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Exponential decay of correlations for surface semiflows with an expanding direction

By

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Abstract

Dolgopyat [4] showed that a class of Axiom A flows has exponential decay of correlations for smooth observables, and Baladi-Vallée [2] gave a nice interpretation of it on suspension semiflows of one-dimensional expanding countable Markov maps. Avila-Gouëzel-Yoccoz [1] extends the result of Baladi-Vallée to higher dimensional systems.

In this paper we show that a class of non-Markov semiflows also has exponential decay of correlations.

We prove that such exponential decay can be shown on an open dense condition for the suspensions of piecewise expanding maps.

1. Introduction

1.1. Definitions

Let $(X, \mathcal{B})$ be a measurable space, $T : X \to X$ be a map on X, and μ be a T-invariant probability measure. For real-valued functions ϕ, ψ on X, the correlation function $\rho_{\phi,\psi}(n)$ is defined as

$$\rho_{\phi,\psi}(n) = \int \phi \cdot \psi \circ T^n d\mu - \int \phi d\mu \int \psi d\mu.$$

The functions ϕ, ψ are called observables.

It is well known that the asymptotic behavior of the correlation function indicates chaotic or statistical properties of the iteration of T. In particular, a dynamical system given by T is mixing if $\rho_{\phi,\psi}(n)$ converges to zero as $n \to \infty$. The correlation function is said to enjoy exponential decay if there is a constant α in $(0, 1)$, and for each ϕ, ψ there is $C(\phi, \psi)$ so that for all $n \in \mathbb{Z}_+$

$$|\rho_{\phi,\psi}(n)| \leq C(\phi, \psi)\alpha^n$$

holds.

A correlation function is also defined for a continuous-time dynamical system. Let F_t be a measurable flow or semiflow on X (where t is the time variable), and μ be an F_t-invariant measure. When ϕ, ψ are real-valued functions

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on X, the correlation function $\rho_{\phi,\psi}(t)$ is defined as

$$\rho_{\phi,\psi}(t) = \int \phi \cdot \psi \circ F_t d\mu - \int \phi d\mu \int \psi d\mu.$$

The concepts of mixing and exponential decay of correlations are also defined for continuous-time dynamical systems analogously to the discrete-time systems.

The study of statistical properties of continuous-time dynamical systems is more difficult and less well understood than discrete-time dynamical systems, because continuous-time dynamical systems have at least one dimensional neutral direction, which is the flow direction.

The main purpose of this paper is to give a sufficient condition of the exponential decay of correlations for suspension semiflows over piecewise expanding maps without finite or countable Markov partitions. Here the suspension semiflow is a special class of continuous-time dynamical systems which is defined as follows. Let $T : X \to X$ be a measurable map and μ be a T-invariant measure. Let $r : X \to \mathbb{R}_+$ be a positive function with $\inf r > 0$ and assume $\int r d\mu < \infty$. Then, we define a new space X^r as $\{(x, s) \in X \times \mathbb{R}_+ \mid 0 \leq s \leq r(x)\}/\sim$, where the equivalence relation $\sim$ is generated by $(x, r(x)) \sim (T(x), 0)$, and on the space X^r the semiflow F_t is defined as $F_t(x, s) = (x, s + t)/\sim$. We can naturally define an F_t-invariant probability measure μ^r as $\mu \times \mathrm{Leb}/\int r d\mu$, where Leb is the Lebesgue measure on $\mathbb{R}$. This measure μ^r is frequently used for suspension semiflows because it has a good relation to ergodic properties of (T, μ). For example, if (T, μ) is ergodic, then so is (F_t, μ^r). The study of suspension semiflows is important since we can understand many continuous-time dynamical systems as suspension semiflows of the Poincaré maps of the dynamical systems. Meanwhile, it should be noted that if r is constant, (F_t, μ^r) cannot be mixing even if (T, μ) is mixing.

The so-called "cocycle condition", which is given in the next subsection, is often used for verifying the decay of correlations. It is, however, not easy to use the cocycle condition in the case of non-Markov expanding semiflows. In this paper, we introduce a new equivalent condition called the backward orbit condition and use it to study the correlation decay of such semiflows.

1.2. Decay of correlation for countable Markov flows

In a paper [3], Chernov proved the stretched exponential decay of correlations for geodesic flows on a Riemannian surface with non-constant negative curvature. A condition called the Uniform Non-Integrability (UNI) plays an important role in his paper. Later Dolgopyat [4] proved the exponential decay of correlations for a larger class of flows using UNI and the Perron-Frobenius operator. Pollicott [6] brought this method into the study of suspension semiflows on surfaces with finite Markov partitions, and Baladi and Vallée [2] extended it to those with countable Markov partitions. Avila, Gouëzel and Yoccoz [1] extended it to higher dimensional semiflows with Markov property and proved that the UNI condition is equivalent to the cocycle condition. Tsujii [7] proved the similar theorem by a different method.

