{R}^{\Lambda}_{\geq 0}$. The map

$$J(\Lambda) \to \mathrm{Sd}(\Lambda), \qquad (n,g) \mapsto C(n,g)$$

is bijective. For $(n, g) \in J(\Lambda)$, the dimension of $C(n, g)$ is equal to n.

Let $Q = (Q(w))_w \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$. For each $w \in \mathbf{Z}$, we regard $Q(w)$ as a totally ordered set by Proposition 2.1.13.

Let $\Lambda = \bigsqcup_{w\in\mathbf{Z}} Q(w)$. Define a subcone C of $\mathbf{R}^{\Lambda}_{\geq 0} = \prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$ by

$$C = \Big\{ \big((a_{w,j})_{j\in Q(w)}\big)_w \in \prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0} \;\Big|\; a_{w,j} \leq a_{w,j'} \\ \text{if } w \in \mathbf{Z},\ j, j' \in Q(w), \text{ and } j \geq j' \Big\}.$$

Let

$$\mathrm{Sd}'(\Lambda) = \big\{\sigma \in \mathrm{Sd}(\Lambda) \;\big|\; \sigma \subset C\big\} \subset \mathrm{Sd}(\Lambda),$$

$$J'(\Lambda) = \big\{(n,g) \in J(\Lambda) \;\big|\; g(w,j) \leq g(w,j') \text{ if } w \in \mathbf{Z}, j, j' \in Q(w) \text{ and } j \geq j'\big\} \\ \subset J(\Lambda).$$

Here and hereafter, $g(w, -)$ denotes the restriction of the map g on $Q(w) \subset \Lambda$ for any w. Then

$$\mathrm{Sd}'(\Lambda) = \big\{C(n,g) \;\big|\; (n,g) \in J'(\Lambda)\big\},$$

and $\mathrm{Sd}'(\Lambda)$ is a subdivision of C.

We have an isomorphism of cones

(1) $$\mathbf{R}^{\Lambda}_{\geq 0} = \prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0} \xrightarrow{\sim} C, \quad b \mapsto c,$$

where $c_{w,j} := \sum_{k\in Q(w), k\geq j} b_{w,k}$ for $w \in \mathbf{Z}$ and $j \in Q(w)$.

Let Σ_Q be the subdivision of the cone $\mathbf{R}^{\Lambda}_{\geq 0} = \prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$ corresponding to the subdivision $\mathrm{Sd}'(\Lambda)$ of the cone C via the above isomorphism (1).

3.5.6.
Let $\overline{\mathcal{W}} \to \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$ be the map defined in Section 3.3.1.

For $Q = (Q(w))_w \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, let $\overline{\mathcal{W}}(Q) \subset \overline{\mathcal{W}}$ be the set of all $\Phi \in \overline{\mathcal{W}}$ such that $\Phi(w) \subset Q(w)$ for any $w \in \mathbf{Z}$.

PROPOSITION 3.5.7
Let $Q = (Q(w))_w \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$. Then we have a bijection

$$\overline{\mathcal{W}}(Q) \to \Sigma_Q, \qquad \Phi \mapsto \sigma_\Phi,$$

where σ_Φ is the set of all elements $((b_{w,j})_{j\in Q(w)})_{w\in\mathbf{Z}}$ of $\prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$ satisfying the following condition (1).

(1) *Let $w, w' \in \mathbf{Z}$, $j \in Q(w)$, $j' \in Q(w')$. Assume that, for any $M \in \Phi$ such that $j \leq M(\mathrm{gr}^W_w)$, we have $j' \leq M(\mathrm{gr}^W_{w'})$ (see Proposition 2.1.13). Then*

$$\sum_{k\in Q(w),k\geq j} b_{w,k} \leq \sum_{k\in Q(w'),k\geq j'} b_{w',k}.$$

REMARK
Condition (1) is equivalent to the following conditions (1a) and (1b):

(1a) $b_{w,j} = 0$ unless there is an $M \in \Phi$ such that $j = M(\mathrm{gr}^W_w)$;

(1b) $b_{w,j} = b_{w',j'}$ if there is an $M \in \Phi$ such that $j = M(\mathrm{gr}^W_w)$ and $j' = M(\mathrm{gr}^W_{w'})$.

Proof
By the construction in Section 3.5.5, we have bijections $J'(\Lambda) \simeq \mathrm{Sd}'(\Lambda) \simeq \Sigma_Q$. Under these bijections, the above σ_Φ is equal to the element of Σ_Q corresponding to the element $C(n,g) \in \mathrm{Sd}'(\Lambda)$, where (n,g) is the element of $J'(\Lambda)$ ($\Lambda = \bigsqcup_{w\in\mathbf{Z}} Q(w)$) defined as follows. Let n be the cardinality of Φ, that is, $n = \dim \sigma_\Phi$. Let $M^{(1)} = (M^{(1)}(w))_w, \ldots, M^{(n)} = (M^{(n)}(w))_w$ be all the members of Φ such that $M^{(1)}(w) \leq \cdots \leq M^{(n)}(w)$ for any $w \in \mathbf{Z}$ with respect to the ordering in Proposition 2.1.13. Then, for $w \in \mathbf{Z}$ and $j \in Q(w)$, define

$$g(w,j) = \sharp\{k \mid 1 \leq k \leq n, M^{(k)}(w) \geq j\}.$$

By Lemma 3.3.2, this map $\overline{\mathcal{W}}(Q) \to J'(\Lambda), \Phi \mapsto (n,g)$, is bijective. □

LEMMA 3.5.8
Let $Q \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, let $p \in S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$, let q be a point of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ lying over p, let $\Phi = \overline{\mathcal{W}}(q)$ (see Section 3.2.2), and let $\sigma_q = \sigma_\Phi \in \Sigma_Q$ (see Proposition 3.5.7). Let $P'(\sigma_q) \subset M^{\mathrm{gp}}_{S,>0,p}$ be as in Section 3.1.13. That is, for S and $\mathcal{S}$ in Section 3.5.4, let $\mathcal{S}(\sigma_q)$ be the subset of $\mathcal{S}^{\mathrm{gp}}$ consisting of all elements m of $\mathcal{S}^{\mathrm{gp}}$ such that the homomorphism $\mathcal{S}^{\mathrm{gp}} \to \mathbf{R}$ defined by any

element of σ_q sends m into $\mathbf{R}_{\geq 0}$, let $P(\sigma_q)$ be the image of $\mathcal{S}(\sigma_q)$ in $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_p$, and let $P'(\sigma_q)$ be the inverse image of $P(\sigma_q)$ in $M_{S,>0,p}^{\mathrm{gp}}$. Then we have

$$P'(\sigma_q) = \{ f \in M_{S,>0,p}^{\mathrm{gp}} \mid f(\tau_q(t)\mathbf{r}_q) \text{ converges in } \mathbf{R}_{\geq 0} \}, \tag{1}$$

$$P'(\sigma_q)^\times = \{ f \in M_{S,>0,p}^{\mathrm{gp}} \mid f(\tau_q(t)\mathbf{r}_q) \text{ converges to an element of } \mathbf{R}_{>0} \}. \tag{2}$$

Here $\mathbf{r}_q$ is a point on the torus orbit associated to q, $\tau_q : \mathbf{R}_{\geq 0}^\Phi \to \mathrm{Aut}(\mathrm{gr}^W)$ is $\bar{\tau}_{q'}$ in Section 3.2.3 for a point $q' \in D_{\mathrm{SL}(2)}$ lying over q, and t tends to 0^Φ.

Proof

In the notation of Section 3.5.4, $P'(\sigma_q) \subset M_{S,>0,p}^{\mathrm{gp}}$ is written as

$$P'(\sigma_q) = \bigcup_{m \in \mathcal{S}(\sigma_q)} \mathcal{O}_{S,>0,p}^\times \prod_{w \in \mathbf{Z}, j \in Q(w)} \beta_{w,j}^{m(w,j)},$$

where $m = \big((m(w,j))_{j \in Q(w)}\big)_{w \in \mathbf{Z}}$. This coincides with the right-hand side of (1) by Proposition 3.2.6(ii). Since $P'(\sigma_q)^\times = P'(\sigma_q) \cap P'(\sigma_q)^{-1}$, (2) follows. □

THEOREM 3.5.9

Let $Q \in \prod_{w \in \mathbf{Z}} \mathcal{W}(\mathrm{gr}_w^W)$.

(i) *Let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Sigma_Q)$ be the log modification (see Proposition 3.1.12) of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ corresponding to the subdivision Σ_Q of the cone $\prod_{w \in \mathbf{Z}} \mathbf{R}_{\geq 0}^{Q(w)}$ in Section 3.5.5. Then we have a bijection*

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Sigma_Q)$$

which sends a point q of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ lying over $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ to the point (p, σ_q, h_q) (see Section 3.1.13) of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Sigma_Q)$ lying over p, where σ_q is as in Lemma 3.5.8 and h_q is the homomorphism defined by

$$h_q : P'(\sigma_q)^\times \to \mathbf{R}_{>0}, \qquad f \mapsto \lim_{t \to 0^\Phi} f(\tau_q(t)\mathbf{r}_q),$$

where $\mathbf{r}_q$, τ_q and $\Phi = \overline{\mathcal{W}}(q)$ are as in Lemma 3.5.8.

(ii) *Let $\Phi \in \overline{\mathcal{W}}(Q)$, and let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(\Phi) \subset D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ be the image of $D_{\mathrm{SL}(2)}^{II}(\Phi)$. Then $D_{\mathrm{SL}(2)}^{II}(\Phi)$ coincides with the inverse image of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(\Phi)$ in $D_{\mathrm{SL}(2)}$. Furthermore, let $\sigma_\Phi \in \Sigma_Q$ be as in Proposition 3.5.7; then $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(\Phi)$ coincides with the part of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ which corresponds to the part $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\sigma_\Phi)$ of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Sigma_Q)$ under the bijection in (i).*

Proof

Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, let A be the fiber of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ on p, and let B be the set of all pairs (Φ, Z), where Φ is an element of $\overline{\mathcal{W}}$ whose image in $\prod_w \mathcal{W}(\mathrm{gr}_w^W)$ is $\big(\mathcal{W}(p(\mathrm{gr}_w^W))\big)_w$ and Z is an $\mathbf{R}_{>0}^\Phi$-orbit in $D(\mathrm{gr}^W)$ contained in $\prod_w Z_w$, where Z_w is the torus orbit associated to $p(\mathrm{gr}_w^W)$. Then we have a bijection from A to B given by $q \mapsto (\Phi, Z)$, where $\Phi = \overline{\mathcal{W}}(q)$ and Z is the torus orbit associated to q.

Assume that $Q(w) = \mathcal{W}(p(\mathrm{gr}_w^W))$ for all w. Then, once $\Phi \in \overline{\mathcal{W}}(Q)$ is fixed, the set B_Φ of all Z such that $(\Phi, Z) \in B$ is a $\bigl((\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)})/\mathbf{R}_{>0}^{\Phi}\bigr)$-torsor. On the other hand, let σ be the cone corresponding to Φ, and let C_Φ be the set of all homomorphisms $P'(\sigma)^\times \to \mathbf{R}_{>0}$ which extend the evaluation $\mathcal{O}_{>0,p}^\times \to \mathbf{R}_{>0}$ at p. Then C_Φ is also a $\bigl((\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)})/\mathbf{R}_{>0}^{\Phi}\bigr)$-torsor with respect to the following action. By the canonical isomorphism $M_{>0,p}^{\mathrm{gp}}/\mathcal{O}_{>0,p}^\times \simeq \prod_{w\in\mathbf{Z}} \mathbf{Z}^{Q(w)}$, we have an isomorphism

$$\mathrm{Hom}(M_{>0,p}^{\mathrm{gp}}/\mathcal{O}_{>0,p}^\times, \mathbf{R}_{>0}) \simeq \prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)}$$

which induces an isomorphism between quotient groups

$$\mathrm{Hom}\bigl(P'(\sigma)^\times/\mathcal{O}_{>0,p}^\times, \mathbf{R}_{>0}\bigr) \simeq \Bigl(\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)}\Bigr)/\mathbf{R}_{>0}^{\Phi}.$$

Since C_Φ is a $\mathrm{Hom}(P'(\sigma)^\times/\mathcal{O}_{>0,p}^\times, \mathbf{R}_{>0})$-torsor in the evident way, it is a $\bigl((\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)})/\mathbf{R}_{>0}^{\Phi}\bigr)$-torsor. Let A_Φ be the subset of A consisting of all $q \in A$ such that $\mathcal{W}(q) = \Phi$. Then the bijection $A \to B$ induces a bijection $A_\Phi \to B_\Phi$. The map $A_\Phi \to C_\Phi$ which sends $q \in A_\Phi$ to the homomorphism $P'(\sigma)^\times \to \mathbf{R}_{>0}, f \mapsto \lim_{t\to 0^\Phi} f(\tau_q(t)\mathbf{r}_q)$ (see Lemma 3.5.8) induces a map $B_\Phi \to C_\Phi$ which is compatible with the action of $\bigl(\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)}\bigr)/\mathbf{R}_{>0}^{\Phi}$. Since B_Φ and C_Φ are $\bigl((\prod_{w\in\mathbf{Z}} \mathbf{R}_{>0}^{Q(w)})/\mathbf{R}_{>0}^{\Phi}\bigr)$-torsors, this map $B_\Phi \to C_\Phi$ is bijective. Hence the map $A_\Phi \to C_\Phi$ is bijective.

