3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}$, $\mathfrak{g}_0$ and $\mathfrak{g}_{ed}$, III: $G = E_8$

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Abstract In the articles [4] and [7], we completed the determination of group realizations $\mathfrak{g}_{ev}$ and $\mathfrak{g}_0$ of 2-graded decompositions $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of exceptional Lie algebras $\mathfrak{g}$ for the universal exceptional Lie groups. In the present article, which is a continuation of [5] and [8], we determine group realizations of subalgebras $\mathfrak{g}_{ev}$, $\mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of 3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ for the universal exceptional Lie groups of type E_8.

Introduction

The 3-graded decompositions of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l},$$

are classified, and the types of subalgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ and $\mathfrak{g}_{ed} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3$ are determined. Table 1 shows the results of $\mathfrak{g}_{ev}, \mathfrak{g}_0$, and $\mathfrak{g}_{ed}$ for the exceptional Lie algebras of type E_8 (see [3]).

In the articles [5] and [8], we gave the group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$, and $\mathfrak{g}_{ed}$ for the connected exceptional universal linear Lie groups G of type G_2, F_4, E_6, and E_7. In this article, for the connected exceptional universal linear Lie groups G of type E_8, we realize the subgroups G_{ev}, G_0, and G_{ed} of G corresponding to $\mathfrak{g}_{ev}, \mathfrak{g}_0$, and $\mathfrak{g}_{ed}$ of $\mathfrak{g} = \mathrm{Lie}\, G$. Our results are shown in Table 2.

This article is a continuation of [5] and [8], and we use the same notation as in [5] and [8]. So the numbering of sections and theorems starts from Section 5.

Together with the preceding articles [5] and [8] and the present article, the group realization of Hara's table (see [3]) with respect to 3-graded decompositions of exceptional simple Lie algebras by the connected exceptional universal linear Lie groups has been completed.

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Table 1

Case 1	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim\mathfrak{g}_1, \dim\mathfrak{g}_2, \dim\mathfrak{g}_3$
	${\mathfrak{e}_8}^C$	$\mathfrak{sl}(2,C)\oplus{\mathfrak{e}_7}^C$ $\mathfrak{sl}(3,C)\oplus{\mathfrak{e}_6}^C$	$\mathfrak{sl}(2,C)\oplus C\oplus{\mathfrak{e}_6}^C$ $54,\ 27,\ 2$
	$\mathfrak{e}_{8(8)}$	$\mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{e}_{7(7)}$ $\mathfrak{sl}(3,\boldsymbol{R})\oplus\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(2,\boldsymbol{R})\oplus\boldsymbol{R}\oplus\mathfrak{e}_{6(6)}$ $54,\ 27,\ 2$
	$\mathfrak{e}_{8(-24)}$	$\mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{e}_{7(-25)}$ $\mathfrak{sl}(3,\boldsymbol{R})\oplus\mathfrak{e}_{6(-26)}$	$\mathfrak{sl}(2,\boldsymbol{R})\oplus\boldsymbol{R}\oplus\mathfrak{e}_{6(-26)}$ $54,\ 27,\ 2$

Case 2	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim\mathfrak{g}_1, \dim\mathfrak{g}_2, \dim\mathfrak{g}_3$
	${\mathfrak{e}_8}^C$	$\mathfrak{so}(16,C)$ $\mathfrak{sl}(9,C)$	$C\oplus\mathfrak{sl}(8,C)$ $56,\ 28,\ 8$
	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(8,8)$ $\mathfrak{sl}(9,\boldsymbol{R})$	$\boldsymbol{R}\oplus\mathfrak{sl}(8,\boldsymbol{R})$ $56,\ 28,\ 8$

Table 2

Case 1	G	G_{ev} G_{ed}	G_0
	${E_8}^C$	$(SL(2,C)\times{E_7}^C)/\boldsymbol{Z}_2$ $(SL(3,C)\times{E_6}^C)/\boldsymbol{Z}_3$	$(SL(2,C)\times C^*\times{E_6}^C)/\boldsymbol{Z}_6$
	$E_{8(8)}$	$(SL(2,\boldsymbol{R})\times E_{7(7)})/\boldsymbol{Z}_2\times 2$ $SL(3,\boldsymbol{R})\times E_{6(6)}$	$(SL(2,\boldsymbol{R})\times\boldsymbol{R}^+\times E_{6(6)})\times 2$
	$E_{8(-24)}$	$(SL(2,\boldsymbol{R})\times E_{7(-25)})/\boldsymbol{Z}_2\times 2$ $SL(3,\boldsymbol{R})\times E_{6(-26)}$	$(SL(2,\boldsymbol{R})\times\boldsymbol{R}^+\times E_{6(-26)})\times 2$

Case 2	G	G_{ev} G_{ed}	G_0
	${E_8}^C$	$Ss(16,C)$ $SL(9,C)/\boldsymbol{Z}_3$	$(C^*\times SL(8,C))/\boldsymbol{Z}_{24}$
	$E_{8(8)}$	$Ss(8,8)\times 2$ $SL(9,\boldsymbol{R})\times 3$	$(\boldsymbol{R}^+\times SL(8,\boldsymbol{R}))\times 3$

5. Group E_8

5.1. Lie groups of type E_8 and their Lie algebras

In a C-vector space ${\mathfrak{e}_8}^C$ and $\boldsymbol{R}$-vector spaces $\mathfrak{e}_{8(8)}, \mathfrak{e}_{8(-24)}$,

$$ {\mathfrak{e}_8}^C = {\mathfrak{e}_7}^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus C\oplus C\oplus C, $$

$$ \mathfrak{e}_{8(8)} = \mathfrak{e}_{7(7)}\oplus\mathfrak{P}'\oplus\mathfrak{P}'\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R}, $$

$$ \mathfrak{e}_{8(-24)} = \mathfrak{e}_{7(-25)}\oplus\mathfrak{P}\oplus\mathfrak{P}\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R}, $$

we define a Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] \\ = (\Phi, P, Q, r, s, t),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\ Q = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1, \\ P = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1, \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1, \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1, \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1; \end{cases}$$

then this becomes a simple Lie algebra of types ${E_8}^C, E_{8(8)}$, and $E_{8(-24)}$, respectively.

We define a C-linear transformation γ of ${\mathfrak{e}_8}^C$ by

$$\gamma(\Phi, P, Q, r, s, t) = (\gamma\Phi\gamma, \gamma P, \gamma Q, r, s, t),$$

where γ of the right-hand side is the same as $\gamma \in {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$, and the complex conjugation in ${\mathfrak{e}_8}^C$ is denoted by τ:

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t).$$

The connected universal linear Lie groups ${E_8}^C, E_{8(8)}$, and $E_{8(-24)}$ of type E_8 are given, respectively, by

$$\begin{aligned} {E_8}^C &= \{\alpha \in \mathrm{Iso}_C({\mathfrak{e}_8}^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}, \\ E_{8(8)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{e}_{8(8)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}, \\ E_{8(-24)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{e}_{8(-24)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}. \end{aligned}$$

The group ${E_8}^C$ is simply connected. From the definitions of the groups above, we have the following.

PROPOSITION 5.1
We have

$$E_{8(8)} \cong ({E_8}^C)^{\tau\gamma}, \qquad E_{8(-24)} = ({E_8}^C)^{\tau}.$$

For $\alpha \in {E_7}^C$, the mapping $\widetilde{\alpha} : {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ is defined by

$$\widetilde{\alpha}(\Phi, P, Q, r, s, t) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, s, t);$$

then $\widetilde{\alpha} \in {E_8}^C$, so α and $\widetilde{\alpha}$ are identified. The group ${E_8}^C$ contains ${E_7}^C$ as a subgroup by

$${E_7}^C = \{\widetilde{\alpha} \in {E_8}^C \mid \alpha \in {E_7}^C\}.$$

Especially, elements υ, λ, and ι of ${E_7}^C$ ($\upsilon(X, Y, \xi, \eta) = (-X, -Y, -\xi, -\eta), \lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi), \iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta)$) are also elements of ${E_8}^C$.

5.2. Subgroups of type ${A_1}^C \oplus {E_7}^C, {A_1}^C \oplus C \oplus {E_6}^C$, and ${A_2}^C \oplus {E_6}^C$ of ${E_8}^C$

We define C-linear transformations $\widetilde{\lambda}$ and w of ${\mathfrak{e}_8}^C = {\mathfrak{e}_7}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ by

$$\widetilde{\lambda}(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

$$\begin{aligned} w(\Phi, P, Q, r, s, t) &= w\big(\Phi(\phi, A, B, \nu), (X, Y, \xi, \eta), (Z, W, \zeta, \mu), r, s, t\big) \\ &= \big(\Phi(\phi, \omega A, \omega^2 B, \nu), (\omega X, \omega^2 Y, \xi, \eta), (\omega Z, \omega^2 W, \zeta, \mu), r, s, t\big), \end{aligned}$$

$\omega = e^{2\pi i/3}$, respectively. Then $\widetilde{\lambda}, w \in {E_8}^C$ and $\widetilde{\lambda}^2 = 1, w^3 = 1$.

In the Lie algebra ${\mathfrak{e}_8}^C$, let

$$Z = \big(\Phi(0, 0, 0, -3), 0, 0, 0, 0, 0\big).$$

Hereafter (see Theorems 5.2.1 and 5.4.1) in $\mathfrak{P}^C$ and ${\mathfrak{e}_8}^C$, we use the following notation:

$$\dot{X} = (X, 0, 0, 0), \qquad \underset{\cdot}{Y} = (0, Y, 0, 0), \qquad \dot{\xi} = (0, 0, \xi, 0), \qquad \underset{\cdot}{\eta} = (0, 0, 0, \eta),$$

$$\Phi = (\Phi, 0, 0, 0, 0, 0), \qquad P^- = (0, P, 0, 0, 0, 0), \qquad Q_- = (0, 0, Q, 0, 0, 0),$$

$$\widetilde{r} = (0, 0, 0, r, 0, 0), \qquad s^- = (0, 0, 0, 0, s, 0), \qquad t_- = (0, 0, 0, 0, 0, t).$$

Moreover, we mix and combine the above notation. For example,

$$\dot{X}^- = \big(0, (X, 0, 0, 0), 0, 0, 0, 0\big), \qquad \underset{\cdot}{W}_- = \big(0, 0, (0, W, 0, 0), 0, 0, 0\big).$$

THEOREM 5.2.1

The 3*-graded decomposition of the Lie algebra* $\mathfrak{e}_{8(8)} = ({\mathfrak{e}_8}^C)^{\tau\gamma}$ *(or* ${\mathfrak{e}_8}^C$*),*

$$\mathfrak{e}_{8(8)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = (\Phi(0, 0, 0, -3), 0, 0, 0, 0, 0)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01}, \quad 0 \le k < 4 \le l \le 7, G_{kl} \text{ otherwise}, \\ \widetilde{A}_1(e_k), \widetilde{A}_2(e_k), \widetilde{A}_3(e_k), \widetilde{F}_1(e_k), \widetilde{F}_2(e_k), \widetilde{F}_3(e_k), \quad 0 \le k \le 3, \\ i\widetilde{A}_1(e_k), i\widetilde{A}_2(e_k), i\widetilde{A}_3(e_k), i\widetilde{F}_1(e_k), i\widetilde{F}_2(e_k), i\widetilde{F}_3(e_k), \quad 4 \le k \le 7, \\ (E_1 - E_2)^\sim, (E_2 - E_3)^\sim, \mathbf{1}, \tilde{1}, 1^-, 1_-, \end{array} \right\} 82,$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} \dot{E}_1^-, \dot{E}_2^-, \dot{E}_3^-, \dot{F}_1(e_k)^-, \dot{F}_2(e_k)^-, \dot{F}_3(e_k)^-, \quad 0 \le k \le 3, \\ i\dot{F}_1(e_k)^-, i\dot{F}_2(e_k)^-, i\dot{F}_3(e_k)^-, \quad 4 \le k \le 7, \\ \dot{E}_{1-}, \dot{E}_{2-}, \dot{E}_{3-}, \dot{F}_1(e_k)_-, \dot{F}_2(e_k)_-, \dot{F}_3(e_k)_-, \quad 0 \le k \le 3, \\ i\dot{F}_1(e_k)_-, i\dot{F}_2(e_k)_-, i\dot{F}_3(e_k)_-, \quad 4 \le k \le 7, \end{array} \right\} 54,$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} \widehat{E}_1, \widehat{E}_2, \widehat{E}_3, \widehat{F}_1(e_k), \widehat{F}_2(e_k), \widehat{F}_3(e_k), \quad 0 \le k \le 3, \\ i\widehat{F}_1(e_k), i\widehat{F}_2(e_k), i\widehat{F}_3(e_k), \quad 4 \le k \le 7, \end{array} \right\} 27,$$

$$\mathfrak{g}_{-3} = \{\underset{\cdot}{1}^-, \underset{\cdot}{1}_-\}\ 2,$$

$$\mathfrak{g}_1 = \widetilde{\lambda}(\mathfrak{g}_{-1}), \mathfrak{g}_2 = \widetilde{\lambda}(\mathfrak{g}_{-2}), \mathfrak{g}_3 = \widetilde{\lambda}(\mathfrak{g}_{-3}).$$

Since $(\exp\Phi(0,0,0,-3\nu))(X,Y,\xi,\eta) = (e^{\nu}X, e^{-\nu}Y, e^{-3\nu}\xi, e^{3\nu}\eta), \nu \in C$, we have

$$\exp\Big(\frac{2\pi i}{2}Z\Big) = \upsilon, \qquad \exp\Big(\frac{2\pi i}{4}Z\Big) = \upsilon\iota, \quad \exp\Big(\frac{2\pi i}{3}Z\Big) = w.$$

Now, let

$$z_2 = \exp\Big(\frac{2\pi i}{2}\,\mathrm{ad}\,Z\Big), \quad z_4 = \exp\Big(\frac{2\pi i}{4}\,\mathrm{ad}\,Z\Big), \quad z_3 = \exp\Big(\frac{2\pi i}{3}\,\mathrm{ad}\,Z\Big).$$