Here, we recall the result of Avila, Gouëzel and Yoccoz [1]. In later sec-

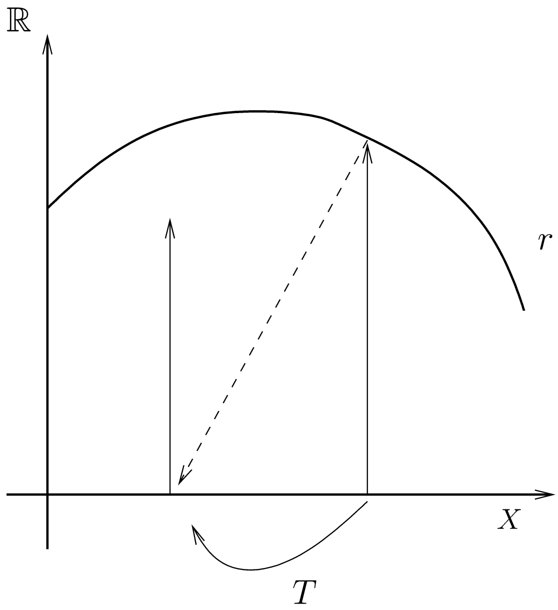

Figure 1. X^r and F_t

tions, we shall use their result to obtain a new result which is applicable to a different class of maps. Their result is about suspension semiflows of maps on n-dimensional manifolds, but we only use it for maps on an interval.

Let $0 < \cdots < c_i < c_{i+1} < \cdots < 1$ be a finite or countable partition of $I = [0,1]$ and $T : I \to I$ be a C^2 uniformly expanding map with respect to this partition, i.e. $T|_{(c_i,c_{i+1})}$ is C^2 for each i, $T|_{(c_i,c_{i+1})}$ extends to a homeomorphism from $[c_i, c_{i+1}]$ onto I, and there are $C \geq 1$ and $\rho < 1$ so that $|h(x)-h(y)| \leq C\rho^n$ for every inverse branch h of T^n. Let r be a piecewise C^1 roof function, i.e. $r|_{(c_i,c_{i+1})}$ is C^r on each interval (c_i, c_{i+1}) and $\inf r > 0$. Let $\mathcal{H}_n$ be the set of inverse branches of T^n and $\mathcal{H}$ denotes $\mathcal{H}_1$. Let F_t be a suspension semiflow given by T and r. For $n \geq 1$, write $r^{(n)} = r + r \circ T + \cdots + r \circ T^{n-1}$.

We consider the following four conditions.

Condition 1 There is a constant $\hat{K} \geq 0$ such that the inequality $|h''| \leq \hat{K}|h'|$ holds for every $h \in \mathcal{H}$.

Condition 2 There is a negative constant σ_0 such that

$$\sum_{h\in\mathcal{H}} \sup \exp(-\sigma(r \circ h))|h'| < \infty$$

holds for any $\sigma > \sigma_0$.

Condition 3 $|r' \circ h||h'| \leq \hat{K}$ for any $h \in \mathcal{H}$.

Condition 4 (T, r) satisfies the cocycle condition, i.e. there is no $u : I \to \mathbb{R}$ and $K : I \to \mathbb{R}$ satisfying $r = u \circ T - u + K$, where u is C^1 and K is constant on each interval (c_i, c_{i+1}).

Theorem A (Avila, Gouëzel and Yoccoz). *If T and r satisfy the above conditions, then there is a unique T-invariant probability measure μ which is absolutely continuous with respect to the Lebesgue measure. The measure μ satisfies $\int r d\mu < \infty$. Moreover the correlation function for the suspension semiflow F_t with respect to the induced measure μ^r decays exponentially fast for any C^1 observables.*

1.3. Statements of result

The following two theorems are the main results of this paper.

1.3.1. An equivalent condition to the cocycle condition Our first result is to give an equivalent condition to the cocycle condition, which we call the backward orbit condition.

Let T be a C^2 uniformly expanding map with respect to a finite or countable partition $0 < \cdots < c_i < c_{i+1} < \cdots < 1$. Let $r : I \to \mathbb{R}_+$ be a piecewise C^1 roof function.

We say (T, r) satisfies *the backward orbit condition* if there exist two backward orbits $\{a_n\}_{n=-\infty}^{0}$ and $\{b_n\}_{n=-\infty}^{0}$ of x_0, namely, $x_0 = a_0 = b_0$, $T(a_n) = a_{n+1}$ and $T(b_n) = b_{n+1}$, such that the inequality

$$\sum_{k=0}^{\infty} \frac{r'(b_{-k})}{(T^k)'(b_{-k})} \neq \sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})} \tag{1.1}$$

holds.