Theorem 3.5.9 follows from this. □

3.5.10.
We regard $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ as follows. For $Q \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}_w^W)$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ is regarded as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ via the bijection in Theorem 3.5.9. If $Q' \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}_w^W)$ and $Q'(w) \subset Q(w)$ for all $w \in \mathbf{Z}$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q')$ is open in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ and the structure of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q')$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ coincides with the one induced from that of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$, as is easily seen. Hence there is a unique structure on $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ for which $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ are open and which induces on each $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim(Q)$ the above structure as an object of $\mathcal{B}_{\mathbf{R}}(\log)$.

PROPOSITION 3.5.11
Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. Then the following two conditions are equivalent.

(1) *The fiber of the surjection $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ over p consists of one element.*

(2) *There are at most one $w \in \mathbf{Z}$ such that the element $p(w)$ of $D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)$ does not belong to $D(\mathrm{gr}_w^W)$.*

Proof
This is seen easily by the proof of Theorem 3.5.9. □

From this the next corollary follows.

COROLLARY 3.5.12

The following three conditions are equivalent.

(1) *The map* $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ *is bijective.*

(2) *The morphism* $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ *is an isomorphism of local ringed spaces over* $\mathbf{R}$.

(3) *There are at most one* $w \in \mathbf{Z}$ *such that* $D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w) \neq D(\mathrm{gr}^W_w)$.

3.5.13.

Consider $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ for the five Examples I–V in Section 1.1.1.

For Examples I–IV, we have $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ by Corollary 3.5.12.

EXAMPLE V

Let M be the increasing filtration on gr^W_0 defined by

$$M_{-3} = 0 \subset M_{-2} = M_{-1} = \mathbf{R}e'_1 \subset M_0 = M_1 = M_{-1} + \mathbf{R}e'_2 \subset M_2 = \mathrm{gr}^W_0 .$$

Let M' be the increasing filtration on gr^W_1 defined by

$$M'_{-1} = 0 \subset M'_0 = M'_1 = \mathbf{R}e'_4 \subset M'_2 = \mathrm{gr}^W_1 .$$

Let $Q = \{Q(w)\}_{w\in\mathbf{Z}}$ be the following: $Q(0) := \{M\}$, $Q(1) := \{M'\}$, and $Q(w)$ is the empty set for $w \in \mathbf{Z} \smallsetminus \{0,1\}$. Let $\Lambda := \{M, M'\}$.

Then the subdivision Σ_Q of $\mathbf{R}^{\Lambda}_{\geq 0} = \prod_{w\in\mathbf{Z}} \mathbf{R}^{Q(w)}_{\geq 0}$ in Section 3.5.5 is just the barycentric subdivision of $\mathbf{R}^2_{\geq 0}$. In the notation in Section 3.5.5, $0 \leq n \leq 2$ and g is a function $\Lambda \to \{0,\dots,n\}$, and hence the fan Σ_Q consists of the vertex $\{(0,0)\}$ and the following cones according to Cases $m = 1,2,3,4,5$ in Section 2.3.9:

(0) $n = 0$, $g(M) = g(M') = 0$, and $C(0,g) = \{(0,0)\}$,
(1) $n = 1$, $g(M) = 1$, $g(M') = 0$, and $C(1,g) = \mathbf{R}_{\geq 0} \times \{0\}$,
(2) $n = 1$, $g(M) = 0$, $g(M') = 1$, and $C(1,g) = \{0\} \times \mathbf{R}_{\geq 0}$,
(3) $n = 1$, $g(M) = g(M') = 1$, and $C(1,g) = \{(a_\lambda)_\lambda \in \mathbf{R}^2_{\geq 0} \mid a_M = a_{M'}\}$,
(4) $n = 2$, $g(M) = 2$, $g(M') = 1$, and $C(2,g) = \{(a_\lambda)_\lambda \in \mathbf{R}^2_{\geq 0} \mid a_M \geq a_{M'}\}$,
(5) $n = 2$, $g(M) = 1$, $g(M') = 2$, and $C(2,g) = \{(a_\lambda)_\lambda \in \mathbf{R}^2_{\geq 0} \mid a_M \leq a_{M'}\}$.

Let B be the closure of $\mathbf{R}^2_{>0}$ in the corresponding blowing up of $\mathbf{C}^2$ at $(0,0)$.

Let $S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$. Then the inverse image $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q)$ of S via the projection $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ (see Section 3.5.3) is the log modification $S(\Sigma_Q)$ in Proposition 3.1.12 (see Theorem 3.5.9(i)), and we have the following commutative diagram:

$$\begin{array}{ccc} S(\Sigma_Q) = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q) & \simeq & B \times \mathbf{R}^2 \times \{\pm 1\} \\ \downarrow & & \downarrow \\ S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) & \simeq & \mathbf{R}^2_{\geq 0} \times \mathbf{R}^2 \times \{\pm 1\}. \end{array}$$

In the above isomorphism for $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q)$, the class p_m in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q)$ of the SL(2)-orbit in Case m in Section 2.3.9 corresponds to the point $(b_m,(0,0),1)$ of $B\times\mathbf{R}^2\times\{\pm1\}$, where b_m is the following point of B: b_1 is the limit of $(t,1)\in\mathbf{R}^2_{>0}$ for $t\to0$, b_2 is the limit of $(1,t)$ for $t\to0$, b_3 is the limit of (t,t) for $t\to0$, b_4 is the limit of (t_0t_1,t_1) for $t_0,t_1\to0$, and b_5 is the limit of (t_0,t_0t_1) for $t_0,t_1\to0$.

PROPOSITION 3.5.14

The map $D^{II}_{\mathrm{SL}(2)}\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ *is a morphism of* $\mathcal{B}_{\mathbf{R}}(\log)$.

The proof is given together with that of Theorem 3.5.15 below.

THEOREM 3.5.15

Fix any $F\in D(\mathrm{gr}^W)$, *let* $L=\mathcal{L}(F)$, *and let* $\bar{L}$ *be the compactified vector space associated to the graded vector space* L *of weights* ≤-2. *Then* $D^{II}_{\mathrm{SL}(2)}$ *is an* $\bar{L}$*-bundle over* $\mathrm{spl}(W)\times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ *in* $\mathcal{B}_{\mathbf{R}}(\log)$.

For the definition of the compactified vector space $\bar{L}$, see the explanation after Proposition 3.2.6 (see [KNU2, Section 7] for details).

Proof of Proposition 3.5.14 and Theorem 3.5.15

We deduce Proposition 3.5.14 and Theorem 3.5.15 from Theorem 3.4.4.

Let $p\in D^{II}_{\mathrm{SL}(2)}$, and let p' be the image of p in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$. Let $\mathbf{r}\in D$ be a point on the torus orbit associated to p, and let $\bar{\mathbf{r}}$ be the image of $\mathbf{r}$ in $D(\mathrm{gr}^W)$. It is sufficient to show that for some open neighborhood U of p' in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$, if we denote the inverse image of U in $D^{II}_{\mathrm{SL}(2)}$ by $\tilde{U}$, then $\tilde{U}$ is open in $D^{II}_{\mathrm{SL}(2)}$, the projection $\tilde{U}\to U$ is a morphism of $\mathcal{B}_{\mathbf{R}}(\log)$, and $\tilde{U}$ is isomorphic to $U\times\mathrm{spl}(W)\times\bar{L}$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ over $U\times\mathrm{spl}(W)$.

For $w\in\mathbf{Z}$, let $p_w=p(\mathrm{gr}^W_w)$ and $\mathbf{r}_w=\mathbf{r}(\mathrm{gr}^W_w)$. Take (R_w,S_w) for $(p_w,\mathbf{r}_w)$ as a pair in Section 3.4.1. Let $\Phi=\overline{\mathcal{W}}(p)$ and $Q(w)=\mathcal{W}(p_w)$. Let R' be an $\mathbf{R}$-subspace of $\prod_w\mathrm{Lie}(\tilde\rho_w(\mathbf{R}^{Q(w)}_{>0}))$ such that $\prod_w\mathrm{Lie}(\tilde\rho_w(\mathbf{R}^{Q(w)}_{>0}))=\mathrm{Lie}(\tilde\rho(\mathbf{R}^{\Phi}_{>0}))\oplus R'$. Let $R=\big(\prod_w R_w\big)\oplus R'$ and $S=\prod_w S_w$. Then (R,S) is a pair for $(p,\mathbf{r})$ as in Section 3.4.1.

Let $\bar{Y}(p,\mathbf{r},S)$ (resp., $\bar{Y}(p,\mathbf{r},R,S)$) be the subset of $Z(p)\times S$ (resp., $Z(p,R)\times S$) consisting of all elements (t,f,g,h,k) ($(t,f,g,h)\in Z(p)$ (resp., $\in Z(p,R)$), $k\in S$) which satisfy condition (4) in Section 3.4.2. We define the structure of $\bar{Y}(p,\mathbf{r},S)$ (resp., $\bar{Y}(p,\mathbf{r},R,S)$) as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ just in the same way as in the definition for $Y^{II}(p,\mathbf{r},S)$ (resp., $Y^{II}(p,\mathbf{r},R,S)$) in Section 3.4.2. Note that we have evident isomorphisms in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$Y^{II}(p,\mathbf{r},S)\simeq\bar{Y}(p,\mathbf{r},S)\times\bar{L}\times\mathfrak{g}_{\mathbf{R},u},\qquad Y^{II}(p,\mathbf{r},R,S)\simeq\bar{Y}(p,\mathbf{r},R,S)\times\bar{L}\times\mathfrak{g}_{\mathbf{R},u}.$$

Let $\bar{Y}_0(p,\mathbf{r},S)$ (resp., $\bar{Y}_0(p,\mathbf{r},R,S)$) be the open set of $\bar{Y}(p,\mathbf{r},S)$ (resp., $\bar{Y}(p,\mathbf{r},R,S)$) consisting of all elements (t,f,g,h,k) such that $t\in\mathbf{R}^{\Phi}_{>0}$.

For an open neighborhood U of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$ (resp., $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times R\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$), we define $\bar{Y}(p,\mathbf{r},S,U)$ (resp., $\bar{Y}(p,\mathbf{r},R,S,U)$) as the open set of $\bar{Y}(p,\mathbf{r},S)$ (resp., $\bar{Y}(p,\mathbf{r},R,S)$) consisting of all elements

(t,f,g,h,k) such that $(f,g,h,k)\in U$. Let $\bar{Y}_0(p,\mathbf{r},S,U)=\bar{Y}_0(p,\mathbf{r},S)\cap\bar{Y}(p,\mathbf{r},S,U)$ (resp., $\bar{Y}_0(p,\mathbf{r},R,S,U)=\bar{Y}_0(p,\mathbf{r},R,S)\cap\bar{Y}(p,\mathbf{r},R,S,U)$).

CLAIM 1

For a sufficiently small open neighborhood U of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times R\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$, there is an open immersion $\bar{Y}(p,\mathbf{r},R,S,U)\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ in $\mathcal{B}_{\mathbf{R}}(\log)$ whose restriction to $\bar{Y}_0(p,\mathbf{r},R,S,U)$ is given as $(t,f,g,h,k)\mapsto\bar{\tau}_p(t)\exp(g)\exp(k)\bar{\mathbf{r}}\in D(\mathrm{gr}^W)$ and which sends $(0^{\Phi},0,0,0,0)\in\bar{Y}(p,\mathbf{r},R,S,U)$ to p'.

We give the proof of Claim 1 later. We need one more claim.

CLAIM 2

Let $q\in D_{\mathrm{SL}(2)}$, and let $(q',s)\in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}\times\mathrm{spl}(W)$ be the image of q. Then the fiber on (q',s) in $D_{\mathrm{SL}(2)}$ regarded as a topological subspace of $D^I_{\mathrm{SL}(2)}$ (resp., $D^{II}_{\mathrm{SL}(2)}$) is homeomorphic to $\bar{L}$.

Claim 2 is shown easily.