Then, since $({\mathfrak{e}_8}^C)_{ev} = ({\mathfrak{e}_8}^C)^{z_2} = ({\mathfrak{e}_8}^C)^{\upsilon}, ({\mathfrak{e}_8}^C)_0 = ({\mathfrak{e}_8}^C)^{z_4} = ({\mathfrak{e}_8}^C)^{\upsilon\iota}, ({\mathfrak{e}_8}^C)_{ed} = ({\mathfrak{e}_8}^C)^{z_3} = ({\mathfrak{e}_8}^C)^{w}$, we determine the structures of groups

$$({E_8}^C)_{ev} = ({E_8}^C)^{z_2} = ({E_8}^C)^{\upsilon},$$
$$({E_8}^C)_0 = ({E_8}^C)^{z_4} = ({E_8}^C)^{\upsilon\iota},$$
$$({E_8}^C)_{ed} = ({E_8}^C)^{z_3} = ({E_8}^C)^{w}.$$

We define a mapping $\psi : SL(2,C) \to {E_8}^C, A \to \psi(A)$, where $\psi(A)$ is the C-linear transformation of ${\mathfrak{e}_8}^C$ defined by

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & b1 & 0 & 0 & 0 \\ 0 & c1 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & ad+bc & -ac & bd \\ 0 & 0 & 0 & -2ab & a^2 & -b^2 \\ 0 & 0 & 0 & 2cd & -c^2 & d^2 \end{pmatrix},$$

and we define a mapping $\phi : C^* \to {E_7}^C, \theta \to \phi(\theta)$, where $\phi(\theta)$ is the C-linear transformation of $\mathfrak{P}^C$ defined by

$$\phi(\theta)(X,Y,\xi,\theta) = (\theta X, \theta^{-1}Y, \theta^{-3}\xi, \theta^3\eta).$$

THEOREM 5.2.2

We have the following:

(1) $({E_8}^C)_{ev} \cong (SL(2,C) \times {E_7}^C)/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E,1),(-E,-1)\}$,

(2) $({E_8}^C)_0 \cong (SL(2,C) \times C^* \times {E_6}^C)/\boldsymbol{Z}_6, \boldsymbol{Z}_6 = \boldsymbol{Z}_2 \times \boldsymbol{Z}_3, \boldsymbol{Z}_2 = \{(E,1,1), (-E,-1,1)\}, \boldsymbol{Z}_3 = \{(E,1,1),(E,\omega,\phi(\omega^2)),(E,\omega^2,\phi(\omega))\}$,

(3) $({E_8}^C)_{ed} \cong (SL(3,C) \times {E_6}^C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(E,1),(\omega E,\omega^2 1),(\omega^2 E,\omega 1)\}$.

Proof

(1) We define a mapping $\varphi_{ev} : SL(2,C) \times {E_7}^C \to ({E_8}^C)^{\upsilon} = ({E_8}^C)_{ev}$ by

$$\varphi_{ev}(A,\beta) = \psi(A)\beta;$$

φ_{ev} is well defined because $\psi(A) \in ({E_8}^C)^{\upsilon}$. Since $\psi(A)$ and $\beta \in {E_7}^C$ commute, φ_{ev} is a homomorphism. $\operatorname{Ker}\varphi_{ev} = \{(E,1),(-E,-1)\} = \boldsymbol{Z}_2$. Since $({E_8}^C)^{\upsilon}$ is connected and $\dim_C(\mathfrak{sl}(2,C) \oplus {\mathfrak{e}_7}^C) = 3 + 133 = 136 = 82 + 27 \times 2 = \dim_C(({\mathfrak{e}_8}^C)_{ev}) = \dim_C(({\mathfrak{e}_8}^C)^{\upsilon})$ (see Theorem 5.2.1), φ_{ev} is surjective. Thus we have the isomorphism $({E_8}^C)_{ev} = ({E_8}^C)^{\upsilon} \cong (SL(2,C) \times {E_7}^C)/\boldsymbol{Z}_2$.

(2) Since the group ${E_7}^C$ has subgroups C^* and ${E_6}^C$ (see [6, Theorem 4.4.4]), we define a mapping $\varphi_0 : SL(2,C) \times C^* \times {E_6}^C \to ({E_8}^C)^{\upsilon\iota} = ({E_8}^C)_0$ by

$$\varphi_0(A,\theta,\beta) = \psi(A)\phi(\theta)\beta$$

as the restriction mapping of φ_{ev}. So φ_0 is well defined and a homomorphism. Since $(\upsilon\iota)^2 = \upsilon$, $({E_8}^C)^{\upsilon\iota}$ is a subgroup of $({E_8}^C)^{\upsilon}$. Now, for $\alpha \in ({E_8}^C)^{\upsilon\iota} \subset ({E_8}^C)^{\upsilon}$, there exist $A \in SL(2,C)$ and $\beta' \in {E_7}^C$ such that $\alpha = \varphi_{ev}(A,\beta')$ from (1). Moreover, from the condition $(\upsilon\iota)\alpha(\upsilon\iota)^{-1} = \alpha$, that is, $(\upsilon\iota)\varphi_{ev}(A,\beta')(\upsilon\iota)^{-1} = \varphi_{ev}(A,\beta')$, we have $\varphi_{ev}(A,\iota\beta'\iota^{-1}) = \varphi_{ev}(A,\beta')$. Hence

$$\begin{cases} A = A, \\ \iota\beta'\iota^{-1} = \beta', \end{cases} \qquad \text{or} \qquad \begin{cases} A = -A, \\ \iota\beta'\iota^{-1} = -\beta'. \end{cases}$$

In the former case, $A \in SL(2,C), \beta' \in ({E_7}^C)^{\iota} \cong (C^* \times {E_6}^C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(1,1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\}$ (see [6, Theorem 4.4.4]), so β' is expressed as $\beta' = \varphi(\theta)\beta, \theta \in C^*, \beta \in {E_6}^C$. The latter case is impossible because $A = 0$. It is easy to see that

$$\begin{aligned} \operatorname{Ker}\varphi_0 &= \big\{(E,1,1), (E,\omega,\phi(\omega^2)), (E,\omega^2,\phi(\omega)), \\ &\qquad (-E,-1,1), (E,-\omega,\phi(\omega^2)), (-E,-\omega^2,\phi(\omega))\big\} \\ &= \big\{(E,1,1), (-E,-1,1)\big\} \\ &\qquad \times \big\{(E,1,1), (E,\omega,\phi(\omega^2)), (E,\omega^2,\phi(\omega)\big\} \\ &= \boldsymbol{Z}_2 \times \boldsymbol{Z}_3. \end{aligned}$$

Thus we have the isomorphism $({E_8}^C)_0 = ({E_8}^C)^{\upsilon\iota} \cong (SL(2,C) \times C^* \times {E_6}^C)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_3)$.

(3) The determination of the group $({E_8}^C)^w$ is essentially done in Gomyo [1]. However, we write the result again. We construct one more C-Lie algebra $\check{\mathfrak{e}}_8{}^C$ of type ${E_8}^C$.

We first consider a $27 \times 3 = 81$ dimensional C-vector space

$$(\mathfrak{J}^C)^3 = \left\{ \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \,\middle|\, X_i \in \mathfrak{J}^C \right\}.$$

In $(\mathfrak{J}^C)^3$, we define an inner product $(\boldsymbol{X},\boldsymbol{Y})$, a Hermitian inner product $\langle \boldsymbol{X},\boldsymbol{Y}\rangle$, a cross product $\boldsymbol{X} \times \boldsymbol{Y}$, an element $\boldsymbol{X} \cdot \boldsymbol{Y}$ of $\mathfrak{sl}(3,C)$, and an element $\boldsymbol{X} \vee \boldsymbol{Y}$ of ${\mathfrak{e}_6}^C$, respectively, by

$$(\boldsymbol{X},\boldsymbol{Y}) = (X_1,Y_1) + (X_2,Y_2) + (X_3,Y_3) \in C,$$

$$\langle \boldsymbol{X},\boldsymbol{Y}\rangle = \langle X_1,Y_1\rangle + \langle X_2,Y_2\rangle + \langle X_3,Y_3\rangle \in C,$$

$$\boldsymbol{X} \times \boldsymbol{Y} = \begin{pmatrix} X_2 \times Y_3 - Y_2 \times X_3 \\ X_3 \times Y_1 - Y_3 \times X_1 \\ X_1 \times Y_2 - Y_1 \times X_2 \end{pmatrix} \in (\mathfrak{J}^C)^3,$$

$$\boldsymbol{X}\cdot\boldsymbol{Y}=\begin{pmatrix}(X_1,Y_1) & (X_1,Y_2) & (X_1,Y_3)\\ (X_2,Y_1) & (X_2,Y_2) & (X_2,Y_3)\\ (X_3,Y_1) & (X_3,Y_2) & (X_3,Y_3)\end{pmatrix}-\frac{1}{3}(\boldsymbol{X},\boldsymbol{Y})E\in\mathfrak{sl}(3,C),$$

$$\boldsymbol{X}\vee\boldsymbol{Y}=X_1\vee Y_1+X_2\vee Y_2+X_3\vee Y_3\in{\mathfrak{e}_6}^C,$$

where $\boldsymbol{X}=\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix},\boldsymbol{Y}=\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}\in(\mathfrak{J}^C)^3$. Further, for $\phi\in\mathrm{Hom}_C(\mathfrak{J}^C), D=(d_{ij})\in M(3,C)$, and $\boldsymbol{X}=\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\in(\mathfrak{J}^C)^3$, we define $\phi\boldsymbol{X}, D\boldsymbol{X}\in(\mathfrak{J}^C)^3$ naturally by

$$\phi(\boldsymbol{X})=\begin{pmatrix}\phi X_1\\ \phi X_2\\ \phi X_3\end{pmatrix},\quad D\boldsymbol{X}=\begin{pmatrix}d_{11}X_1+d_{12}X_2+d_{13}X_3\\ d_{12}X_1+d_{22}X_2+d_{23}X_3\\ d_{31}X_1+d_{32}X_2+d_{33}X_3\end{pmatrix}.$$

PROPOSITION 5.2.3 (GOMYO [1, Theorem 3.1])

In an $8+78+81+81=248$ *dimensional* C*-vector space*

$$\check{\mathfrak{e}}_8{}^C=\mathfrak{sl}(3,C)\oplus{\mathfrak{e}_6}^C\oplus(\mathfrak{J}^C)^3\oplus(\mathfrak{J}^C)^3,$$

we define a Lie bracket $[R_1,R_2]$ *by*

$$[(D_1,\phi_1,\boldsymbol{X}_1,\boldsymbol{Y}_1),(D_2,\phi_2,\boldsymbol{X}_2,\boldsymbol{Y}_2)]=(D,\phi,\boldsymbol{X},\boldsymbol{Y}),$$

$$\begin{cases}D=[D_1,D_2]+\frac{1}{4}\boldsymbol{X}_1\cdot\boldsymbol{Y}_2-\frac{1}{4}\boldsymbol{X}_2\cdot\boldsymbol{Y}_1,\\ \phi=[\phi_1,\phi_2]+\frac{1}{2}\boldsymbol{X}_1\vee\boldsymbol{Y}_2-\frac{1}{2}\boldsymbol{X}_2\vee\boldsymbol{Y}_1,\\ \boldsymbol{X}=\phi_1\boldsymbol{X}_2-\phi_2\boldsymbol{X}_1+D_1\boldsymbol{X}_2-D_2\boldsymbol{X}_1-\boldsymbol{Y}_1\times\boldsymbol{Y}_2,\\ \boldsymbol{Y}=-{}^t\phi_1\boldsymbol{Y}_2+{}^t\phi_2\boldsymbol{Y}_1-{}^tD_1\boldsymbol{Y}_2+{}^tD_2\boldsymbol{Y}_1+\boldsymbol{X}_1\times\boldsymbol{X}_2;\end{cases}$$

then $\check{\mathfrak{e}}_8{}^C$ *becomes a* C*-Lie algebra of type* ${E_8}^C$*.*

Proof

Let ${\mathfrak{e}_8}^C={\mathfrak{e}_7}^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus C\oplus C\oplus C$ be the usual C-Lie algebra of type ${E_8}^C$. We define a mapping $f:{\mathfrak{e}_8}^C\to\check{\mathfrak{e}}_8{}^C$ by

$$f\big(\varPhi(\phi,A,B,\nu),(X,Y,\xi,\eta),(Z,W,\zeta,\mu),r,s,t\big)$$
$$=\left(\begin{pmatrix}\frac{2}{3}\nu & -\frac{1}{2}\xi & \frac{1}{2}\zeta\\ \frac{1}{2}\mu & -\frac{1}{3}\nu-r & t\\ \frac{1}{2}\eta & s & -\frac{1}{3}\nu+r\end{pmatrix},\phi,\begin{pmatrix}-2A\\Z\\X\end{pmatrix},\begin{pmatrix}-2B\\Y\\-W\end{pmatrix}\right);$$

then we can prove that f is an isomorphism as Lie algebras by straightforward calculations. Thus we have the isomorphism ${\mathfrak{e}_8}^C\cong\check{\mathfrak{e}}_8{}^C$. □

Now, let $\check{E}_8{}^C$ be the automorphism group of $\check{\mathfrak{e}}_8{}^C$, that is,

$$\check{E}_8{}^C=\big\{\alpha\in\mathrm{Iso}_C(\check{\mathfrak{e}}_8{}^C)\,\big|\,\alpha[R_1,R_2]=[\alpha R_1,\alpha R_2]\big\}.$$

The group ${E_8}^C$ is isomorphic to the group ${\check{E}_8}^C$ by the correspondence $\alpha \in {E_8}^C \to f\alpha f^{-1} \in {\check{E}_8}^C$. Then the transformation w of ${\mathfrak{e}_8}^C$ is transfered to the following transformation w of ${\check{\mathfrak{e}}_8}^C$:

$$w(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (D, \phi, \omega\boldsymbol{X}, \omega^2\boldsymbol{Y}).$$

So, we determine the structure of the group $({\check{E}_8}^C)^w$ instead of the group $({E_8}^C)^w$.