Theorem 1.1. *If T and r satisfy the conditions* 1, 2, 3 *in Theorem* A, *then the cocycle condition is equivalent to the backward orbit condition.*

We consider this theorem useful, because the backward orbit condition is often easier to apply to suspension semiflows than the cocycle condition, as is seen in the proof of Theorem 1.2.

1.3.2. About non-Markov system Our second result is about non-Markov systems. The main idea lies in the fact that we can find a Markov structure in a (non-Markov) piecewise uniformly expanding map, and hence in the suspension semiflows. This idea is inspired by the tower dynamics [8].

Let $0 = c_0 < \cdots < c_N = 1$ be a finite partition of I. Let $T : I \to I$ be a piecewise C^2 map and $r : I \to \mathbb{R}_+$ be a piecewise C^1 map, i.e. $T|_{(c_i, c_{i+1})}$ is C^2 and $r|_{(c_i, c_{i+1})}$ is C^1 for each i. Assume $1 < \beta \leq |T'| \leq \gamma$, $\inf r > 0$, $\sup r < \infty$, and $\sup |r'| < \infty$. Let $\mathcal{H}$ be the collection of $(T|_{[c_i, c_{i+1}]})^{-1}$.

We consider the following structure. Assume there are a subinterval of I, say $\hat{I} = [\hat{a}, \hat{b}]$, a finite or countable partition of $\hat{I}$ given by $\hat{a} < \cdots < d_i < d_{i+1} < \cdots < \hat{b}$, and a function $R : \hat{I} \to \mathbb{Z}_+$ such that $R|_{(d_i, d_{i+1})}$ is constant for each i. Let $R_i = R|_{(d_i,d_{i+1})}$. Define $T^R(x) = T^{R(x)}(x)$ and assume T^R is a C^2 uniformly expanding map with respect to the partition of $\hat{I}$. $(T, \{(d_i, d_{i+1})\}_i, R)$ is called a tower.

We consider a backward orbit of T^R, $\{a_k^R\}_{k=-\infty}^0$, i.e. $T^R(a_k^R) = a_{k+1}^R$ for every $k < 0$. From the definition of a tower, there are an integer sequence $\{p_k\}_{k=-\infty}^0$ and a backward orbit of T, $\{a_n\}_{n=-\infty}^0$, such that

$$a_0^R = a_0$$

and

$$\begin{aligned} a_k^R &\in [d_{p_k}, d_{p_k+1}] \\ a_k^R &= a_{-(R_{p_0}+\cdots+R_{p_{k-1}})} \end{aligned} \tag{1.2}$$

hold for every k. We call $\{a_n\}_{n=-\infty}^0$ a backward orbit corresponding to $\{a_k^R\}_{k=-\infty}^0$.

Here we also define the backward orbit condition for the tower. We say T and r satisfy the backward orbit condition for the tower if there are a point $x_0 \in \hat{I}$, two backward orbits of T^R, $\{\hat{a}_n\}$ and $\{\hat{b}_n\}$ of x_0, and their corresponding backward orbits $\{a_n\}_{n=-\infty}^0$ and $\{b_n\}_{n=-\infty}^0$ such that

$$\sum_{k=0}^{\infty} \frac{r'(b_{-k})}{(T^k)'(b_{-k})} \neq \sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})} \tag{1.3}$$

holds.

Theorem 1.2. *For a piecewise C^2 expanding map T, a piecewise C^1 roof function r, and a tower $(T, \{(d_i, d_{i+1})\}_i, R)$, assume the following three conditions:*

Condition A *There is a constant K such that $|h''| \le K|h'|$ holds for any $h \in \mathcal{H}$.*

Condition B *There is $\beta' < \beta$ such that $\sum_i \beta'^{-R_i} < +\infty$ holds.*

Condition C *T and r satisfy the backward orbit condition.*

Then there is a T-invariant probability measure μ on I which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Moreover the correlation function for the suspension semiflow F_t given by (T, r) with μ^r decays exponentially fast for any C^1 observables.

Note that the condition B in the above theorem is satisfied if there is a constant $A > 0$ such that $Ai < R_i$ for every i.

The result in this paper has an advantage over the previous existing results in that we can apply it to the non-Markov (or hidden Markov) semiflows. The cocycle condition is a good characterization of exponential decay of correlations on Markov semiflows, but we cannot directly interpret the cocycle condition on the semiflow with tower structures. On the other hand, it is easy to consider the backward orbit condition on the semiflow with tower structure as well as Markov semiflows. Indeed, in the next subsection, we will give a sufficient conditions for exponential decay of correlation for the suspension semiflows over the *beta*-transformations as an application of the backward orbit condition to non-Markov semiflows.