We show that Proposition 3.5.14 and Theorem 3.5.15 follow from Claims 1 and 2. Let U be a sufficiently small open neighborhood of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times R\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)\times S$, let U' be the image of the open immersion $Y^{II}(p,\mathbf{r},R,S,U)\to D^{II}_{\mathrm{SL}(2)}$ (see Theorem 3.4.4), and let U'' be the image of the open immersion $\bar{Y}(p,\mathbf{r},R,S,U)\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ (see Claim 1). Then $U'\to U''$ is a morphism of $\mathcal{B}_{\mathbf{R}}(\log)$ since $Y^{II}(p,\mathbf{r},R,S,U)\to\bar{Y}(p,\mathbf{r},R,S,U)$, which is identified with the projection $\bar{Y}(p,\mathbf{r},R,S,U)\times\bar{L}\times\mathfrak{g}_{\mathbf{R},u}\to\bar{Y}(p,\mathbf{r},R,S,U)$, is a morphism of $\mathcal{B}_{\mathbf{R}}(\log)$. The map $U'\to U''\times\mathrm{spl}(W)$ is a trivial $\bar{L}$-bundle since $Y^{II}(p,\mathbf{r},R,S,U)\to\bar{Y}(p,\mathbf{r},R,S,U)\times\mathrm{spl}(W)$ is identified with the projection $\bar{Y}(p,\mathbf{r},R,S,U)\times\bar{L}\times\mathrm{spl}(W)\to\bar{Y}(p,\mathbf{r},R,S,U)\times\mathrm{spl}(W)$. Hence this morphism is proper. Let V be the inverse image of $U''\times\mathrm{spl}(W)$ under the canonical map $D^{II}_{\mathrm{SL}(2)}\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}\times\mathrm{spl}(W)$. We prove $V=U'$. Indeed, since U' is proper over $U''\times\mathrm{spl}(W)$, U' is open and closed in V. Since all fibers of $V\to U''\times\mathrm{spl}(W)$ are connected by Claim 2, and since $U'\to U''\times\mathrm{spl}(W)$ is surjective, we have $V=U'$. Hence V is open in $D^{II}_{\mathrm{SL}(2)}$, $V\to U''$ is a morphism of $\mathcal{B}_{\mathbf{R}}(\log)$, and $V\to U''\times\mathrm{spl}(W)$ is a trivial $\bar{L}$-bundle.

We prove Claim 1.

For each $w\in\mathbf{Z}$, let $Q(w)\in\mathcal{W}(\mathrm{gr}^W_w)$ be the image of Φ. For each $w\in\mathbf{Z}$, by Theorem 3.4.4 for the pure case, there is an open neighborhood U_w of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times S_w$ such that we have a morphism $Y^{II}(p_w,\mathbf{r}_w,S_w,U_w)\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ which sends $(t,f,g,h,k)\in Y^{II}_0(p_w,\mathbf{r}_w,S_w,U_w)$ to $\tau_{p_w}(t)\exp(g)\exp(k)\mathbf{r}_w$, which induces an open immersion $Y^{II}(p_w,\mathbf{r}_w,R_w,S_w,U'_w)\to D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ ($U'_w:=U_w\cap(\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times R_w\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times S_w)$) and which sends $(0^{Q(w)},0,0,0,0)\in Y^{II}(p_w,\mathbf{r}_w,R_w,S_w,U'_w)$ to p_w. By Lemma 3.4.13 for the pure case, for some open neighborhood $U''_w\subset U_w$ of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)\times S_w$, we have a morphism $Y^{II}(p_w,\mathbf{r}_w,S_w,U''_w)\to Y^{II}(p_w,\mathbf{r}_w,R_w,S_w,U'_w)$ which commutes with the morphisms to $D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$. Let $\bar{Y}(p,\mathbf{r},S)\to$

$Y^{II}(p_w, \mathbf{r}_w, S_w)$ be the morphism $(t,f,g,h,k) \mapsto (t(\mathrm{gr}^W_w), f(\mathrm{gr}^W_w), g(\mathrm{gr}^W_w), h(\mathrm{gr}^W_w), k(\mathrm{gr}^W_w))$, where $t(\mathrm{gr}^W_w)$ denotes the image of t under the homomorphism $\mathbf{R}^{\Phi}_{\geq 0} \to \mathbf{R}^{Q(w)}_{\geq 0}$ of multiplicative monoids induced by the map $\Phi \to Q(w)$. Then if U is a sufficiently small open neighborhood of zero in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times R \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$, the image of $\bar{Y}(p, \mathbf{r}, S, U)$ under this morphism is contained in $Y^{II}(p_w, \mathbf{r}_w, S_w, U''_w)$ for any w. Hence we have a composite morphism

$$\xi : \bar{Y}(p, \mathbf{r}, S, U) \to \prod_w Y^{II}(p_w, \mathbf{r}_w, S_w, U''_w) \to \prod_w Y^{II}(p_w, \mathbf{r}_w, R_w, S_w, U'_w).$$

Let P be the fiber product of

$$\prod_w Y^{II}(p_w, \mathbf{r}_w, R_w, S_w, U'_w) \to \prod_w \mathbf{R}^{Q(w)}_{\geq 0} \leftarrow \mathbf{R}^{\Phi}_{\geq 0} \times^{\mathbf{R}^{\Phi}_{>0}} \Big(\prod_w \mathbf{R}^{Q(w)}_{>0}\Big)$$

in $\mathcal{B}_{\mathbf{R}}(\log)$. Here $\mathbf{R}^{\Phi}_{\geq 0} \times^{\mathbf{R}^{\Phi}_{>0}} \big(\prod_w \mathbf{R}^{Q(w)}_{>0}\big)$ is the quotient of $\mathbf{R}^{\Phi}_{\geq 0} \times \big(\prod_w \mathbf{R}^{Q(w)}_{>0}\big)$ under the action of $\mathbf{R}^{\Phi}_{>0}$ given by $(x,y) \mapsto (ax, a^{-1}y)$ $(a \in \mathbf{R}^{\Phi}_{>0})$. Then P is identified with the fiber product of

$$\prod_w Y^{II}(p_w, \mathbf{r}_w, R_w, S_w, U'_w) \to \prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)\big(Q(w)\big) \leftarrow D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi).$$

Hence we have an open immersion $P \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$.

We have a unique morphism

$$\xi^{\sim} : \bar{Y}(p, \mathbf{r}, S, U) \to P$$

in $\mathcal{B}_{\mathbf{R}}(\log)$ which is compatible with ξ. It is induced from ξ and from the morphism $\bar{Y}(p, \mathbf{r}, S, U) \to \mathbf{R}^{\Phi}_{\geq 0} \times^{\mathbf{R}^{\Phi}_{>0}} \big(\prod_w \mathbf{R}^{Q(w)}_{>0}\big)$ which sends (t,f,g,h,k) to tt', where $t' \in \prod_w \mathbf{R}^{Q(w)}_{>0}$ is the $\big(\prod_w \mathbf{R}^{Q(w)}_{>0}\big)$-component of $\xi(1,g,g,g,k)$.

CLAIM 3

If U is a sufficiently small open neighborhood of zero in $T := \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times R \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$, the morphism $\bar{Y}(p, \mathbf{r}, R, S, U) \to P$ induced by $\xi^{\sim}$ is an open immersion.

By Claim 3, the open immersion stated in Claim 1 is obtained as the composite $\bar{Y}(p, \mathbf{r}, R, S, U) \to P \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$. It remains to prove Claim 3.

For an open neighborhood U of zero in T, let $P(U)$ be the open set of P consisting of all elements (t,f,g,h,k) $\big(t \in \mathbf{R}^{\Phi}_{\geq 0} \times^{\mathbf{R}^{\Phi}_{>0}} \big(\prod_w \mathbf{R}^{Q(w)}_{>0}\big),\ f, h \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W),\ g \in \prod_w R_w,\ k \in \prod_w S_w\big)$ such that $t = t' \exp(a)$ for some $t' \in \mathbf{R}^{\Phi}_{\geq 0}$ and for some $a \in R'$ satisfying $(f, a+g, h, k) \in U$. Then, for a given open neighborhood U of zero in T, there is an open neighborhood U' of zero in T such that the map $\xi^{\sim}$ induces a morphism $\bar{Y}(p, \mathbf{r}, R, S, U') \to P(U)$. On the other hand, if U is an open neighborhood of zero in T, then for a sufficiently small open neighborhood U' of zero in T, we have a morphism $P(U') \to \bar{Y}(p, \mathbf{r}, R, S, U)$. This morphism is obtained as the composite $P(U') \to \bar{Y}(p, \mathbf{r}, S, U'') \to \bar{Y}(p, \mathbf{r}, R, S, U)$. Here U'' is a suitable open neighborhood of zero in T. The first arrow is

$(t'\exp(a), f, g, h, k) \mapsto (t', f', g', h', k)$, where f', g', h' are near to f, g, h, respectively, and defined by $\exp(g') = \exp(a)\exp(g)$, $\exp(f') = \exp(f)\exp(a)$, $\exp(h') = \exp(2a)\exp(g)\exp(-a)$. The second arrow is a morphism constructed in the same way as in the proof of Lemma 3.4.13. For an open neighborhood U of zero in T, the composite $\bar{Y}(p, \mathbf{r}, R, S, U'') \to P(U') \to \bar{Y}(p, \mathbf{r}, R, S, U)$ and the composite $P(U'') \to \bar{Y}(p, \mathbf{r}, R, S, U') \to P(U)$ are inclusion maps. Here U' and U'' are open neighborhoods of zero in T, U' is sufficiently small relative to U, and U'' is sufficiently small relative to U'. This proves Claim 3. □

THEOREM 3.5.16

The canonical map

$$D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$$

is proper.

Proof

The map $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ is proper by Theorem 3.5.15. The map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is proper (see Theorem 3.5.9, Section 3.5.10). □

THEOREM 3.5.17

Let Γ be a subgroup of $G_{\mathbf{Z}}$. For $* = I, II$, we have the following.

(i) *The action of Γ on $D^*_{\mathrm{SL}(2)}$ is proper, and the quotient space $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ is Hausdorff.*

(ii) *Assume that Γ is neat. Let $\gamma \in \Gamma$, $p \in D_{\mathrm{SL}(2)}$, and assume $\gamma p = p$. Then $\gamma = 1$.*

(iii) *Assume that Γ is neat. Then the quotient $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ belongs to $\mathcal{B}_{\mathbf{R}}(\log)$, and the projection $D^*_{\mathrm{SL}(2)} \to \Gamma\backslash D^*_{\mathrm{SL}(2)}$ is a local isomorphism of objects of $\mathcal{B}_{\mathbf{R}}(\log)$.*

Here in (iii), we define the sheaf of real analytic functions on $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ and the log structure with sign on $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ in the natural way. That is, for an open set U of $\Gamma\backslash D^*_{\mathrm{SL}(2)}$, a real-valued function f on U is said to be real analytic if the pullback of f on the inverse image of U in $D^*_{\mathrm{SL}(2)}$ is real analytic. The log structure M of $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ is defined to be the sheaf of real analytic functions whose pullbacks on $D^*_{\mathrm{SL}(2)}$ belong to the log structure of $D^*_{\mathrm{SL}(2)}$. The subgroup sheaf $M^{\mathrm{gp}}_{>0}$ of M^{gp} is defined to be the part of M^{gp} consisting of the local sections whose pullbacks to $D^*_{\mathrm{SL}(2)}$ belong to the $M^{\mathrm{gp}}_{>0}$ of $D^*_{\mathrm{SL}(2)}$.

Recall that a subgroup Γ of $G_{\mathbf{Z}}$ is said to be *neat* if, for any $\gamma \in \Gamma$, the subgroup of $\mathbf{C}^\times$ generated by all eigenvalues of the action of γ on $H_{0,\mathbf{C}}$ is torsion free. If Γ is neat, then Γ is torsion free. There exists a neat subgroup of $G_{\mathbf{Z}}$ of finite index (see [Bo]).

Proof of Theorem 3.5.17

The proof is similar to [KNU2, Section 9], where we considered D_{BS}.

(i) $D^{II}_{\mathrm{SL}(2)}$ is Hausdorff because $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is Hausdorff (see [KU2]), and the map $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is proper (see Theorem 3.5.16). It follows that $D^{I}_{\mathrm{SL}(2)}$ is also Hausdorff.

Let Γ_u be the kernel of $\Gamma \to \mathrm{Aut}(\mathrm{gr}^W)$. The properness of the action of Γ on $D^{II}_{\mathrm{SL}(2)}$ is reduced to the properness of the action of Γ/Γ_u on $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, which is proved in [KU2], and to the properness of the action of Γ_u on $\mathrm{spl}(W)$. The properness of that on $D^{I}_{\mathrm{SL}(2)}$ follows from this because $D^{I}_{\mathrm{SL}(2)}$ is Hausdorff.

Since the action of Γ on $D^{*}_{\mathrm{SL}(2)}$ for $* = I, II$ is proper, the quotient space $\Gamma \backslash D^{*}_{\mathrm{SL}(2)}$ is Hausdorff.

(ii) The pure case is proved in [KU2]. The general case is reduced to the pure case since the action of Γ_u on $\mathrm{spl}(W)$ is fixed point free.