We first define a mapping $\varphi_1 : SL(3, C) \to ({\check{E}_8}^C)^w$ by

$$\varphi_1(A)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (ADA^{-1}, \phi, A\boldsymbol{X}, {}^tA^{-1}\boldsymbol{Y}).$$

We have to prove that $\varphi_1(A) \in ({\check{E}_8}^C)^w$. Indeed, since the action of $D_1 = (D_1, 0, 0, 0) \in \mathfrak{sl}(3, C) \subset ({\check{\mathfrak{e}}_8}^C)^w$ is given by

$$\big(\mathrm{ad}(D_1)\big)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = \big((\mathrm{ad}\, D_1)D, 0, D_1\boldsymbol{X}, -{}^tD_1\boldsymbol{Y}\big),$$

we have

$$\begin{aligned}&\big(\exp \mathrm{ad}(D_1)\big)(D, \phi, \boldsymbol{X}, \boldsymbol{Y})\\ &\quad = \big((\exp D_1)D(\exp D_1)^{-1}, \phi, (\exp D_1)\boldsymbol{X}, {}^t(\exp D_1)^{-1}\boldsymbol{Y}\big).\end{aligned}$$

Hence, for $A = \exp D_1 \in SL(3, C)$, we have $\varphi_1(A) = (\exp \mathrm{ad}(D_1)) \in {\check{E}_8}^C$. Evidently, $w\varphi_1(A) = \varphi_1(A)w$; hence we have $\varphi_1(A) \in ({\check{E}_8}^C)^w$. Next, we define a mapping $\varphi_2 : {E_6}^C \to ({\check{E}_8}^C)^w$ by

$$\varphi_2(\beta)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (D, \beta\phi\beta^{-1}, \beta\boldsymbol{X}, {}^t\beta^{-1}\boldsymbol{Y}).$$

We have to prove that $\varphi_2(\beta) \in ({\check{E}_8}^C)^w$. Indeed, since the action of $\phi' = (0, \phi', 0, 0) \in ({\check{\mathfrak{e}}_8}^C)^w$ is given by

$$(\mathrm{ad}\, \phi')(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = \big(0, (\mathrm{ad}\, \phi')\phi, \phi'\boldsymbol{X}, -{}^t\phi'\boldsymbol{Y}\big),$$

we have

$$\big(\exp \mathrm{ad}(\phi')\big)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = \big(D, (\exp \phi')\phi(\exp \phi')^{-1}, (\exp \phi')\boldsymbol{X}, {}^t(\exp \phi')^{-1}\boldsymbol{Y}\big).$$

Hence, for $\beta = \exp \phi'$, we have $\varphi_2(\beta) = (\exp \mathrm{ad}(\phi')) \in {\check{E}_8}^C$. Evidently, $w\varphi_2(\beta) = \varphi_2(\beta)w$; hence we have $\varphi_2(\beta) \in ({\check{E}_8}^C)^w$.

Now, we define a mapping $\varphi_{ed} : SL(3, C) \times {E_6}^C \to ({\check{E}_8}^C)^w = ({\check{E}_8}^C)_{ed}$ by

$$\varphi_{ed}(A, \beta) = \varphi_1(A)\varphi_2(\beta).$$

Since $\varphi_1(A)$ and $\varphi_2(\beta)$ commute, φ_{ed} is a homomorphism. It is not difficult to show that $\mathrm{Ker}\, \varphi_{ed} = \{(E, 1), (\omega E, \omega^2 1), (\omega^2 E, \omega 1)\} = \boldsymbol{Z}_3$. Since $({\check{E}_8}^C)^\omega$ is connected and $\dim_C(\mathfrak{sl}(3, C) \oplus {\mathfrak{e}_6}^C) = 8 + 78 = 86 = \dim_C(({\mathfrak{e}_8}^C)_{ed})$ (see Theorem 5.2.1) $= \dim_C(({\check{\mathfrak{e}}_8}^C)^w)$, φ_{ed} is surjective. Thus we have $({E_8}^C)_{ed} \cong ({\check{E}_8}^C)_{ed} = ({\check{E}_8}^C)^w \cong (SL(3, C) \times {E_6}^C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(E, 1), (\omega E, \omega^2 1), (\omega^2 E, \omega 1)\}$. $\square$

5.3. Subgroups of type $A_1 \oplus E_{7(7)}, A_1 \oplus \boldsymbol{R} \oplus E_{6(6)}$, and $A_2 \oplus E_{6(6)}$ of $E_{8(8)}$

In this section, we use Lie algebras $\mathfrak{e}_{8(8)}, {\mathfrak{e}_8}^C$ and Lie groups $E_{8(8)}, {E_8}^C$ defined in Section 5.1 and ${\check{E}_8}^C$ defined in Section 5.2.

Since $(\mathfrak{e}_{8(8)})_{ev} = ({\mathfrak{e}_8}^C)_{ev} \cap ({\mathfrak{e}_8}^C)^{\tau\gamma} = ({\mathfrak{e}_8}^C)^{\upsilon} \cap ({\mathfrak{e}_8}^C)^{\tau\gamma}, (\mathfrak{e}_{8(8)})_0 = ({\mathfrak{e}_8}^C)_0 \cap ({\mathfrak{e}_8}^C)^{\tau\gamma} = ({\mathfrak{e}_8}^C)^{\upsilon\iota} \cap ({\mathfrak{e}_8}^C)^{\tau\gamma}, (\mathfrak{e}_{8(8)})_{ed} = ({\mathfrak{e}_8}^C)_{ed} \cap ({\mathfrak{e}_8}^C)^{\tau\gamma} = ({\mathfrak{e}_8}^C)^w \cap ({\mathfrak{e}_8}^C)^{\tau\gamma}$, we determine the structures of groups

$$(E_{8(8)})_{ev} = ({E_8}^C)_{ev} \cap ({E_8}^C)^{\tau\gamma} = ({E_8}^C)^{\upsilon} \cap ({E_8}^C)^{\tau\gamma},$$
$$(E_{8(8)})_0 = ({E_8}^C)_0 \cap ({E_8}^C)^{\tau\gamma} = ({E_8}^C)^{\upsilon\iota} \cap ({E_8}^C)^{\tau\gamma},$$
$$(E_{8(8)})_{ed} = ({E_8}^C)_{ed} \cap ({E_8}^C)^{\tau\gamma} = ({E_8}^C)^{w} \cap ({E_8}^C)^{\tau\gamma}.$$

THEOREM 5.3.1

We have the following:

(1) $(E_{8(8)})_{ev} \cong (SL(2,\boldsymbol{R}) \times E_{7(7)})/\boldsymbol{Z}_2 \times \{1, l\}, \boldsymbol{Z}_2 = \{(E,1),(-E,-1)\}$,
(2) $(E_{8(8)})_0 \cong (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}) \times \{1, l_0\}$,
(3) $(E_{8(8)})_{ed} \cong SL(3,\boldsymbol{R}) \times E_{6(6)}$.

Proof

(1) For $\alpha \in (E_{8(8)})_{ev} \subset ({E_8}^C)_{ev} = ({E_8}^C)^{\upsilon}$, there exist $A \in SL(2,C)$ and $\beta \in {E_7}^C$ such that $\alpha = \varphi_{ev}(A,\beta) = \psi(A)\beta$ (see Theorem 5.2.2(1)). From the condition $\tau\gamma\alpha\gamma\tau = \alpha$, that is, $\tau\gamma\psi(A)\beta\gamma\tau = \psi(A)\beta$, we have $\psi(\tau A)\tau\gamma\beta\gamma\tau = \psi(A)\beta$. Hence

$$\begin{cases} \tau A = A, \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases} \qquad \text{or} \qquad \begin{cases} \tau A = -A, \\ \tau\gamma\beta\gamma\tau = -\beta. \end{cases}$$

In the former case, from $\tau A = A$, we have $A \in SL(2,\boldsymbol{R})$, and from $\tau\gamma\beta\gamma\tau = \beta$, we have $\beta \in ({E_7}^C)^{\tau\gamma} \cong E_{7(7)}$ (see [6, Theorem 4.3.2]). In the latter case, $A = iI (I = \mathrm{diag}(1,-1)), \beta = \iota$ satisfy the conditions, and we denote $\varphi_{ev}(iI,\iota)$ by l. Thus we have the isomorphism $(E_{8(8)})_{ev} \cong ((SL(2,\boldsymbol{R}) \times E_{7(7)}) \cup l(SL(2,\boldsymbol{R}) \times E_{7(7)}))/\boldsymbol{Z}_2 = (SL(2,\boldsymbol{R}) \times E_{7(7)})/\boldsymbol{Z}_2 \times \{1, l\}, \boldsymbol{Z}_2 = \{(E,1),(-E,-1)\}$.

(2) For $\alpha \in (E_{8(8)})_0 \subset ({E_8}^C)_0 = ({E_8}^C)^{\upsilon\iota}$, there exist $A \in SL(2,C), \theta \in C^*$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_0(A,\theta,\beta) = \psi(A)\phi(\theta)\beta$ (see Theorem 5.2.2(2)). From the condition $\tau\gamma\alpha\gamma\tau = \alpha$, that is, $\tau\gamma\psi(A)\phi(\theta)\beta\gamma\tau = \psi(A)\phi(\theta)\beta$, we have $\psi(\tau A)\phi(\tau\theta)\tau\gamma\beta\gamma\tau = \psi(A)\phi(\theta)\beta$. Hence

$$\text{(i)} \begin{cases} \tau A = A, \\ \tau\theta = \theta, \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau A = A, \\ \tau\theta = \omega\theta, \\ \tau\gamma\beta\gamma\tau = \phi(\omega^2)\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau A = A, \\ \tau\theta = \omega^2\theta, \\ \tau\gamma\beta\gamma\tau = \phi(\omega)\beta, \end{cases} \qquad \text{(iv)} \begin{cases} \tau A = -A, \\ \tau\theta = -\theta, \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases}$$

$$\text{(v)} \begin{cases} \tau A = -A, \\ \tau\theta = -\omega\theta, \\ \tau\gamma\beta\gamma\tau = \phi(\omega^2)\beta, \end{cases} \qquad \text{(vi)} \begin{cases} \tau A = -A, \\ \tau\theta = -\omega^2\theta, \\ \tau\gamma\beta\gamma\tau = \phi(\omega)\beta. \end{cases}$$

Case (i). From $\tau A = A, \tau\theta = \theta$, we have $A \in SL(2, \boldsymbol{R}), \theta \in \boldsymbol{R}^*$, and from $\tau\gamma\beta\gamma\tau = \beta$, we have $\beta \in ({E_6}^C)^{\tau\gamma} \cong E_{6(6)}$. Hence the group of case (i) is isomorphic to

$$\big(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times E_{6(6)}\big)/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \big\{(E, 1, 1), (-E, -1, 1)\big\}.$$

The mapping $g : SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times E_{6(6)} \to SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}$,

$$g(A, \theta, \beta) = \begin{cases} (A, \theta, \beta) & \text{if} \quad \theta > 0, \\ (-A, -\theta, \beta) & \text{if} \quad \theta < 0 \end{cases}$$

induces the isomorphism $SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)} \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times E_{6(6)})/\boldsymbol{Z}_2$. Therefore the group of case (i) is isomorphic to $SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}$.

Case (ii). We have $\varphi_0(E, \omega, \phi(\omega^2)) = \psi(E)\phi(\omega)\phi(\omega^2) = 1$.

Case (iii). We have $\varphi_0(E, \omega^2, \phi(\omega)) = \psi(E)\phi(\omega^2)\phi(\omega) = 1$.

Case (iv). We have $\varphi_0(iI, i, 1) = l_0$ (hereafter we denote $\varphi_0(iI, i, 1)$ by l_0).

Case (v). We have $\varphi_0(iI, i\omega, \phi(\omega^2)) = \varphi_0(iI, i, 1)\varphi_0(E, \omega, \phi(\omega^2)) = l_0$.

Case (vi). We have $\varphi_0(iI, i\omega^2, \phi(\omega)) = \varphi_0(iI, i, 1)\varphi_0(E, \omega^2, \phi(\omega)) = l_0$.

Thus we have the isomorphism $(E_{8(8)})_0 \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}) \cup l_0(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}) = (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(6)}) \times \{1, l_0\}$.

(3) Under the isomorphism between ${\mathfrak{e}_8}^C$ and ${\check{\mathfrak{e}}_8}^C$ given in the proof of Theorem 5.2.2(3), the transformation γ and the complex conjugation τ of ${\mathfrak{e}_8}^C$ are transfered to the following transformation γ and the complex conjugation τ of ${\check{E}_8}^C$:

$$\gamma(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (D, \gamma\phi\gamma, \gamma\boldsymbol{X}, \gamma\boldsymbol{Y}),$$
$$\tau(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (\tau D, \tau\phi\tau, \tau\boldsymbol{X}, \tau\boldsymbol{Y}),$$

respectively. Hence instead of $(E_{8(8)})_{ed} = ({E_8}^C)_{ed} \cap ({E_8}^C)^{\tau\gamma}$, we consider $(\check{E}_{8(8)})_{ed} = ({\check{E}_8}^C)_{ed} \cap ({\check{E}_8}^C)^{\tau\gamma}$. Now, for $\alpha \in (\check{E}_{8(8)})_{ed} \subset ({\check{E}_8}^C)_{ed} = ({\check{E}_8}^C)^w$, there exist $A \in SL(3, C)$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_{ed}(A, \beta) = \varphi_1(A)\varphi_2(\beta)$ (see Theorem 5.2.2(3)). From the condition $\gamma\tau\alpha\tau\gamma = \alpha$, that is, $\gamma\tau\varphi_1(A)\varphi_2(\beta)\tau\gamma = \varphi_1(A)\varphi_2(\beta)$, we have $\varphi_1(\tau A)\varphi_2(\tau\gamma\beta\gamma\tau) = \varphi_1(A)\varphi_2(\beta)$. Hence

$$\text{(i)} \begin{cases} \tau A = A, \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau A = \omega A, \\ \tau\gamma\beta\gamma\tau = \omega^2\beta, \end{cases} \qquad \text{or} \qquad \text{(iii)} \begin{cases} \tau A = \omega^2 A, \\ \tau\gamma\beta\gamma\tau = \omega\beta. \end{cases}$$

Case (i). From $\tau A = A$, we have $A \in SL(3, \boldsymbol{R})$, and from $\tau\gamma\beta\gamma\tau = \beta$, we have $\beta \in ({E_6}^C)^{\tau\gamma} \cong E_{6(6)}$.

Case (ii). We have $\varphi_{ed}(\omega E, \omega^2 1)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (\omega D\omega^{-1}, \omega^2\phi\omega^{-2}, \omega\omega^2\boldsymbol{X}, \omega^{-1}\omega^{-2}\boldsymbol{Y}) = (D, \phi, \boldsymbol{X}, \boldsymbol{Y})$, that is, $\varphi_{ed}(\omega E, \omega^2 1) = 1$.

Case (iii). We have $\varphi_{ed}(\omega^2 E, \omega 1)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (\omega^2 D\omega^{-2}, \omega\phi\omega^{-1}, \omega^2\omega\boldsymbol{X}, \omega^{-2}\omega^{-1}\boldsymbol{Y}) = (D, \phi, \boldsymbol{X}, \boldsymbol{Y})$; that is, $\varphi_{ed}(\omega^2 E, \omega 1) = 1$.