1.4. Applications

The backward orbit condition is an open dense condition, and hence exponential decay of correlations is generically observed among the suspension semiflows for C^2 expanding maps. Theorem 1.2 extends this fact to non-Markov dynamical systems.

As an example, we consider the β-transformation. Let T be the β-transformation for some $\beta \in (1,2)$, i.e. T is a map from I to I given by

$$T(x) = \begin{cases} \beta x & (0 \le x < \frac{1}{\beta}) \\ \beta x - 1 & (\frac{1}{\beta} \le x \le 1), \end{cases}$$

and $r : I \to \mathbb{R}_+$ be a C^2 function with $\inf r > 0$ and $\sup r < \infty$. Let F_t be the suspension semiflow given by T and r. We can easily find a tower structure in the β-transformation (see Appendix for more details).

Theorem 1.3. *Let F_t be the suspension semiflow given by the β-transformation. If the roof function r is convex, then the correlation functions for F_t with C^1 observables decay exponentially fast.*

We have to show the existence of two backward orbits in Theorem 1.2, which are in fact chosen to be the two backward orbits of 0, namely, one of which is 0, 0, 0, ..., and the other is 0, $\frac{1}{\beta}, \ldots$. Construction of the tower structure and its properties are discussed in Appendix.

Another application is about genericity of roof functions. The next theorem shows that there are a lot of expanding semiflows with exponential decay of correlations.

Theorem 1.4. *Let $T : I \to I$ be the β-transformation for some $\beta \in (1,2)$, and $\mathcal{T}$ be the set of all real-valued C^1 functions on I with its infimum is greater than 0.*

Then there exists an open dense subset $\mathcal{S}$ in $\mathcal{T}$ with C^1 norm such that for all $r \in \mathcal{S}$ the correlation functions for the suspension semiflows F_t^r given by T and r decay exponentially fast for C^1 observables on I^r.

We can easily show that the backward orbit condition in Theorem 1.2 is an open dense condition from which Theorem 1.4 follows.

2. The backward orbit condition

In this section, we will prove the following lemma from which Theorem 1.1 immediately follows.

Lemma 2.1. *The following two statements are equivalent under the conditions* 1, 2, 3 *of Theorem* A.

(i) *There are a* C^1 *function* $u : I \to \mathbb{R}$ *and a piecewise constant function* $w : I \to \mathbb{R}$ *which satisfies* $r = u \circ T - u + w$.

(ii) *For every* $a_0 \in I$, $\sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})}$ *is constant for all backward orbits* $\{a_n\}_{n=-\infty}^{0}$ *of* a_0.

Proof. It is easy to show (ii) from (i), since $r' = u' \circ T \cdot T' - u'$ implies

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})} &= \sum_{k=0}^{\infty} \left(\frac{u'(Ta_{-k})T'(a_{-k})}{(T^k)'(a_{-k})} - \frac{u'(a_{-k})}{(T^k)'(a_{-k})} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{u'(Ta_{-k})T'(a_{-k})}{(T^{k-1})'(Ta_{-k})T'(a_{-k})} - \frac{u'(a_{-k})}{(T^k)'(a_{-k})} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{u'(a_{-k+1})}{(T^{k-1})'(a_{-k+1})} - \frac{u'(a_{-k})}{(T^k)'(a_{-k})} \right) \\ &= \frac{u'(T(a_0))}{(T)'(T(a_0))},\end{aligned} \tag{2.1}$$

and the last quantity depends only on a_0, independent of choices of backward orbits.

To show the converse, define a function $v : I \to \mathbb{R}$ as follows. For every $a \in I$, choose one backward orbit of a, $\{a_n\}$, and define $v(a)$ as $v(a) = \sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})}$, which is independent of choices of backward orbits due to (ii). Therefore this function is well-defined, and satisfies $r' = v \circ T \cdot T' - v$ from the similar calculation of (2.1). Therefore, take u as the primitive function of v which satisfies (i). $\square$

3. Piecewise expanding maps and suspension semiflows

In this section we will prove Theorem 1.2.

3.1. Exponential decay of correlation for T^R and r^R

Let $r^R(x) = r^{(R(x))}(x)$. Let $\mathcal{H}^R$ be the set of inverse branches of T^R, $\mathcal{H}^R_n$ be the set of inverse branches of $(T^R)^n$, and $\hat{h}_i = (T^{R_i}|_{(c_i,c_{i+1})})^{-1}$. It is obvious that $\hat{h}_i = h_{j_{R_i-1}} \circ \cdots \circ h_{j_0}$, where $h_{j_{R_i-1}}, \ldots, h_{j_0}$ are the elements of $\mathcal{H}$. Let $\hat{F}_t$ be the suspension semiflow given by T^R and r^R.