(iii) By (i) and (ii), the map $D^{*}_{\mathrm{SL}(2)} \to \Gamma \backslash D^{*}_{\mathrm{SL}(2)}$ is a local homeomorphism. The assertion (iii) follows from this. □

3.6. Examples

We consider $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ for Examples I–V in Section 1.1.1.

3.6.1.

We consider $D^{II}_{\mathrm{SL}(2)}$.

We use the notation in Section 1.1.1. As in Section 1.2.9, we denote by L the graded vector space $\mathcal{L}(F) = L^{-1,-1}_{\mathbf{R}}(F) \subset \mathcal{L}$ with $F \in D(\mathrm{gr}^W)$, which is independent of the choice of F for Examples I–V. Recall that $D^{II}_{\mathrm{SL}(2)}$ is an $\bar{L}$-bundle over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ (see Theorem 3.5.15) and that for Examples I–IV, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ (see Corollary 3.5.12). We describe the structure of the open set $D^{II}_{\mathrm{SL}(2)}(\Phi)$ of $D^{II}_{\mathrm{SL}(2)}$ for some $\Phi \in \overline{\mathcal{W}}$.

Let $\bar{\mathfrak{h}} = \{x + iy \mid x, y \in \mathbf{R}, 0 < y \le \infty\} \supset \mathfrak{h}$. We regard $\bar{\mathfrak{h}}$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ via $\bar{\mathfrak{h}} \simeq \mathbf{R}_{\ge 0} \times \mathbf{R}, x + iy \mapsto (1/\sqrt{y}, x)$ (cf. Section 3.2.13).

EXAMPLE I

We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccc} D & \simeq & \mathrm{spl}(W) \times L \\ \cap & & \cap \\ D^{II}_{\mathrm{SL}(2)} & \simeq & \mathrm{spl}(W) \times \bar{L} \end{array}$$

where the upper isomorphism is that of Section 1.2.9. Here $\mathrm{spl}(W) \simeq \mathbf{R}$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = D(\mathrm{gr}^W)$ which is just a one-point set, $L \simeq \mathbf{R}$ with weight -2, and $\bar{L}$ is isomorphic to the interval $[-\infty, \infty]$ endowed with the real analytic structure as in [KNU2, Example 7.5], with $w = -2$ which contains $\mathbf{R} = L$ in the natural way (see Section 1.2.9).

EXAMPLE II

Let $Q = \{W'\} \in \mathcal{W}(\mathrm{gr}^W_{-1}) = \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, where

$$W'_{-3} = 0 \subset W'_{-2} = W'_{-1} = \mathbf{R}e'_1 \subset W'_0 = \mathrm{gr}^W_{-1}.$$

The isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-1}) \simeq \mathfrak{h}$ extends to $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$.

Let Φ be the unique nonempty element of $\overline{\mathcal{W}}(Q)$. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \\ \cap & & \cap \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}} \end{array}$$

Recall that $\mathrm{spl}(W) \simeq \mathbf{R}^2$ (see Section 1.2.9). In this diagram, the upper isomorphism is that of Section 1.2.9. The lower isomorphism is induced by the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$.

The specific examples of SL(2)-orbits of rank 1 in Section 2.3.9, Example II have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}}$ are $i\infty$.

EXAMPLE III

Let $Q = \{W'\} \in \mathcal{W}(\mathrm{gr}^W_{-3}) = \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, where

$$W'_{-5} = 0 \subset W'_{-4} = W'_{-3} = \mathbf{R}e'_1 \subset W'_{-2} = \mathrm{gr}^W_{-3}\,.$$

The isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-3}) \simeq \mathfrak{h}$ extends to $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$.

Let Φ be the unique nonempty element of $\overline{\mathcal{W}}(Q)$. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \times L & & \big(s, x+iy, (d_1, d_2)\big) \\ \cap & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} & & \big(s, x+iy, (y^{-2}d_1, y^{-1}d_2)\big) \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^2$, $L \simeq \mathbf{R}^2$ with weight -3, and $(d_1, d_2) \in \mathbf{R}^2 = L$ (see Section 1.2.9). In this diagram, the upper isomorphism is that of Section 1.2.9. The lower isomorphism is induced by the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$, and the following morphism $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$. It is induced by $\nu_{\alpha,\beta}$, where $\alpha_{-3} : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_{-3})$ is defined by $\alpha_{-3}(t)e'_1 = t^{-4}e'_1$, $\alpha_{-3}(t)e'_2 = t^{-2}e'_2$, and $\beta : D(\mathrm{gr}^W_{-3}) = \mathfrak{h} \to \mathbf{R}_{>0}$ is the distance to Φ-boundary defined by $x + iy \mapsto 1/\sqrt{y}$ (see Section 3.2.13). Note that the right vertical arrow is *not* the evident map, as indicated.

The SL(2)-orbits in Section 2.3.9, Example III, Case 1 (resp., Case 2, resp., Case 3) have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}} \times \bar{L}$ belong to $\{i\infty\} \times L$ (resp., $\{i\} \times (\bar{L} \smallsetminus L)$, resp., $\{i\infty\} \times (\bar{L} \smallsetminus L)$).

EXAMPLE IV

Let $Q = \{W'\} \in \mathcal{W}(\mathrm{gr}^W_{-1}) = \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, where

$$W'_{-3} = 0 \subset W'_{-2} = W'_{-1} = \mathbf{R}e'_2 \subset W'_0 = \mathrm{gr}^W_{-1}\,.$$

The isomorphism $D(\mathrm{gr}^W) = D(\mathrm{gr}^W_{-1}) \simeq \mathfrak{h}$ extends to $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$.

Let Φ be the unique nonempty element of $\overline{\mathcal{W}}(Q)$. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \times L & & (s, x+iy, d) \\ \cap & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} & & (s, x+iy, y^{-1}d) \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^5$, $L \simeq \mathbf{R}$ with weight -2, and $d \in \mathbf{R} = L$ (see Section 1.2.9). In this diagram, the upper isomorphism is that of Section 1.2.9. The lower isomorphism is induced from the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$, $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}$ and the following morphism $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$. It is induced by $\nu_{\alpha,\beta}$ (see Propositions 3.2.6, 3.2.7, Section 3.2.8, Proposition 3.2.9, Theorem 3.2.10), where $\alpha_{-1} : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_{-1})$ is defined by

$$\alpha_{-1}(t)e'_2 = t^{-2}e'_2, \alpha_{-1}(t)e'_3 = e'_3$$

and $\beta : D(\mathrm{gr}^W_{-1}) = \mathfrak{h} \to \mathbf{R}_{>0}$ is the distance to Φ-boundary defined by $x + iy \mapsto 1/\sqrt{y}$ (see Section 3.2.13). Note that the right vertical arrow is *not* the inclusion map, as indicated.

The SL(2)-orbits in Section 2.3.9, Example IV, Case 1 (resp., Case 2, resp., Case 3) have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}} \times \bar{L}$ belong to $\{i\infty\} \times L$ (resp., $\{i\} \times (\bar{L} \smallsetminus L)$, resp., $\{i\infty\} \times (\bar{L} \smallsetminus L)$).

EXAMPLE V

Let $Q \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, and let the log modification B of $\mathbf{R}^2_{\geq 0}$ be as in Section 3.5.13. The isomorphism $D(\mathrm{gr}^W) \simeq \mathfrak{h}^{\pm} \times \mathfrak{h}$ (see Section 1.2.9) extends to an isomorphism $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \simeq \bar{\mathfrak{h}}^{\pm} \times \bar{\mathfrak{h}}$ ($\bar{\mathfrak{h}}^{\pm}$ is the disjoint union of $\bar{\mathfrak{h}}^{+} = \bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^{-} = \{x + iy \mid x \in \mathbf{R}, 0 > y \geq -\infty\}$ ($\mathfrak{h}^{+} \simeq \mathfrak{h}^{-}, x + iy \mapsto -x - iy$)), and this composite isomorphism is extended to an isomorphism $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q) \simeq B \times \mathbf{R}^2 \times \{\pm 1\}$ (see Section 3.5.13).

Let Φ be the maximal element of $\overline{\mathcal{W}}(Q)$. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h}^{\pm} \times \mathfrak{h} & & (s, x+iy, x'+iy') \\ \cap & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times B \times \mathbf{R}^2 \times \{\pm 1\} & & \big(s, 1/\sqrt{|y|}, 1/\sqrt{y'}, x, x', \mathrm{sign}(y)\big) \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^6$ (see Section 1.2.9). In this diagram, the upper isomorphism is that of Section 1.2.9. The lower isomorphism is induced from the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(Q) \simeq B \times \mathbf{R}^2 \times \{\pm 1\}$.

The SL(2)-orbits in Section 2.3.9, Example V have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in B are described in Section 3.5.13.

3.6.2.

We consider $D^{I}_{\mathrm{SL}(2)}$. For Examples I–IV, $D^{I}_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$ by Proposition 3.4.29.

EXAMPLE V

Let $\Psi = \{W'\} \in \mathcal{W}$, where

$$W'_{-3} = 0 \subset W'_{-2} = W'_{-1} = \mathbf{R}e_1 \subset W'_0 = W'_{-1} + \mathbf{R}e_2$$
$$\subset W'_1 = W'_0 + \mathbf{R}e_4 + \mathbf{R}e_5 \subset W'_2 = H_{0,\mathbf{R}}.$$

(This W' is $W^{(1)}$ in Section 2.3.9, Example V, Case 1.) Let $\bar{\Psi} = \{W'(\mathrm{gr}^W)\} \in \overline{\mathcal{W}}$. Then $D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$ is the open set of $D^{II}_{\mathrm{SL}(2)}(\Phi)$ in Section 3.6.1, Example V corresponding to the subcone $\mathbf{R}_{\geq 0} \times \{0\}$ of $\mathbf{R}^2_{\geq 0}$.

We compare $D^{I}_{\mathrm{SL}(2)}(\Psi)$ and $D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$. For $j = 1, 2, 3$, let

$$A_j = \mathrm{Hom}_{\mathbf{R}}(\mathrm{gr}^W_1, \mathbf{R}e_j).$$

We have an isomorphism of real analytic manifolds

$$\mathrm{spl}(W) \xrightarrow{\sim} \prod_{j=1}^{3} A_j, \qquad s \mapsto (a_j)_{1 \leq j \leq 3},$$

$$\text{where} \quad s(v) \equiv \sum_{j=1}^{3} a_j(v) \bmod \mathbf{R}e_4 + \mathbf{R}e_5 \text{ for } v \in \mathrm{gr}^W_1 .$$

Let

$$(A_3 \times \bar{\mathfrak{h}}^{\pm})' := \big\{(v, x+iy) \in A_3 \times \bar{\mathfrak{h}}^{\pm} \bigm| v = 0 \text{ if } y = \pm\infty\big\} \subset A_3 \times \bar{\mathfrak{h}}^{\pm}.$$

Then we have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccc} D & \simeq & \Big(\prod_{j=1}^{3} A_j\Big) \times \mathfrak{h}^{\pm} \times \mathfrak{h} \\ \cap & & \cap \\ D^{II}_{\mathrm{SL}(2)}(\bar{\Psi}) & \simeq & \Big(\prod_{j=1}^{3} A_j\Big) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h} \end{array}$$

In this diagram, the upper isomorphism is induced by the isomorphism in Section 1.2.9 and the above isomorphism $\mathrm{spl}(W) \simeq \prod_{j=1}^{3} A_j$. On the other hand, we have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccccc} D & \simeq & \Big(\prod_{j=1}^{3} A_j\Big) \times \mathfrak{h}^{\pm} \times \mathfrak{h} & & (a_1, a_2, a_3, x+iy, \tau) \\ \cap & & \downarrow & & \downarrow \\ D^{I}_{\mathrm{SL}(2)}(\Psi) & \simeq & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & & (a_1, a_2, |y|^{1/2} a_3, x+iy, \tau) \end{array}$$

In this diagram, the upper isomorphism is the same as in the first diagram. The lower isomorphism is induced from the canonical morphisms $D^{I}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \to A_1 \times A_2$ and $D^{I}_{\mathrm{SL}(2)}(\Psi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\bar{\Psi}) \simeq \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$, and the following morphism $D^{I}_{\mathrm{SL}(2)}(\Psi) \to A_3$. It is the composite

$$D^{I}_{\mathrm{SL}(2)}(\Psi) \xrightarrow{\text{by } \nu_{\alpha,\beta}} D \xrightarrow{\mathrm{spl}_W} \mathrm{spl}(W) \simeq \prod_{j=1}^{3} A_j \to A_3,$$

where $\nu_{\alpha,\beta}$ is the morphism described in Propositions 3.2.6, 3.2.7, Section 3.2.8, Proposition 3.2.9, and Theorem 3.2.10. Here $\alpha : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the splitting of Ψ defined by $\alpha(t)e_1 = t^{-2}e_1$, $\alpha(t)e_2 = e_2$, $\alpha(t)e_3 = t^2 e_3$, $\alpha(t)e_4 = te_4$, $\alpha(t)e_5 = te_5$, and $\beta : D \to \mathbf{R}_{>0}$ is the distance to Ψ-boundary defined as the composite $D \to D(\mathrm{gr}_0^W) \simeq \mathfrak{h}^{\pm} \to \mathbf{R}_{>0}$, where the last arrow is $x + iy \mapsto 1/\sqrt{|y|}$.