Thus we have the isomorphism $(E_{8(8)})_{ed} \cong (\check{E}_{8(8)})_{ed} \cong SL(3, \boldsymbol{R}) \times E_{6(6)}$. $\square$

5.4. Subgroups of type $A_1 \oplus E_{7(-25)}, A_1 \oplus \boldsymbol{R} \oplus E_{6(-26)}$, and $A_2 \oplus E_{6(-26)}$ of $E_{8(-24)}$

In this section, we use Lie algebras $\mathfrak{e}_{8(-24)}, {\mathfrak{e}_8}^C$ and Lie groups $E_{8(-24)}$, ${E_8}^C$ defined in Section 5.1 and ${\check{E}_8}^C$ defined in Section 5.2.

THEOREM 5.4.1

The 3*-graded decomposition of the Lie algebra* $\mathfrak{e}_{8(-24)} = ({\mathfrak{e}_8}^C)^\tau$ *(or* ${\mathfrak{e}_8}^C$*),*

$$\mathfrak{e}_{8(-24)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = (\Phi(0,0,0,-3),0,0,0,0,0)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{kl}, \quad 0 \le k < l \le 7, \\ \widetilde{A}_1(e_k), \widetilde{A}_2(e_k), \widetilde{A}_3(e_k), \widetilde{F}_1(e_k), \widetilde{F}_2(e_k), \widetilde{F}_3(e_k), \quad 0 \le k \le 7, \\ (E_1 - E_2)^\sim, (E_2 - E_3)^\sim, \mathbf{1}, \tilde{1}, 1^-, 1_-, \end{array} \right\} 82,$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} \dot{E}_1^-, \dot{E}_2^-, \dot{E}_3^-, \dot{F}_1(e_k)^-, \dot{F}_2(e_k)^-, \dot{F}_3(e_k)^-, \quad 0 \le k \le 7, \\ \dot{E}_{1-}, \dot{E}_{2-}, \dot{E}_{3-}, \dot{F}_1(e_k)_-, \dot{F}_2(e_k)_-, \dot{F}_3(e_k)_-, \quad 0 \le k \le 7, \end{array} \right\} 54,$$

$$\mathfrak{g}_{-2} = \{\widehat{E}_1, \widehat{E}_2, \widehat{E}_3, \widehat{F}_1(e_k), \widehat{F}_2(e_k), \widehat{F}_3(e_k), \quad 0 \le k \le 7\}\ 27,$$

$$\mathfrak{g}_{-3} = \{\underset{\cdot}{1}^-, \underset{\cdot}{1}_-\}\ 2,$$

$$\mathfrak{g}_1 = \widetilde{\lambda}(\mathfrak{g}_{-1}), \mathfrak{g}_2 = \widetilde{\lambda}(\mathfrak{g}_{-2}), \mathfrak{g}_3 = \widetilde{\lambda}(\mathfrak{g}_{-3}).$$

Since $(\mathfrak{e}_{8(-24)})_{ev} = ({\mathfrak{e}_8}^C)_{ev} \cap ({\mathfrak{e}_8}^C)^\tau = ({\mathfrak{e}_8}^C)^\upsilon \cap ({\mathfrak{e}_8}^C)^\tau, (\mathfrak{e}_{8(-24)})_0 = ({\mathfrak{e}_8}^C)_0 \cap ({\mathfrak{e}_8}^C)^\tau = ({\mathfrak{e}_8}^C)^{\upsilon\iota} \cap ({\mathfrak{e}_8}^C)^\tau, (\mathfrak{e}_{8(-24)})_{ed} = ({\mathfrak{e}_8}^C)_{ed} \cap ({\mathfrak{e}_8}^C)^\tau = ({\mathfrak{e}_8}^C)^w \cap ({\mathfrak{e}_8}^C)^\tau$, we determine the structures of groups

$$(E_{8(-24)})_{ev} = ({E_8}^C)_{ev} \cap ({E_8}^C)^\tau = ({E_8}^C)^\upsilon \cap ({E_8}^C)^\tau,$$

$$(E_{8(-24)})_0 = ({E_8}^C)_0 \cap ({E_8}^C)^\tau = ({E_8}^C)^{\upsilon\iota} \cap ({E_8}^C)^\tau,$$

$$(E_{8(-24)})_{ed} = ({E_8}^C)_{ed} \cap ({E_8}^C)^\tau = ({E_8}^C)^w \cap ({E_8}^C)^\tau.$$

THEOREM 5.4.2

We have the following:

(1) $(E_{8(-24)})_{ev} \cong (SL(2,\boldsymbol{R}) \times E_{7(-25)})/\boldsymbol{Z}_2 \times \{1, l\}, \boldsymbol{Z}_2 = \{(E,1), (-E,-1)\}$,
(2) $(E_{8(-24)})_0 \cong (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, l_0\}$,
(3) $(E_{8(-24)})_{ed} \cong SL(3,\boldsymbol{R}) \times E_{6(-26)}$.

Proof

(1) For $\alpha \in (E_{8(-24)})_{ev} \subset ({E_8}^C)_{ev} = ({E_8}^C)^\upsilon$, there exist $A \in SL(2,C)$ and $\beta \in {E_7}^C$ such that $\alpha = \varphi_{ev}(A,\beta) = \psi(A)\beta$ (see Theorem 5.2.2(1)). From the condition $\tau\alpha\tau = \alpha$, that is, $\tau\psi(A)\beta\tau = \psi(A)\beta$, we have $\psi(\tau A)\tau\beta\tau = \psi(A)\beta$. Hence

$$\begin{cases} \tau A = A, \\ \tau\beta\tau = \beta, \end{cases} \quad \text{or} \quad \begin{cases} \tau A = -A, \\ \tau\beta\tau = -\beta. \end{cases}$$

In the former case, from $\tau A=A$, we have $A \in SL(2,\boldsymbol{R})$, and from $\tau\beta\tau=\beta$, we have $\beta \in ({E_7}^C)^\tau \cong E_{7(-25)}$ (see [6, Theorem 4.3.2]). In the latter case, $A = iI, (I = \mathrm{diag}(1,-1)), \beta = \iota$ satisfy the conditions, and $l = \psi(iI)\iota$. Thus we have the isomorphism $(E_{8(-24)})_{ev} \cong ((SL(2,\boldsymbol{R}) \times E_{7(-25)}) \cup l(SL(2,\boldsymbol{R}) \times E_{7(-25)}))/\boldsymbol{Z}_2 = (SL(2,\boldsymbol{R}) \times E_{7(-25)})/\boldsymbol{Z}_2 \times \{1, l\}, \boldsymbol{Z}_2 = \{(E,1),(-E,-1)\}$.

(2) For $\alpha \in (E_{8(-24)})_0 \subset ({E_8}^C)_0 = ({E_8}^C)^{\upsilon\iota}$, there exist $A \in SL(2,C), \theta \in C^*$, and $\beta \in {E_6}^C$ such that $\alpha = \varphi_0(A,\theta,\beta) = \psi(A)\phi(\theta)\beta$ (see Theorem 5.2.2(2)). From the condition $\tau\alpha\tau = \alpha$, that is, $\tau\psi(A)\phi(\theta)\beta\tau = \psi(A)\phi(\theta)\beta$, we have $\psi(\tau A)\phi(\tau\theta)\tau\beta\tau = \psi(A)\phi(\theta)\beta$. Hence

$$\text{(i)} \begin{cases} \tau A = A, \\ \tau\theta = \theta, \\ \tau\beta\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau A = A, \\ \tau\theta = \omega\theta, \\ \tau\beta\tau = \phi(\omega^2)\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau A = A, \\ \tau\theta = \omega^2\theta, \\ \tau\beta\tau = \phi(\omega)\beta, \end{cases} \qquad \text{(iv)} \begin{cases} \tau A = -A, \\ \tau\theta = -\theta, \\ \tau\beta\tau = \beta, \end{cases}$$

$$\text{(v)} \begin{cases} \tau A = -A, \\ \tau\theta = -\omega\theta, \\ \tau\beta\tau = \phi(\omega^2)\beta, \end{cases} \qquad \text{(vi)} \begin{cases} \tau A = -A, \\ \tau\theta = -\omega^2\theta, \\ \tau\beta\tau = \phi(\omega)\beta. \end{cases}$$

Case (i). From $\tau A = A, \tau\theta = \theta$, we have $A \in SL(2,\boldsymbol{R}), \theta \in \boldsymbol{R}^*$, and from $\tau\beta\tau = \beta$, we have $\beta \in ({E_6}^C)^\tau = E_{6(-26)}$. Hence the group of case (i) is $(SL(2,\boldsymbol{R}) \times \boldsymbol{R}^* \times E_{6(-26)})/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E,1,1),(-E,-1,1)\}$. By the analogous argument in the proof of Theorem 5.3.1(2), we have $(SL(2,\boldsymbol{R}) \times \boldsymbol{R}^* \times E_{6(-26)})/\boldsymbol{Z}_2 \cong SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(-26)}$.

Case (ii). We have $\varphi_0(E,\omega,\phi(\omega^2)) = \psi(E)\phi(\omega)\phi(\omega^2) = 1$.

Case (iii). We have $\varphi_0(E,\omega^2,\phi(\omega)) = \psi(E)\phi(\omega^2)\phi(\omega) = 1$.

Case (iv). We have $\varphi_0(iI,i,1) = l_0$.

Case (v). We have $\varphi_0(iI,i\omega,\phi(\omega^2)) = \varphi_0(iI,i,1)\varphi_0(E,\omega,\phi(\omega^2)) = l_0$.

Case (vi). We have $\varphi_0(iI,i\omega^2,\phi(\omega)) = \varphi_0(iI,i,1)\varphi_0(E,\omega^2,\phi(\omega)) = l_0$.

Thus we have the isomorphism $(E_{8(-24)})_0 \cong (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(-26)}) \cup l_0(SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(-26)}) = (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, l_0\}$.

(3) From the opening statement in the proof of Theorem 5.3.1(3), we use $(\check{E}_{8(-24)})_{ed} = ({\check{E}_8}^C)_{ed} \cap ({\check{E}_8}^C)^\tau = ({\check{E}_8}^C)^w \cap ({\check{E}_8}^C)^\tau$ instead of the group $(E_{8(-24)})_{ed} = ({E_8}^C)_{ed} \cap ({E_8}^C)^\tau = ({E_8}^C)^w \cap ({E_8}^C)^\tau$. Now, for $\alpha \in (\check{E}_{8(-24)})_{ed} \subset ({\check{E}_8}^C)^w$, there exists $A \in SL(3,C)$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_{ed}(A,\beta) = \varphi_1(A)\varphi_2(\beta)$ (see Theorem 5.2.2(3)). From the condition $\tau\alpha\tau = \alpha$, that is, $\tau\varphi_1(A)\varphi_2(\beta)\tau = \varphi_1(A)\varphi_2(\beta)$, we have $\varphi_1(\tau A)\varphi_2(\tau\beta\tau) = \varphi_1(A)\varphi_2(\beta)$. Hence

$$\text{(i)} \begin{cases} \tau A = A, \\ \tau\beta\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau A = \omega A, \\ \tau\beta\tau = \omega^2\beta, \end{cases} \qquad \text{or} \qquad \text{(iii)} \begin{cases} \tau A = \omega^2 A, \\ \tau\beta\tau = \omega\beta. \end{cases}$$

Case (i). From $\tau A = A$, we have $A \in SL(3, \boldsymbol{R})$, and from $\tau\beta\tau = \beta$, we have $\beta \in ({E_6}^C)^\tau = E_{6(-26)}$.

Case (ii). We have $\varphi_{ed}(\omega E, \omega^2 1)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (\omega D\omega^{-1}, \omega^2\phi\omega^{-2}, \omega\omega^2\boldsymbol{X}, \omega^{-1}\omega^{-2}\boldsymbol{Y}) = (D, \phi, \boldsymbol{X}, \boldsymbol{Y})$, that is, $\varphi_{ed}(\omega E, \omega^2 1) = 1$.

Case (iii). We have $\varphi_{ed}(\omega^2 E, \omega 1)(D, \phi, \boldsymbol{X}, \boldsymbol{Y}) = (\omega^2 D\omega^{-2}, \omega\phi\omega^{-1}, \omega^2\omega\boldsymbol{X}, \omega^{-2}\omega^{-1}\boldsymbol{Y}) = (D, \phi, \boldsymbol{X}, \boldsymbol{Y})$, that is, $\varphi_{ed}(\omega^2 E, \omega 1) = 1$.

Thus we have the isomorphism $(E_{8(-24)})_{ed} \cong (\check{E}_{8(-24)})_{ed} \cong SL(3, \boldsymbol{R}) \times E_{6(-26)}$. □

5.5. Subgroups of type $C \oplus {A_7}^C$ and ${A_8}^C$ of ${E_8}^C$

In this section, we use another C-Lie algebra $\tilde{\mathfrak{e}}_8{}^C$ of type ${E_8}^C$ constructed by Gomyo [1]. We review notation in the definition of $\tilde{\mathfrak{e}}_8{}^C$.

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the canonical C-basis of n-dimensional C-vector space C^n, and let $(\boldsymbol{x}, \boldsymbol{y})$ be the inner product in C^n satisfying $(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$. In the exterior C-vector space $\Lambda^k(C^n)$, we define an inner product by

$$(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k, \boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_k) = \det\big((\boldsymbol{x}_i, \boldsymbol{y}_j)\big), \quad k \geq 1,$$

$$(a, b) = ab, \quad a, b \in \Lambda^0(C^n) = C.$$

Then $\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}, 1 \leq i_1 < \cdots < i_k \leq n$, form an orthonormal C-basis of $\Lambda^k(C^n)$. For $\boldsymbol{u} \in \Lambda^k(C^n)$, we define an element $*\boldsymbol{u} \in \Lambda^{n-k}(C^n)$ satisfying

$$(*\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \wedge \boldsymbol{v}, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n), \quad \boldsymbol{v} \in \Lambda^{n-k}(C^n).$$

Then $*$ induces a C-linear isomorphism $* : \Lambda^k(C^n) \to \Lambda^{n-k}(C^n)$.