First we will prove that T^R and r^R satisfy the conditions of Theorem A. Obviously T^R is uniformly expanding.

Let $h_{i_n} \circ \cdots \circ h_{i_0}$ be an element of $\mathcal{H}^R$. The condition 1 is verified by proving the followinglemma.

Lemma 3.1. *The inequality*

$$|(h_{i_n} \circ \cdots \circ h_{i_0})''(x)| \leq \frac{K}{1-\beta^{-1}} |(h_{i_n} \circ \cdots \circ h_{i_0})'(x)|$$

holds.

Proof. Let f and g be C^2, then

$$\begin{aligned} (f \circ g)'' &= (f' \circ g \cdot g')' \\ &= f'' \circ g \cdot (g')^2 + f' \circ g \cdot g'' \end{aligned} \tag{3.1}$$

To prove the lemma, the above equality is used repeatedly.

$$\begin{aligned}
&|(h_{i_n} \circ \cdots \circ h_{i_0})''(x)| \\
\leq\ &|h''_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x) \cdot ((h_{i_{n-1}} \circ \cdots \circ h_{i_0})'(x))^2| \\
&+ |h'_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x) \cdot (h_{i_{n-1}} \circ \cdots \circ h_{i_0})''(x)| \\
&\text{(from (3.1) and the triangle inequality)} \\
\leq\ &K\beta^{-(n-1)}|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)| \\
&+ |h'_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x)| \cdot |(h_{i_{n-1}} \circ \cdots \circ h_{i_0})''(x)| \\
&\text{(from the condition A and the Leibniz rule)} \\
\leq\ &K\beta^{-(n-1)}|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)| \\
&+ |h'_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x)| \cdot \\
&\big(|h''_{i_{n-1}} \circ h_{i_{n-2}} \circ \cdots \circ h_{i_0}(x) \cdot ((h_{i_{n-2}} \circ \cdots \circ h_{i_0})'(x))^2| \\
&+ |h'_{i_{n-1}} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x) \cdot (h_{i_{n-2}} \circ \cdots \circ h_{i_0})''(x)|\big) \\
&\text{(from (3.1) and the triangle inequality again)} \\
\leq\ &K\beta^{-(n-1)}|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)| \\
&+ K\beta^{-(n-2)}|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)| \\
&+ |h'_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x)| \cdot |h'_{i_{n-1}} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_0}(x)| \\
&\cdot |(h_{i_{n-2}} \circ \cdots \circ h_{i_0})''(x)| \\
\leq\ &\quad \vdots \\
\leq\ &K(\beta^{-(n-1)} + \cdots + 1)|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)| \\
\leq\ &\frac{K}{1-\beta^{-1}}|(h_{i_n} \circ \cdots \circ h_{i_0})'(x)|.
\end{aligned}$$

This is the desired inequality. $\square$

For the condition 2, let $\sigma_0 = \frac{1}{\sup r}(\log \beta' - \log \beta)$. If $\sigma_0 < \sigma$, we have

$$\begin{aligned}
&\sum_i \sup((\exp(-\sigma \cdot r^R \circ \hat{h}_i))|\hat{h}_i'|) \\
\leq &\sum_i (\exp(-\sigma_0 \cdot \sup_{I_i} r^R))|\hat{h}_i'| \\
\leq &\sum_i (\exp(-\sigma_0 \cdot \sup r \cdot R_i))\beta^{-R_i} \\
\leq &\sum_i (\beta^{-1}\exp(-\sigma_0 \cdot \sup r))^{R_i} \\
= &\sum_i \beta'^{-R_i} < +\infty,
\end{aligned}$$

which proves the condition 2.

The condition 3 is proved as

$$\begin{aligned}
|(r^R \circ \hat{h}_i)'| &= |(r^{(R_i)} \circ \hat{h}_i)'| \\
&= |((r + r\circ T + \cdots + r \circ T^{R_i - 1}) \circ \hat{h}_i)'| \\
&\leq \sup |r'|(\beta^{-R_i} + \cdots + \beta^{-1}) \\
&\leq \sup |r'| \cdot \frac{\beta^{-1}}{1-\beta^{-1}}.
\end{aligned}$$