Note that the right vertical arrow of the above commutative diagram is *not* the inclusion map, as indicated.

The lower isomorphisms in the above two commutative diagrams form a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$,

$$\begin{array}{ccccc} D^{I}_{\mathrm{SL}(2)}(\Psi) & \simeq & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & \ni & (a_1, a_2, a_3, x+iy, \tau) \\ \downarrow & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\bar{\Psi}) & \simeq & \Big(\prod_{j=1}^{3} A_j\Big) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h} & \ni & (a_1, a_2, |y|^{-1/2}a_3, x+iy, \tau) \end{array}$$

Here the left vertical arrow is the inclusion map. The right vertical arrow is *not* the evident map, as indicated.

The SL(2)-orbits in Section 2.3.9, Example V, Case 1 have classes in $D^{I}_{\mathrm{SL}(2)}(\Psi)$ whose images in $\bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$ are $(i\infty, i)$.

3.7. $D_{\mathrm{BS,val}}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$

We outline the definitions of $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS,val}}$ in the fundamental diagram in Section 0.2, which connect $D_{\mathrm{SL}(2)}$ and D_{BS}. The detailed studies of these spaces will be given later in this series of articles.

3.7.1.

Let S be an object of $\mathcal{B}_{\mathbf{R}}(\log)$ (see Section 3.1). Then we have a local ringed space S_{val} over S with a log structure with sign. This is the real analytic analogue of the complex analytic theory considered in [KU3, Section 3.6]. In the case when we have a chart $\mathcal{S} \to M_{S,>0}$ with $\mathcal{S}$ an fs monoid,

$$S_{\mathrm{val}} = \varprojlim_{\Sigma} S(\Sigma),$$

where Σ ranges over all finite rational subdivisions of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$ (see Proposition 3.1.12). The general case is reduced to this case by gluing (cf. [KU3, Section 3.6]).

3.7.2.

For $* = I, II$, define $D^{*}_{\mathrm{SL}(2),\mathrm{val}} = (D^{*}_{\mathrm{SL}(2)})_{\mathrm{val}}$. In the pure case, as topological spaces they coincide with the topological space $D_{\mathrm{SL}(2),\mathrm{val}}$ in [KU2].

3.7.3.

$D_{\mathrm{BS,val}}$ is defined similarly; that is, $D_{\mathrm{BS,val}} = (D_{\mathrm{BS}})_{\mathrm{val}}$. Here we use the log structure with sign of D_{BS} induced by $\bar{A}_P \simeq \mathbf{R}^n_{\geq 0}$ and $\bar{B}_P \simeq \mathbf{R}^{n+1}_{\geq 0}$ in the notation in [KNU2, 5.1].

3.7.4.
A canonical injection $D^*_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is defined but not necessarily continuous (for both $* = I$ and II). This is a difference from the pure case, and we try to explain it a little more in the next subsection.

3.8. D_{BS} and $D_{\mathrm{SL}(2)}$

Here in the end of this section, we review some points of our constructions and compare them with the construction of D_{BS} in [KNU2].

3.8.1.
First, see Proposition 1.2.5, which shows that there are three kinds of coordinate functions on D, that is, s, F, and δ. Among these, what is new in the mixed case is s and δ. Thus when we want to endow a partial compactification such as $D_{\mathrm{SL}(2)}$ and D_{BS} with a real analytic structure by extending coordinate functions, we have to treat s and δ. Among these two, s is more important in applications, and the methods to treat s are common to the cases of $D_{\mathrm{SL}(2)}$ and D_{BS}.

3.8.2.
On the other hand, the treatment of the δ-coordinate for $D_{\mathrm{SL}(2)}$ and that for D_{BS} are considerably different (see Section 3.6.1, Examples III, IV, which illustrate the situation of $D_{\mathrm{SL}(2)}$). In there, the third components (δ-coordinates) of the vertical arrows in the diagrams are not the inclusion maps but the twisted ones. In general, the $\bar{L}$-component of the function which gives the real analytic structures on $D_{\mathrm{SL}(2)}$ is not the evident one but the one twisted back by torus actions (cf. Proposition 3.2.6). This twisting is natural in view of the relationship with nilpotent orbits and crucial in the applications (cf. Section 2.5.7).

3.8.3.
In the case of D_{BS}, the δ-coordinate was also naturally twisted, but there is a difference between these two twistings, which explains the discontinuity of $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ in Section 3.7.4.

More precisely, for example, consider Example III in Section 3.6.1. Let p be a point of $D_{\mathrm{SL}(2),\mathrm{val}}$. Then the $\bar{L}$-component of the image of p in $D^{II}_{\mathrm{SL}(2)}$ is in the boundary (i.e., belongs to $\bar{L} \smallsetminus L$) if and only if $W \in \mathcal{W}(p)$, but the $\bar{L}$-component of its image in D_{BS} is in the boundary if and only if p is not split. Hence some arc joining a split point and a nonsplit point in $D_{\mathrm{SL}(2),\mathrm{val}}$ can have a disconnected image on D_{BS}. These equivalences hold for any Hodge types, and we can even prove that for some Hodge types, there are no choices of topologies of $D_{\mathrm{SL}(2)}$ satisfying both the crucial property Section 2.5.7(ii) and the continuities of the maps $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$, and so on, in the fundamental diagram in Section 0.2. These topics will be treated later in this series.

4. Applications

4.1. Nilpotent orbits, SL(2)-orbits, and period maps

In [KNU1], we generalized the SL(2)-orbit theorem in several variables of Cattani, Kaplan, and Schmid [CKS] for degenerations of polarized Hodge structures to an SL(2)-orbit theorem in several variables for degenerations of mixed Hodge structures with polarized graded quotients. Here we interpret it in the style of a result on the extension of a period map into $D_{\mathrm{SL}(2)}$ defined by a nilpotent orbit.

THEOREM 4.1.1

Assume that $(N_1, \dots, N_n, F)$ generates a nilpotent orbit (see Section 2.4.1) and the associated $W^{(j)}(\mathrm{gr}^W)$ is rational (see Section 2.2.2) for any $j = 1, \dots, n$. Then there is a sufficiently small open neighborhood U of $\mathbf{0} := (0, \dots, 0)$ in $\mathbf{R}^n_{\geq 0}$ satisfying the following (i) and (ii).

(i) *The real analytic map*

$$p : U \cap \mathbf{R}^n_{>0} \to D, \qquad t = (t_1, \dots, t_n) \mapsto \exp\Big(\sum_{j=1}^{n} iy_j N_j\Big)F,$$

where $y_j = \prod_{k=j}^{n} t_k^{-2}$, is defined and extends to a real analytic map

$$p : U \to D^I_{\mathrm{SL}(2)}.$$

(ii) *For $c \in U$, $p(c) \in D_{\mathrm{SL}(2)}$ is described as follows. Let $K = \{j \mid 1 \leq j \leq n, c_j = 0\}$, and write $K = \{b(1), \dots, b(m)\}$ with $b(1) < \cdots < b(m)$. Let $b(0) = 0$. For $1 \leq j \leq m$, let $N'_j = \sum_{b(j-1)<k\leq b(j)} \big(\prod_{k\leq \ell < b(j)} c_\ell^{-2}\big) N_k$, where $\prod_{b(j)\leq \ell < b(j)} c_\ell^{-2}$ is considered as 1. Let $F' = \exp\big(i \sum_{b(m)<k\leq n} \big(\prod_{k\leq \ell \leq n} c_\ell^{-2}\big) N_k\big) F$. Then $(N'_1, \dots, N'_m, F')$ generates a nilpotent orbit (see Section 2.4.1), and $p(c)$ is the class of the $\mathrm{SL}(2)$-orbit associated to $(N'_1, \dots, N'_m, F')$ (see Theorem 2.4.2). Hence, when $t \in U$ and $t \to c$, we have the convergence*

$$\exp\Big(\sum_{j=1}^{n} iy_j N_j\Big)F \to \big(\text{class of the } \mathrm{SL}(2)\text{-orbit associated to } (N'_1, \dots, N'_m, F')\big)$$

in $D^I_{\mathrm{SL}(2)}$ and hence in $D^{II}_{\mathrm{SL}(2)}$. In particular, $p(\mathbf{0})$ is the class of the $\mathrm{SL}(2)$-orbit associated to $(N_1, \dots, N_n, F)$.

Proof

For $(N_1, \dots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$, let τ and $((\rho_w, \varphi_w), \mathbf{r}_1, J) \in \mathcal{D}_{\mathrm{SL}(2),n}$ be as in Theorem 2.4.2. Write $J = \{a(1), \dots, a(r)\}$ with $a(1) < \cdots < a(r)$. Let $W^{(j)} = M(N_1 + \cdots + N_j, W)$ $(0 \leq j \leq n)$, where $W^{(0)} := W$. Let $\Psi = \{W^{(a(j))}\}_{1\leq j\leq r}$. Let τ_J be the J-component of τ. Take $\alpha = \tau_J$ as a splitting of Ψ (see Section 3.2.3), and take a distance to Ψ-boundary β (see Section 3.2.4).

For $t = (t_j)_{1\leq j\leq n} \in \mathbf{R}^n_{>0}$, let $t'_J = \big(\prod_{a(j)\leq \ell < a(j+1)} t_\ell\big)_{j\in J}$, where $a(r+1)$ means $n+1$. Let $q(t) = \prod_{a(1)\leq \ell} \tau_\ell(t_\ell)^{-1} p(t)$. Then $q(t) = \tau_J(t'_J)^{-1} p(t)$.

First, we show that $q(t)$ extends to a real analytic map on some open neighborhood U of $\mathbf{0}$ in $\mathbf{R}^n_{\geq 0}$. To see this, we may assume that $a(1) = 1$. Since $\tau(t)$

here coincides with $t(y)$ in [KNU1, Theorem 0.5], in the notation there, we have

$$q(t) = \tau(t)^{-1}p(t) = {}^{e}g(y)\exp\bigl(\varepsilon(y)\bigr)\mathbf{r}.$$

Hence, by [KNU1, Theorem 0.5], the assertion follows. The extended map, also denoted by q, sends $\mathbf{0}$ to $\mathbf{r}_1 \in D$ in Theorem 2.4.2(ii); that is, $q(\mathbf{0}) = \mathbf{r}_1$.

In case where $W \in \Psi$, since $\mathbf{r}_1 \in D_{\mathrm{nspl}}$, shrinking U if necessary, we may assume that $p(t) \in D_{\mathrm{nspl}}$ for any $t \in U \cap \mathbf{R}^n_{>0}$.

CLAIM 1

After further replacing U, the map

$$U \cap \mathbf{R}^n_{>0} \to B := \mathbf{R}^{\Psi}_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}\bigl(W'(\mathrm{gr}^W)\bigr),$$

$$t \mapsto \bigl(\beta(p(t)), \tau_J\beta(p(t))^{-1}p(t), \mathrm{spl}_W(p(t)), (\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(t)(\mathrm{gr}^W)))_{W'}\bigr)$$

extends to a real analytic map $p' : U \to B$ sending $\mathbf{0}$ to $(\mathbf{0}, \tau_J\beta(\mathbf{r}_1)^{-1}\mathbf{r}_1, s, (s^{(W')})_{W'})$. Here s is the limiting splitting of W in [KNU1, Theorem 0.5(1)], which coincides with $\mathrm{spl}_W(\mathbf{r}_1)$ (see Theorem 2.4.2), and $s^{(W')}$ is the splitting of $W'(\mathrm{gr}^W)$ given by $(\rho_w, \varphi_w)_w$ (cf. Proposition 3.2.6(i)).

Since $\beta(p(t)) = \beta(\tau_J(t'_J)q(t)) = t'_J\beta(q(t))$ (see Section 3.2.4), this extends to a real analytic map on some open neighborhood of $\mathbf{0}$ in $\mathbf{R}^n_{\geq 0}$ which sends $\mathbf{0}$ to $\mathbf{0}$.

Since $\tau_J\beta(p(t))^{-1}p(t) = \tau_J\beta(q(t))^{-1}q(t)$, this extends to a real analytic map on some open neighborhood of $\mathbf{0}$ in $\mathbf{R}^n_{\geq 0}$ which sends $\mathbf{0}$ to $\tau_J\beta(\mathbf{r}_1)^{-1}\mathbf{r}_1$.

By [KNU1, Theorem 0.5(2)], $\mathrm{spl}_W(p(t))$ extends to a real analytic map on some open neighborhood of $\mathbf{0}$ in $\mathbf{R}^n_{\geq 0}$ which sends $\mathbf{0}$ to s.