The group $SL(n, C)$ naturally acts on $\Lambda^k(C^n)$ as

$$A(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k) = A\boldsymbol{x}_1 \wedge \cdots \wedge A\boldsymbol{x}_k, \quad A1 = 1.$$

Hence the Lie algebra $\mathfrak{sl}(n, C)$ acts on $\Lambda^k(C^n)$ as

$$D(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k) = \sum_{j=1}^{k} \boldsymbol{x}_1 \wedge \cdots \wedge D\boldsymbol{x}_j \wedge \cdots \wedge \boldsymbol{x}_k, \quad D1 = 0.$$

LEMMA 5.5.1

For $A \in SL(n, C), D \in \mathfrak{sl}(n, C)$, and $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^k(C^n)$, we have

(1) $(A\boldsymbol{u}, {}^tA^{-1}\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}),\ (D\boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{u}, -{}^tD\boldsymbol{v}) = 0,$

(2) $*(A\boldsymbol{u}) = {}^tA^{-1}(*\boldsymbol{u}),\ *(D\boldsymbol{u}) = -{}^tD(*\boldsymbol{u}).$

For $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^k(C^n)$ $(1 \leq k \leq n)$, we define a C-linear mapping $\boldsymbol{u} \times \boldsymbol{v}$ of C^n by

$$(\boldsymbol{u} \times \boldsymbol{v})\boldsymbol{x} = *\big(\boldsymbol{v} \wedge *(\boldsymbol{u} \wedge \boldsymbol{x})\big) + (-1)^{n-k}\frac{n-k}{n}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{x}, \quad \boldsymbol{x} \in \boldsymbol{C}^n.$$

Since $\mathrm{tr}(\boldsymbol{u} \times \boldsymbol{v}) = 0$, $\boldsymbol{u} \times \boldsymbol{v}$ can be regarded as an element of $\mathfrak{sl}(n, C)$ with respect to the canonical C-basis of C^n.

LEMMA 5.5.2

For $A \in SL(n, C), D \in \mathfrak{sl}(n, C)$, and $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^k(C^n)$, we have

(1) $A(\boldsymbol{u}\times\boldsymbol{v})A^{-1} = A\boldsymbol{u}\times {}^tA^{-1}\boldsymbol{v},\ [D,\boldsymbol{u}\times\boldsymbol{v}] = D\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}\times(-{}^tD\boldsymbol{v}),$
(2) ${}^t(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\times\boldsymbol{u},\ \tau(\boldsymbol{u}\times\boldsymbol{v}) = \tau\boldsymbol{u}\times\tau\boldsymbol{v},$
(3) $\mathrm{tr}(D(\boldsymbol{u}\times\boldsymbol{v})) = (-1)^{n-k}(D\boldsymbol{u},\boldsymbol{v}).$

Now, we construct a C-Lie algebra ${\widetilde{\mathfrak{e}_8}}^C$ of type ${E_8}^C$.

PROPOSITION 5.5.3 (GOMYO [1, Theorem 3.2])
In an $80+84+84=248$ dimensional C-vector space

$$ {\widetilde{\mathfrak{e}_8}}^C = \mathfrak{sl}(9,C)\oplus \Lambda^3(C^9)\oplus\Lambda^3(C^9), $$

we define a Lie bracket $[R_1,R_2]$ by

$$ [(D_1,\boldsymbol{u}_1,\boldsymbol{v}_1),(D_2,\boldsymbol{u}_2,\boldsymbol{v}_2)] = (D,\boldsymbol{u},\boldsymbol{v}), $$

$$ \begin{cases} D = [D_1,D_2]+\boldsymbol{u}_1\times\boldsymbol{v}_2-\boldsymbol{u}_2\times\boldsymbol{v}_1, \\ \boldsymbol{u} = D_1\boldsymbol{u}_2-D_2\boldsymbol{u}_1+*(\boldsymbol{v}_1\wedge\boldsymbol{v}_2), \\ \boldsymbol{v} = -{}^tD_1\boldsymbol{v}_2+{}^tD_2\boldsymbol{v}_1-*(\boldsymbol{u}_1\wedge\boldsymbol{u}_2); \end{cases} $$

then ${\widetilde{\mathfrak{e}_8}}^C$ becomes a simple C-Lie algebra.

This C-Lie algebra ${\widetilde{\mathfrak{e}_8}}^C$ has to be type ${E_8}^C$. Let ${\widetilde{E_8}}^C$ be the automorphism group of ${\widetilde{\mathfrak{e}_8}}^C$:

$$ {\widetilde{E_8}}^C = \left\{\alpha\in\mathrm{Iso}_C({\widetilde{\mathfrak{e}_8}}^C)\ \middle|\ \alpha[R_1,R_2] = [\alpha R_1,\alpha R_2]\right\}. $$

Then ${\widetilde{E_8}}^C$ is also a simply connected complex Lie group of type ${E_8}^C$.

We define a C-linear transformation $\widehat{\lambda}$ of ${\widetilde{\mathfrak{e}_8}}^C$ by

$$ \widehat{\lambda}(D,\boldsymbol{u},\boldsymbol{v}) = (-{}^tD,-\boldsymbol{v},-\boldsymbol{u}). $$

Then $\widehat{\lambda}\in{\widetilde{E_8}}^C$ and $\widehat{\lambda}^2=1$. The complex conjugation of ${\widetilde{\mathfrak{e}_8}}^C$ is usually denoted by τ:

$$ \tau(D,\boldsymbol{u},\boldsymbol{v}) = (\tau D,\tau\boldsymbol{u},\tau\boldsymbol{v}). $$

LEMMA 5.5.4 (see GOMYO [1])
The Killing form $\widetilde{B_8}$ of the Lie algebra ${\widetilde{\mathfrak{e}_8}}^C$ is given by

$$ \widetilde{B_8}\big((D_1,\boldsymbol{u}_1,\boldsymbol{v}_1),(D_2,\boldsymbol{u}_2,\boldsymbol{v}_2)\big) = 60\big(\mathrm{tr}(D_1D_2)+(\boldsymbol{u}_1,\boldsymbol{v}_2)+(\boldsymbol{v}_1,\boldsymbol{u}_2)\big). $$

We shall find an $\boldsymbol{R}$-Lie algebra of type $E_{8(8)}$. We define an $\boldsymbol{R}$-Lie algebra $\widetilde{\mathfrak{e}_8}'$ by

$$ \widetilde{\mathfrak{e}_8}' = \mathfrak{sl}(9,\boldsymbol{R})\oplus\Lambda^3(\boldsymbol{R}^9)\oplus\Lambda^3(\boldsymbol{R}^9) = \left({\widetilde{\mathfrak{e}_8}}^C\right)^\tau $$

with the Lie bracket the same as that of ${\widetilde{\mathfrak{e}_8}}^C$.

PROPOSITION 5.5.5
We have that $\widetilde{\mathfrak{e}_8}'$ is an $\boldsymbol{R}$-Lie algebra of type $E_{8(8)}$.

Proof
We find the signature of the Killing form $\widetilde{B}_8{}' = \widetilde{B_8}|\widetilde{\mathfrak{e}}_8{}'$ of $\widetilde{\mathfrak{e}}_8{}'$. Decompose $\widetilde{\mathfrak{e}}_8{}'$ into eigenspaces relative to $\widehat{\lambda}$:

$$\widetilde{\mathfrak{e}}_8{}' = (\widehat{\mathfrak{e}}_8{}')_{\widehat{\lambda}} \oplus (\widetilde{\mathfrak{e}}_8{}')_{-\widehat{\lambda}},$$

$$(\widetilde{\mathfrak{e}}_8{}')_{\widehat{\lambda}} = \{R \in \hat{\mathfrak{e}}_8{}' \mid \widehat{\lambda}R = R\} = \big\{(D, \boldsymbol{u}, -\boldsymbol{u}) \bigm| D \in \mathfrak{sl}(9, \boldsymbol{R}), {}^tD = -D, \boldsymbol{u} \in \Lambda^3(\boldsymbol{R}^9)\big\},$$

$$(\widetilde{\mathfrak{e}}_8{}')_{-\widehat{\lambda}} = \{R \in \widetilde{\mathfrak{e}}_8{}' \mid \widetilde{\lambda}R = -R\} = \big\{(D, \boldsymbol{u}, \boldsymbol{u}) \bigm| D \in \mathfrak{sl}(9, \boldsymbol{R}), {}^tD = D, \boldsymbol{u} \in \Lambda^3(\boldsymbol{R}^9)\big\}.$$

Then, from Lemma 5.5.4, we see that the Killing form $\widetilde{B}_8{}'$ on $(\widetilde{\mathfrak{e}}_8{}')_{\widehat{\lambda}}$ is negative definite and $\widetilde{B}_8{}'$ on $(\widetilde{\mathfrak{e}}_8{}')_{-\widehat{\lambda}}$ is positive definite. Therefore the number of negative eigenvalues of $\widetilde{B}_8{}'$ is $\dim((\widetilde{\mathfrak{e}}_8{}')_{\widehat{\lambda}}) = 44 + 84 = 128$, and the number of positive eigenvalues of $\widetilde{B}_8{}'$ is $\dim((\widetilde{\mathfrak{e}}_8{}')_{-\widehat{\lambda}}) = 36 + 84 = 120$. Therefore the signature of $\widetilde{B}_8{}'$ is $128 - 120 = 8$. Hence the type of $\widetilde{B}_8{}'$ is $E_{8(8)}$. □

Let $\widetilde{E_8}{}'$ be the automorphism group of $\widetilde{\mathfrak{e}_8}{}'$:

$$\widetilde{E_8}{}' = \big\{\alpha \in \mathrm{Iso}_R(\widetilde{\mathfrak{e}_8}{}') \bigm| \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\big\}.$$

Although we cannot give any explicit isomorphism between ${\mathfrak{e}_8}^C$ and $\widetilde{\mathfrak{e}_8}^C$, $\mathfrak{e}_{8(8)}$ and $\widetilde{\mathfrak{e}_8}{}'$, instead of $\widetilde{\mathfrak{e}_8}^C, \widetilde{\mathfrak{e}_8}{}', \widetilde{E_8}^C$, and $\widetilde{E_8}{}'$, we use the same notation as ${\mathfrak{e}_8}^C, \mathfrak{e}_{8(8)}$, ${E_8}^C$, and $E_{8(8)}$ of Sections 5.1.

In the C-Lie algebra ${\mathfrak{e}_8}^C = \mathfrak{sl}(9, C) \oplus \Lambda^3(C^9) \oplus \Lambda^3(C^9)$, let

$$Z = \frac{1}{3}\big(\mathrm{diag}(-8, 1, 1, 1, 1, 1, 1, 1, 1), 0, 0\big).$$

THEOREM 5.5.6
The 3*-graded decomposition of the Lie algebra* $\mathfrak{e}_{8(8)} = ({\mathfrak{e}_8}^C)^\tau$ *(or* ${\mathfrak{e}_8}^C$*),*

$$\mathfrak{e}_{8(8)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}\, Z, Z = \frac{1}{3}(\mathrm{diag}(-8, 1, 1, 1, 1, 1, 1, 1, 1), 0, 0)$*, is given by*

$$\mathfrak{g}_0 = \big\{(E_{ii} - E_{99}, 0, 0), 1 \le i \le 8, (E_{kl}, 0, 0), 2 \le k \le 9, 2 \le l \le 9, k \ne l\big\}\ 64,$$

$$\mathfrak{g}_{-1} = \big\{(0, 0, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k), 2 \le i < j < k \le 9\big\}\ 56,$$

$$\mathfrak{g}_{-2} = \big\{(0, \boldsymbol{e}_1 \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k, 0), 2 \le j < k \le 9\big\}\ 28,$$

$$\mathfrak{g}_{-3} = \big\{(E_{1j}, 0, 0), 2 \le j \le 9\big\}\ 8,$$

$$\mathfrak{g}_1 = \widehat{\lambda}(\mathfrak{g}_{-1}), \mathfrak{g}_2 = \widehat{\lambda}(\mathfrak{g}_{-2}), \mathfrak{g}_3 = \widehat{\lambda}(\mathfrak{g}_{-3}).$$

For the characteristic element $Z = \frac{1}{3}(\mathrm{diag}(-8, 1, 1, 1, 1, 1, 1, 1, 1), 0, 0)$, we set

$$z_4 = \exp\Big(\frac{2\pi i}{4}\,\mathrm{ad}\, Z\Big), \qquad z_3 = \exp\Big(\frac{2\pi i}{3}\,\mathrm{ad}\, Z\Big);$$

then we have

$$z_4(D, \boldsymbol{u}, \boldsymbol{v}) = (A_4 D {A_4}^{-1}, A_4 \boldsymbol{u}, {}^t{A_4}^{-1} \boldsymbol{v}), \quad A_4 = \mathrm{diag}({\omega_{12}}^8, \omega_{12}, \omega_{12}, \dots, \omega_{12}),$$

$$z_3(D,\boldsymbol{u},\boldsymbol{v}) = (A_3DA_3{}^{-1}, A_3\boldsymbol{u}, {}^tA_3{}^{-1}\boldsymbol{v}), \quad A_3 = \omega_9 E,$$
$$= (D, \omega_9\boldsymbol{u}, \omega_9{}^{-1}\boldsymbol{v}),$$

where $(D,\boldsymbol{u},\boldsymbol{v}) \in \mathfrak{e}_8{}^C, \omega_{12} = e^{2\pi i/12}, \omega_9 = e^{2\pi i/9}$.

Since $(\mathfrak{e}_8{}^C)_0 = (\mathfrak{e}_8{}^C)^{z_4}, (\mathfrak{e}_8{}^C)_{ed} = (\mathfrak{e}_8{}^C)^{z_3}$, we determine the structures of groups

$$(E_8{}^C)_0 = (E_8{}^C)^{z_4}, \qquad (E_8{}^C)_{ed} = (E_8{}^C)^{z_3}.$$

THEOREM 5.5.7

(1) *As for* $(E_8{}^C)_{ev}$, *we will study this later.*

(2) *We have* $(E_8{}^C)_0 \cong (C^* \times SL(8,C))/\boldsymbol{Z}_{24}, \boldsymbol{Z}_{24} = \boldsymbol{Z}_3 \times \boldsymbol{Z}_8, \boldsymbol{Z}_3 = \{(1,E), (\omega,E),(\omega^2,E)\}$, $\boldsymbol{Z}_8 = \{(\omega_8{}^k, \omega_8{}^kE) \mid k = 0,1,\dots,7\}, \omega = e^{2\pi i/3}, \omega_8 = e^{2\pi i/8}$.

(3) *We have* $(E_8{}^C)_{ed} \cong SL(9,C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{E, \omega E, \omega^2 E\}, \omega = e^{2\pi i/3}$.