Finally, we will prove the condition 4. Observe that an element of $\mathcal{H}_n^R$ can be written as $\hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0}$. Therefore, $(r^R)^{(n)} \circ h$ is expressed as follows:

$$\begin{aligned}
&(r^R)^{(n)} \circ h \\
= &(r^R)^{(n)} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} \\
= &(r^R + r^R \circ T^R + \cdots + r^R \circ (T^R)^{n-1}) \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} \\
= &r^R \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} + \cdots + r^R \circ \hat{h}_{i_0} \\
= &r^R|_{\mathrm{Im}\hat{h}_{i_{n-1}}} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} + \cdots + r^R|_{\mathrm{Im}\hat{h}_{i_0}} \circ \hat{h}_{i_0} \\
= &r^{(R_{i_{n-1}})} \circ \hat{h}_{i_{n-1}} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} + \cdots + r^{(R_{i_0})}\hat{h}_{i_0} \circ \hat{h}_{i_0} \\
= &(r + \cdots + r \circ T^{R_{i_{n-1}}-1}) \circ \hat{h}_{i_{n-1}} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} \\
&+ (r + \cdots + r \circ T^{R_{i_{n-2}}-1}) \circ (T^{R_{i_{n-1}}} \circ \hat{h}_{i_{n-1}}) \circ \hat{h}_{i_{n-2}} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_0} \\
&\quad \vdots \\
&+ (r + \cdots + r \circ T^{R_{i_0}-1}) \circ (T^{R_{i_{n-1}}+\cdots+R_{i_1}} \circ \hat{h}_{i_{n-1}} \circ \cdots \circ \hat{h}_{i_1}) \circ \hat{h}_{i_0} \\
= &(r + \cdots + r \circ T^{R_{i_{n-1}}+\cdots+R_{i_0}}) \circ h \\
= &r^{(R_{i_{n-1}}+\cdots+R_{i_0})} \circ h.
\end{aligned} \tag{3.2}$$

We will choose $\{p_k\}_{k=0}^{\infty}$ inductively as

$$\begin{aligned} &a_0 \in [d_{p_0}, d_{p_0+1}] \\ &a_{-(R_{p_0}+\cdots+R_{p_k})} \in [d_{p_{k+1}}, d_{p_{k+1}+1}]. \end{aligned} \tag{3.3}$$

Then we have

$$\begin{aligned} a_{-R_{p_0}} &= \hat{h}_{p_0}(a_0), \\ a_{-(R_{p_0}+\cdots+R_{p_k})} &= \hat{h}_{p_k}(a_{-R_{p_{k-1}}}), \end{aligned} \tag{3.4}$$

so that the sequence $a_0, a_{-R_{p_0}}, a_{-R_{p_0}-R_{p_1}}, \ldots$ is a backward orbit of T^R. Take $\{q_k\}_{k=0}^{\infty}$ as

$$a_{-k-1} = h_{q_k} \circ \cdots \circ h_{q_0}(x_0), \tag{3.5}$$

then

$$\begin{aligned} \hat{h}_{p_0} &= h_{q_{R_{p_0}}} \circ \cdots \circ h_{q_0} \\ \hat{h}_{p_k} &= h_{q_{R_{p_k}+\cdot+R_{p_0}}} \circ \cdots \circ h_{q_{R_{p_{k-1}}+\cdots+q_{R_{p_0}}+1}}, \end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(r^R)'(a_{-R_{p_0}-\cdots R_{p_k}})}{((T^R)^k)'(a_{-R_{p_0}-\cdots R_{p_k}})} &= \lim_{n\to\infty} ((r^R)^{(n)} \circ \hat{h}_{p_{n-1}} \circ \cdots \circ \hat{h}_{p_0})'(x_0) \\ &= \sum_{k=0}^{\infty} (r \circ h_{q_k} \circ \cdots \circ h_{q_0})'(x_0) \\ &= \sum_{k=0}^{\infty} \frac{r'(a_{-k})}{(T^k)'(a_{-k})}, \end{aligned}$$

from (3.2), (3.4), (3.5) and (3.6). This proves the condition 4.

As a result, T^R and r^R satisfy the assumption of Theorem A, and hence there exists a T^R-invariant probability measure $\hat{\mu}$ which is absolutely continuous with respect to the Lebesgue measure on $\hat{I}$, and the correlation function for $\hat{F}_t$ and $\hat{\mu}^{r^R}$ decays exponentially fast for C^1 observables on $\hat{I}$.

3.2. The relationship between the suspension of the tower and the suspension of the non-Markov map

In this subsection, we will show the existence of an F_t-invariant probability measure μ^r and that the correlation function for F_t and μ^r decays exponentially fast from the result of the previous subsection. Similar idea is also used in [5].

The existence of a T^R-invariant probability measure $\hat{\mu}$, which is absolutely continuous with respect to the Lebesgue measure, is proved in the previous subsection. Uniqueness of the invariant probability measure which is absolutely continuous with respect to the Lebesgue measure and $\sup \frac{d\hat{\mu}}{d\mathrm{Leb}} < \infty$ are shown in [8].