Finally, by [KNU1, Proposition 8.5], $\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(t)(\mathrm{gr}^W))$ extends to a real analytic map on some open neighborhood of $\mathbf{0}$ in $\mathbf{R}^n_{\geq 0}$ which sends $\mathbf{0}$ to $(s^{(W')})_{W'}$.

Next, it is easy to see that $(N'_1, \ldots, N'_m, F')$ generates a nilpotent orbit (see Section 2.4.1) for any c in a sufficiently small U. Since its associated SL(2)-orbits belong to $D^I_{\mathrm{SL}(2)}(\Psi)$, once we prove the following claim the real analytic map $p' : U \to B$ in Claim 1 factors through the image in B of the map $\nu_{\alpha,\beta}$ in Proposition 3.2.7(i).

CLAIM 2

The point $\exp\bigl(\sum_{j=1}^n iy_jN_j\bigr)F$ converges to the class of the SL(2)*-orbit associated to $(N'_1, \ldots, N'_m, F')$ in $D^I_{\mathrm{SL}(2)}$ when $t \in U$ and $t \to c$.*

Thus we reduce both Theorem 4.1.1(i) and (ii) to this claim.

To prove Claim 2, we first consider the case $c = \mathbf{0}$: In this case, the image by $\nu_{\alpha,\beta}$ of the class of the SL(2)-orbit $((\rho_w, \varphi_w)_w, \mathbf{r}_1, J) \in \mathcal{D}_{\mathrm{SL}(2),n}$ associated to $(N_1, \ldots, N_n, F)$ is $\lim_{t_J \to \mathbf{0}_J}(t_J\beta(\mathbf{r}_1), \tau_J\beta(\mathbf{r}_1)^{-1}\mathbf{r}_1, s, (s^{(W')})_{W'})$ by definition of $\nu_{\alpha,\beta}$. On the other hand, $p'(\mathbf{0})$ is $(\mathbf{0}, \tau_J\beta(\mathbf{r}_1)^{-1}\mathbf{r}_1, s, (s^{(W')})_{W'})$ by Claim 1. Since $\nu_{\alpha,\beta}$ is injective (see Proposition 3.2.7(i)), the case where $c = \mathbf{0}$ of Claim 2 follows.

Now we are in the general case. Let $c \in U$, K be as in (ii). Let $t' \in U$ be the element defined by $t'_j = t_j$ if $j \in K$ and by $t_j = c_j$ if $j \notin K$. Then, by the case where $c = \mathbf{0}$, we have the convergence

$$\exp\Big(\sum_{j\in J} iy'_j N'_j\Big)F' \to \big(\text{class of the SL(2)-orbit associated to } (N'_1, \dots, N'_m, F')\big).$$

Together with

$$\begin{aligned}\nu_{\alpha,\beta}\big(\lim_{t\to c} p(t)\big) &= p'(c) = \lim_{t'\to c} p'(t') \\ &= \nu_{\alpha,\beta}\Big(\lim_{t'\to c} \exp\Big(\sum_{j\in J} iy'_j N'_j\Big)F'\Big),\end{aligned}$$

we have the general case of Claim 2. □

4.2. Hodge metrics at the boundary of $D^I_{\mathrm{SL}(2)}$

We expect that $D_{\mathrm{SL}(2)}$ plays a role as a natural space in which real analytic asymptotic behaviors of degenerating objects are well described. In this subsection we illustrate this by taking the degeneration of the Hodge metric as an example, and we explain our previous result on the norm estimate in [KNU1] via $D^I_{\mathrm{SL}(2)}$.

4.2.1.

Let $F \in D$. For $c > 0$, we define a Hermitian form

$$(\ ,\ )_{F,c} : H_{0,\mathbf{C}} \times H_{0,\mathbf{C}} \to \mathbf{C}$$

as follows.

For each $w \in \mathbf{Z}$, let

$$(\ ,\ )_{F(\mathrm{gr}^W_w)} :\ \mathrm{gr}^W_{w,\mathbf{C}} \times \mathrm{gr}^W_{w,\mathbf{C}} \to \mathbf{C}$$

be the Hodge metric $\langle C_{F(\mathrm{gr}^W_w)}(\bullet), \bar{\bullet}\rangle_w$, where $C_{F(\mathrm{gr}^W_w)}$ is the Weil operator. For $v \in H_{0,\mathbf{C}}$ and for $w \in \mathbf{Z}$, let $v_{w,F}$ be the image in $\mathrm{gr}^W_{w,\mathbf{C}}$ of the w-component of v with respect to the canonical splitting of W associated to F. Define

$$(v, v')_{F,c} = \sum_{w\in\mathbf{Z}} c^w (v_{w,F}, v'_{w,F})_{F(\mathrm{gr}^W_w)} \quad (v, v' \in H_{0,\mathbf{C}}).$$

PROPOSITION 4.2.2

Let Ψ be an admissible set of weight filtrations on $H_{0,\mathbf{R}}$ (see Section 3.2.2). Let β be a distance to Ψ-boundary (see Section 3.2.4, Proposition 3.2.5). Assume $W \notin \Psi$ (resp., $W \in \Psi$). For each $W' \in \Psi$, let $\beta_{W'} : D \to \mathbf{R}_{>0}$ (resp., $D_{\mathrm{nspl}} \to \mathbf{R}_{>0}$) be the W'-component of β. For $p \in D$, let

$$(\ ,\ )_{p,\beta} := (\ ,\ )_{p,c} \quad \textit{with } c = \prod_{W'\in\Psi} \beta_{W'}(p)^{-2}.$$

Let $m : \Psi \to \mathbf{Z}$ be a map, let $V = V_m = \bigcap_{W'\in\Psi} W'_{m(W'),\mathbf{C}}$, and let $\mathrm{Her}(V)$ *be the space of all Hermitian forms on V.*

Let $(\ ,\ )_{p,\beta,m} \in \mathrm{Her}(V)$ be the restriction of $\prod_{W'\in\Psi} \beta_{W'}(p)^{2m(W')}(\ ,\ )_{p,\beta}$ to V.

(i) *The real analytic map* $f: D$ *(resp.,* D_{nspl}*)* $\to \mathrm{Her}(V)$*,* $p \mapsto (\ ,\ )_{p,\beta,m}$*, extends to a real analytic map* $f: D^I_{\mathrm{SL}(2)}(\Psi)$ *(resp.,* $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$*)* $\to \mathrm{Her}(V)$*.*

(ii) *For a point* $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ *(resp.,* $p \in D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$*) such that* Ψ *is the set of weight filtrations associated to* p*, the limit of* $(\ ,\ )_{p,\beta,m}$ *at* p *induces a positive definite Hermitian form on the quotient space*

$$V/\Big(\sum_{m'<m}\bigcap_{W'\in\Psi} W'_{m'(W'),\mathbf{C}}\Big),$$

where $m' < m$ *means* $m'(W') \le m(W')$ *for all* $W' \in \Psi$ *and* $m' \ne m$*.*

Proof
We prove (i). Assume $W \notin \Psi$. Fix a splitting $\alpha : (\mathbf{R}^\times)^\Psi \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ of Ψ. Let $p \in D$. Let $v, v' \in V$. Then we have the weight decompositions $v = \sum_{m'\le m} v_{m'}$, $v' = \sum_{m'\le m} v'_{m'}$ with respect to α. Since

$$\begin{aligned}(v,v')_{p,\beta} &= \big(\alpha\beta(p)(\alpha\beta(p))^{-1}v, \alpha\beta(p)(\alpha\beta(p))^{-1}v'\big)_{\alpha\beta(p)(\alpha\beta(p))^{-1}p,\beta}\\ &= \big(\alpha\beta(p)^{-1}v, \alpha\beta(p)^{-1}v'\big)_{\alpha\beta(p)^{-1}p,1},\end{aligned}$$

we have

$$\begin{aligned}&(v,v')_{p,\beta,m}\\ (1)\qquad &= \prod_{W'\in\Psi} \beta_{W'}(p)^{2m(W')}\big(\alpha\beta(p)^{-1}v, \alpha\beta(p)^{-1}v'\big)_{\alpha\beta(p)^{-1}p,1}\\ &= \sum_{m',m''\le m}\prod_{W'\in\Psi} \beta_{W'}(p)^{(2m-m'-m'')(W')}(v_{m'}, v'_{m''})_{\alpha\beta(p)^{-1}p,1}.\end{aligned}$$

This extends to a real analytic function on $D^I_{\mathrm{SL}(2)}(\Psi)$ because $(2m - m' - m'')(W') \ge 0$ for all $W' \in \Psi$, and $D \to D$, $p \mapsto \alpha\beta(p)^{-1}p$, extends to a real analytic map $D^I_{\mathrm{SL}(2)}(\Psi) \to D$ (see Theorem 3.2.10(i)).

In the case $W \in \Psi$, the argument is analogous.

We prove (ii). Let $v, v' \in V$ be as above. Let $\{p_\lambda\}_\lambda$ be a sequence in D which converges to p, and let $q = \lim_\lambda \alpha\beta(p_\lambda)^{-1}(p_\lambda) \in D$. Then, by the result of (i), we have from (1),

$$(2)\qquad \lim_\lambda (v,v')_{p_\lambda,\beta,m} = (v_m, v'_m)_{q,1}.$$

The right-hand side of (2) is nothing but the restriction of the Hermitian metric at $q \in D$ to the m-component with respect to α, which is therefore positive definite. □

4.2.3.
As will be shown in a later part of our series, the norm estimate in [KNU1] for a given variation of mixed Hodge structure $S \to D$ (cf. [KNU1, Section 12]) is incorporated in the diagram

$$U \to D^I_{\mathrm{SL}(2)}(\Psi)\ (\text{resp.}, D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}) \xrightarrow{f} \mathrm{Her}(V).$$

Here U is an open neighborhood of a point of $S_{\text{val}}^{\log}$, the first arrow is induced by an extension of the period map $S_{\text{val}}^{\log} \to \Gamma\backslash D_{\text{SL}(2)}^{I}$, where Γ is an appropriate group (cf. Section 4.4.9), and f is as in Proposition 4.2.2.

4.2.4.

EXAMPLE V

We consider Example V. Here the norm estimate is not continuous on $D_{\text{SL}(2)}^{II}$.

Let Ψ and $\bar{\Psi}$ be as in Section 3.6.2. Fix $u, v \in \mathbf{C}e_4 + \mathbf{C}e_5 \subset W_1'$, and let u', v' be their respective images in gr_1^W. Let $\beta : D \to \mathbf{R}_{>0}$ be the distance to Ψ-boundary which appears in Section 3.6.2.

As in Proposition 4.2.2, the map

$$f : D \to \mathbf{C}, \qquad p \mapsto \beta(p)^2(u, v)_{p,\beta}$$

extends to a real analytic function $f : D_{\text{SL}(2)}^{I}(\Psi) \to \mathbf{C}$. We show, however, that for some choices of u and v, this map $f : D_{\text{SL}(2)}^{I}(\Psi) \to \mathbf{C}$ is not continuous with respect to the topology of $D_{\text{SL}(2)}^{II}$. These can be explained by the following commutative diagram at the end of Section 3.6.2.

$$\begin{array}{ccccc}
\big(a_1, a_2, (a_3, x+iy), \tau\big) & \in & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & \simeq & D_{\text{SL}(2)}^{I}(\Psi) \xrightarrow{f} \mathbf{C} \\
\downarrow & & \downarrow & & \downarrow \\
(a_1, a_2, |y|^{-1/2}a_3, x+iy, \tau) & \in & \Big(\prod_{j=1}^{3} A_j\Big) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h} & \simeq & D_{\text{SL}(2)}^{II}(\bar{\Psi}).
\end{array}$$

Recall that $A_j = \text{Hom}_{\mathbf{R}}(\text{gr}_1^W, \mathbf{R}e_j)$ $(j = 1, 2, 3)$. The composite

$$A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} \simeq D_{\text{SL}(2)}^{I}(\Psi) \xrightarrow{f} \mathbf{C}$$

sends $(a_1, a_2, (a_3, x+iy), \tau)$ to

$$\big(|y|^{-3/2}a_1(u') + |y|^{-1/2}a_2(u') + a_3(u'),$$
$$|y|^{-3/2}a_1(v') + |y|^{-1/2}a_2(v') + a_3(v')\big)_{0,(x+iy)/|y|} + (u', v')_{1,\tau}.$$

Here $(\ ,\ )_{0,(x+iy)/|y|}$ is the Hodge metric on $\text{gr}_{0,\mathbf{C}}^W$ associated to $(x+iy)/|y| \in \mathfrak{h}^{\pm} = D(\text{gr}_0^W)$, and $(\ ,\ )_{1,\tau}$ is the Hodge metric on $\text{gr}_{1,\mathbf{C}}^W$ associated to $\tau \in \mathfrak{h} = D(\text{gr}_1^W)$. On the other hand, the composition

$$\prod_{j=1}^{3} A_j \times \mathfrak{h}^{\pm} \times \mathfrak{h} \simeq D \xrightarrow{f} \mathbf{C},$$

where the first arrow is induced by the lower (not upper) horizontal isomorphism of the above diagram, sends $(a_1, a_2, a_3, x+iy, \tau)$ to

$$\big(|y|^{-3/2}a_1(u') + |y|^{-1/2}a_2(u') + |y|^{1/2}a_3(u'),$$
$$|y|^{-3/2}a_1(v') + |y|^{-1/2}a_2(v') + |y|^{1/2}a_3(v')\big)_{0,(x+iy)/|y|} + (u', v')_{1,\tau}.$$

For some choices of u and v, as is precisely explained below, the last map is not extended continuously to the point $(0, 0, 0, i\infty, i)$ of $\prod_{j=1}^{3} A_j \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$, for this map has the term $|y|^{1/2}$ which diverges at $i\infty$. Since $(0,0,0,i\infty,i)$ is the image of $(0,0,(0,i\infty),i) \in A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h}$ under the left vertical arrow, this shows that for some choices of u and v, $f : D^I_{\mathrm{SL}(2)}(\Psi) \to \mathbf{C}$ is not continuous for the topology of $D^{II}_{\mathrm{SL}(2)}$.