Proof

(2) We define a mapping $\varphi_0 : S(GL(1,C) \times GL(8,C)) \to (E_8{}^C)^{z_4} = (E_8{}^C)_0$ by

$$\varphi_0(A)(D,\boldsymbol{u},\boldsymbol{v}) = (ADA^{-1}, A\boldsymbol{u}, {}^tA^{-1}\boldsymbol{v});$$

φ_0 is well defined. Indeed, by using Lemmas 5.5.1 and 5.5.2, we have

$$\varphi_0(A)[(D_1,\boldsymbol{u}_1,\boldsymbol{v}_1),(D_2,\boldsymbol{u}_2,\boldsymbol{v}_2)] = [\varphi_0(A)(D_1,\boldsymbol{u}_1,\boldsymbol{v}_1), \varphi_0(A)(D_2,\boldsymbol{u}_2,\boldsymbol{v}_2)];$$

that is, $\varphi_0(A) \in E_8{}^C$. Next, since $z_4 = \varphi_0(A_4)$ and $z_4\varphi_0(A) = \varphi_0(A_4)\varphi_0(A) = \varphi_0(A_4A) = \varphi_0(AA_4) = \varphi_0(A)\varphi_0(A_4) = \varphi_0(A)z_4$, we get $\varphi_0(A) \in (E_8{}^C)^{z_4}$. Obviously φ_0 is a homomorphism. It is easy to see that $\mathrm{Ker}\,\varphi_0 = \{E, \omega E, \omega^2 E\} = \boldsymbol{Z}_3$, $(E_8{}^C)^{z_4}$ is connected, $\mathrm{Ker}\,\varphi_0$ is discrete, and $\dim_C\big(\mathfrak{s}(\mathfrak{gl}(1,C) \oplus \mathfrak{gl}(8,C))\big) = (1+64) - 1 = 64 = \dim_C((\mathfrak{e}_8{}^C)_0) = \dim_C((\mathfrak{e}_8{}^C)^{z_4})$ (see Theorem 5.5.6), so φ_0 is surjective. Hence we have

$$(E_8{}^C)^{z_4} \cong S\big(GL(1,C) \times GL(8,C)\big)/\boldsymbol{Z}_3, \quad \boldsymbol{Z}_3 = \{E, \omega E, \omega^2 E\}.$$

Further, the mapping $h : C^* \times SL(8,C) \to S(GL(1,C) \times GL(8,C))$,

$$h(z,B) = \begin{pmatrix} z^{-8} & 0 \\ 0 & zB \end{pmatrix},$$

induces the isomorphism $S(GL(1,C) \times GL(8,C)) \cong (C^* \times SL(8,C))/\boldsymbol{Z}_8$, $\boldsymbol{Z}_8 = \{(\omega_8{}^k, \omega_8{}^kE) \mid k = 0,1,\dots,7\}$, and h satisfies $h(\omega,E) = \omega E$. Thus we have the isomorphism $(E_8{}^C)_0 = (E_8{}^C)^{z_4} \cong (C^* \times SL(8,C))/(\boldsymbol{Z}_3 \times \boldsymbol{Z}_8), \boldsymbol{Z}_3 = \{(1,E), (\omega,E),(\omega^2,E)\}, \boldsymbol{Z}_8 = \{(\omega_8{}^k, \omega_8{}^kE) \mid k = 0,1,\dots,7\}$.

(3) We define a mapping $\varphi_{ed} : SL(9,C) \to (E_8{}^C)^{z_3} = (E_8{}^C)_{ed}$ by

$$\varphi_{ed}(A)(D,\boldsymbol{u},\boldsymbol{v}) = (ADA^{-1}, A\boldsymbol{u}, {}^tA^{-1}\boldsymbol{v}).$$

Then we see that φ_{ed} induces the isomorphism $(E_8{}^C)_{ed} = (E_8{}^C)^{z_3} \cong SL(9,C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{E, \omega E, \omega^2 E\}$ in a way similar to (2) above. $\square$

5.6. Subgroups of type $\boldsymbol{R} \oplus A_{7(7)}$ and $A_{8(8)}$ of $E_{8(8)}$

In this section, we use Lie algebras ${\mathfrak{e}_8}^C, \mathfrak{e}_{8(8)}$ and Lie groups ${E_8}^C, E_{8(8)}$ defined in Section 5.5.

Since $(\mathfrak{e}_{8(8)})_0 = ({\mathfrak{e}_8}^C)_0 \cap ({\mathfrak{e}_8}^C)^\tau = ({\mathfrak{e}_8}^C)^{z_4} \cap ({\mathfrak{e}_8}^C)^\tau, (\mathfrak{e}_{8(8)})_{ed} = ({\mathfrak{e}_8}^C)_{ed} \cap ({\mathfrak{e}_8}^C)^\tau = ({\mathfrak{e}_8}^C)^{z_3} \cap ({\mathfrak{e}_8}^C)^\tau$, we determine the structures of groups

$$(E_{8(8)})_0 = ({E_8}^C)_0 \cap ({E_8}^C)^\tau = ({E_8}^C)^{z_4} \cap ({E_8}^C)^\tau,$$

$$(E_{8(8)})_{ed} = ({E_8}^C)_{ed} \cap ({E_8}^C)^\tau = ({E_8}^C)^{z_3} \cap ({E_8}^C)^\tau.$$

THEOREM 5.6.1

(1) *As for* $(E_{8(8)})_{ev}$, *we will study this later.*
(2) *We have* $(E_{8(8)})_0 \cong (\boldsymbol{R}^+ \times SL(8,\boldsymbol{R})) \times \{1, \zeta, \zeta^2\}$.
(3) *We have* $(E_{8(8)})_{ed} \cong SL(9,\boldsymbol{R}) \times \{1, \zeta, \zeta^2\}$.

Proof

(2) For $\alpha \in (E_{8(8)})_0 \subset ({E_8}^C)_0 = ({E_8}^C)^{z_4}$, there exists $A \in S(GL(1,C) \times GL(8,C))$ such that $\alpha = \varphi_0(A)$ (see Theorem 5.5.7(2)). From the condition $\tau\alpha\tau = \alpha$, that is, $\tau\varphi_4(A)\tau = \varphi_4(A)$, we have $\varphi_0(\tau A) = \varphi_0(A)$. Hence

$$\text{(i) } \tau A = A, \qquad \text{(ii) } \tau A = \omega A, \qquad \text{or} \qquad \text{(iii) } \tau A = \omega^2 A.$$

Case (i). From the condition $\tau A = A$, we have $A \in S(GL(1,\boldsymbol{R}) \times GL(8,\boldsymbol{R}))$. The mapping $h : \boldsymbol{R}^* \times SL(8,\boldsymbol{R}) \to S(GL(1,\boldsymbol{R}) \times GL(8,\boldsymbol{R}))$,

$$h(r,B) = \begin{pmatrix} r^{-8} & 0 \\ 0 & rB \end{pmatrix},$$

induces the isomorphism $S(GL(1,\boldsymbol{R}) \times GL(8,\boldsymbol{R})) \cong (\boldsymbol{R}^* \times SL(8,\boldsymbol{R}))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,E),(-1,-E)\}$. Further, the mapping $k : \boldsymbol{R}^* \times SL(8,\boldsymbol{R}) \to \boldsymbol{R}^+ \times SL(8,\boldsymbol{R})$,

$$k(r,B) = \begin{cases} (r,B) & \text{if } r > 0, \\ (-r,-B) & \text{if } r < 0 \end{cases}$$

induces the isomorphism $\boldsymbol{R}^+ \times SL(8,\boldsymbol{R}) \cong (\boldsymbol{R}^* \times SL(8,\boldsymbol{R}))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,E), (-1,-E)\}$. Hence we have $S(GL(1,\boldsymbol{R}) \times GL(8,\boldsymbol{R})) \cong \boldsymbol{R}^+ \times SL(8,\boldsymbol{R})$.

Case (ii). Since $A = \omega E$ satisfies the condition $\tau A = \omega A$, we have

$$\begin{aligned} \varphi_0(\omega E)(D,\boldsymbol{u},\boldsymbol{v}) &= \big((\omega E)D(\omega E)^{-1}, (\omega E)\boldsymbol{u}, {}^t(\omega E)^{-1}\boldsymbol{v}\big) \\ &= (D, \omega\boldsymbol{u}, \omega^2\boldsymbol{v}) = \zeta(D,\boldsymbol{u},\boldsymbol{v}); \end{aligned}$$

that is, ζ is defined by $\varphi_0(\omega E)$.

Case (iii). Since $A = \omega^2 E$ satisfies the condition $\tau A = \omega^2 A$, in a way similar to case (ii), we have $\varphi_0(\omega^2 E) = \zeta^2$.

Thus we have the isomorphism $(E_{8(8)})_0 \cong (\boldsymbol{R}^+ \times SL(8,\boldsymbol{R})) \cup \zeta(\boldsymbol{R}^+ \times SL(8,\boldsymbol{R})) \cup \zeta^2(\boldsymbol{R}^+ \times SL(8,\boldsymbol{R})) = (\boldsymbol{R}^+ \times SL(8,\boldsymbol{R})) \times \{1, \zeta, \zeta^2\}$.

(3) For $\alpha \in (E_{8(8)})_{ed} \subset ({E_8}^C)_{ed} = ({E_8}^C)^{z_3}$, there exists $A \in SL(9,C)$ such that $\alpha = \varphi_{ed}(A)$ (see Theorem 5.5.7(3)). From the condition $\tau\alpha\tau = \alpha$, that is,

$\tau\varphi_{ed}(A)\tau = \varphi_{ed}(A)$, we have $\varphi_3(\tau A) = \varphi_3(A)$. Hence

$$\text{(i) } \tau A = A, \qquad \text{(ii) } \tau A = \omega A, \qquad \text{or} \qquad \text{(iii) } \tau A = \omega^2 A.$$

Case (i). From the condition $\tau A = A$, we have $A \in SL(9, \boldsymbol{R})$.

Case (ii). Since $A = \omega E$ satisfies the condition $\tau A = \omega A$, we have $\varphi_{ed}(\omega E) = \zeta$ as in Case of (2).

Case (iii). Since $A = \omega^2 E$ satisfies the conditions $\tau A = \omega^2 A$, we have $\varphi_{ed}(\omega^2 E) = \zeta^2$ as in Case (2). Thus we have the isomorphism $(E_{8(8)})_{ed} = ({E_8}^C)^{z_3} \cong SL(9, \boldsymbol{R}) \cup \zeta(SL(9, \boldsymbol{R})) \cup \zeta^2(SL(9, \boldsymbol{R})) = SL(9, \boldsymbol{R}) \times \{1, \zeta, \zeta^2\}$. $\square$

5.7. Subgroup of type ${D_8}^C$ of ${E_8}^C$ and subgroup of type $D_{8(8)}$ of $E_{8(8)}$

In this section, we determine the structures of the groups $({E_8}^C)_{ev}$ (see Theorem 5.5.7(1)) and $(E_{8(8)})_{ev}$ (see Theorem 5.6.1(1)). As we use a realization of semispinor groups $Ss(16, C)$ in ${E_8}^C$ and $Ss(8, 8)$ in $E_{8(8)}$ by Gomyo [2], we review here one more Lie algebra ${\mathfrak{e}_8}^C$ constructed by Gomyo [2].

Let $e_0, e_1, \ldots, e_7$ be the canonical C-basis of the C-vector space $\mathfrak{C}^C$ which is the complexification of the $\boldsymbol{R}$-Calyley algebra $\mathfrak{C}$. In a 16-dimensional C-vector space $(\mathfrak{C}^C)^2$, denote

$$\widetilde{e}_1 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix}, \qquad \widetilde{e}_2 = \begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \ldots, \widetilde{e}_8 = \begin{pmatrix} e_7 \\ 0 \end{pmatrix},$$

$$\widetilde{e}_9 = \begin{pmatrix} 0 \\ e_0 \end{pmatrix}, \qquad \widetilde{e}_{10} = \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \ldots, \widetilde{e}_{16} = \begin{pmatrix} 0 \\ e_7 \end{pmatrix}.$$

We give an inner product $(\widetilde{a}, \widetilde{b})$ in $(\mathfrak{C}^C)^2$ so that $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_{16}$ are an orthonormal C-basis of $(\mathfrak{C}^C)^2$. Let $Cl((\mathfrak{C}^C)^2)$ be the C-Clifford algebra with a C-basis

$$1, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_{16}, \ldots, \widetilde{e}_{k_1} \cdots \widetilde{e}_{k_l} (k_1 < \cdots < k_l), \ldots, \widetilde{e}_1 \widetilde{e}_2 \cdots \widetilde{e}_{16}$$

with relations $\widetilde{e}_k^2 = -1$ and $\widetilde{e}_k \widetilde{e}_l = -\widetilde{e}_l \widetilde{e}_k$ $(k \neq l)$. Now, the complex spinor group $Spin(16, C)$ is defined by

$$Spin(16, C) = \left\{ \widetilde{a}_1 \widetilde{a}_2 \cdots \widetilde{a}_{2q} \in Cl((\mathfrak{C}^C)^2) \,\middle|\, \begin{array}{l} \widetilde{a}_k \in (\mathfrak{C}^C)^2, (\widetilde{a}_k, \widetilde{a}_k) = 1, \\ q = 1, 2, 3, \ldots \end{array} \right\}.$$

It is known that the group $Spin(16, C)$ is connected and is a double covering group of $SO(16, C) = SO((\mathfrak{C}^C)^2)$ by the projection $p : Spin(16, C) \to SO(16, C)$,

$$p(\widetilde{\alpha})\widetilde{x} = \widetilde{\alpha}\widetilde{x}\widetilde{\alpha}^{-1}, \quad \widetilde{x} \in (\mathfrak{C}^C)^2.$$

So $Spin(16, C)$ is simply connected. In $Cl((\mathfrak{C}^C)^2)$, let

$$\widetilde{\zeta} = \widetilde{e}_1 \widetilde{e}_2 \cdots \widetilde{e}_{15} \widetilde{e}_{16}.$$

Then $\widetilde{\zeta} \in Spin(16, C)$ and $\widetilde{\zeta}^2 = 1$. The center of the group $Spin(16, C)$ is given by

$$z\big(Spin(16, C)\big) = \{1, -1, \widetilde{\zeta}, -\widetilde{\zeta}\}.$$

The complex semispinor group $Ss(16, C)$ is defined by

$$Ss(16, C) = Spin(16, C)/\{1, \widetilde{\zeta}\}.$$

It is known that $Spin(16,C)/\{1,-1\} \cong SO(16,C)$ and $Ss(16,C) \not\cong SO(16,C)$.

