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On parabolic geometry, II

By

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Abstract

Let G be a simple linear algebraic group defined over $\mathbb{C}$ and P a parabolic subgroup of it. Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P over a smooth complex projective variety M. We prove that (M, E_P, ω) is holomorphically isomorphic to the standard parabolic geometry $(G/P, G, \omega_0)$ whenever M is rationally connected. We then show that this is indeed the case if M has Picard number one and contains a (possibly singular) rational curve. This last result is a generalization of the main result of [3], where we dealt with the case $G = \mathrm{PGL}(d, \mathbb{C}), G/P = \mathbb{P}^{d-1}_{\mathbb{C}}$.

1. Introduction

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$ and $P \subset G$ a parabolic subgroup. A parabolic geometry of type G/P is roughly a manifold equipped with an infinitesimal model of G/P (we recall the definition of a parabolic geometry in Section 2 below). Our aim here is to investigate holomorphic parabolic geometries such that the underlying complex manifold admits a nonconstant holomorphic map from $\mathbb{CP}^1$. We first recall the main result of [3].

Let G be $\mathrm{PGL}(d, \mathbb{C})$ and P_0 a parabolic subgroup of it such that $G/P_0 = \mathbb{P}^{d-1}_{\mathbb{C}}$. Then we prove in [3] the following:

Theorem 1.1. *Let (M, E_{P_0}, ω) be a holomorphic parabolic geometry of type $\mathrm{PGL}(d, \mathbb{C})/P_0$ such that M is a smooth complex projective variety. If there is a nonconstant holomorphic map $\mathbb{CP}^1 \longrightarrow M$, then M is biholomorphic to the projective space $\mathbb{CP}^{d-1}$.*

In this article we consider general parabolic geometries and prove the following proposition (see Proposition 4.1):

Proposition 1.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P on a smooth complex projective variety M. If M has Picard number*

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one and contains a (possibly singular) rational curve, then (M, E_P, ω) is the tautological parabolic geometry of type G/P, meaning $M = G/P$, $E_P = G$ and ω is the Maurer–Cartan form on G.

Proposition 1.1 is derived using the following main theorem proved here (Theorem 3.1).

Theorem 1.2. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P such that M is a rationally connected smooth projective variety. Then (M, E_P, ω) is the tautological parabolic geometry of type G/P.*

2. Parabolic geometry and the tangent bundle

Let G be a simple linear algebraic group defined over $\mathbb{C}$, and let $\mathfrak{g}$ denote the Lie algebra of G (by G simple we mean that G is connected and $\mathfrak{g}$ is simple). Fix a proper parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p}$. Let M be a connected complex manifold, and let $\psi : E_P \longrightarrow M$ be a holomorphic principal P–bundle (so P acts freely on the right of E_P with M being the quotient). For each point $g \in P$, let

$$\tau_g : E_P \longrightarrow E_P, \quad z \longmapsto \tau_g(z) = zg \tag{2.1}$$

denote the translation map. Given any $v \in \mathfrak{p}$, the exponential map defines the one–parameter subgroup $\{\exp(tv)\}_{t\in\mathbb{C}} \subset P$ and induces a vector field $\zeta_v \in H^0(E_P, \Theta_{E_P})$ given by

$$\zeta_{v,z}(f) = \left.\frac{df(z\exp(tv))}{dt}\right|_{t=0}, \quad z \in E_P, f \in \mathcal{O}_{E_P,z}.$$

A *holomorphic Cartan connection* on E_P is a $\mathfrak{g}$–valued holomorphic one–form

$$\omega \in H^0(E_P, \Omega^1_{E_P} \otimes_{\mathbb{C}} \mathfrak{g}) = \mathrm{Hom}(\Theta_{E_P}, \mathcal{O}_{E_P} \otimes_{\mathbb{C}} \mathfrak{g})$$

which satisfies the following three conditions:

- $\omega : \Theta_{E_P} \longrightarrow \mathcal{O}_{E_P} \otimes_{\mathbb{C}} \mathfrak{g}$ is an isomorphism,
- $\omega(\zeta_{v,z}) = v$, for $v \in \mathfrak{p}$ and $z \in E_P$, where ζ_v is the vector field defined above, and
- $\tau_g^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for $g \in P$, meaning if we write $\omega = \sum_i \omega_i \otimes v_i$, where $\omega_i \in \Omega_{E_P}$ and $v_i \in \mathfrak{g}$, then $\tau_g^*\omega = \sum_i \omega_i \otimes \mathrm{Ad}(g^{-1})(v_i)$.

See [5, p. 99–100], [11, p. 184].

On the standard principal P–bundle $E_P = G \longrightarrow G/P$, there is a standard Cartan connection ω_0. In fact, we have a natural isomorphism $\Theta_G = \mathcal{O}_G \otimes_{\mathbb{C}} \mathfrak{g}$ given by the left–invariant vector fields on G. Then define $\omega_0 : \Theta_G \longrightarrow \mathcal{O}_G \otimes_{\mathbb{C}} \mathfrak{g}$ to be this isomorphism (ω_0 is also called the *Maurer–Cartan form*). It is easy to check that ω_0 satisfies the three conditions above.

We call $(G/P, G, \omega_0)$ the *tautological parabolic geometry.*

A general parabolic geometry of type G/P is defined as follows.

Definition 2.1. A *holomorphic parabolic geometry* of type G/P is a triple (M, E_P, ω), where
- M is a connected complex manifold,
- E_P is a holomorphic principal P–bundle over M, and
- ω is a Cartan connection on E_P.

Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P.

Consider the right action of G on $E_P \times G$ given by $(z,h)g = (zg, g^{-1}h)$ and the associated quotient

$$E_G := (E_P \times G)/P. \tag{2.2}$$

The projection $E_P \longrightarrow M$ gives a projection $E_G \longrightarrow M$, and the right–translation action of G on itself defines an action of G on E_G, thus making $E_G \longrightarrow M$ a holomorphic principal G–bundle. We have an inclusion map

$$E_P \hookrightarrow E_G \tag{2.3}$$

that sends any z to the equivalence class of (z, e), where e is the identity element of G.

We recall that the $\mathfrak{g}$–valued one–form ω defines a holomorphic connection on the principal G–bundle E_G [11, p. 365, Proposition 3.1]; see [1] for holomorphic connections. This holomorphic connection, which we will denote by D^ω, is constructed as follows.

Let $\omega_0 : \Theta_G \longrightarrow G \times \mathfrak{g}$ be the Maurer–Cartan form defined earlier. Consider the $\mathfrak{g}$–valued holomorphic one–form $\widetilde{\omega} := p_1^*\omega + p_2^*\omega_0$ on $E_P \times G$, where p_1 (respectively, p_2) is the projection of $E_P \times G$ to E_P (respectively, G). This form $\widetilde{\omega}$ descends to a $\mathfrak{g}$–valued holomorphic one–form on the quotient space E_G in (2.2). The conditions on ω ensure that this descended one–form defines a holomorphic connection on the principal G–bundle E_G. The above mentioned holomorphic connection D^ω is defined to be the one given by this descended one–form.

Let

$$\mathrm{ad}(E_G) = (E_G \times \mathfrak{g})/G \longrightarrow M \tag{2.4}$$

be the adjoint vector bundle of the principal G–bundle E_G in (2.2). We recall that two points (z_1, v_1) and (z_2, v_2) of $E_G \times \mathfrak{g}$ are identified in the quotient space $\mathrm{ad}(E_G)$ if and only if there is an element $g_0 \in G$ such that $(z_1, v_1) = (z_2 g_0, \mathrm{Ad}(g_0^{-1})(v_2))$. Similarly, let

$$\mathrm{ad}(E_P) = (E_P \times \mathfrak{p})/P \longrightarrow M \tag{2.5}$$

be the adjoint bundle of the principal P–bundle E_P (as before, $\mathfrak{p}$ is the Lie algebra of P).