More precisely, take u and v such that there exists $b \in A_3$ for which $(b(u'), b(v'))_{0,i} \neq 0$. Let c be a real number such that $0 < c < 1/2$. Then, as $y \to \infty$, $(0,0,y^{c-1/2}b, iy, i) \in \prod_{j=1}^3 A_j \times \mathfrak{h}^{\pm} \times \mathfrak{h}$ converges to $(0,0,0,i\infty,i) \in \prod_{j=1}^3 A_j \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$. However, f sends the image of $(0,0,y^{c-1/2}b, iy, i)$ in D under the lower isomorphism of the diagram to $(y^c b(u'), y^c b(v'))_{0,i} + (u',v')_{1,i}$, which diverges.

4.3. Hodge filtrations at the boundary

4.3.1.
In Section 4.3, let $X = D^I_{\mathrm{SL}(2)}$ or $D^{II}_{\mathrm{SL}(2)}$.

Let $\mathcal{O}_X$ be the sheaf of real analytic functions on X, and let $\alpha : M_X \to \mathcal{O}_X$ be the log structure with sign on X. We define a sheaf of rings $\mathcal{O}'_X$ on X by $\mathcal{O}'_X := \mathcal{O}_X[q^{-1} \mid q \in \alpha(M_X)] \supset \mathcal{O}_X$. Let $\mathcal{O}'_{X,\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathcal{O}'_X$. The following theorem shows that the Hodge filtration over $\mathcal{O}'_{X,\mathbf{C}}$ extends to the boundary of X.

THEOREM 4.3.2
Let X be one of $D^I_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, and let $\mathcal{O}'_X$ be as in Section 4.3.1.

Then, for each $p \in \mathbf{Z}$, there is a unique $\mathcal{O}'_{X,\mathbf{C}}$-submodule F^p of $\mathcal{O}'_{X,\mathbf{C}} \otimes_{\mathbf{Z}} H_0$ which is locally a direct summand and whose restriction to D coincides with the filter F^p of $\mathcal{O}_{X,\mathbf{C}} \otimes_{\mathbf{Z}} H_0$.

Proof
It is sufficient to prove the case $X = D^{II}_{\mathrm{SL}(2)}$ because the assertion for $X = D^I_{\mathrm{SL}(2)}$ follows from that for $X = D^{II}_{\mathrm{SL}(2)}$ by pulling back.

Assume $X = D^{II}_{\mathrm{SL}(2)}$. Let F be the universal Hodge filtration on D, and write $F = s(\theta(F',\delta))$ ($s \in \mathrm{spl}(W)$, $F' \in D(\mathrm{gr}^W)$, $\delta \in \mathcal{L}(F')$) as in Proposition 1.2.5. Let Φ be an admissible set of weight filtrations on gr^W (see Section 3.2.2), let α be a splitting of Φ, and let β be a distance to Φ-boundary as in Proposition 3.2.5(ii). We observe

$$(1) \qquad s\big(\theta(F',\delta)\big) = s\big(\theta(\alpha\beta(F')(\alpha\beta(F'))^{-1}F', \mathrm{Ad}(\alpha\beta(F'))\,\mathrm{Ad}(\alpha\beta(F'))^{-1}\delta)\big).$$

By Proposition 3.2.6(ii), $(\alpha\beta(F')^{-1}F', \mathrm{Ad}(\alpha\beta(F'))^{-1}\delta)$, and s extend real analytically over the Φ-boundary. Let $G' = \prod_w \mathrm{Aut}(\mathrm{gr}^W_w)$, and consider the splitting $\alpha : \mathbf{G}_m^{\Phi} \to G'$. Then the section $\beta(F')$ of $\mathbf{G}_m^{\Phi}(\mathcal{O}'_X)$ on $D^{II}_{\mathrm{SL}(2)}(\Phi)$ is sent to a section $\alpha\beta(F')$ of $G'(\mathcal{O}'_X)$ over $D^{II}_{\mathrm{SL}(2)}(\Phi)$. Thus $F = s(\theta(F',\delta))$ extends uniquely to a filtration of $\mathcal{O}'_{X,\mathbf{C}} \otimes H_0$ consisting of $\mathcal{O}'_{X,\mathbf{C}}$-submodules which are locally direct summands. □

4.3.3.
REMARKS

(i) For $D_{\mathrm{SL}(2),\mathrm{val}}$, D_{BS}, $D_{\mathrm{BS},\mathrm{val}}$, theorems similar to Theorem 4.3.2 are analogously proved.

(ii) The Hodge decomposition and the Hodge metric also extend over the boundary after tensoring with $\mathcal{O}'_{X,\mathbf{C}}$. In the pure case, this together with the period map $S^{\log}_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ (see Section 4.2.3) explains the existence of the log C^∞ Hodge decomposition in [KMN].

4.4. Example IV and height pairing

We consider Example IV. The space $D_{\mathrm{SL}(2)} = D^{I}_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$ in this example is related to the asymptotic behavior of the Archimedean height pairing for elliptic curves in degeneration (see [P2], [C], [Si]). We describe which kind of SL(2)-orbits appear in such a geometric situation of degeneration.

The following observations were obtained in discussions with Spencer Bloch.

4.4.1.
Recall (see [A]) that the *Archimedean height pairing* for an elliptic curve E over $\mathbf{C}$ is $\langle Z, W\rangle \in \mathbf{R}$ defined for divisors Z, W on E of degree zero such that $|Z| \cap |W| = \emptyset$ ($|Z|$ here denotes the support of Z), characterized by the following properties (1)–(4).

(1) If $|Z| \cap |W| = |Z'| \cap |W| = \emptyset$, then $\langle Z + Z', W\rangle = \langle Z, W\rangle + \langle Z', W\rangle$.

(2) We have $\langle Z, W\rangle = \langle W, Z\rangle$.

(3) If f is a meromorphic function on E such that $|(f)| \cap |W| = \emptyset$ and if $W = \sum_{w\in|W|} n_w(w)$, then $\langle (f), W\rangle = -(2\pi)^{-1} \sum_{w\in|W|} n_w \log(|f(w)|)$.

(4) The map $(E(\mathbf{C}) \smallsetminus |W|) \times (E(\mathbf{C}) \smallsetminus |W|) \to \mathbf{R}$, $(a, b) \mapsto \langle (a) - (b), W\rangle$, is continuous.

4.4.2.
Consider Example IV.

Let $\tau \in \mathfrak{h}$, and let E_τ be the elliptic curve $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$.

For divisors Z, W on E_τ of degree zero such that $|Z| \cap |W| = \emptyset$, we define an element

$$p(\tau, Z, W) \in G_{\mathbf{Z},u}\backslash D$$

as follows.

For $\tau \in \mathfrak{h}$ and $z \in \mathbf{C}$, let

$$\theta(\tau, z) = \prod_{n=0}^{\infty}(1 - q^n t) \cdot \prod_{n=1}^{\infty}(1 - q^n t^{-1}), \quad \text{where } q = e^{2\pi i\tau}, t = e^{2\pi i z}.$$

We have

$$(1) \qquad \theta(\tau, z+1) = \theta(\tau, z), \quad \theta(\tau, z+\tau) = -e^{-2\pi i z}\theta(\tau, z).$$

Write

$$Z=\sum_{j=1}^{r} m_j(p_j), \qquad W=\sum_{j=1}^{s} n_j(q_j)$$

($p_j, q_j \in E_\tau$, $m_j, n_j \in \mathbf{Z}$, $\sum_{j=1}^{r} m_j = 0$, $\sum_{j=1}^{s} n_j = 0$), and write

$$p_j = \big(z_j \bmod (\mathbf{Z}\tau+\mathbf{Z})\big), \qquad q_j = \big(w_j \bmod (\mathbf{Z}\tau+\mathbf{Z})\big)$$

with $z_j, w_j \in \mathbf{C}$. Define

(2) $$p(\tau, Z, W) = \text{class of } F(\tau, w, \lambda, z) \in G_{\mathbf{Z},u}\backslash D$$

$$\text{with} \quad z=\sum_{j=1}^{r} m_j z_j, \quad w=\sum_{j=1}^{s} n_j w_j, \quad \lambda = (2\pi i)^{-1}\log\Big(\prod_{j,k}\theta(\tau, z_j - w_k)^{m_j n_k}\Big),$$

and with $F(\tau, w, \lambda, z) \in D$ as in Section 1.1.1, Example IV. This element $p(\tau, Z, W)$ of $G_{\mathbf{Z},u}\backslash D$ is well defined: as is easily seen using (1), the right-hand side of (2) does not change when we replace $((z_j)_j, (w_j)_j)$ by $((z'_j)_j, (w'_j)_j)$ such that $z'_j \equiv z_j \bmod \mathbf{Z}\tau + \mathbf{Z}$ and $w'_j \equiv w_j \bmod \mathbf{Z}\tau + \mathbf{Z}$ for any j. For example, in the case where $z'_\ell = z_\ell + \tau$ for some ℓ, $z'_j = z_j$ for the other $j \neq \ell$, and $w'_j = w_j$ for any j, by (1), the right-hand side of (2) given by $(z'_j)_j, (w'_j)_j$ is the class of $F(\tau, w, \lambda + m_\ell w, z + m_\ell\tau) = \gamma F(\tau, w, \lambda, z)$, where γ is the element of $G_{\mathbf{Z},u}$ which sends e_j $(j = 1, 2, 3)$ to e_j and e_4 to $e_4 - m_\ell e_3$.

4.4.3.
Let $L = \mathcal{L}(F)$ with $F \in D(\mathrm{gr}^W)$, which is independent of F, and let $\delta : D \to L = \mathbf{R}$ be the δ-component (see Proposition 1.2.5). Note that

$$\delta\big(F(\tau, w, \lambda, z)\big) = \mathrm{Im}(\lambda) - \mathrm{Im}(z)\,\mathrm{Im}(w)/\mathrm{Im}(\tau)$$

(see Section 1.2.9, Example IV).

LEMMA 4.4.4
The map $\delta : D \to \mathbf{R}$ factors through the projection $D \to G_{\mathbf{Z},u}\backslash D$, and we have

$$\delta\big(p(\tau, Z, W)\big) = \langle Z, W\rangle,$$

where $\langle Z, W\rangle \in \mathbf{R}$ is the Archimedean height pairing (see Section 4.4.1).

4.4.5.
The equality in Lemma 4.4.4 is well known. It has also the following geometric (cohomological) interpretation.

Let E be an elliptic curve over $\mathbf{C}$, and let Z and W be divisors of degree zero on E such that $|Z| \cap |W| = \emptyset$. We assume $Z \neq 0$, $W \neq 0$.

Let $U = E \smallsetminus |Z|$, $V = E \smallsetminus (|Z| \cup |W|)$, and let $j : V \to U$ be the inclusion map. Write $Z = \sum_{z\in|Z|} m_z(z)$, $W = \sum_{w\in|W|} n_w(w)$. We have exact sequences of mixed Hodge structures

$$0 \to H^1(E, \mathbf{Z})(1) \to H^1(U, \mathbf{Z})(1) \to \mathbf{Z}^{|Z|} \to H^2(E, \mathbf{Z})(1) \to 0,$$

$$0 \to H^0(U,\mathbf{Z})(1) \to \mathbf{Z}(1)^{|W|} \to H^1(U, j_!\mathbf{Z})(1) \to H^1(U,\mathbf{Z})(1) \to 0.$$

Note that the map $\mathbf{Z}^{|Z|} \to H^2(E,\mathbf{Z})(1) = \mathbf{Z}$ is identified with the degree map. Let $A \subset B \subset H^1(U, j_!\mathbf{Z})(1)$ be sub mixed Hodge structures defined as follows. A is the image of $\{x = (x_w)_w \in \mathbf{Z}(1)^{|W|} \mid \sum_w n_w x_w = 0\}$ under $\mathbf{Z}(1)^{|W|} \to H^1(U, j_!\mathbf{Z})(1)$. B is the inverse image of $\{(m_z x)_z \mid x \in \mathbf{Z}\}$ under the composition $H^1(U, j_!\mathbf{Z})(1) \to H^1(U,\mathbf{Z})(1) \to \mathbf{Z}^{|Z|}$. Let $H = B/A$. Then we have the induced injective homomorphism $a : \mathbf{Z}(1) \to H$, the induced surjective homomorphism $b : H \to \mathbf{Z}$, and $\mathrm{Ker}(b)/\mathrm{Im}(a) = H^1(E,\mathbf{Z})(1)$. A well-known cohomological interpretation of the height pairing $\langle Z, W\rangle$ is

$$\langle Z, W\rangle = \delta(H).$$

On the other hand, in the case $E = E_\tau$, as is well known,

$$p(\tau, Z, W) = \mathrm{class}(H).$$

This explains Lemma 4.4.4.