In the C-Lie algebra $\mathfrak{so}(8,C) = \mathfrak{so}(\mathfrak{C}^C) = \{X \in \mathrm{Hom}_C(\mathfrak{C}^C) \mid (Xx,y) + (x, Xy) = 0, x,y \in \mathfrak{C}^C\}$, G_{kl} $(0 \le k \le 7, 0 \le l \le 7, k \ne l)$ is defined as a C-endomorphism of $\mathfrak{C}^C$ satisfying

$$G_{kl}e_l = e_k, \qquad G_{kl}e_k = -e_l, \qquad G_{kl}e_j = 0 \quad \text{otherwise},$$

then $G_{kl}, 0 \le k < l \le 7$ is C-basis of $\mathfrak{so}(8,C)$. (These G_{kl} are already used in Theorems 5.2.1 and 5.4.1.) Next, $F_{kl} \in \mathfrak{so}(8,C)$ $(0 \le k \le 7, 0 \le l \le 7, k \ne l)$ is defined as

$$F_{kl}x = \frac{1}{2}e_k(\overline{e}_l x), \quad x \in \mathfrak{C}^C.$$

Now, we define C-linear transformations μ, κ, and ν of $\mathfrak{so}(8,C)$ by

$$\mu G_{kl} = F_{kl}, \qquad (\kappa X)x = \overline{X\overline{x}}, \quad x \in \mathfrak{C}^C, \ \nu = \mu\kappa.$$

Then μ, κ, and ν are outer automorphisms of $\mathfrak{so}(8,C)$.

For $x, y \in \mathfrak{C}^C$, we define a C-linear transformation $x \times y$ of $\mathfrak{C}^C$ by

$$(x \times y)z = (y,z)x - (x,z)y, \quad z \in \mathfrak{C}^C.$$

Let $\mathfrak{so}(16,C) = \{D \in \mathrm{Hom}((\mathfrak{C}^C)^2) \mid (D\widetilde{x}, \widetilde{y}) + (\widetilde{x}, D\widetilde{y}) = 0, \widetilde{x}, \widetilde{y} \in (\mathfrak{C}^C)^2\} = \{D \in M(16,C) \mid {}^tD + D = 0\}$. We define a C-bilinear mapping $\times : (\mathfrak{C}^C \otimes \mathfrak{C}^C) \times (\mathfrak{C}^C \otimes \mathfrak{C}^C) \to \mathfrak{so}(16,C)$ by

$$(x_1 \otimes y_1, 0) \times (x_2 \otimes y_2, 0) = \begin{pmatrix} (y_1,y_2)\pi(x_1 \times x_2) & 0 \\ 0 & (x_1,x_2)\pi(y_1 \times y_2) \end{pmatrix},$$

$$(0, z_1 \otimes u_1) \times (0, z_2 \otimes u_2) = \begin{pmatrix} (u_1,u_2)\nu^2(z_1 \times z_2) & 0 \\ 0 & (z_1,z_2)\nu^2(u_1 \times u_2) \end{pmatrix},$$

$$(x \otimes y, 0) \times (0, z \otimes u) = \begin{pmatrix} 0 & \frac{1}{2}(x\overline{z})^t(y\overline{u}) \\ -\frac{1}{2}(y\overline{u})^t(x\overline{z}) & 0 \end{pmatrix},$$

$$(0, z \otimes u) \times (x \otimes y, 0) = \begin{pmatrix} 0 & -\frac{1}{2}(x\overline{z})^t(y\overline{u}) \\ \frac{1}{2}(y\overline{u})^t(x\overline{z}) & 0 \end{pmatrix}.$$

We define a representation ρ of $Spin(16,C)$ on $(\mathfrak{C}^C \otimes \mathfrak{C}^C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C)$ (called the half-spinor representation of $Spin(16,C)$) by

$$\begin{aligned} &\rho\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)(x \otimes y, 0) \\ &\quad = \big(-a_1(\overline{a}_2 x) \otimes y - x \otimes b_1(\overline{b}_2 y), \overline{a}_1 x \otimes \overline{b}_2 y - \overline{a}_2 x \otimes \overline{b}_1 y\big), \\ &\rho\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)(0, z \otimes u) \\ &\quad = \big(-a_1 z \otimes b_2 u + a_2 z \otimes b_1 u, -\overline{a}_1(a_2 z) \otimes u - z \otimes \overline{b}_1(b_2 u)\big), \\ &\rho(\widetilde{a}_1\widetilde{a}_2 \cdots \widetilde{a}_{2m-1}\widetilde{a}_{2m}) = \rho(\widetilde{a}_1\widetilde{a}_2) \cdots \rho(\widetilde{a}_{2m-1}\widetilde{a}_{2m}). \end{aligned}$$

Then the differential representation $d\rho$ of $\mathfrak{so}(16,C)$ on $(\mathfrak{C}^C \otimes \mathfrak{C}^C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C)$ has the following property:

$$d\rho\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right)(x \otimes y, z \otimes u) = \big((\mu X)x \otimes y + x \otimes (\mu Y)y, (\nu X)z \otimes u + z \otimes (\nu Y)u\big).$$

Under preliminaries above, we have the following proposition.

PROPOSITION 5.7.1 (GOMYO [2, Theorem 3.4])

In a $120+64+64=248$ *dimensional* C*-vector space*

$$\widehat{\mathfrak{e}}_8{}^C = \mathfrak{so}(16,C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C),$$

we define a Lie bracket $[R_1, R_2]$ *by*

$$[(D_1,P_1),(D_2,P_2)] = \big([D_1,D_2] - P_1 \times P_2, d\rho(D_1)P_2 - d\rho(D_2)P_1\big);$$

then $\widehat{\mathfrak{e}}_8{}^C$ *becomes a simple* C*-Lie algebra.*

This C-Lie algebra $\widehat{\mathfrak{e}}_8{}^C$ has to be of type $E_8{}^C$. Let $\widehat{E}_8{}^C$ be the automorphism group of $\widehat{\mathfrak{e}}_8{}^C$:

$$\widehat{E}_8{}^C = \big\{\alpha \in \mathrm{Iso}_C(\widehat{\mathfrak{e}}_8{}^C) \mid \alpha[R_1,R_2] = [\alpha R_1, \alpha R_2]\big\}.$$

Then $\widehat{E}_8{}^C$ is also a simply connected complex Lie group of type $E_8{}^C$. So we use notations $\mathfrak{e}_8{}^C$ and $E_8{}^C$ instead of $\widehat{\mathfrak{e}}_8{}^C$ and $\widehat{E}_8{}^C$.

In the C-algebra $\mathfrak{e}_8{}^C = \mathfrak{so}(16,C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C)$, let

$$Z = \big(\mathrm{diag}(iJ,iJ,iJ,iJ,-iJ,iJ,iJ,iJ),0,0\big), \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let G_{kl} be an element of $\mathfrak{so}(16,C) = \{D \in M(16,C) \mid {}^tD + D = 0\}$ such that $G_{kl} = E_{kl} - E_{lk}$ (where E_{kl} is a matrix of $M(16,C)$ when the (k,l)-entry is 1 and the others are zero). Then $G_{kl}, 0 \le k < l \le 15$ is a C-basis of $\mathfrak{so}(16,C)$. The complex conjugation in $\mathfrak{e}_8{}^C$ is usually denoted by τ:

$$\tau(D, x \otimes y, z \otimes u) = (\tau D, \tau x \otimes \tau y, \tau z \otimes \tau u).$$

THEOREM 5.7.2

The 3*-graded decomposition of the Lie algebra* $\mathfrak{e}_8{}^C = \mathfrak{so}(16,C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C) \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C)$,

$$\mathfrak{e}_8{}^C = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}\, Z, Z = (\mathrm{diag}(iJ,iJ,iJ,iJ,-iJ,iJ,iJ,iJ),0,0)$*, is given by*

$$\mathfrak{g}_0 = \left\{\begin{array}{l} (G_{k,k+1},0,0), k=0,2,4,\dots,14, \\ (G_{k,k+j} + G_{k+1,k+1+j},0,0), \\ \qquad k=0,2,4,\dots,14, j=k+2,k+4,\dots,14, k,j \ne 8, \\ (G_{k,8} - G_{k+1,9},0,0), k=0,2,4,6, G_{8,k} - G_{9,k+1}, k=10,12,14, \\ (G_{k,k+1+j} - G_{k+1,k+j},0,0), \\ \qquad k=0,2,4,\dots,14, j=k+2,k+4,\dots,14, k,j \ne 8, \\ (G_{k,8} + G_{k+1,9},0,0), k=0,2,4,6, G_{8,k} + G_{9,k+1}, k=10,12,14 \end{array}\right\}_{64},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} (0, (e_0 \otimes e_0 + e_1 \otimes e_1) - i(e_0 \otimes e_1 - e_1 \otimes e_0), 0), \\ (0, (e_1 \otimes e_k + e_0 \otimes e_{k+1}) - i(e_0 \otimes e_k - e_1 \otimes e_{k+1}), 0), \\ \qquad k = 2, 4, 6, \\ (0, (e_l \otimes e_0) + i(e_l \otimes e_1), 0), (0, (e_l \otimes e_k) - i(e_l \otimes e_{k+1}), 0), \\ \qquad k = 2, 4, 6, l = 2, 3, \ldots 7, \\ (0, 0, (e_0 \otimes e_0 + e_1 \otimes e_1) + i(e_0 \otimes e_1 - e_1 \otimes e_0)), \\ (0, 0, (e_k \otimes e_0 - e_{k+1} \otimes e_1) + (e_k \otimes e_1 + e_{k+1} \otimes e_0)), \\ \qquad k = 2, 4, 6, \\ (0, 0, (e_l \otimes e_0) + i(e_l \otimes e_1), (e_l \otimes e_k) - i(e_l \otimes e_{k+1})), \\ \qquad k = 2, 4, 6, l = 2, 3, \ldots 7 \end{array} \right\}_{58},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} (G_{k,l} - G_{k+1,l+1} + i(G_{k,l+1} + G_{k+1,l}), 0, 0), \\ \qquad k = 0, 2, \ldots, 12, l = k+2, k+4, \ldots, 14, k, l \neq 8 \\ (G_{k,8} - G_{k+1,9} - i(G_{k,9} + G_{k+1,8}), 0, 0), \\ \qquad k = 0, 2, \ldots, 14, k \neq 8 \end{array} \right\}_{28},$$

$$\mathfrak{g}_{-3} = \left\{ \begin{array}{l} (0, (e_0 \otimes e_0 - e_1 \otimes e_1) + i(e_0 \otimes e_1 + e_1 \otimes e_0), 0), \\ (0, (e_0 \otimes e_k - e_0 \otimes e_{k+1}) - i(e_0 \otimes e_{k+1} + e_1 \otimes e_k), 0), \\ \qquad k = 2, 4, 6, \\ (0, 0, (e_0 \otimes e_0 - e_1 \otimes e_1) + i(e_0 \otimes e_1 + e_1 \otimes e_0)), \\ (0, 0, (e_k \otimes e_0 + e_{k+1} \otimes e_1) - i(e_{k+1} \otimes e_0 - e_k \otimes e_1)), \\ \qquad k = 2, 4, 6, \end{array} \right\}_{8},$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1}), \qquad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2}), \qquad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3}).$$

Proof
Noting that

$$\begin{aligned} \mu\big(i(G_{01} + G_{23} + G_{45} + G_{67})\big) &= 2iG_{01}, \\ \mu\big(i(-G_{01} + G_{23} + G_{45} + G_{67})\big) &= i(G_{01} - G_{23} - G_{45} - G_{67}), \\ \nu\big(i(G_{01} + G_{23} + G_{45} + G_{67})\big) &= i(G_{01} - G_{23} - G_{45} - G_{67}), \\ \nu\big(i(-G_{01} + G_{23} + G_{45} + G_{67})\big) &= 2iG_{01}, \end{aligned}$$

we can prove this theorem by direct calculations. □

We define a C-linear transformation ε of ${\mathfrak{e}_8}^C$ by

$$\varepsilon(D, x \otimes y, z \otimes u) = (D, -x \otimes y, -z \otimes u).$$

Then $\varepsilon \in {E_8}^C$ and $\varepsilon^2 = 1$.

Now, for the characteristic element $Z = (\mathrm{diag}(iJ, iJ, iJ, iJ, -iJ, iJ, iJ, iJ), 0, 0)$, we have the following proposition.