We consider the following system $\bar{T} : \Delta \to \Delta$ and $\bar{r} : \Delta \to \mathbb{R}_+$, which may be thought of as an intermediate step from (T, r) to (T^R, r^R).

$$\Delta = \{(x, n) \in \hat{I} \times \mathbb{Z} \mid 0 \le n < R(x)\},$$
$$\bar{T}(x, n) = \begin{cases} (x, n+1) & (n \le R(x) - 2) \\ (T^{R(x)}x, 0) & (n = R(x) - 1), \end{cases}$$
$$\bar{r}(x, n) = r(T^n x).$$

Let $\bar{F}_t$ be the suspension semiflow of $\bar{T}$ and $\bar{r}$. Define $\pi : \Delta \to I$, $\bar{\pi} : \Delta^{\bar{r}} \to I^r$,

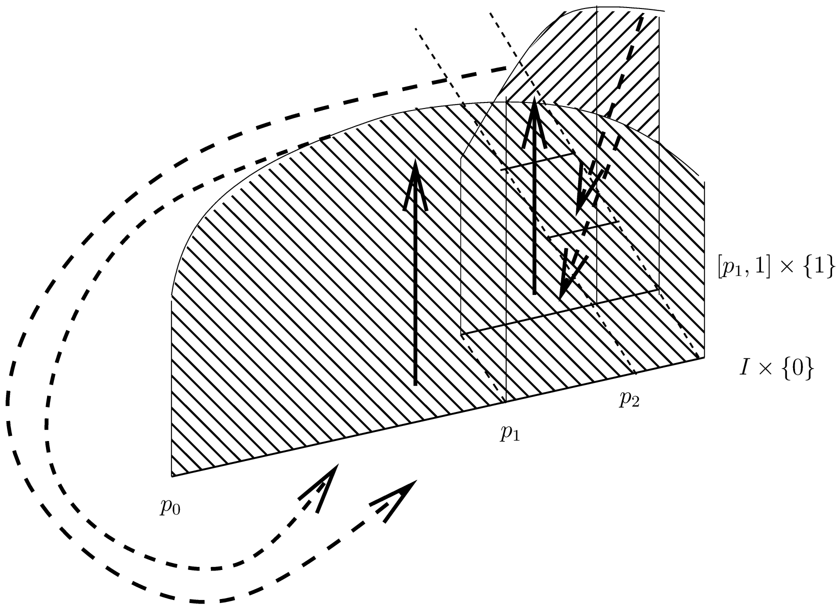

Figure 2. $\Delta^{\bar{r}}$ and $\bar{F}_t$

and $\hat{\pi} : \hat{I}^{r^R} \to \Delta^{\bar{r}}$ as

$$\pi(x, n) = T^n x,$$
$$\bar{\pi}(x, n, t) = (T^n x, t),$$
$$\hat{\pi}(x, t) = (x, n, t - r^{(n)}(x)) \qquad (r^{(n)}(x) \le t < r^{(n+1)}(x)),$$

then we have

$$\pi \circ \bar{T} = T \circ \pi, \tag{3.7}$$
$$\bar{\pi} \circ \bar{F}_t = F_t \circ \bar{\pi}, \tag{3.8}$$
$$\hat{\pi} \circ \hat{F}_t = \bar{F}_t \circ \hat{\pi}, \tag{3.9}$$

and $\hat{\pi}$ is a diffeomorphism from $\hat{I}^{r^R}$ to $\Delta^{\bar{r}}$. In other words, $\hat{F}_t$ is conjugate to $\bar{F}_t$ and $\bar{F}_t$ is semiconjugate to F_t.

Next we will define some invariant measures and show the relation of $\hat{\mu}$ to such measures. We define a measure $\bar{\mu}_0$ on Δ as

$$\bar{\mu}_0|_{[d_n,d_{n+1}]\times\{k\}}(A\times\{k\}) = \hat{\mu}(A)$$

on $[d_n, d_{n+1}]\times\{k\}$ for all n and k. The measure $\bar{\mu}_0$ is $\bar{T}$-invariant because in the case $k\geq 1$,

$$\bar{\mu}_0(\bar{T}^{-1}A\times\{k\}) = \bar{\mu}_0(A\times\{k-1\}) = \hat{\mu}(A) = \bar{\mu}_0(A\times\{k\})$$

and in the case $k=0$,

$$\begin{aligned}
\bar{\mu}_0(\bar{T}^{-1}A\times\{0\}) &= \bar{\mu}_0(\sqcup_n (T^{-R_n}A)\cap I_n\times\{R_n-1\}) \\
&= \sum_n \bar{\mu}((T^{-R_n}A)\cap[d_n,d_{n+1}]\times\{R_n-1\}) \\
&= \sum_n \hat{\mu}((T^{-R_n}A)\cap[d_n,d_{n+1}]) \qquad (3.10)\\
&= \hat{\mu}(\sqcup_n (T^{-R_n}A)\cap[d_n,d_{n+1}]) \\
&= \hat{\mu}((T^R)^{-1}A) = \hat{\mu}(A) = \bar{\mu}_0(A\times\{0\}).
\end{aligned}$$