4.4.6.
We consider degeneration.

Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$, and let $\Delta^* = \Delta \smallsetminus \{0\}$. Fix an integer $c \geq 1$, and consider the family of elliptic curves over Δ^* whose fiber over $e^{2\pi i\tau/c}$ ($\mathrm{Im}(\tau) > 0$) is $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. This family has a Néron model E_c over Δ whose fiber over $0 \in \Delta$ is canonically isomorphic to $\mathbf{C}^\times \times \mathbf{Z}/c\mathbf{Z}$ as a Lie group. If $a \in \mathbf{Q}$ and $ca \in \mathbf{Z}$, and if u is a holomorphic function $\Delta \to \mathbf{C}^\times$, there is a section of E_c over Δ whose restriction to Δ^* is given by $e^{2\pi i\tau/c} \mapsto (a\tau + f(e^{2\pi i\tau/c}) \bmod \mathbf{Z}\tau + \mathbf{Z})$ with $f = (2\pi i)^{-1}\log(u)$ and whose value at $0 \in \Delta$ is $(u(0), ca \bmod c\mathbf{Z}) \in \mathbf{C}^\times \times \mathbf{Z}/c\mathbf{Z}$. Any section of E_c over Δ is obtained in this way.

Let $\Gamma \subset G_{\mathbf{Z}}$ be the subgroup consisting of all elements γ which satisfy $\gamma(e_j) - e_j \in \bigoplus_{1\leq k<j} \mathbf{Z}e_k$ for $j = 1,2,3,4$. Note that $\Gamma \supset G_{\mathbf{Z},u}$. Note also that $\delta : D \to L = \mathbf{R}$ factors through the projection $D \to \Gamma\backslash D$.

Fix $m_j, n_k \in \mathbf{Z}$, $a_j, b_k \in \mathbf{Q}$ $(1 \leq j \leq r,\ 1 \leq k \leq s)$ such that $\sum_j m_j = 0$ and $\sum_k n_k = 0$, $ca_j, cb_k \in \mathbf{Z}$ for any j,k, and take holomorphic functions $u_j, v_k : \Delta \to \mathbf{C}^\times$ $(1 \leq j \leq r, 1 \leq k \leq s)$. Assume that, for any j,k, the section p_j of E_c defined by (a_j, u_j) and the section q_k of E_c defined by (b_k, v_k) do not meet over Δ. Consider the morphism

$$p : \Delta^* \to \Gamma\backslash D, \qquad e^{2\pi i\tau/c} \mapsto \Big(p\Big(\tau, \sum_j m_j(p_j), \sum_k n_k(q_k)\Big) \bmod \Gamma\Big)$$

with

$$p_j := \big(a_j\tau + f_j(e^{2\pi i\tau/c}) \bmod \mathbf{Z}\tau + \mathbf{Z}\big),$$
$$q_k := \big(b_k\tau + g_k(e^{2\pi i\tau/c}) \bmod \mathbf{Z}\tau + \mathbf{Z}\big),$$

where

$$f_j := (2\pi i)^{-1}\log(u_j), \qquad g_k := (2\pi i)^{-1}\log(v_k).$$

4.4.7.
Let $\Delta^{\log} = |\Delta| \times \mathbf{S}^1$, where $|\Delta| := \{r \in \mathbf{R} \mid 0 \le r < 1\}$, $\mathbf{S}^1 := \{u \in \mathbf{C}^\times \mid |u| = 1\}$. We have a projection $\Delta^{\log} \to \Delta, (r,u) \mapsto ru$ ($r \in |\Delta|$, $u \in \mathbf{S}^1$) and an embedding $\Delta^* \to \Delta^{\log}, ru \mapsto (r,u)$ ($r \in |\Delta|$, $r \neq 0$, $u \in \mathbf{S}^1$).

We define the sheaf of C^∞-functions on $\Delta^{\log}$ as follows. For an open set U of $\Delta^{\log}$ and a real-valued function h on U, h is C^∞ if and only if the following (1) holds. Let U' be the inverse image of U in $\mathbf{R}_{\ge 0} \times \mathbf{R}$ under the surjective map $\mathbf{R}_{\ge 0} \times \mathbf{R} \to \Delta^{\log}, (t,x) \mapsto (e^{-1/t^2}, e^{2\pi i x})$.

(1) The pullback of h on U' extends, locally on U', to a C^∞-function on some open neighborhood of U' in $\mathbf{R}^2$.

Roughly speaking, a function h on $\Delta^{\log}$ is C^∞ if $h(e^{2\pi i(x+iy)})$ ($x \in \mathbf{R}, 0 < y \le \infty$) is a C^∞-function in x and $1/\sqrt{y}$.

The restriction of this sheaf of C^∞-functions on $\Delta^{\log}$ to the open set Δ^* coincides with the usual sheaf of C^∞-functions on Δ^*.

PROPOSITION 4.4.8
Let $\Phi \in \overline{\mathcal{W}}$ be as in Section 3.6.1, Example IV.

(i) *The map $p : \Delta^* \to \Gamma\backslash D$ in Section 4.4.6 extends to a C^∞ map $\Delta^{\log} \to \Gamma\backslash D^{II}_{\mathrm{SL}(2)}(\Phi)$. That is, we have a commutative diagram of local ringed spaces over $\mathbf{R}$*

$$\begin{array}{ccc} \Delta^* & \xrightarrow{p} & \Gamma\backslash D \\ \cap & & \cap \\ \Delta^{\log} & \to & \Gamma\backslash D^{II}_{\mathrm{SL}(2)}(\Phi). \end{array}$$

(ii) *Let $B_2(x)$ be the second Bernoulli polynomial $x^2 - x + 1/6$. For $x \in \mathbf{R}$, $\{x\}$ denotes the unique real number such that $0 \le \{x\} < 1$ and $\{x\} \equiv x \bmod \mathbf{Z}$.*

Then the composite $\Delta^* \xrightarrow{p} \Gamma\backslash D \xrightarrow{\delta} L = \mathbf{R}$ has the form

$$e^{2\pi i(x+iy)/c} \mapsto \frac{1}{2}\Big(\sum_{j,k} m_j n_k B_2(\{a_j - b_k\})\Big)y + h(e^{2\pi i(x+iy)/c})$$

for some C^∞-function h on $\Delta^{\log}$.

(iii) *Let*

$$D^{II}_{\mathrm{SL}(2)}(\Phi) \simeq \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L}$$

be the lower isomorphism in the commutative diagram in Section 3.6.1, Example IV. Then the projection $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$ factors through $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \Gamma\backslash D^{II}_{\mathrm{SL}(2)}(\Phi)$, and the composite $\Delta^{\log} \xrightarrow{p} \Gamma\backslash D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$ sends any point of $\Delta^{\log} \smallsetminus \Delta^*$ to

$$\frac{1}{2}\Big(\sum_{j,k} m_j n_k B_2(\{a_j - b_k\})\Big) \in \mathbf{R} = L \subset \bar{L}.$$

In (ii) and (iii), $B_2(x)$ can be replaced by the polynomial $x^2 - x$. The constant term of $B_2(x)$ does not play a role, for $\sum_{j,k} m_j n_k = 0$.

Note that the restriction of the map $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$ in (iii) to D is *not* $\delta : D \to L$ but *is* $p \mapsto \mathrm{Ad}\big(\alpha\beta(p(\mathrm{gr}^W))\big)^{-1}\delta(p)$, where α and β are as in Section 3.6.1, Example IV.

Proof of Proposition 4.4.8
We may and do assume $0 \le a_j < 1$ and $0 \le b_k < 1$. Let $J = \{(j,k) \mid 1 \le j \le r, 1 \le k \le s,\ a_j < b_k\}$. Then, for each j and k, the function

$$e^{2\pi i\tau/c} \mapsto \theta\big(\tau, (a_j - b_k)\tau + f_j(e^{2\pi i\tau/c}) - g_k(e^{2\pi i\tau/c})\big)$$

on Δ is meromorphic and its order of zero at $0 \in \Delta$ is $(a_j - b_k)c$ if $(j,k) \in J$ and is zero otherwise. By using this and by using the description of $\mathrm{spl}_W : D \to \mathrm{spl}(W)$ in Section 1.2.9, Example IV, we see that the composite

$$\Delta^* \xrightarrow{p} \Gamma\backslash D \underset{\simeq}{\xrightarrow{1.2.9}} \Gamma\backslash\big(\mathrm{spl}(W) \times \mathfrak{h}\big) \times L$$

has the property that the part $\Delta^* \to \Gamma\backslash(\mathrm{spl}(W) \times \mathfrak{h})$ extends to a C^∞-function $\Delta^{\log} \to \Gamma\backslash(\mathrm{spl}(W) \times \bar{\mathfrak{h}})$ and that the part $\Delta^* \to L = \mathbf{R}$ has the form $e^{2\pi i\tau/c} \mapsto \big(-(\sum_j m_j a_j)(\sum_k n_k b_k) + \sum_{(j,k)\in J} m_j n_k (a_j - b_k)\big)\,\mathrm{Im}(\tau) + h(e^{2\pi i\tau/c})$, where h is a C^∞-function on $\Delta^{\log}$. Note that

$$\begin{aligned} &-\Big(\sum_j m_j a_j\Big)\Big(\sum_k n_k b_k\Big) + \sum_{(j,k)\in J} m_j n_k (a_j - b_k) \\ &= \frac{1}{2}\Big(\sum_{j,k} m_j n_k B_2(\{a_j - b_k\})\Big). \end{aligned}$$

Hence, for the lower isomorphism $D^{II}_{\mathrm{SL}(2)}(\Phi) \simeq \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L}$ in the diagram in Section 3.6.1, Example IV, the composite $\Delta^* \to \bar{L}$ is written as $e^{2\pi i\tau/c} \mapsto (1/2)(\sum_{j,k} m_j n_k B_2(\{a_j - b_k\})) + (\mathrm{Im}(\tau))^{-1} h(e^{2\pi i\tau/c})$, where $(\mathrm{Im}(\tau))^{-1}h$ is a C^∞-function on $\Delta^{\log}$ which has value zero on $\Delta^{\log} \smallsetminus \Delta^*$. These imply the assertions. □

4.4.9.
The above Proposition 4.4.8 implies a special case of the height estimate by Pearlstein [P2].

The lower map in the diagram in Proposition 4.4.8(i) is an example of the extended period map (cf. Section 4.2.3). In a forthcoming part of this series of articles, the existence of the extended period map $X^{\log}_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ (X is a log smooth fs log analytic space) will be proved generally for a variation of mixed Hodge structure on $U = X_{\mathrm{triv}}$ with polarized graded quotients with global monodromy in an appropriate group Γ which has unipotent local monodromy along $D = X \smallsetminus U$ and is admissible at the boundary. This will be accomplished by the CKS map $D^\sharp_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ in the fundamental diagram in Section 0.2 (see [KU3, Section 8.4.1], for the pure case), and imply the height estimate of Pearlstein for more general cases.

Correction to Part I. There are some mistakes in calculating examples in Part I (see [KNU2, Section 10]). First, the r^{-2}'s in Section 10.2.1 should be r^{-1}. (Note that we gave the real analytic structure on $\bar{A}_P$ in the notation in [KNU2, Section 2.6] by using the fundamental roots.) There are similar mistakes also in Section 10.3; that is, r should be replaced by $r^{1/2}$ in the third last line in p. 219 of [KNU2], which should be $(x+ir^{-1},\ldots)$, in the second line in p. 220: $(s_1,s_2,x,r,d)\mapsto x+ir^{-1}$, and in the second last line in p. 220: $t(r)(e_1)=r^{-1/2}e_1$, $t(r)(e_2)=r^{1/2}e_2$.

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The poem at the beginning is a translation by Professor Luc Illusie of a Japanese poem composed by two of the authors (K. Kato and S. Usui). These poems were placed at the beginning of [KU3]. We put the French version here again as it well captures the spirit of this article.

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