The inclusion of $\mathfrak{p}$ in $\mathfrak{g}$ and the inclusion map in (2.3) together define an inclusion map $\mathrm{ad}(E_P) \hookrightarrow \mathrm{ad}(E_G)$. The isomorphism $\omega : \Theta_{E_P} \longrightarrow \mathcal{O}_{E_P} \otimes_{\mathbb{C}} \mathfrak{g}$ yields a holomorphic isomorphism of vector bundles

$$\beta : \mathrm{ad}(E_G)/\mathrm{ad}(E_P) \longrightarrow \Theta_M. \tag{2.6}$$

A holomorphic connection on a principal bundle induces holomorphic connections on all the associated fiber bundles. Let $\widetilde{D}^\omega$ be the holomorphic connection on $\mathrm{ad}(E_G)$ induced by the holomorphic connection D^ω on the principal G–bundle E_G.

Since Θ_M is a quotient of $\mathrm{ad}(E_G)$ (see (2.6)), we have the following

Proposition 2.1. *The holomorphic tangent bundle Θ_M is a quotient bundle of a vector bundle admitting a holomorphic connection. More precisely, Θ_M is a quotient of the vector bundle $\mathrm{ad}(E_G)$ equipped with the holomorphic connection $\widetilde{D}^\omega$.*

3. Rationally connected varieties

A smooth complex projective variety Z is said to be rationally connected if for any two given points of Z, there exists an irreducible rational curve on Z that contains them; see [8, p. 433, Theorem 2.1] and [8, p. 434, Definition–Remark 2.2] for other equivalent conditions. Note that a rationally connected projective manifold is connected.

Theorem 3.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P such that M is rationally connected. Then $M \cong G/P$, and (M, E_P, ω) is holomorphically isomorphic to the tautological parabolic geometry $(G/P, G, \omega_0)$.*

Proof. Consider the holomorphic connection D^ω on the principal G–bundle E_G (see (2.2)) constructed in Section 2. If the curvature of the connection D^ω vanishes (i.e., if D^ω is flat), then we have the developing map from the universal cover $\widetilde{M}$ of M

$$M \xleftarrow{\gamma} \widetilde{M} \xrightarrow{\delta} G/P$$

such that

- the developing map δ is a local biholomorphism, and
- the pair $(\gamma^* E_P, \gamma^*\omega)$ coincide with the pullback by δ of the tautological principal P–bundle $G \longrightarrow G/P$ equipped with the tautological Cartan connection.

Since M is rationally connected, it follows that M is simply connected [4, p. 545, Theorem 3.5], [7, p. 362, Proposition 2.3]. Therefore, to prove the theorem it suffices to show that the connection D^ω on E_G is flat. Indeed, if D^ω is flat, then the developing map δ gives the required isomorphism.

In [2] it was shown that any holomorphic connection on a rationally connected variety is flat (see [2, p. 160, Theorem 3.1]). In particular, D^ω is flat. This completes the proof of the theorem. $\square$

Since a complex Fano manifold is rationally connected (see [9, p. 766, Theorem 0.1]), Theorem 3.1 has the following corollary:

Corollary 3.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P. If M is a complex Fano manifold, then*

$$M \cong G/P,$$

and (M, E_P, ω) is holomorphically isomorphic to the tautological parabolic geometry $(G/P, G, \omega_0)$.

4. Parabolic geometry and rational curves

The Picard number of a complex projective manifold Z is the rank of the Néron–Severi group

$$\mathrm{NS}(Z) := (H^2(Z, \mathbb{Z})/\mathrm{Torsion}) \cap H^{1,1}(Z) \subset H^2(Z, \mathbb{C}).$$

Proposition 4.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type G/P on a smooth complex projective variety M. If M has Picard number one and contains at least one (possibly singular) rational curve, then $M \cong G/P$, and (M, E_P, ω) is holomorphically isomorphic to the tautological parabolic geometry $(G/P, G, \omega_0)$.*

Proof. Let

$$\phi : \mathbb{CP}^1 \longrightarrow M$$

be a nonconstant holomorphic morphism. Consider the pulled back vector bundle

$$\phi^* \mathrm{ad}(E_G) \longrightarrow \mathbb{CP}^1 \tag{4.1}$$

equipped with the holomorphic connection $\phi^* \widetilde{D}^\omega$ (see Proposition 2.1).