PROPOSITION 5.7.3
We have

$$\exp\Big(\frac{2\pi i}{2} \operatorname{ad} Z\Big) = \varepsilon.$$

Proof
Since Z is a central element of $(\mathfrak{so}(16,C),0,0)$, the action of $\exp(\pi i\,\mathrm{ad}\,Z)$ on $(\mathfrak{so}(16,C),0,0)$ is trivial. Next,

$$\begin{aligned} i\,\mathrm{ad}\,Z(x\otimes y,0) &= \big(-\mu(G_{01}+G_{23}+G_{45}+G_{67})x\otimes y \\ &\qquad -x\times\mu(G_{01}+G_{23}+G_{45}+G_{67})y,0\big) \\ &= \big(-2G_{01}x\otimes y - x\otimes(G_{01}-G_{23}-G_{45}-G_{67})y,0\big) \\ &= \big(\mathrm{diag}(2J,0,0,0)x\otimes y + x\otimes \mathrm{diag}(J,-J,-J,-J)y,0\big). \end{aligned}$$

Hence, for $t\in\boldsymbol{R}$, we have

$$\begin{aligned} &\big(\exp(ti\,\mathrm{ad}\,Z)\big)(x\otimes y,0) \\ &\quad= \big(\mathrm{diag}(R(2t),E,E,E)x\otimes \mathrm{diag}(R(t),R(-t),R(-t),R(-t))y,0\big), \end{aligned}$$

where $R(t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$. Setting $t=\pi$, we have

$$\begin{aligned} &\big(\exp(\pi i\,\mathrm{ad}\,Z)\big)(x\otimes y,0) \\ &\quad= \big(\mathrm{diag}(E,E,E,E)x\otimes \mathrm{diag}(-E,-E,-E,-E)y,0\big) \\ &\quad= \big(x\otimes(-y),0\big) = (-x\otimes y,0). \end{aligned}$$

Similarly, we obtain

$$\big(\exp(\pi i\,\mathrm{ad}\,Z)\big)(0,z\otimes u) = \big(0,z\otimes(-u)\big) = (0,-z\otimes u).$$

Thus we have

$$\begin{aligned} &\big(\exp(\pi i\,\mathrm{ad}\,Z)\big)(D,x\otimes y,z\otimes u) \\ &\quad= (D,-x\otimes y,-z\otimes u) = \varepsilon(D,x\otimes y,z\otimes u), \end{aligned}$$

that is, $\exp((2\pi i/2)\,\mathrm{ad}\,Z)=\varepsilon$. □

Set $z_2=\exp((2\pi i/2)\,\mathrm{ad}\,Z)=\varepsilon$. Then since $({\mathfrak{e}_8}^C)_{ev}=({\mathfrak{e}_8}^C)^{z_2}=({\mathfrak{e}_8}^C)^{\varepsilon}$, we determine the structure of the group

$$({E_8}^C)_{ev}=({E_8}^C)^{z_2}=({E_8}^C)^{\varepsilon}.$$

THEOREM 5.7.4
We have

$$({E_8}^C)_{ev}\cong Ss(16,C).$$

Proof
We define a mapping $\varphi_{ev}: Spin(16,C)\to({E_8}^C)^{\varepsilon}=({E_8}^C)_0$ by

$$\varphi_{ev}(\widetilde{\alpha})(D,P)=\big(p(\widetilde{\alpha})Dp(\widetilde{\alpha})^{-1},\rho(\widetilde{\alpha})P\big).$$

Since $\varphi_{ev}(-1)=\varepsilon$, for $\widetilde{\alpha}\in Spin(16,C)$ we have $\varphi_{ev}(\widetilde{\alpha})\varepsilon=\varphi_{ev}(\widetilde{\alpha})\varphi_{ev}(-1)=\varphi_{ev}(\widetilde{\alpha}(-1))=\varphi_{ev}((-1)\widetilde{\alpha})=\varphi_{ev}(-1)\varphi_{ev}(\widetilde{\alpha})=\varepsilon\varphi_{ev}(\widetilde{\alpha})$, that is, $\varphi(\widetilde{\alpha})\in({E_8}^C)^{\varepsilon}$. Hence φ_{ev} is well defined. Since $({E_8}^C)^{\varepsilon}$ is connected and $\dim_C(({\mathfrak{e}_8}^C)^{\varepsilon})=$

$\dim_C(({\mathfrak{e}_8}^C)_{ev}) = 64 + 28 \times 2$ (see Theorem 5.7.2) $= 120 = \dim_C(\mathfrak{spin}(16, C))$, $\operatorname{Ker}\varphi_{ev}$ is discrete, so $\operatorname{Ker}\varphi_{ev}$ is contained in the center of $Spin(16, C)$: $\operatorname{Ker}\varphi_{ev} \subset z(Spin(16, C)) = \{1, -1, \widetilde{\zeta}, -\widetilde{\zeta}\}$. However

$$\varphi_{ev}(1) = \varphi_{ev}(\widetilde{\zeta}) = 1 \qquad \text{and} \qquad \varphi_{ev}(-1) = \varphi_{ev}(-\widetilde{\zeta}) = \varepsilon,$$

so $\operatorname{Ker}\varphi = \{1, \widetilde{\zeta}\}$. Again, since $({E_8}^C)^\varepsilon$ is connected and $\dim_C(({\mathfrak{e}_8}^C)^\varepsilon) = \dim_C(\mathfrak{spin}(16, C))$, φ is surjective. Thus we have the isomorphism $({E_8}^C)_{ev} = ({E_8}^C)^\varepsilon \cong Spin(16, C)/\{1, \widetilde{\zeta}\} = Ss(16, C)$. $\square$

Next, we define the semispinor group $Ss(8, 8)$. Let $I_8 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$, $E \in M(8, C)$, and define $Spin(8, 8)$ by

$$Spin(8, 8) = \bigl\{\widetilde{\alpha} \in Spin(16, C) \bigm| (\tau I_8)\widetilde{\alpha} = \widetilde{\alpha}\bigr\},$$

where τ is the complex conjugation in $Cl((\mathfrak{C}^C)^2)$. Then $Spin(8, 8)$ is a connected (but not simply connected) group, and $Ss(8, 8)$ is defined by

$$Ss(8, 8) = Spin(8, 8)/\{1, \widetilde{\zeta}\},$$

which is a double-covering group of the identity-connected component group $SO(8, 8)^0$ of $SO(8, 8) = \{A \in SO(16, C) \mid \tau(I_8 A I_8) = A\}$.

We define C-linear transformations ε_1 and ε_2 of ${\mathfrak{e}_8}^C$ by

$$\varepsilon_1(D, x \otimes y, z \otimes u) = (I_8 D I_8, -x \otimes y, z \otimes u),$$

$$\varepsilon_2(D, x \otimes y, z \otimes u) = (I_8 D I_8, x \otimes y, -z \otimes u).$$

Then $\varepsilon_1, \varepsilon_2 \in {E_8}^C, {\varepsilon_1}^2 = {\varepsilon_2}^2 = 1$, and $\varepsilon, \varepsilon_1, \varepsilon_2$ commute with each other.

We find an $\boldsymbol{R}$-Lie algebra of type $E_{8(8)}$. We define an $\boldsymbol{R}$-Lie algebra ${\mathfrak{e}_8}'$ by

$${\mathfrak{e}_8}' = \mathfrak{so}(8, 8) \oplus (i\mathfrak{C} \otimes \mathfrak{C}) \oplus (\mathfrak{C} \otimes \mathfrak{C}) = ({\mathfrak{e}_8}^C)^{\tau\varepsilon_1}$$

with the Lie bracket the same as that of ${\mathfrak{e}_8}^C$.

LEMMA 5.7.5 (GOMYO [2, Proposition 3.5])

The Killing form B_8 of the Lie algebra ${\mathfrak{e}_8}^C$ is given by

$$\begin{aligned} &B_8\bigl((D_1, (x_1 \otimes y_1, z_1 \otimes u_1)), (D_2, (x_2 \otimes y_2, z_2 \otimes u_2))\bigr) \\ &\qquad = 30\operatorname{tr}(D_1 D_2) - 60\bigl((x_1, x_2)(y_1, y_2) + (z_1, z_2)(u_1, u_2)\bigr). \end{aligned}$$

PROPOSITION 5.7.6

We have that ${\mathfrak{e}_8}'$ is an $\boldsymbol{R}$-Lie algebra of type $E_{8(8)}$.

Proof

We find the signature of the Killing form ${B_8}' = B_8 \mid {\mathfrak{e}_8}'$ of ${\mathfrak{e}_8}'$. Decompose ${\mathfrak{e}_8}'$ into eigenspaces relative to τ:

$${\mathfrak{e}_8}' = ({\mathfrak{e}_8}')_\tau \oplus ({\mathfrak{e}_8}')_{-\tau},$$

$$\begin{aligned}(\mathfrak{e}_8{}')_\tau &= \{R \in \mathfrak{e}_8{}' \mid \tau R = R\}\\ &= \big\{(D,0,Q) \bigm| D \in \mathfrak{so}(8,8), \tau D = D, Q \in \mathfrak{C} \otimes \mathfrak{C}\big\},\end{aligned}$$

$$\begin{aligned}(\mathfrak{e}_8{}')_{-\tau} &= \{R \in \mathfrak{e}_8{}' \mid \tau R = -R\}\\ &= \big\{(D,iP,0) \bigm| D \in \mathfrak{so}(8,8), \tau D = -D, P \in \mathfrak{C} \otimes \mathfrak{C}\big\}.\end{aligned}$$

Then, from Lemma 5.7.5, we see that the Killing form $B_8{}'$ on $(\mathfrak{e}_8{}')_\tau$ is positive definite and $B_8{}'$ on $(\mathfrak{e}_8{}')_{-\tau}$ is negative definite. Therefore the number of positive eigenvalues of $B_8{}'$ is $\dim((\mathfrak{e}_8{}')_\tau) = 54 + 64 = 120$, and the number of negative eigenvalues of $B_8{}'$ is $\dim((\mathfrak{e}_8{}')_{-\tau}) = 64+64 = 128$. Therefore the signature of $B_8{}'$ is $128 - 120 = 8$. Hence the type of $B_8{}'$ is $E_{8(8)}$. □

Let $E_8{}'$ be the automorphism group of $\mathfrak{e}_8{}'$:

$$E_8{}' = \big\{\alpha \in \mathrm{Iso}_R(\mathfrak{e}_8{}') \bigm| \alpha[R_1,R_2] = [\alpha R_1, \alpha R_2]\big\}.$$

Although we cannot give any explicit isomorphism between $E_8{}'$ and $E_{8(8)}$ of Section 5.1, hereafter we denote $\mathfrak{e}_8{}'$ by $\mathfrak{e}_{8(8)}$ and $E_8{}'$ by $E_{8(8)}$.

PROPOSITION 5.7.7

The involution $\tau\varepsilon_1$ leaves $(\mathfrak{e}_8{}^C)_{ev}$ invariant.

Proof

We can easily check that $(\mathfrak{e}_8{}^C)_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ of Theorem 5.7.2 is left invariant under the action of $\tau\varepsilon_1$ of $\mathfrak{so}(16,C)$. So we have this proposition. □

From Proposition 5.7.7, we have $(\mathfrak{e}_{8(8)})_{ev} = (\mathfrak{e}_8{}^C)_{ev} \cap (\mathfrak{e}_8{}^C)^{\tau\varepsilon_1} = (\mathfrak{e}_8{}^C)^{\varepsilon} \cap (\mathfrak{e}_8{}^C)^{\tau\varepsilon_1}$. So we determine the structure of the group

$$(E_{8(8)})_{ev} = (E_8{}^C)_{ev} \cap (E_8{}^C)^{\tau\varepsilon_1} = (E_8{}^C)^{\varepsilon} \cap (E_8{}^C)^{\tau\varepsilon_1}.$$

THEOREM 5.7.8

We have

$$(E_{8(8)})_{ev} \cong Ss(8,8) \times \{1, J\varepsilon_2\}.$$

Proof

For $\alpha \in (E_{8(8)})_{ev} \subset (E_8{}^C)_{ev} = (E_8{}^C)^{\varepsilon}$, there exists $\widetilde{\alpha} \in Spin(16,C)$ such that $\alpha = \varphi_{ev}(\widetilde{\alpha})$ (see Theorem 5.7.5). From the condition $\tau\varepsilon_1\alpha\varepsilon_1\tau = \alpha$, that is, $\tau\varepsilon_1\varphi_{ev}(\widetilde{\alpha})\varepsilon_1\tau = \varphi_{ev}(\widetilde{\alpha})$, we have $\varphi_{ev}(\tau(I_8\widetilde{\alpha})) = \varphi_{ev}(\widetilde{\alpha})$. Hence

$$\text{(i) } (\tau I_8)\widetilde{\alpha} = \widetilde{\alpha} \qquad \text{or} \qquad \text{(ii) } (\tau I_8)\widetilde{\alpha} = \widetilde{\zeta}\widetilde{\alpha}.$$

Case (i). From the condition $(\tau I_8)\widetilde{\alpha} = \widetilde{\alpha}$, we have $\widetilde{\alpha} \in Spin(8,8)$.

Case (ii). We easily obtain that $\widetilde{\alpha} = \widetilde{j}$ satisfies condition (ii), where

$$\widetilde{j} = \begin{pmatrix} \frac{1}{\sqrt{2}}e_0 \\ \frac{1}{\sqrt{2}}e_0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e_1 \\ \frac{1}{\sqrt{2}}e_1 \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{\sqrt{2}}e_7 \\ \frac{1}{\sqrt{2}}e_7 \end{pmatrix} \in Spin(16,C).$$

Here we define a transformation J of ${\mathfrak{e}_8}^C$ by

$$J(D, x \otimes y, z \otimes u) = (J_8 D {J_8}^{-1}, y \otimes x, u \otimes z),$$

where $J_8 = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, $E \in M(8, C)$. Then we have $\varphi_{ev}(\widetilde{j}) = J\varepsilon_2$.
Thus we have the isomorphism $(E_{8(8)})_0 = (({E_8}^C)^{\tau\varepsilon_1})^{\varepsilon} \cong Ss(8,8) \cup J\varepsilon_2(Ss(8, 8)) = Ss(8,8) \times \{1, J\varepsilon_2\}$. □

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References

[1] S. Gomyo, *Realization of maximal subgroups of rank 8 of simply connected compact simple Lie groups of type* E_8, Tsukuba J. Math. **21** (1997), 595–616.

[2] ———, *Realizations of subgroups of type* D_8 *of connected exceptional simple Lie groups of type* E_8, Tsukuba J. Math. **23** (1999), 585–615.

[3] M. Hara, *Real semisimple graded Lie algebras of the third kind* (in Japanese), master's thesis, Shinshu University, Nagano, Japan, 2000.

[4] T. Miyashita and I. Yokota, 2-*graded decompositions of exceptional Lie algebras* $\mathfrak{g}$ *and group realizations of* $\mathfrak{g}_{ev}, \mathfrak{g}_0$*, III:* $G = E_8$, Japan. J. Math. (N.S.) **26** (2000), 31–50.

[5] ———, 3-*graded decompositions of exceptional Lie algebras* $\mathfrak{g}$ *and group realizations of* $\mathfrak{g}_{ev}, \mathfrak{g}_0$ *and* $\mathfrak{g}_{ed}$*, II:* $G = E_7$*, case 1*, J. Math. Kyoto Univ. **46** (2006), 383–413; *cases 2, 3, and 4* **46** (2006), 805–832; *case 5*, **47** (2007), 121–128.

[6] I. Yokota, *Realization of involutive automorphisms* σ *and* G^σ *of exceptional linear Lie groups G, II:* $G = E_7$, Tsukuba J. Math. **14** (1990), 379–404.

[7] ———, 2-*graded decompositions of exceptional Lie algebras* $\mathfrak{g}$ *and group realizations of* $\mathfrak{g}_{ev}, \mathfrak{g}_0$*, I:* $G = G_2, F_4, E_6$, Japan. J. Math. (N.S.) **24** (1998), 257–296; *II:* $G = E_7$, **25** (1999), 154–179.

[8] ———, 3-*graded decompositions of exceptional Lie algebras* $\mathfrak{g}$ *and group realizations of* $\mathfrak{g}_{ev}, \mathfrak{g}_0$ *and* $\mathfrak{g}_{ed}$*, I:* $G = G_2, F_4, E_6$, J. Math. Kyoto Univ. **41** (2001), 449–475.

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