Since $\sup\frac{d\hat{\mu}}{dLeb}<\infty$, $\bar{\mu}_0(\Delta)<\infty$ holds and hence we can define a normalized measure of $\bar{\mu}_0$, which is denoted by $\bar{\mu}$. Let μ be $\pi_*\bar{\mu}$. Because $\bar{T}$ is semiconjugate to T, this measure is a T-invariant probability measure. We can prove

$$\bar{\pi}_*\bar{\mu}^{\bar{r}} = \mu^r,$$

similarly to the proof of (3.10). We can also show

$$\hat{\pi}_*\hat{\mu}^{r^R} = \bar{\mu}^{\bar{r}} \qquad (3.11)$$

because $\hat{\pi}$ is a diffeomorphism and $|D\hat{\pi}|=1$.

Letting $\hat{\phi} := \phi\circ\bar{\pi}\circ\hat{\pi}$ and $\hat{\psi} := \psi\circ\bar{\pi}\circ\hat{\pi}$, we can verify that the correlation functions for F_t with observable ϕ and ψ are equal to the correlation functions for $\hat{F}_t$ with observable $\hat{\phi}$ and $\hat{\psi}$ respectively, since

$$\begin{aligned}
\int \phi\cdot(\psi\circ F_s)\,d\mu^r &= \int \phi\cdot(\psi\circ F_s)\,d(\bar{\pi}\circ\hat{\pi})_*\mu^r \\
&= \int (\phi\circ\bar{\pi}\circ\hat{\pi})\cdot(\psi\circ F_s\circ\bar{\pi}\circ\hat{\pi})\,d\hat{\mu}^{r^R} \\
&= \int \hat{\phi}\cdot(\psi\circ\bar{\pi}\circ\hat{\pi}\circ\hat{F}_s)\,d\hat{\mu}^{r^R} \\
&= \int \hat{\phi}\cdot(\hat{\psi}\circ\hat{F}_s)\,d\hat{\mu}^{r^R}.
\end{aligned}$$

From this equality and the result of subsection 3.1, we conclude that the correlation functions for F_t and μ^r decay exponentially fast.

Appendix

Here, we construct the tower structure of the β-transformation.

Assume $T^n(1) \neq \frac{1}{\beta}$ for every n. We define $\{R_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ inductively as follows. Let $p = \frac{1}{\beta}$, $d_0 = 0$, $d_1 = p$, and $R_0 = 1$. Take $R_{i+1} > R_i$ so as to satisfy

$$T^{R_i}(1) < p, \ldots, T^{R_{i+1}-2}(1) < p, T^{R_{i+1}-1}(1) > p.$$

Since T is piecewise monotone, there is a unique $d_{i+2} > d_{i+1}$ that satisfies

$$T^{R_i}(d_{i+2}) < p, \ldots, T^{R_{i+1}-2}(d_{i+2}) < p, T^{R_{i+1}-1}(d_{i+2}) = p.$$

The partition $\{[d_i, d_{i+i}]\}_i$ and the sequence $\{R_i\}_i$ define the tower from the following lemma.

Lemma 3.2. $d_i \to 1$ *as* $i \to \infty$.

Proof. It suffices to show $1 - d_{i+1} \leq d_{i+1} - d_i$.

From the definition of R_i and d_i,

$$T^{R_{i-1}}|_{(d_i,1]}(x) = \beta^{R_{i-1}}(x - d_i)$$

holds.

Since

$$T^{R_{i-1}}(1) < p, \ldots, T^{R_i-2}(1) < p, T^{R_i-1}(1) > p,$$

we have

$$T^{R_i}|_{(d_i,1]}(x) = \begin{cases} \beta^{R_i}(x - d_i) & (x < d_{i+1}) \\ \beta^{R_i}(x - d_{i+1}) & (x > d_{i+1}) \end{cases}$$

for each i. We also have

$$\lim_{x \to d_{i+1}-0} T^{R_i}(x) = 1.$$

since $T^{R_i-1}(d_{i+1}) = p$.

Therefore we have

$$\beta^{R_i}(d_{i+1} - d_i) = 1$$
$$T^{R_i}(1) = \beta^{R_i}(1 - d_{i+1}) < 1.$$

These two estimations lead $1 - d_{i+1} \leq d_{i+1} - d_i$. □

Note that $R_{i+1} > R_i$ immediately implies $R_i \geq i$ for every i and this is sufficient for the condition B in Theorem 1.2.

If there exists a positive integer n satisfying $T^n(1) = \frac{1}{\beta}$, we can find a finite partition of I as above and we can construct the tower more easily.

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