We note that a holomorphic vector bundle V over $\mathbb{CP}^1$ admits a holomorphic connection if and only if V is holomorphically trivial (any holomorphic connection on a Riemann surface is automatically flat, and $\mathbb{CP}^1$ is simply connected). In particular, $\phi^* \mathrm{ad}(E_G)$ in (4.1) is a trivial vector bundle. Therefore, any quotient bundle of $\phi^* \mathrm{ad}(E_G)$ is generated by its global sections.

From Proposition 2.1 it follows that the pullback $\phi^* \Theta_M$ is a quotient of $\phi^* \mathrm{ad}(E_G)$. Therefore, $\phi^* \Theta_M$ is generated by its global sections. A theorem due to Grothendieck says that any holomorphic vector bundle over $\mathbb{CP}^1$ splits into a direct sum of holomorphic line bundles [6, p. 126, Théorème 2.1]. Let

$$\phi^* \Theta_M = \bigoplus_{i=1}^{d_0} \xi_i \tag{4.2}$$

be a decomposition into a direct sum of holomorphic line bundles, where $d_0 = \dim_{\mathbb{C}} M$. Since $\phi^* \Theta_M$ is generated by its global sections, it follows that

$$\mathrm{degree}(\xi_i) \geq 0 \tag{4.3}$$

for all $i \in [1, d_0]$.

On the other hand, since ϕ is a nonconstant map, the differential of the map ϕ

$$d\phi : \Theta_{\mathbb{CP}^1} \longrightarrow \phi^* \Theta_M$$

is a nonzero homomorphism. There is no nonzero holomorphic homomorphism from $\Theta_{\mathbb{CP}^1}$ to the trivial line bundle over $\mathbb{CP}^1$, hence not all ξ_i can be trivial. Therefore, from (4.3) and (4.2) we conclude that degree$(\phi^*\Theta_M) > 0$.

In view of the above inequality, from the assumption that the Picard number of M is one it follows that the anti–canonical line bundle $\bigwedge^{d_0}\Theta_M \longrightarrow M$ is ample. Now Corollary 3.1 completes the proof of the proposition. $\square$

We will give an example of parabolic geometry.

Let $\Gamma \subset G$ be a discrete subgroup and $U \subset G/P$ an analytic open subset such that the following three conditions hold:

- the left–translation action of Γ on G/P leaves U invariant,
- the action of Γ on U is free, and
- $\Gamma\backslash U$ is compact.

So $\Gamma\backslash U$ is a compact complex manifold. Let $\widetilde{U} \subset G$ be the inverse image of U. Consider the parabolic geometry $(U, \widetilde{U}, \omega_0|_{\widetilde{U}})$ of type G/P obtained by restricting the tautological parabolic geometry $(G/P, G, \omega_0)$. The $\mathfrak{g}$–valued form $\omega_0|_{\widetilde{U}}$ descends to $\Gamma\backslash\widetilde{U}$, and the triple $(\Gamma\backslash U, \Gamma\backslash\widetilde{U}, \omega_0')$, where ω_0' is the descended $\mathfrak{g}$–valued one–form on $\Gamma\backslash\widetilde{U}$, is a holomorphic parabolic geometry of type G/P on the compact complex manifold $\Gamma\backslash U$.

McKay proved that if (M, E_P, ω) is a holomorphic parabolic geometry of type G/P on a compact Kähler manifold M with $c_1(M) = 0$, then M is covered by a torus [10, Theorem 1].

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